OPTIMAL PROJECTIVE SPECTRA

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Abstract. We give another proof of the consistency of $\aleph_1 = \mathfrak{a} < \mathfrak{s} = \mathfrak{c} = \aleph_2$ by showing Shelah's original forcing to obtain the inequality satisfies a stronger combinatorial property than preserving the unboundedness of the ground model reals. Namely we show the forcing and its iterations preserve tight mad families, and use this result to furthermore show the above inequality is compatible with a Δ^1_3 wellorder of the reals in a model with $\mathfrak{c} = \aleph_2$, the witness to $\mathfrak{a} = \aleph_1$ is coanalytic, and there exists a Π^1_2 tight mad family of size $\mathfrak{c}$. In each of these cases the projective complexity is minimal.

1. Introduction

The purpose of this paper is to reveal a combinatorial property of a rather well-known proper forcing notion introduced by Shelah in 1984 (see Definition 18), designed to give the consistency of $\aleph_1 = \mathfrak{b} < \mathfrak{s} = \aleph_2$, thereby establishing the consistency of $\mathfrak{b}$ and $\mathfrak{s}$. Namely we show this partial order satisfies an iterable preservation property (see Definition 3) introduced by Guzman, Hrušák, and Tellez, which guarantees the preservation of tight mad families under countable support iterations.

Proposition (See Proposition 29)**.** Let $\mathbb{Q}$ be the forcing notion of Definition 18, and let $A \in V$ be a tight mad family. Let G be $\mathbb{Q}$-generic over V. Then $(\mathcal{A} \text{ is a tight mad family})^{V[G]}$.

The notion $\mathbb{Q}$ of [She84, Definition 6.8] is the first instance of a so-called creature forcing, which now refers to a broad class of forcing notions (see [RS99]); it adds a generic real unsplit by the ground model reals in a similar manner as Mathias forcing, though unlike Mathias forcing $\mathbb{Q}$ is almost ω^ω-bounding. This latter fact implies that in a countable support iteration of $\mathbb{Q}$, the ground model reals remain an unbounded family, thus providing a witness to $\mathfrak{b} = \aleph_1$, while the former fact implies that in an ω_2-length countable support iteration of $\mathbb{Q}$ we have $\mathfrak{s} \geq \aleph_2$. Since ZFC proves $\mathfrak{b} \leq \mathfrak{a}$ and countable support iterations of $\mathbb{Q}$ preserve a witness to $\mathfrak{a} = \aleph_1$, the fact $(\mathfrak{b} = \aleph_1)^{V[G]}$ is a consequence of the ZFC inequality and the above preservation theorem. This gives an alternative proof of the consistency of $\mathfrak{b} = \aleph_1 < \mathfrak{s} = \aleph_2$ first established in [She84, Theorem 3.1].

This will be applied to give the following result at the intersection of descriptive set theory and set theory of the reals. Using a countable support iteration of S-proper forcing notions we show:

2020 *Mathematics Subject Classification.* 03E35, 03E17, 03E55.

Key words and phrases. projective well-orders; tight mad families; finite logarithmic measures.

The authors would like to thank the Austrian Science Fund FWF for the generous support through grant number START Y1012-N35.

Theorem (Theorem 86)**.** It is consistent with $\aleph_1 = \mathfrak{a} < \mathfrak{s} = \mathfrak{c} = \aleph_2$ that there exists a Δ^1_3 wellorder of the reals, a coanalytic tight mad family of size $\aleph_1$, and a Π^1_2 tight mad family of size $\aleph_2$.

By classical theorems of Mansfield, Mathias [Mat77], and Mansfield-Solovay, respectively, these projective definitions are of minimal descriptive complexity, i.e. they are optimal. Theorem 86 will be proved through a series of intermediary steps, beginning with the presentation of the construction of a Δ^1_3 wellorder of the reals of Fischer and Friedman [FF10]. This construction gave the first instance of a model with a Δ^1_3 wellorder and a nontrivial theory of cardinal characteristics, and improved Harrington's result [Har77] on the consistency of a $\boldsymbol{\Delta^1_3}$ wellorder together with $\mathfrak{c}$ arbitrarily large and Martin's Axiom (MA).

Theorem (Theorem 63)**.** It is consistent with $\aleph_1 = \mathfrak{a} < \mathfrak{s} = \mathfrak{c} = \aleph_2$ that there exists a Δ^1_3-definable wellorder of the reals and moreover $\mathfrak{a} = \aleph_1$ is witnessed by a Π^1_1 tight mad family.

In the recent literature there is interest in the projective definability of witnesses to cardinal characteristics; the relevant body of work includes, for example, [BFB22], [FFZ11], [FS21], [FFK13], [BK13], [FFST25] or [FSS25]. We will extend this line of inquiry by asking about the projective definability of witnesses for all cardinals κ belonging to the set

$$\text{spec}(\mathfrak{a}) = \{|\mathcal{A}| \mid \mathcal{A} \subseteq [\omega]^\omega, \mathcal{A} \text{ is mad}\},$$

analagously to Hechler's [Hec72] generalization of the study of the cardinal $\mathfrak{a} = \min(\text{spec}(\mathfrak{a}))$. Towards this end we consider an S-proper countable support iteration given Friedman and Zdomskyy [FZ10] (see Definition 70), whose purpose is to show the consistency of $\mathfrak{b} = \mathfrak{c} = \aleph_2$ with a Π^1_2-definable tight mad family. We show the iterand of their construction responsible for adding the Π^1_2 tight mad family of size $\mathfrak{c}$ strongly preserves ground model tight mad families (Proposition 80), and thereby obtain:

Theorem (Theorem 85)**.** It is consistent that $\mathfrak{a} = \aleph_1 < \mathfrak{c} = \aleph_2$, and there exists a Π^1_1-definable tight mad family of size $\aleph_1$, and a Π^1_2-definable tight mad family of size $\aleph_2$.

By a result of Raghavan [Rag09] and the Mansfield-Solovay theorem, this is the best possible projective definability of a tight mad family of size strictly greater than $\aleph_1$.

In Section 2 we introduce the forcing notion $\mathbb{Q}$, the notion of a tight mad family and strong preservation of tightness, and prove Proposition 29. In Section 3 we define the countable support iteration of Fischer and Friedman for adding a Δ^1_3 wellorder, and prove Theorem 63. The following Section 4 gives the definition of the Friedman-Zdomskyy forcing, shows it preserves tight mad families, and proves Theorem 85. In Section 5 we show how these results are weaved together to yield Theorem 86. The last section includes remarks and open questions.

2. Creature Forcing

Define the relation $\leq^*$ of *eventual domination* on the collection ω^ω by letting $f \leq^* g$ if and only if there exists $n \in \omega$ such that for all $m \geq n$, $f(m) \leq g(m)$. Define the relation *splits* on the collection $[\omega]^\omega$ by letting a split b if and only if both $b \cap a$ and $b \setminus a$ are infinite. In 1984 Shelah answered a question of Nyikos by showing that it is consistent that the minimal size of a family $\mathcal{F} \subseteq \omega^\omega$ which is unbounded with respect to $\leq^*$ can be strictly less than the minimal size of a splitting family $\mathcal{S} \subseteq [\omega]^\omega$, that is, a family $\mathcal{S}$ with the property that for any infinite $b \subseteq \omega$ there is $a \in \mathcal{S}$ which splits b. We denote with $\mathfrak{b}$ and $\mathfrak{s}$ these respective cardinalities:

$$\mathfrak{b} = \min\{|\mathcal{F}| \mid \mathcal{F} \subseteq \omega^\omega \text{ is } \leq^* \text{-unbounded}\},$$
$$\mathfrak{s} = \min\{|\mathcal{S}| \mid \mathcal{S} \subseteq [\omega]^\omega \text{ is splitting}\}.$$

That $\mathfrak{s} < \mathfrak{b}$ is consistent was already shown in 1980 by Balcar and Simon [BPS80, Remark 4.7], during an investigation of topological properties of the space of all ultrafilters on ω. Shelah [She84, Section 4] also shows that $\mathfrak{s} < \mathfrak{b}$ also holds in a generic extension by a finite support iteration of Hechler forcing of length κ, for $\kappa \geq \omega_2$ a regular cardinal.

ZFC proves the following inequalities,

$$\begin{array}{ccccc} \mathfrak{s} & \longrightarrow & \mathfrak{d} & \longrightarrow & \mathfrak{c} \\ \uparrow & & \uparrow & & \uparrow \\ \aleph_1 & \longrightarrow & \mathfrak{b} & \longrightarrow & \mathfrak{a} \end{array}$$

where an arrow between the cardinals represents the relation "less than or equal to". The cardinal $\mathfrak{d}$, the dual of $\mathfrak{b}$, is the minimal size of a *dominating family*, that is, a family $\mathcal{F} \subseteq \omega^\omega$ such that for all $f \in \omega^\omega$ there is $g \in \mathcal{F}$ such that $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. Hechler forcing is the standard forcing notion for increasing $\mathfrak{d}$, however it also increases the size of $\mathfrak{b}$. A family $\mathcal{A} \subseteq [\omega]^\omega$ is *almost disjoint* if $a \cap b$ is finite for every $a, b \in \mathcal{A}$, and such a familly is *maximal almost disjoint (mad)* if $\mathcal{A}$ is maximal with respect to inclusion among all almost disjoint families; equivalently, for every $b \in [\omega]^\omega$ there exists $a \in \mathcal{A}$ such that $a \cap b$ is infinite. We will always consider infinite mad families. The cardinal $\mathfrak{a}$ denotes the minimal size of a mad family; the cardinals $\mathfrak{a}$ and $\mathfrak{d}$ are independent.

One approach to obtaining a model in which $\mathfrak{s} = \aleph_2$ is with a countable support iteration of Mathias forcing $\mathbb{M}$, this being the partial order consisting of $(s, A) \in [\omega]^{<\omega} \times [\omega]^\omega$, with $\max s < \min A$, and the extension relation defined as $(t, B) \leq (s, A)$ if and only if t end-extends s, $B \subseteq A$, and $t \setminus s \subseteq A$. For any $X \subseteq [\omega]^\omega$, the set

$$D_X = \{(s, A) \in \mathbb{M} \mid A \subseteq X \vee A \subseteq \omega \setminus X\}$$

is dense in $\mathbb{M}$, and for this reason Mathias forcing adds a real $a \in [\omega]^\omega$ such that one of $a \cap X$ or $a \setminus X$ is finite, when X is any ground model infinite subset of ω. This implies the ground model $[\omega]^\omega$ cannot be a splitting family.

Mathias forcing cannot be used to show the consistency of $\mathfrak{b} < \mathfrak{s}$. The issue is that if G is $\mathbb{M}$-generic and $a := \bigcup\{s \mid \exists A(s, A) \in G\}$, then a is an infinite subset of ω such that the enumeration function $e_a \colon \omega \to a$ is a *dominating real over* V, meaning e_a has the property that $f \leq^* e_a$ whenever $f \in \omega^\omega \cap V$ is a ground model function, and so provides a witness that $\omega^\omega \cap V$ is no longer $\leq^*$-unbounded in the $\mathbb{M}$-generic extension. In other words, the Mathias reals grow too fast to preserve the unboundedness of $\omega^\omega \cap V$, and in a countable support iteration of $\mathbb{M}$ the bounding number $\mathfrak{b}$ is of size $\aleph_2 = \mathfrak{c}$.

To remediate this problem Shelah's original "creature forcing" of [She84], henceforth denoted by $\mathbb{Q}$ (Definition 18 below), slows the growth of the generic.[1] Rather than taking just $A \subseteq [\omega]^\omega$ as the second coordinate of the condition, this which guides the construction of the generic real, one considers infinite sequences of finite subsets of ω, $\langle s_i : i \in \omega \rangle$ such that $\max s_i < \min s_{i+1}$, and equips each s_i with a measure $h_i \colon \mathcal{P}(s_i) \to \omega$ (see Definition 14 of a logarithmic measure). This allows for careful selection of the integers with which one extends a given finite approximation s, so that the resulting generic real does not bound all ground model $f \in \omega^\omega$ simultaneously, and is a real not split by the ground model $[\omega]^\omega$ for similar reasons as with $\mathbb{M}$.

A proper forcing $\mathbb{P}$ such that in any $\mathbb{P}$-generic extension, the set of ground model reals $\omega^\omega \cap V$ remains unbounded with respect to $\leq^*$, is known as *weakly ω^ω-bounding*. However this property is not preserved under countable support iterations, and therefore to show a countable support iteration of length ω_2 of $\mathbb{Q}$ preserves unboundedness of the ground model reals it suffices to show that $\mathbb{Q}$ has the slightly stronger property of being *almost ω^ω-bounding*:

Definition 1. A forcing notion $\mathbb{P}$ is *almost ω^ω-bounding* if and only if for every $p \in \mathbb{P}$ and every $\mathbb{P}$ name $\dot{f}$ for an element of ω^ω, there exists $g \in \omega^\omega \cap V$ with the property that for every $A \in [\omega]^\omega$, there is $q_A \leq p$ such that

$$q_A \Vdash \exists^\infty n \in A(\dot{f}(n) \leq \check{g}(n)).$$

Countable support iterations of almost ω^ω-bounding forcings are weakly ω^ω-bounding; this is shown in [Abr10, Theorem 4.4], and Section 4 of [Abr10] contains more in depth discussion of these notions.

Aside from establishing the independence of $\mathfrak{s}$ and $\mathfrak{b}$, another important contribution of [She84] is responding to a question of Mathias, who asked if it is consistent that in a model of $\mathfrak{b} < \mathfrak{s}$, also $\aleph_1 < \mathfrak{a}$; Shelah gives a positive answer, but it is only after a modification of his original forcing $\mathbb{Q}$ that this can be shown. Indeed, he shows $\mathfrak{a} = \aleph_1$ in the countable support iteration of $\mathbb{Q}$ by directly constructing the specific mad family in the ground model, utilizing that CH holds for this construction, and then shows this particular mad family is indestructible under $\mathbb{Q}$ and its iterations.

In 1995 Alan Dow [Dow95] constructs a mad family that is indestructible with respect to countable support iterations of Miller forcing. Miller forcing, also known as rational perfect set forcing, was introduced by Arnold Miller in 1984 [Mil84] with the purpose of increasing the dominating number $\mathfrak{d}$ while keeping $\mathfrak{b}$ small. It is the partial order consisting of trees $T \subseteq \omega^{<\omega}$ such that for every $t \in T$ there exists $s \in T$ extending t and such that s has infinitely many immediate successors in T; the extension relation is given by $T \leq S$ if and only if T is a subtree of S. Miller forcing is proper [Dow95, Proposition 8.11], and like the creature forcing of [She84] this forcing is almost ω^ω-bounding [Dow95, Lemma 8.13]. Unlike Shelah's creature forcing however, Miller forcing does not add an unsplit real [Mil84, Proposition 3.3].

Since 1984 the theory of preservation in forcing has considerably developed; see, for example, [She17] or [Gol93]. Likewise has been developed the theory of *indestructibility* of various witnesses to cardinal characteristics, in particular the indestructibility of mad families; see [BY05], [Hru01],

[1] The term *creature forcing* now refers to a broad class of forcing notions; see [RS99].

or [HF03]. In 2020, Guzman, Hrušák, and Ferreira study the preservation of tight mad families, where:

Definition 2. An almost disjoint family $\mathcal{A} \subseteq [\omega]^\omega$ is *tight* if for all countable collections $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})^+$, there exists a single $a \in \mathcal{A}$ such that $a \cap b$ is infinite for each $b \in \mathcal{B}$.

Above, $\mathcal{I}(\mathcal{A})^+$ denotes the complements of the sets belonging to the ideal generated by $\mathcal{A}$ and the finite subsets of ω:

$$\mathcal{I}(\mathcal{A}) = \{b \subseteq \omega \mid \exists F \in [\mathcal{A}]^{<\omega} \ (b \subseteq^* \bigcup F\}.$$

Notice that if an almost disjoint family is tight, then it is immediately maximal. This strengthening initially appeared in the work of Malykhin [Mal89] in 1989, under the name of ω-mad families, and the connection of these families to Cohen-indestructibility was studied by Kurilić in 2001 [Kur01]. In fact, the existence of tight mad families is equivalent to the existence of Cohen-indestructible mad families: on the one hand, tight mad families are Cohen-indestructible, while on the other hand if $\mathcal{A}$ is a Cohen-indestrictible mad family, there exists $B \in \mathcal{I}(\mathcal{A})$ such that $\mathcal{A} \restriction B := \{A \cap B \mid A \in \mathcal{A}\}$ is a tight mad family; see [Kur01, Theorem 4]. The existence of tight mad families also follows from the assumption $\mathfrak{b} = \mathfrak{c}$ and a certain parametrized $\Diamond$-principle (see [HF03]). Nonetheless it remains a long standing open question if ZFC proves the existence of tight mad families.

One of the important contributions of [GHT20] to the study of tight mad families was the introduction of a property of proper forcings sufficient for guaranteeing the preservation of tight mad families.

Definition 3 ([GHT20, Definition 7.1])**.** Let $\mathcal{A}$ be a tight mad family. We say a proper forcing notion $\mathbb{P}$ *strongly preserves the tightness of* $\mathcal{A}$ if for every $p \in \mathbb{P}$ and every countable elementary $\mathcal{M} \prec H_\theta$, where θ is a sufficiently large regular cardinal so that $\mathbb{P}, \mathcal{A}, p \in M$, and for every $B \in \mathcal{I}(\mathcal{A})$ such that $B \cap Y$ is infinite for all $Y \in \mathcal{I}(\mathcal{A})^+ \cap \mathcal{M}$, there exists $q \leq p$ such that $q \in (\mathcal{M}, \mathbb{P})$-generic and $q \Vdash$ "$\forall \dot{Z} \in (\mathcal{I}(\mathcal{A})^+ \cap M[\dot{G}])(|\dot{Z} \cap B| = \omega)$". Such a q is called an $(\mathcal{M}, \mathbb{P}, \mathcal{A}, B)$*-generic condition.* [2]

It is easy to see that if $\mathbb{P}$ is proper and preserves the tightness of some $\mathcal{A}$, then $\mathcal{A}$ remains tight in any $\mathbb{P}$-generic extension. The important feature of strong preservation of tightness is its preservation under countable support iterations. Similarly to the proof that properness is preserved under countable support iterations (see [Abr10] or [She82]), the authors of [GHT20] show this is the case.

Lemma 4 ([GHT20, Lemma 6.3])**.** Let $\mathcal{A}$ be a tight mad family, let $\mathbb{P}$ be a proper forcing strongly preserving the tightness of $\mathcal{A}$, and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a forcing notion such that $\Vdash_{\mathbb{P}}$"$\dot{\mathbb{Q}}$ strongly preserves the tightness of $\check{\mathcal{A}}$". Then $\mathbb{P} * \dot{\mathbb{Q}}$ strongly preserves the tightness of $\mathcal{A}$. Furthermore if $B \in \mathcal{I}(\mathcal{A})$, and $\mathcal{M}$ is a countable elementary submodel of H_θ with $\mathcal{A}, \mathbb{P}, \dot{\mathbb{Q}} \in \mathcal{M}$, if $p \in \mathbb{P}$ is

[2]Recall an $(\mathcal{M}, \mathbb{P})$-generic condition is a condition $q \in \mathbb{P}$ such that $q \Vdash D \cap \dot{G} \cap M \neq \emptyset$ whenever $D \in \mathcal{M}$ is a dense open subset of $\mathbb{P}$, and $\dot{G}$ denotes the canonical name for the $\mathbb{P}$-generic filter. A forcing $\mathbb{P}$ is proper if for any $\mathcal{M} \prec H_\theta$ the set of $(\mathcal{M}, \mathbb{P})$-generic conditions is dense below any $p \in \mathcal{M} \cap \mathbb{P}$.

$(\mathcal{M}, \mathbb{P}, \mathcal{A}, B)$-generic and $\dot{q}$ is a $\mathbb{P}$-name for a condition in $\dot{Q}$ such that $p \Vdash$"$\dot{q}$ is $(\mathcal{M}[\dot{G}], \dot{\mathbb{Q}}, \mathcal{A}, B)$-generic", then $(p, \dot{q})$ is an $(\mathcal{M}, \mathbb{P} * \dot{\mathbb{Q}}, \mathcal{A}, B)$-generic condition.

Proposition 5 ([GHT20, Proposition 6.4])**.** Let $\mathcal{A}$ be a tight mad family, and let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \gamma, \beta < \gamma \rangle$ be a countable support iteration of proper forcings such that $\Vdash_{\mathbb{P}_\alpha}$"$\dot{\mathbb{Q}}_\alpha$ strongly preserves the tightness of $\check{\mathcal{A}}$". Let $B \in \mathcal{I}(\mathcal{A})$, and let $\mathcal{M}$ be a countable elementary submodel of H_θ, where θ a sufficiently large regular cardinal, such that $\mathcal{A}, \mathbb{P}, \gamma \in M$. Then for every $\alpha \in M \cap \gamma$ and for every $p \in \mathbb{P}_\alpha$ such that p is $(\mathcal{M}, \mathbb{P}_\alpha, \mathcal{A}, B)$-generic, if $\dot{q}$ is a $\mathbb{P}_\alpha$-name such that $p \Vdash_{\mathbb{P}_\alpha}$"$\dot{q} \in \mathbb{P}_\gamma \cap \mathcal{M}$ and $\dot{q} \restriction \alpha \in G_\alpha$", then there exists $\overline{p} \in \mathbb{P}_\gamma$ such that $\overline{p} \restriction \alpha = p$ and $\overline{p} \Vdash_{\mathbb{P}_\gamma} \dot{q} \in \dot{G}$.

Lemma 6 ([GHT20, Corollary 6.5])**.** Let γ be an ordinal, and let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \gamma, \beta < \gamma \rangle$ be a countable support iteration such that for all $\alpha < \gamma$, $\Vdash_{\mathbb{P}_\alpha}$" $\dot{\mathbb{Q}}_\alpha$ strongly preserves the tightness of $\check{\mathcal{A}}$". Then $\mathbb{P}$ strongly preserves the tightness of $\mathcal{A}$.

Definition 3 serves as a key tool in showing a forcing notion does not increase the size of $\mathfrak{a}$, and therefore also preserves the size of $\mathfrak{b}$. Some examples of forcings exhibiting this property are Miller forcing, Miller partition forcing, and Sacks forcing. Our main theorem includes Shelah's creature forcing to this list, and provides an alternative proof of Shelah's result that $\mathfrak{b} < \mathfrak{s}$ in generic extension via a countable support iteration of $\mathbb{Q}$.

Proposition 7 (See Proposition 29)**.** Shelah's forcing $\mathbb{Q}$ strongly preserves the tightness of tight mad families from the ground model.

Theorem 8. *Let V be a model of CH and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ be a countable support iteration with such that $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$ name for $\mathbb{Q}$. If G is $\mathbb{P}_{\omega_2}$-generic over V, then in $V[G]$ it holds that $\aleph_1 = \mathfrak{b} = \mathfrak{a} < \mathfrak{s} = \aleph_2$.*

The standard approach in the literature thus far for showing a proper forcing $\mathbb{P}$ strongly preserves tightness as in Definition 3, is to modify the construction of the $(\mathcal{M}, \mathbb{P})$-generic in the following way. Suppose $\mathcal{M}$ is a countable elementary submodel of H_θ, for θ sufficiently large, containing $\mathbb{P}, \mathcal{A}$ and $p \in \mathcal{M} \cap \mathbb{P}$. We inductively construct a sequence $\langle p_n : n \in \omega \rangle \subseteq \mathcal{M} \cap \mathbb{P}$ below p with $p_{n+1} \leq p_n$ for each $n \in \omega$, and in order to obtain an $(\mathcal{M}, \mathbb{P})$-generic condition, we only demand that at each inductive stage n, $p_n \in D_n$ where D_n comes from an enumeration of the dense open subsets of $\mathbb{P}$ which are in $\mathcal{M}$. To furthermore obtain an $(\mathcal{M}, \mathbb{P}, \mathcal{A}, B)$-generic condition, we also require that p_{n+1} forces "$(\dot{Z}_n \cap B) \setminus n \neq \emptyset$", where $\dot{Z}_n$ comes from a fixed, repetitive enumeration of the $\mathbb{P}$-names in $\mathcal{M}$ for an elements of $\mathcal{I}(\mathcal{A})^+$, and $B \in \mathcal{I}(\mathcal{A})$ is the fixed witness that intersects each Y in the countable set $\mathcal{I}(A)^+ \cap \mathcal{M}$ in an infinite set. This is achieved by what we shall call an *outer hull argument.*

Definition 9. If $\mathbb{P}$ is a forcing notion, $p \in \mathbb{P}$ and $\dot{Z}$ is a $\mathbb{P}$-name for a subset of ω, the *outer hull of $\dot{Z}$ with respect to p* is the set

$$W_p := \{m \in \omega \mid \exists r \leq p \ (r \Vdash m \in \dot{Z})\}.$$

Fact 10 ([GHT20, Lemma 6.2])**.** For an almost disjoint family $\mathcal{A}$ and $\mathbb{P}$ a forcing notion, if $\dot{Z}$ is a $\mathbb{P}$-name for an element of $\mathcal{I}(\mathcal{A})^+$, then for any $p \in \mathbb{P}$, $W_p \in \mathcal{I}(\mathcal{A})^+$.

The above follows from noting that $\dot{Z}$ will be forced by p to be a subset of W_p. Thus in order to obtain a $p_{n+1} \in \mathbb{P}$ forcing that there exists m such that $m > n$ and $m \in \dot{Z}_n \cap B$, it suffices to note that $W_{p_n} \in \mathcal{I}(\mathcal{A})^+$ and so as $B \cap W_{p_n}$ is infinite, and so there is $m \in B$ with $m > n$ and $m \in W_{p_n} \cap \mathcal{M}$; this latter fact yields the desired p_{n+1}.

However, in the case of the creature forcing $\mathbb{Q}$ as well as the Friedman–Zdomskyy forcing, we must take more care into what type of extension we are taking and want to restrict the existential quantifier in the definition of the outer hull. Thus we consider a refined notion of the outer hull, and must prove that Fact 10 still holds for these refinements. These proofs can be found as Claim 30 and Lemma 77 in the current section and in Section 4, respectively.

Before defining $\mathbb{Q}$ and proving Proposition 29, we show that in some sense, the indestructible mad families constructed by Shelah and Dow are canonical examples of tight mad families. The following construction originates in the proof of [She84, Theorem 3.1]; see also [She17, Theorem 7.1, Chapter VI].

Proposition 11. Assume CH. Then there exists a tight mad family.

Proof. Fix an enumeration

$$\{\langle B_n^\alpha : n \in \omega\rangle \mid \alpha < \omega_1\}$$

of sequences $\langle B_n^\alpha : n \in \omega\rangle$, with $B_n^\alpha \in [\omega]^{<\omega} \setminus \{\emptyset\}$ and $B_n^\alpha \cap B_m^\alpha = \emptyset$ for all distinct $m, n \in \omega$.

By induction on $\alpha < \omega_1$, recursively define a family $\mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\}$ as follows. First, choose $\{A_n \mid n \in \omega\}$ to be any partition of ω into infinite sets.

For $\alpha \in [\omega, \omega_1)$, choose $A_\alpha \subseteq \omega$ so that A_α is almost disjoint from A_β for each $\beta < \alpha$, and for each $\beta < \alpha$:

<u>If</u> for all $k \in \omega$ and $\alpha_j < \alpha$ for $j < k$, for all $m \in \omega$ the set

$$\{n \in \omega \mid \min(B_n^\beta) > m \wedge \ B_n^\beta \cap \bigcup_{j<k} A_{\alpha_j} = \emptyset\}$$

is infinite,

<u>then:</u>

(1) there exist infinitely many $n \in \omega$ such that $B_n^\beta \subseteq A_\alpha$, and
(2) for all $k \in \omega$ and $\alpha_j \leq \alpha$ for $j < k$, the set $\{n \in \omega \mid B_n^\beta \cap (\bigcup_{j<k} A_{\alpha_j}) = \emptyset\}$ is infinite.

This can be done by induction, using $\mathfrak{c} = \aleph_1$. Let $\mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\}$. Let us see that $\mathcal{A}$ is tight.

Let $\{C_j \mid j \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$. As each C_j is an infinite subset of ω, it is a countable union of singletons, $C_j = \bigcup_{n\in\omega}\{C_j(n)\}$, where $C_j(n)$ denotes the $(n+1)$-st smallest element of C_j. Then there exists $\beta_j < \omega_1$ such that $\langle\{C_j(n)\} : n \in \omega\rangle = \langle B_n^{\beta_j} : n \in \omega\rangle$. Let $\delta < \omega_1$ be such that $\beta_j < \delta$ for every $j \in \omega$, and so

$$\{\langle B_n^{\beta_j} : n \in \omega\rangle \mid j \in \omega\} \subseteq \{\langle B_n^i : n \in \omega\rangle \mid i < \delta\}.$$

Then at stage δ of the construction, for every $j \in \omega$ and $\alpha_0, \dots \alpha_{k-1} < \delta$, there are infinitely many $n \in \omega$ such that $C_j(n) > m$ and

$$C_j(n) \cap \bigcup_{\ell < k} A_{\alpha_\ell} = \emptyset,$$

so by (1) of the construction, there exist infinitely many n such that $\{C_j(n)\} = B_n^{\beta_j} \subseteq A_\delta$. Therefore for each $j \in \omega$, $C_j \cap A_\delta$ is infinite, so $\mathcal{A}$ is indeed tight.

□

Dow [Dow95, Proposition 8.24] shows that a generic extension over a model of CH by an ω_2-length iteration of Miller forcing, $\aleph_1 = \mathfrak{s} = \mathfrak{a} = \mathfrak{b} < \mathfrak{d} = \aleph_2$. In particular he shows that Miller forcing is almost ω^ω-bounding, giving $\mathfrak{b} = \aleph_1$, though to show $\mathfrak{a} = \aleph_1$ the approach is, like Shelah's [She84], by directly constructing a mad family in the ground model and proving this family is indestructible by iterations of Miller forcing. One difference between Dow's construction and that of Shelah's is that the former only requires the assumption $\mathfrak{p} = \mathfrak{c}$ instead of full CH. The cardinal invariant $\mathfrak{p}$ refers to the *pseudointersection number*; it is the minimal cardinality of a collection $\mathcal{F} \subseteq [\omega]^\omega$ with the strong finite intersection property (SFIP), meaning for any finite $F \subseteq \mathcal{F}$ the intersection $\bigcap F$ is infinite, and moreover there does not exist $y \in [\omega]^\omega$ such that $y \subseteq^* x$ for every $x \in \mathcal{F}$, where $y \subseteq^* x$ if and only if $y \setminus x$ is finite. Every set of cardinality $\kappa < \mathfrak{p}$ with the SFIP has a pseudointersection; this is the key to Dow's construction.

Proposition 12 ([Dow95, Lemma 2.3])**.** Assuming $\mathfrak{p} = \mathfrak{c}$, there exists a mad family $\mathcal{A} = \{A_\alpha \mid \alpha < \mathfrak{c}\}$ such that for any fixed enumeration
$\{\langle B_n^\alpha : n \in \omega\rangle \mid \alpha < \mathfrak{c}\}$ of $[[\omega]^{<\omega}]^\omega$, for any $\alpha < \mathfrak{c}$:
If there exists $\beta < \alpha$ such that

$$(*) : \text{for all } I \in \mathcal{I}(\textstyle\bigcup_{\beta<\alpha} A_\beta),\ B_n^\beta \cap I = \emptyset \text{ for infinitely many } n \in \omega,$$

then there are infinitely many n such that $B_n^\beta \subseteq A_\alpha$.

Proof. We define $\mathcal{A}$ inductively by first letting $\{A_n \mid n \in \omega\}$ be any partition of ω into infinite sets. Suppose A_β has been constructed for all $\beta < \alpha$, and let $\mathcal{I}_\alpha$ denote the ideal $\mathcal{I}(\bigcup_{\beta<\alpha} A_\beta)$.

We can assume without loss of generality that $(*)$ holds for all $\beta < \alpha$. Next, for each finite partial function $s \colon \omega \to \alpha$, define F_s to be the set consisting of all $x \in [\omega]^{<\omega}$ such that

- $x \cap \bigcup_{i \in \mathrm{dom}(s)} A_{s(i)} = \emptyset$, and
- $\forall i \in \mathrm{dom}(s)\ \exists m > \max \mathrm{dom}(s)\ (B_m^{s(i)} \subseteq x)$.

Let $\mathcal{F} = \{F_s \mid s \colon \omega \to \alpha$, s is a finite partial function$\}$. Then $\mathcal{F} \subseteq [[\omega]^{<\omega}]^\omega$ has the SFIP. Indeed, let s, t be distinct finite partial functions from ω to α, and define $h \colon \omega \to \alpha$ such that

$$h(i) = \begin{cases} s(i) & \text{if } i \in \mathrm{dom}(s); \\ t(j) & \text{if } j \in \mathrm{dom}(t) \text{ and } i = \max \mathrm{dom}(s) + 1 + j. \end{cases}$$

Then $F_s \cap F_t = F_h \in \mathcal{F}$.

Therefore $\mathcal{F}$ is a set with the SFIP and $|\mathcal{F}| = |\alpha^{<\omega}| < \mathfrak{c} = \mathfrak{p}$, so by definition of $\mathfrak{p}$ there exists an infinite $Y \subseteq [\omega]^{<\omega}$ such that $Y \subseteq^* F$ for each $F \in \mathcal{F}$. Let $A_\alpha := \bigcup Y$. To see A_α is as desired, let $\beta < \alpha$, and let $s\colon \omega \to \alpha$ be such that $s(0) = \beta$. Then

$$\begin{aligned} n \in A_\beta \cap A_\alpha &\Leftrightarrow \exists x \in Y (n \in x \cap A_\beta) \\ &\Rightarrow \exists x \in Y (n \in x \wedge x \notin F_s) \subseteq Y \setminus F_s \end{aligned}$$

and as the last set is finite, so is $A_\beta \cap A_\alpha$.

Moreover we can show that the set $\{n \in \omega \mid B^\beta_n \subseteq A_\alpha\}$ is infinite. Note that as $Y \cap F_s$ is infinite, it is in particular nonempty, and if $y \in Y \cap F_s$ then there is $m > \max \mathrm{dom}(s)$ with $B^{s(0)}_m = B^\beta_m \subseteq y \subseteq A_\alpha$. Because $\max \mathrm{dom}(s)$ can be taken to be any $n \in \omega$, as we only require $s(0) = \beta$, the integer m above can be taken arbitrarily large. □

Proposition 13. Assume $\mathfrak{p} = \mathfrak{c}$ and let $\mathcal{A} = \{A_\alpha \mid \alpha < \mathfrak{c}\}$ be the mad family constructed above. Then $\mathcal{A}$ is tight.

Proof. The proof proceeds almost identically to that of Proposition 11. Let $\{C_j \mid j \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$, and for each $j \in \omega$ find $\alpha_j < \mathfrak{c}$ such that

$$\langle \{C_j(n)\} : n \in \omega \rangle = \langle B^{\alpha_j}_n : n \in \omega \rangle.$$

Then $\{\alpha_j \mid j \in \omega\}$ is bounded below $\mathfrak{c}$, there exists $\beta < \mathfrak{c}$ such that $\alpha_j < \beta$ for all $j \in \omega$. Therefore

$$\{\langle \{C_j(n)\} : n \in \omega \rangle \mid j \in \omega\} \subseteq \{\langle B^\alpha_n : n \in \omega \rangle \mid \alpha < \beta\}.$$

For each $\alpha_j < \beta$, it cannot be that there is $I \in \mathcal{I}_\beta$ such that $B^{\alpha_j}_n = \{C_j(n)\} \cap I \neq \emptyset$ for all but finitely many $n \in \omega$, as this implies $C_j \subseteq^* I$ and so $C_j \in \mathcal{I}_\beta$. But this in turn implies $C_j \in \mathcal{I}(\mathcal{A})$, contradicting the hypothesis on C_j. Therefore it must be that $\{n \in \omega \mid B^{\alpha_j}_n \cap I = \emptyset\}$ is infinite for each $I \in \mathcal{I}_\beta$ and each $j \in \omega$, and therefore by construction of A_β there are infinitely many n such that $B^{\alpha_j}_n = \{C_j(n)\} \subseteq A_\beta$. □

The above together with the fact Miller forcing strongly preserves tightness gives a different way of seeing $a < \mathfrak{d}$ in the Miller model, the result of Dow [Dow95, Proposition 8.24].

We proceed with the proof of Proposition 29; for this we introduce the definition of the original creature forcing, beginning with the introduction of logarithmic measures. We follow the presentation as given in Abraham [Abr10]; see also [Fis08].

Definition 14. For s a subset of ω, a *logarithmic measure on* s is a function $h\colon [s]^{<\omega} \to \omega$ such that for all $A, B \in [s]^{<\omega}$ and $\ell \in \omega$, if $h(A \cup B) \geq \ell + 1$, then either $h(A) \geq \ell$ or $h(B) \geq \ell$. A *finite logarithmic measure* is a pair (s, h) such that $s \subseteq \omega$ is finite and h is a logarithmic measure on s.

A standard induction gives the following.

Lemma 15 ([She84]; [Fis08, Lemma 2.1.3])**.** If h is a logarithmic measure on s and $h(A_0 \cup \ldots \cup A_{n-1}) > \ell$, then there exists $j < n$ such that $h(A_j) \geq \ell - j$.

The *level* of a finite logarithmic measure (s, h) is the value $h(s)$ and is denoted level(h). If $A \subset s$ such that $h(A) > 0$, then A is called *h-positive*.

Definition 16. Let $P \subseteq [\omega]^{<\omega}$ be an upwards closed collection. The logarithmic measure h on $[\omega]^{<\omega}$ *induced by* P is defined inductively on the cardinality of $s \in [\omega]^{<\omega}$ as follows:

(1) $h(e) \geq 0$ for all $e \in [\omega]^{<\omega}$;
(2) $h(e) > 0$ if and only if $e \in P$;
(3) For all $\ell \geq 1$, $h(e) \geq \ell + 1$ if and only if $|e| > 1$ and for all $e_0, e_1 \subseteq e$ such that $e = e_0 \cup e_1$, then $h(e_0) \geq \ell$ or $h(e_1) \geq \ell$.

Then $h(e) = \ell$ if and only if $\ell \in \omega$ is maximal such that $h(e) \geq \ell$.

Note that if h is as above and e is such that $h(e) \geq \ell$, then $h(a) \geq \ell$ for all sets $a \supseteq e$. In the following we will always assume an induced logarithmic measure is *non-atomic*, meaning there are no h-positive singletons. This assumption is necessary for the proof of the next lemma, which gives a condition on P which implies that the logarithmic measure it induces will take arbitrarily high values, and this in turn allows for the construction of pure extensions with desired properties.

Lemma 17 ([Abr10, Lemma 4.7], [Fis08, Lemma 2.1.9])**.** Let $P \subseteq [\omega]^{<\omega}$ be an upwards closed collection of nonempty sets, and let h be the induced logarithmic measure. Suppose that:

($\dagger$) for every $n \in \omega$ and for every finite partition $\omega = A_0 \cup \ldots \cup A_{n-1}$, there exists $i < n$ such that $[A_i]^{<\omega}$ contains some $x \in P$.

Then for every $n, k \in \omega$, and finite partition of ω into sets $A_0 \cup \ldots \cup A_{n-1}$, there exists $i < n$ and $x \subseteq A_i$ such that $h(x) \geq k$.

The above can be proved by induction on $k \in \omega$, arguing by contradiction and appealing to König's lemma; see Fischer [Fis08, Lemma 2.1.9, Lemma 2.1.10]. We now define the main forcing notion of interest.

Definition 18 ([She84, Definition 2.8], [Fis08, Definition 1.3.5])**.** Let $\mathbb{Q}$ be the partial order consisting of pairs $p = (u, T)$ such that $u \subseteq \omega$ is finite and T is a sequence $T = \langle t_i : i \in \omega \rangle$, where for all $i \in \omega$, t_i is a pair $t_i = (s_i, h_i)$, where h_i is a finite logarithmic measure on s_i, and such that:

(1) $\max(u) < \min(s_0)$;
(2) $\max(s_i) < \min(s_{i+1})$ for all $i \in \omega$;
(3) $\langle h(s_i) : i \in \omega \rangle$ is unbounded and $h_i(s_i) < h_{i+1}(s_{i+1})$ for all $i \in \omega$.

For T as above, let $\text{int}(t_i) = s_i$, and $\text{int}(T) = \bigcup_{i \in \omega} \text{int}(t_i)$. When $e \subseteq \text{int}(t_i)$ is such that $h_i(e) > 0$, we say that e is *t_i-positive*. For conditions $(u_0, T_0), (u_1, T_1) \in \mathbb{Q}$, writing $T_j = \langle t_i^j : i \in \omega \rangle$, $t_i^j = (s_i^j, h_i^j)$ for $j < 2$, define $(u_1, T_1) \leq (u_0, T_0)$, if and only if:

(5) u_1 end-extends u_0 and $u_1 \setminus u_0 \subseteq \text{int}(T_0)$;
(6) $\text{int}(T_1) \subseteq \text{int}(T_0)$ and there is a sequence $\langle B_i : i \in \omega \rangle$ of finite subsets of ω such that $\max(B_i) < \min(B_{i+1})$ and for each $i \in \omega$, $s_i^1 \subseteq \bigcup_{j \in B_i} s_j^0$;
(7) For all $i \in \omega$ and $e \subseteq s_i^1$, if $h_i^1(e) > 0$ then there exists $j \in B_i$ such that $h_j^0(e \cap s_j^0) > 0$.

In the case $u_1 = u_0$, call (u_1, T_1) a *pure extension* of (u_0, T_0).

If $(\emptyset, T)$ is a condition in $\mathbb{Q}$, we will identify $(\emptyset, T)$ and T, and this is meant by $T \in \mathbb{Q}$. For a condition $T = \langle t_i : i \in \omega \rangle \in \mathbb{Q}$ and $k \in \omega$, let

$$i_T(k) = \min\{i \in \omega \mid k < \min(\mathrm{int}(t_i))\}$$

and write $T \setminus k = \langle t_i : i \geq i_T(k) \rangle$. Then $T \setminus k \in \mathbb{Q}$ and $T \setminus k \leq T$. Similarly if $u \subseteq \omega$ is finite, $T \setminus \max(u) \in \mathbb{Q}$. By a slight abuse of notation, we will understand by $(u, T \setminus u)$ to mean the condition $(u, T \setminus \max(u))$ in the case $\max(u) \geq \min(\mathrm{int}(t_0))$.

That $\mathbb{Q}$ satisfies Axiom A (see [Abr10, Definition 2.3]) and hence is proper can be established by the following.

Definition 19. For $n \in \omega$ and $(u_0, T_0), (u_1, T_1) \in \mathbb{Q}$, let $\leq_0$ be the usual partial order on $\mathbb{Q}$. Let us write $T_j = \langle t_i^j : i \in \omega \rangle$ for $j < 2$. Define

(1) $(u_1, T_1) \leq_1 (u_0, T_0)$ iff $(u_1, T_1) \leq_0 (u_0, T_0)$ and $u_1 = u_0$, and
(2) for $n \geq 1$ let $(u_1, T_1) \leq_{n+1} (u_0, T_0)$ iff $(u_1, T_1) \leq_1 (u_0, T_0)$ and $t_i^1 = t_i^0$ for all $i < n$.

In particular, $(u_1, T_1) \leq_1 (u_0, T_0)$ if and only if (u_1, T_1) is a pure extension of (u_0, T_0).

Definition 20. Given a sequence $\langle p_i : i \in \omega \rangle \subseteq \mathbb{Q}$, $p_i = (u, T_i)$, $T_i = \langle t_j^i : j \in \omega \rangle$ such that $p_{i+1} \leq_{i+1} p_i$ for all $i \in \omega$, define the *fusion* of $\langle p_i : i \in \omega \rangle$ to be the condition $q := (u, \langle t_j : j \in \omega \rangle)$ such that $t_j := t_j^{j+1}$ for all $j \in \omega$.

If q is the fusion of $\langle p_i : i \in \omega \rangle$, then $q \leq_{i+1} p_i$ for all $i \in \omega$. The following notion is crucial both for proving that $\mathbb{Q}$ is proper and for our preservation result.

Definition 21. For $(u, T) \in \mathbb{Q}$, with $T = \langle t_i : i \in \omega \rangle$, and D an open dense subset of $\mathbb{Q}$, we say (u, T) is *preprocessed* for D *and* $k \in \omega$ if for every $v \subseteq k$ such that v end-extends u, if $(v, \langle t_j : j \geq k \rangle)$ has a pure extension in D, then already $(v, \langle t_j : j \geq k \rangle) \in D$.

Note that if $(u, T) \in \mathbb{Q}$ is preprocessed for D and k, then any extension of (u, T) is also preprocessed for D and k.

Lemma 22 (Fischer [Fis08, Lemma 1.3.9])**.** For every open dense subset $D \subseteq \mathbb{Q}$, every $k \in \omega$, and every $p \in \mathbb{Q}$, there exists $q \in \mathbb{Q}$ such that $q \leq_{k+1} p$ and q is preprocessed for D and k.

As a consequence of the existence of fusion for $\mathbb{Q}$:

Lemma 23. For every open dense $D \subseteq \mathbb{Q}$ and every $p \in \mathbb{Q}$ there exists a pure extension $q \leq p$ such that q is preprocessed for D and every $k \in \omega$.

Let $\dot{C}$ be a $\mathbb{Q}$-name for a subset of ω, and let $j \in \omega$. Let $\dot{C}(j)$ denote the name for the j-th element of $\dot{C}$. We say a condition p *decides* $\dot{C}(j)$ if there exists $\ell \in \omega$ such that $p \Vdash \dot{C}(j) = \check{\ell}$. For such $\dot{C}$ and $j \in \omega$, let

$$E_{C(j)} = \{p \in \mathbb{Q} \mid p \text{ decides } \dot{C}(j)\}.$$

Note that when $p \in \mathbb{Q}$ forces that $\dot{C}$ is infinite, the set $E_{C(j)}$ is open dense below p in $\mathbb{Q}$.

Lemma 24. Let T be a pure condition in $\mathbb{Q}$, $\dot{C}$ a $\mathbb{Q}$-name for an infinite subset of ω, $n, j \in \omega$, and fix $v \subseteq n$. Then there exists $R = \langle r_i : i \in \omega \rangle \in \mathbb{Q}$ such that $R \leq T$ and for all $i \in \omega$ and r_i-positive $s \subseteq \mathrm{int}(r_i)$, there exists $w \subseteq s$ such that $(v \cup w, R \setminus \max(s))$ decides $\dot{C}(j)$.

Proof. Let $T = \langle t_i : i \in \omega \rangle$ with $t_i = (s_i, h_i)$. By Lemma 23 we can suppose that T is preprocessed for $E_{C(j)}$ and every $k \in \omega$. Let $\mathcal{P}_v(C(j))$ denote the those $x \in [\mathrm{int}(T)]^{<\omega}$ such that:

(1) For some $k \in \omega$, $x \cap \mathrm{int}(t_k)$ is t_k-positive;
(2) There exists $w \subseteq x$ such that $(v \cup w, \langle t_i : i > \max(x) \rangle)$ decides $\dot{C}(j)$.

Note that $\mathcal{P}_v(C(j))$ is upwards closed. Let $h \colon [\omega]^{<\omega} \to \omega$ be the logarithmic measure induced by $\mathcal{P}_v(C(j))$. We will show that h takes arbitrarily high values by showing property (†) of Lemma 17 holds.

Fix $M \in \omega$ and a finite partition $\omega = A_0 \cup \ldots A_{M-1}$. First,

Claim 25. There exists $N < M$ and an extension $T' \leq T$ such that $\mathrm{int}(T') \subseteq A_N$

Proof. Suppose towards contradiction that for all $N < M$ there is no extension $T' \leq T$ such that $\mathrm{int}(T') \subseteq A_N$. This implies that for any $T' = \langle t'_i : i \in \omega \rangle \leq T$, with $t'_i = (s'_i, h'_i)$, for all $N < M$ the sequence $\langle h'_i(s'_i \cap A_N) : i \in \omega \rangle \notin \mathbb{Q}$, and therefore it must be that $\langle h'_i(s'_i) \cap A_N) : i \in \omega \rangle$ is bounded. As $T' \leq T$ then also $\langle h_i(s_i \cap A_N) : i \in \omega \rangle$ is bounded, so let $J_N \in \omega$ be such that $h_i(s_i \cap A_N) \leq J_N$ for every $i \in \omega$. Let $J := \max_{N<M} J_N$. Since $T \in \mathbb{Q}$, by (3) of Definition 18 there exists $i \in \omega$ such that $h_i(s_i) > J + M$. But then $h_i(\bigcup_{N<M} s_i \cap A_N) \geq J + M + 1$ so by Lemma 15 there exists $N < M$ such that $h_i(s_i \cap A_N) \geq J + M - N \geq J + 1 > J$, a contradiction. □

Therefore fix T' and $N < M$ as given by the above claim. Since $(v, T' \setminus v) \in \mathbb{Q}$, there exists $w \subseteq \mathrm{int}(T' \setminus v)$ and R such that $(v \cup w, R)$ is a condition in $\mathbb{Q}$ extending $(v, T' \setminus v)$ and $(v \cup w, R)$ decides the value of $\dot{C}(j)$. Since $w \subseteq \mathrm{int}(T' \setminus v)$ is finite and using the definition of the extension relation, there exists m_0, m_1 such that $w \subseteq \bigcup_{m \in [m_0, m_1]} \mathrm{int}(t'_m)$. Let $x := \bigcup_{m \in [m_0, m_1]} \mathrm{int}(t'_m)$ and note $x \in [A_N]^{<\omega}$. As $T' \setminus v \in \mathbb{Q}$ we may assume there is at least one $m \in [m_0, m_1]$ such that $\mathrm{int}(t'_m)$ is t'_m-positive. Then $T' \setminus v \leq T \setminus v$ so there exists $k \in \omega$ such that $h_k(\mathrm{int}(t'_m) \cap \mathrm{int}(t_k)) = h_k(x \cap \mathrm{int}(t_k)) > 0$. Therefore $x \subseteq \mathrm{int}(T') \subseteq A_N$ satisfies (1) in the definition of $\mathcal{P}_v(C(j))$.

We can also show that (2) holds: we have that $w \subseteq x$ was such that $(v \cup w, R) \in E_{C(j)}$, but T was preprocessed for $E_{C(j)}$ and $\max w$, and $(v \cup w, T \setminus w)$ has a pure extension into $E_{C(j)}$ so already $(v \cup w, T \setminus w) \in E_{C(j)}$. Altogether we have found $x \in \mathcal{P}_v(C(j)) \cap [A_N]^{<\omega}$, verifying (†).

We can now define $R = \langle r_n : n \in \omega \rangle$, where $r_n = (x_n, g_n)$, inductively as follows. Clearly $\mathcal{P}_v(C(j))$ is nonempty, as we just showed above, so pick $x_0 \in \mathcal{P}_v(C(j))$, and let $g_0 := h \restriction \mathcal{P}(x_0)$. Assuming $r_i = (x_i, g_i)$ defined for all $i \leq n$ so that $\max(x_i) < \min(x_{i+1})$ and $g_i(x_i) < g_{i+1}(x_{i+1})$ for $i < n$, since h takes arbitrarily high values there is $x_{n+1} \in \mathcal{P}_v(C(j))$ with $h(x_{n+1}) > h(x_n)$. We can assume $\max(x_n) < \min(x_{n+1})$, since otherwise h is bounded. Define $g_{n+1} := h \restriction \mathcal{P}(x_{n+1})$. This completes the definition of R.

Then $R \in \mathbb{Q}$, and R extends T: as each $x_n = \mathrm{int}(r_n)$ is a finite subset of $\mathrm{int}(T)$ there are $m_0^n < m_1^n \in \omega$ with $x_n \subseteq \bigcup_{i \in [m_0^n, m_1^n]} \mathrm{int}(t_i)$, so the sequence $\langle [m_0^n, m_1^n] : n \in \omega \rangle$ witnesses (6) and (7) of Definition 18. Indeed, if $s \subseteq x_n$ is such that $g_n(s) = h(s) > 0$, by (1) of the definition of $\mathcal{P}_v(C(j))$ there is k such that $s \cap \mathrm{int}(t_k)$ is t_k-positive, and necessarily $k \in [m_0^n, m_1^n]$. Moreover, R is as desired, since if $s \subseteq x_i$ is r_i-positive, by (2) of the definition of $\mathcal{P}_v(C(j))$ there is $w \subseteq s$ such that

$(v \cup w, \langle t_i : i > \max(s)\rangle)$ decides $C(j)$, but $(v \cup w, \langle r_i : i > \max(s)\rangle) \leq (v \cup w, \langle t_i : i > \max(s)\rangle)$ and so makes the same decision.

□

Remark 26. If a condition R in $\mathbb{Q}$ has the property as in the conclusion of the above lemma, then any further extension retains this same property. Indeed, if $T = \langle t_i : i \in \omega\rangle \leq R$ and $s \subseteq \text{int}(t_i)$ is t_i-positive, then there is $j \in \omega$ such that $s \cap \text{r}_\text{j}$ is r_j-positive. Therefore there is $w \subseteq s \cap \text{int}(r_j)$ so that $(v \cup w, R \setminus \max(s \cap x_j))$ decides $C(j)$. So in particular $w \subseteq s \subseteq \text{int}(t_i)$ is such that $(v \cup w, T \setminus \max(s))$ makes the same decision about $C(j)$, since it is an extension of $(v \cup w, T \setminus \max(s \cap x_j)) \leq (v \cup w, R \setminus \max(s \cap x_j))$.

This is important for establishing the next lemma:

Lemma 27. For any $T \in \mathbb{Q}$, $n, j \in \omega$ and $\dot{C}$ a $\mathbb{Q}$-name for an infinite subset of ω, there exists $R = \langle r_i : i \in \omega\rangle \in \mathbb{Q}$ such that $R \leq T$ and for all $v \subseteq n$, for all $i \in \omega$ and r_i-positive $s \subseteq \text{int}(r_i)$, there is $w \subseteq s$ such that $(v \cup w, R \setminus s)$ decides $\dot{C}(j)$.

Proof. Fix $T \in \mathbb{Q}$, fix $n, j \in \omega$, and let $\dot{C}$ be a $\mathbb{Q}$-name for an element of $[\omega]^\omega$. Let $\{v_k \mid j < 2^n\}$ enumerate all subsets of n, and consider v_0. By Lemma 24 there is $T_0 = \langle t^0_i : i \in \omega\rangle \leq T$ such that for all $i \in \omega$ and t^0_i-positive sets $s \subseteq \text{int}(t^0_i)$, there is $w_0 \subseteq s$ such that $(v_0 \cup w_0, T_0 \setminus s)$ decides $\dot{C}(j)$. Next considering v_1 and T_0, by Lemma 24 and Remark 26 there is $T_1 \leq T_0$ such that for all $\ell < 2$ and v_ℓ, for all $i \in \omega$ and t^1_i-positive set $s \subseteq \text{int}(t^1_i)$, there is $w_\ell \subseteq s$ such that $(v_\ell \cup w_\ell, T_1 \setminus s)$ decides $\dot{C}(j)$. Continuing in this way, for each $k < 2^n$ we obtain $T_k \in \mathbb{Q}$ extending T such that for all $\ell \leq k$, for all $i \in \omega$ and t^k_i-positive $s \subseteq \text{int}(t^k_i)$, there exists $w_\ell \subseteq s$ such that $(v_\ell \cup w_\ell, T_k \setminus s)$ decides $\dot{C}(j)$. Then in particular $R := T_{2^n-1} \in \mathbb{Q}$ satisfies the conclusion of the lemma, since if v is any subset of n, there is $k < 2^n$ such that $v = v_k$, and as $R \leq T_k$, for every $i \in \omega$ and r_i-positive $s \subseteq \text{int}(r_i)$ there is $w \subseteq s$ so that $(v \cup w, R \setminus s)$ decides $\dot{C}(j)$. □

Corollary 28. For any $(u, T) \in \mathbb{Q}$ and any $n, j \in \omega$, there exists $(u, R) \leq_{n+1} (u, T)$ such that for all $v \subseteq n$, for all $i \geq n$ and for every $s \subseteq \text{int}(r_i)$ which is r_i-positive, there exists $w \subseteq s$ such that $(v \cup w, R \setminus s)$ decides the value of $\dot{C}(j)$.

Proof. Fix $(u, T) \in \mathbb{Q}$ and $n, j \in \omega$, and write $T = \langle t_i : i \in \omega\rangle$. Consider $(\emptyset, \langle t_i : i \geq n\rangle) \in \mathbb{Q}$. By Lemma 27 there exists $R' = \langle r'_i : i \in \omega\rangle \leq \langle t_i : i \geq n\rangle$ with the property that for any $v \subseteq n$, any $i \geq n$ and r'_i-positive $s \subseteq \text{int}(r_i)$, there is $w \subseteq s$ so that $(v \cup w, R' \setminus s)$ decides $\dot{C}(j)$.

Define $R = \langle r_i : i \in \omega\rangle$ by letting $r_i = t_i$ for $i < n$, and for $i \geq n$ let $r_i = r'_i$. Then $(u, R) \leq_{n+1} (u, T)$ is as desired. □

The central result of this section is the following.

Proposition 29. For every $p \in \mathbb{Q}$ and $M \prec H_\theta$ countable elementary submodel, where θ is sufficiently large, containing $\mathbb{Q}, p, \mathcal{A}$, and every $B \in \mathcal{I}(\mathcal{A})$ such that $|B \cap Y| = \aleph_0$ for all $Y \in \mathcal{I}(\mathcal{A})^+ \cap M$, and $\dot{Z}$ a $\mathbb{Q}$-name for an element of $\mathcal{I}(\mathcal{A})^+$ in M, there exists a pure extension $q \leq p$ such that q is $(M, \mathbb{Q}, \mathcal{A}, B)$-generic.

Proof. Fix θ, M and B as above, and let (u, T_0) be a condition in $\mathbb{Q} \cap M$. Let $\{D_n \mid n \in \omega\}$ enumerate all open dense subsets of $\mathbb{Q}$ in M, and let $\{\dot{Z}_n \mid n \in \omega\}$ enumerate all $\mathbb{Q}$-names for subsets of ω in M which are forced to be in $\mathcal{I}(\mathcal{A})^+$ such that each name appears infinitely often. We inductively define a sequence $\langle q_n : n \in \omega\rangle$ of conditions in $\mathbb{Q} \cap M$, where $q_0 = (u, T_0)$, $q_n = (u, T_n)$ with $T_n = \langle t_i^n : i \in \omega\rangle$, such that the following are satisfied:

(1) For all $n \in \omega$, $q_{n+1} \leq_{n+1} q_n$;
(2) q_{n+1} is preprocessed for D_n and every $k \in \omega$;
(3) For all $v \subseteq n$, for all $i \geq n$ and $s \subseteq \operatorname{int}(t_i^{n+1})$ which is t_i^{n+1}-positive, if v is an end-extension of u, then for some $w \subseteq s$, $((v \cup w), \langle t_j^{n+1} : j > \max(s)\rangle) \Vdash (\dot{Z}_n \cap B) \setminus n \neq \emptyset$.

Suppose q_n has been constructed thus far; we will define q_{n+1}. Consider the condition $(u, \langle t_i^n : i \geq n\rangle) \leq q_n$. By Lemma 23, there exists a pure extension $(u, \langle t_i' : i \geq n\rangle) \leq (u, \langle t_i^n : i \geq n\rangle)$ which is preprocessed for D_n and every $k \in \omega$. Let $q_n^0 = (u, \langle t_i^{n,0} : i \in \omega\rangle)$, where for $i < n$, $t_i^{n,0} = t_i^n$, and $t_i^{n,0} = t_i'$ for $i \geq n$. Then $q_n^0 \leq_{n+1} q_n$ and q_n^0 satisfies (2).

Next, consider the set

$$\begin{aligned} W_{n+1} = \{m \in \omega \mid & \exists r = (u, \langle t_i' : i \in \omega\rangle) \leq_{n+1} q_n^0 \text{ satisfying:} \\ & \forall v \subseteq n\ \forall i \geq n\ \forall s \subseteq \operatorname{int}(t_i')\ [s \text{ is } t_i'\text{-positive} \Rightarrow \exists w \subseteq s \\ & (v \cup w, \langle t_i : i > \max(s)\rangle) \Vdash m \in \dot{Z}_n\} \end{aligned}$$

Claim 30. $W_{n+1} \in \mathcal{I}(\mathcal{A})^+ \cap M$.

Proof. We have that $W_{n+1} \in M$ since it is definable from the forcing relation and from $\mathbb{Q}$, q_n^0, and $\dot{Z}_n$, which are all assumed to be elements of M.

To see $W_{n+1} \notin \mathcal{I}(\mathcal{A})$, let F be a finite subset of $\mathcal{A}$; we show $W_{n+1} \setminus \bigcup F$ is infinite. Since $\Vdash_{\mathbb{Q}} \dot{Z}_n \in \mathcal{I}(\mathcal{A})^+$, in particular q_n^0 forces that $\dot{Z}_n$ is not finitely covered by $\bigcup F$, or in other words q_n^0 forces that the set $\dot{Z}_n \setminus \bigcup F$ is infinite. Let $\dot{C}$ be a $\mathbb{Q}$-name for the set $\dot{Z}_n \setminus \bigcup F$.

Now as $q_n^0 \in \mathbb{Q}$, and $\dot{C}$ is a $\mathbb{Q}$-name for an infinite subset of ω, for all $k \in \omega$, Corollary 28 gives $q^j \leq_{n+1} q_n^0$, where $q^j = (u, R_j)$ and $R_j = \langle r_i^j : i \in \omega\rangle$ such that for all $v \subseteq n$, for all $i \geq n$ and r_i^j-positive $s \subseteq \operatorname{int}(r_i^j)$, there is $w \subseteq s$ such that $(v \cup w, R_j \setminus s)$ decides $\dot{C}(j)$ and $j > k$. So there exists $m_j \in \omega$ such that

$$(v \cup w, R_j \setminus s) \Vdash \dot{C}(j) = \check{m}_j.$$

Note that if $m_j = \dot{C}(j)$, then $m_j \geq j > k$. Therefore for all $k \in \omega$ there exists $m_j > k$ such that $m_j \in W_{n+1}$, witnessed by q^j as above, and moreover $m \notin \bigcup F$ since $q^j \Vdash \check{m}_j \notin \bigcup F$. Thus $W_{n+1} \setminus \bigcup F$ is infinite, and as $F \in [\mathcal{A}]^{<\omega}$ was arbitrary this proves the claim. □

By assumption on the set B, there exists $m_{n+1} \in W_{n+1} \cap B$ such that $m > n$. Let $r = (u, \langle t_i' : i \in \omega\rangle) \leq_{n+1} q_n^0$ be given by $m_{n+1} \in W_{n+1}$, and define $q_{n+1} = (u, \langle t_i^{n+1} : i \in \omega\rangle)$ such that $t_i^{n+1} = t_i'$ for all $i \in \omega$. Since $r \leq_{n+1} q_n^0 \leq_{n+1} q_n$, we have that $q_{n+1} \leq_{n+1} q_n$. This completes the inductive construction.

Let $q = (u, T)$ be the fusion of the q_n's (see Definition 20), so $T = \langle t_i : i \in \omega\rangle$ with $t_i = t_i^{i+1}$ for all $i \in \omega$.

We show q is $(M, \mathbb{Q})$-generic by showing that for all $n \in \omega$, the set $D_n \cap M$ is dense below q. Towards this end let $r = (v, R)$ be an arbitrary extension of q; as D_n is dense there exists $w \subseteq \operatorname{int}(R)$ such that $(v \cup w, R \setminus \max(w)) \in D_n$. Then as $(v \cup w, R \setminus \max(w)) \leq (u, T_{n+1} \setminus \max(w)) \leq q_{n+1}$ and q_{n+1} is preprocessed for D_n and $\max(w)$, already $r' = (v \cup w, T_{n+1} \setminus \max(w)) \in D_n$. Then $r' \in M$ and r, r' are compatible, as witnessed by the condition $(v \cup w, R \setminus \max(w))$. Therefore q is an $(M, \mathbb{Q})$-generic condition.

Next, we show $q \Vdash |\dot{Z}_n \cap B| = \omega$ for every $n \in \omega$. Fix n, and let $(v, R) \leq q$ be arbitrary; it suffices to show that for every $k \in \omega$ there exists an extension of (v, R) which forces $(\dot{Z}_n \cap B) \setminus k \neq \emptyset$. Find $i \in \omega$ such that $i > k$, $v \subseteq i$, $\dot{Z}_n = \dot{Z}_i$ and $s = \operatorname{int}(R) \cap \operatorname{int}(t_i)$ is t_i-positive. The fact that s is $t_i = t_i^{i+1}$-positive and $r \leq q \leq q_{i+1}$ implies, by item (3), that there exists $w \subseteq s$ such that

$$(v \cup w), \langle t_j^{i+1} : j > \max(s) \rangle) \Vdash m_{i+1} \in \dot{Z}_i,$$

where $m_{i+1} \in B$ and $m_{i+1} \geq i > k$.

Then $(v \cup w, R \setminus s)$ is an extension both of (v, R) and of the condition $(v \cup w, \langle t_j^{i+1} : j > \max(s) \rangle)$, so by the latter,

$$(v \cup w, R \setminus s) \Vdash m_{i+1} \in (B \cap \dot{Z}_i) \setminus k = (B \cap \dot{Z}_n) \setminus k.$$

This completes the proof that q is an $(M, \mathbb{Q}, \mathcal{A}, B)$-generic condition, giving that $\mathcal{A}$ remains a tight mad family in any $\mathbb{Q}$-generic extension. □

Lemma 31. Let G be $\mathbb{Q}$-generic over V, let $\mathcal{S} \subseteq [\omega]^\omega$ be an element of V, and let $a = \{s \subseteq \omega \mid \exists T (s, T) \in G\}$. Then for all $b \in \mathcal{S}$, $(b \text{ does not split } a)^{V[G]}$.

Proof. For G, $\mathcal{S}$, and a as above, for any $b \in \mathcal{S}$ the set

$$D_b = \{(s, T) \in \mathbb{Q} \mid \operatorname{int}(T) \subseteq b \vee \operatorname{int}(T) \subseteq \omega \setminus b\}$$

is dense in $\mathbb{Q}$. This uses the fact $b \cup (\omega \setminus b)$ is a finite partition of ω and so any condition (s, T) admits a pure extension (s, T') such that $\operatorname{int}(T') \subseteq b$ or $\operatorname{int}(T') \subseteq \omega \setminus b$; see Claim 25.

□

Theorem 32 ([She84, Theorem 3.1])**.** *Assume CH, and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ be a countable support iteration such that for all $\alpha < \omega_2$, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for the partial order $\mathbb{Q}$ of Definition 18. Let G be $\mathbb{P}_{\omega_2}$-generic over V. Then $V[G] \models \aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_2$.*

Proof. Let $\mathcal{A} \in V$ be a tight mad family; such a mad family exists since CH holds in V. Let G be $\mathbb{P}_{\omega_2}$-generic over V. For all $\alpha < \omega_2$, by Proposition 29, $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ is a proper forcing which strongly preserves the tightness of $\mathcal{A}$. Then $\mathbb{P}_{\omega_2}$ is a countable support iteration of proper forcings and hence is proper (see [Abr10, Theorem 2.7]), and so $\aleph_1^V = \aleph_1^{V[G]}$. Moreover, using Lemma 6, we also have that $\mathbb{P}_{\omega_2}$ strongly preserves the tightness and hence maximality of $\mathcal{A}$. Therefore $V[G] \models \mathfrak{a} = \aleph_1$, since $(\mathcal{A} \text{ is mad and } |\mathcal{A}| = \aleph_1)^{V[G]}$.

Let $\mathcal{S} \subseteq [\omega]^\omega$ be family of cardinality $< \aleph_2$. Then there exists $\alpha < \omega_2$ such that $\mathcal{S} \in V[G_\alpha]$, where $G_\alpha = G \cap \mathbb{P}_\alpha$ is $\mathbb{P}_\alpha$-generic over V. This uses the fact that since CH holds in the ground model, $\mathbb{P}_{\omega_2}$ is $\aleph_2$-cc (see [Abr10, Theorem 2.10]). By definition of $\dot{\mathbb{Q}}_\alpha$ and by Lemma 31, in $V[G_{\alpha+1}]$, $\mathcal{S}$ is not a splitting family, so also this holds in $V[G]$. Therefore $(\mathfrak{s} = \aleph_2)^{V[G]}$. □

3. Sacks coding and tightness

Projective wellorderings of the reals are indicative of to what extent regularity properties hold for the projective classes, as the definable wellordering yields examples of nonregular sets—such as a non-Lebesgue measurable set, a set without the property of Baire, or in general any other interesting set arising from an application of the Axiom of Choice—which are of the same descriptive complexity as that of the wellorder. Accordingly, projectively definable wellorderings are typically antagonistic with large cardinal assumptions and compactness principles, as these assert regularity properties and determinacy for the projective pointclasses.

Recall that under $V = L$ there exists a Δ^1_2 wellorder of the reals [Göd39]; this complexity is optimal by the Lebesgue measurability of analytic sets. Conversely, Mansfield's theorem states that if there exists a Σ^1_2 wellordering of the reals, then all reals are constructible. Using a finite support iteration of ccc forcings, Harrington [Har77, Theorem B] showed that a $\boldsymbol{\Delta}^1_3$ wellordering is consistent with $\neg\mathsf{CH}$ and Martin's Axiom (MA); Friedman and Caicedo showed that the Bounded Proper Forcing Axiom (BFPA) and the assumption $\omega_1 = \omega_1^L$ imply the existence of a Σ^1_3 wellorder of the reals [CF11].

However in these last two constructions, the forcing axioms rendered all cardinal characteristics equivalent to $\mathfrak{c}$, and the question of projective wellorderings of the reals in models with nontrivial structure of cardinal characteristics of the continuum was first addressed by Fischer and Friedman in 2010 [FF10]. Using a countable support iteration of S-proper forcing notions they showed a Δ^1_3 wellorder of the reals is compatible with $\mathfrak{c} = \aleph_2$ and each of the following inequalities: $\mathfrak{d} < \mathfrak{c}$, $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$, $\mathfrak{b} < \mathfrak{g}$. This was made possible by defining a new forcing notion, *Sacks coding*, which uses Sacks reals to code the wellorder and gives a way of forcing a Δ^1_3 wellorder with an ω^ω-bounding iteration.

Mad families of size continuum can be obtained giving a wellorder of the continuum, and so it is natural to ask about their definability in the sense of classical descriptive set theory. Mathias [Mat77] was the first to do this, when in 1969 he showed no analytic almost disjoint family can be maximal. Subsequent work revealed further the antagonism between mad families and determinacy assumptions: under $\mathsf{ZF} + \mathsf{DC}$ + Projective Determinacy (PD) there are no projectively definable mad families, under $V = L(\mathbb{R}) + \mathsf{AD}$, no mad family can be an element of $L(\mathbb{R})$, and there are no mad families in Solovay's model.

On the other hand, there exist Σ^1_2 mad families under $V = L$. This was significantly improved by Miller [Mil89], who constructed in L a coanalytic mad family, along with various other combinatorial objects such as maximal independent families and Hamel bases. His technique originates from a robust coding method of Erdös, Kunen, and Mauldin [EKM81], which has by now been systematized into a black-box theorem yielding Π^1_1 combinatorially significant sets of reals in L; see [Vid14]. Törnquist showed the assumption $V = L$ is not necessary to obtain the coanalytic mad family, namely that it is sufficient to assume there exists a Σ^1_2 mad family.

It is interesting to ask whether in ZFC alone one can construct a cardinal preserving forcing iteration yielding a generic extension with Δ^1_3 wellorder in the presence of $\neg\mathsf{CH}$, while simultaneously controlling values of cardinal characteristics as well as the definability of the witnesses to those values. In 2022 Bergfalk, Fischer, and Switzer established that a Δ^1_3 wellorder of the reals is consistent with $\mathfrak{a} = \mathfrak{u} < \mathfrak{i}$, $\mathfrak{a} = \mathfrak{i} < \mathfrak{u}$, and $\mathfrak{a} < \mathfrak{u} = \mathfrak{i}$, with the added feature that the witnesses to the cardinal characteristics of value $\aleph_1$ can be taken to be coanalytic. They do this by showing various preservation properties of the Sacks coding forcing, and in particular they show this coding strongly preserves tightness of tight mad families from the ground model ([BFB22, Lemma 4.3], given in Fact 51, item (5) below). In this section we use this last preservation result as well as Proposition 29 to show the following:

Theorem 33 (See Theorem 63)**.** *It is consistent with a Δ^1_3 wellorder of the reals and $\mathfrak{c} = \aleph_2$ that $\aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_2$, and $\mathfrak{a} = \aleph_1$ is witnessed by a Π^1_1-definable tight mad family.*

The obtention of a new cardinal inequality together with a projective wellorder of the reals answers (1) of Question 7 from [FF10], while the presence of a projective witness for $\mathfrak{a} = \aleph_1$ of minimal complexity responds to (2) of Question 7 from [FF10].

Our strategy for the above theorem will be to define a countable support iteration in a model of $V = L$, in which there exists a Π^1_1-definable tight mad family $\mathcal{A}$ by [BFB22, Lemma 4.2]. Each iterand will be a $\mathbb{P}_\alpha$-name for an S-proper forcing (see Definition 36 below) which strongly preserves the tightness of $\mathcal{A}$, and along the way we construct the wellorder $<_G = \bigcup_{\alpha<\omega_2} <_\alpha$ by defining the initial segments $<_\alpha$, where $<_\alpha$ wellorders the reals of $L^{\mathbb{P}_\alpha}$. We let $\dot{<}_\alpha$ denote a $\mathbb{P}_\alpha$-name for $<_\alpha$. This ordering $<_\alpha$ can be naturally defined using the wellordering of the nice $\mathbb{P}_\alpha$-names for the reals of $L^{\mathbb{P}_\alpha}$; using appropriate bookeeping, at stage α we add a Sacks-generic real coding a pair of reals $x, y \in V^{\mathbb{P}_\alpha}$ such that $x <_\alpha y$. The way in which the α-th generic real codes these intial stages of the iteration is done so to yield the Δ^1_3-definability of the wellorder, in the following way.

At stage α, the Sacks-generic real r_α will code a countable sequence of generic club subsets of ω_1,

$$\vec{C}_\alpha = \langle C_{\alpha+m} : m \in \Delta(x * y)\rangle$$

as well as a set $Y \subset \omega_1$, which have just been added to $L^{\mathbb{P}_\alpha}$. The set $\Delta(x * y) \subseteq \omega$ is a recursive coding of the pair (x, y), and by adding $\vec{C}_\alpha$, indicates a pattern of stationary/nonstationary in the sequence

$$\langle S_{\alpha+m} : m \in \omega\rangle \in L,$$

where $S_{\alpha+m} \subseteq \omega_1^L$ is stationary costationary in L, and $C_{\alpha+m} \cap S_{\alpha+m} = \emptyset$. Note that, if also r_α codes the pair (x, y), then

$$L[r_\alpha] \models \Delta(x * y) \subseteq \{m \in \omega \mid S_{\alpha+m} \text{ is nonstationary}\}.$$

If it can also be assured that for no $m \notin \Delta(x * y)$, the set $S_{\alpha+m}$ loses its stationarity in the final extension (an "accidental stationary kill"; this appears as Claim 61 below), the idea is that in the final generic extension $L[\langle r_\alpha : \alpha < \omega_2\rangle]$, the wellorder $<_G$ has the following definition.

$$x <_G y \Leftrightarrow \exists \alpha < \omega_2 \; L[r_\alpha] \models \Delta(x * y) = \{m \in \omega \mid S_{\alpha+m} \text{ is nonstationary}\}.$$

To make the above a (lightface) projective definition of $<_G$, the sets Y is added after $\vec{C}_\alpha$ in order to localize the generic nonstationarity of each $S_{\alpha+m}$ to a large class of countable transitive ZF^-

models (Definition 42 below); this uses what is sometimes referred to as "David's Trick". Roughly, this allows us to bound the initial existential quantification to the $\omega_2^{\mathcal{M}}$, where $\mathcal{M}$ belongs to the said class of countable transitive models, and $\mathcal{M}$ contains the real r_α. This idea will also be explained in Section 4, when we add a generic Π^1_2 tight mad family of size $\mathfrak{c}$.

A very important fact we will use in our complexity calculations is that the satisfaction relation $\models$ for countable models of set theory is Δ^1_1-definable; this follows since a countable model of set theory is essentially a pair (ω, E), such that E is a binary relation on ω and (ω, E) is a model of set theory. All of this information can be coded by a single real; details are given in [MW85, Example 1.20].

We next give the definitions of the three forcing notions involved in the definition of the countable support iteration giving a Δ^1_3 wellorder of large continuum in [FF10] and then outline this construction, with the appropriate modification allowing for Theorem 63.

3.1. **Club shooting.** Baumgartner, Harrington, and Kleinberg [BHK76] introduced a cardinal preserving forcing notion which, given a stationary costationary $S \subseteq \omega_1$, adds a closed unbounded $C \subseteq \omega_1$ such that $C \cap S = \emptyset$. The forcing is often referred to as *club shooting*.

Definition 34. Let $S \subseteq \omega_1$ be a stationary costationary set. Define $Q(S)$ to be the partial order consisting of closed, bounded subsets of $\omega_1 \setminus S$, ordered by end-extension.

Lemma 35. The following hold:

(1) $Q(S)$ is ω-distributive, thus does not add new reals.[3]
(2) Let G be $Q(S)$-generic, and let $C_G = \bigcup\{d \in Q(S) \mid d \in G\}$. Then C_G is a club in $\omega_1^{V[G]}$, and witnesses $(\check{S}$ is nonstationary$)^{V[G]}$.

Proof. See, for example, [Jec03, Chapter 25] or [Cum10, Section 6]. □

By item (2) above, $Q(S)$ is not a proper forcing notion, however, it still retains many of the desirable properties of a proper forcing.

Definition 36. Let $S \subseteq \omega_1$ be a stationary set. A forcing notion $\mathbb{P}$ is *S-proper* if for all countable elementary submodels $M \prec H_\theta$, with θ sufficiently large, and such that $M \cap \omega_1 \in S$, for every $p \in \mathbb{P} \cap M$ there is $q \leq p$ which is $(M, \mathbb{P})$-generic.[4]

A proof of the following can be found in [Gol98, Theorem 3.7]

Lemma 37. Suppose $S \subseteq \omega_1$ is stationary and $\mathbb{P}$ is an S-proper forcing notion. Then $\mathbb{P}$ preserves ω_1 as well as the stationarity of any stationary subset of S.

Lemma 38. $Q(S)$ is $(\omega_1 \setminus S)$-proper.

[3]A forcing notion $\mathbb{P}$ is *ω-distributive* if the intersection of countably many dense open subsets of $\mathbb{P}$ is again dense open. Equivalently, any ω-distributive forcing adds no new countable sequences of ordinals.

[4]A condition $p \in \mathbb{P}$ is called $(M, \mathbb{P})$-generic if for all dense open subsets $D \subseteq \mathbb{P}$ such that $D \in M$, the set $D \cap M$ is predense below p. A forcing notion is proper if the set of $(M, \mathbb{P})$-generic conditions is dense below any $p \in \mathbb{P} \cap M$, for any countable elementary $M \prec H_\theta$ with $\mathbb{P} \in M$.

Importantly, S-properness is preserved under countable support iterations; this is proved as in the case of properness (see for example, [Abr10, Theorem 2.7]). That this proof also works to yield the second clause below was noted in [BFB22, Lemma 14].

Lemma 39. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ be a countable support iteration such that for all $\alpha < \delta$, $\Vdash_{\mathbb{P}_\alpha}$"$\dot{\mathbb{Q}}_\alpha$ is an S-proper poset". Then $\mathbb{P}_\delta$ is S-proper. Moreover, if $\mathcal{A}$ is a tight mad family in the ground model and for all $\alpha < \omega_2$, $\Vdash_{\mathbb{P}_\alpha}$"$\dot{\mathbb{Q}}_\alpha$ strongly preserves the tightness of $\check{\mathcal{A}}$", then also $\mathbb{P}_\delta$ strongly preserves the tightness of $\mathcal{A}$.

The next two lemmas are shown in the case of proper forcings in [Abr10, Theorem 2.10] and [Abr10, Theorem 2.12], respectively.

Lemma 40. Assume CH, and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ be a countable support iteration of S-proper posets of length $\delta \leq \omega_2$, such that for all $\alpha < \delta$, $\Vdash_{\mathbb{P}_\alpha}$"$|\dot{\mathbb{Q}}_\alpha| = \omega_1$". Then $\mathbb{P}_\delta$ is $\aleph_2$-cc.

Lemma 41. Assume CH, and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ be a countable support iteration of S-proper posets of length $\delta < \omega_2$, such that for all $\alpha < \delta$, $\Vdash_{\mathbb{P}_\alpha}$"$|\dot{\mathbb{Q}}_\alpha| = \omega_1$". Then CH holds in $V^{\mathbb{P}_\delta}$.

3.2. **Localization.** The next forcing to do this has roots in René David's work [Dav82] on absolute Π^1_2 singletons, these being nonconstructible reals which are the unique solution to a Π^1_2 formula. The forcing notion below will allow for the localization of the generic clubs to a large class of countable transitive ZF^- models, where ZF^- denotes ZF without the Powerset Axiom.

Definition 42. A transitive model $\mathcal{M}$ of ZF^- is called *suitable* if $\omega_2^{\mathcal{M}}$ exists and $\omega_2^{\mathcal{M}} = \omega_2^{L^{\mathcal{M}}}$.

Throughout the rest of this section we assume V is a generic extension of L via a cofinality preserving forcing extension.

Definition 43. For $X \subseteq \omega_1$ and a Σ_1-sentence $\varphi(\omega_1, X)$ with parameters ω_1 and X such that φ holds in all suitable models $\mathcal{M}$ with $\omega_1, X \in \mathcal{M}$, denote by $\mathcal{L}(\varphi)$ the set of all functions $r\colon |r| \to 2$ where $|r| = \mathrm{dom}(r)$ is a countable limit ordinal, and such that:

(1) if $\gamma < |r|$ then $\gamma \in X$ if and only if $r(2\gamma) = 1$;
(2) if $\gamma \leq |r|$ and $\mathcal{M}$ is a countable suitable model such that $\gamma = \omega_1^{\mathcal{M}}$ and $r \restriction \gamma \in \mathcal{M}$, then $\mathcal{M} \models \varphi(\gamma, X \cap \gamma)$.

The extension relation is end-extension.

Each $r \in \mathcal{L}(\varphi)$ is an approximation to the characteristic function of a subset $Y \subseteq \omega_1$ such that $\mathrm{Even}(Y) = \{\gamma \mid 2\gamma \in Y\} = X$. The "odd part" of r, i.e. the values r takes on ordinals of the form $2\gamma + 1$, is used for the following.

Lemma 44 ([FF10, Lemma 1])**.** For every $r \in \mathcal{L}(\varphi)$ and countable limit ordinal $\gamma > |r|$, there exists $r' \leq r$ such that $|r'| = \gamma$.

Corollary 45 ([FF10, Lemma 2])**.** If G is $\mathcal{L}(\varphi)$-generic and $\mathcal{M}$ is a countable suitable model such that $\bigcup G \restriction \omega_1^{\mathcal{M}} \in \mathcal{M}$, then $\mathcal{M} \models \varphi(\omega_1^{\mathcal{M}}, X \cap \omega_1^{\mathcal{M}})$.

Lemma 46 ([FF10, Lemma 3, Lemma 4])**.** $\mathcal{L}(\varphi)$ is proper, and moreover does not add new reals.

3.2.1. *Sacks Coding.* Sacks coding, or coding with perfect trees, was first defined by Fischer and Friedman [FF10]. Recall a tree $T \subseteq 2^{<\omega}$ is *perfect* if for all $s \in T$ there exists an extension t of s such that $t \in T$ and $t^\frown(i) \in T$ for each $i < 2$. *Sacks forcing* is the partial order consisting of perfect trees $T \subseteq 2^{<\omega}$, and a condition S extends a condition T if S is a subtree of T. Sacks forcing is proper, ω^ω-bounding, adds a generic real which is the unique branch through each T in the generic filter, and is often viewed as a very minimally destructive way of forcing $\neg\mathsf{CH}$. The Sacks coding forcing defined below will inherit these properties, and in particular can be seen as a minimally destructive way of forcing a Δ^1_3 wellorder in the presence of large continuum.

Throughout this section we assume $V = L[Y]$, where $Y \subseteq \omega_1$ is generic over L for a cardinal preserving forcing notion.

Definition 47. Fix $n^* \in \omega$. By induction on $i < \omega_1$, define a sequence $\overline{\mu} = \langle \mu_i : i \in \omega_1 \rangle$ such that μ_i is an ordinal and $(|\mu_i| = \aleph_0)^L$, for each $i < \omega_1$. Let $\mu_0 = \emptyset$, and supposing $\langle \mu_j : j < i \rangle$ has been defined, let μ_i be the least ordinal $\mu > \sup_{j<i} \mu_j$ such that :

1. $L_\mu[Y \cap i] \prec_{\Sigma^1_{n^*}} L_{\omega_1}[Y \cap i]$;
2. $L_\mu[Y \cap i] \models \mathsf{ZF}^-$;
3. $L_\mu[Y \cap i] \models$ "ω is the largest cardinal".

Let $\mathcal{B}_i := L_{\mu_i}[Y \cap i]$. For $r \in 2^\omega$, we say that r *codes* Y *below* i if for all $j < i$,

$$j \in Y \Leftrightarrow \underbrace{L_{\mu_j}[Y \cap j][r]}_{=\mathcal{B}_j[r]} \models \mathsf{ZF}^-.$$

For a tree $T \subseteq 2^{<\omega}$, let $|T|$ denote the least $i < \omega_1$ such that $T \in \mathcal{B}_i$.

Definition 48. ([FF10, Definition 2]) *Sacks coding* is the partial order $C(Y)$ consisting of perfect trees $T \subseteq 2^{<\omega}$ such that r codes Y below $|T|$ whenever r is a branch through T. For $T_0, T_1 \in C(Y)$, let $T_1 \leq T_0$ if and only if T_1 is a subtree of T_0.

Remark 49. The hidden parameter n^* above is needed for the preservation results pertaining to definable combinatorial objects, e.g. item (5) of Fact 51 below. Specifically, n^* is chosen to an upper bound on the complexity of the formula expressing all relevant combinatorial properties which we want to reflect down to the models $\mathcal{B}_i$. In the present case of preserving a Π^1_1 tight mad family, $n^* = 5$ is sufficient. See also [BFB22, Remark 1]

Remark 50. Let G be $C(Y)$-generic over $L[Y]$. Since the definition of $C(Y)$ is absolute, it still holds in $L[Y][G]$ that if T is any condition in $C(Y)$ and r is a branch through T,

$$L[Y][G] \models Y \cap |T| = \{j < |T| \mid \mathcal{B}_j[r] \models \mathsf{ZF}^-\}$$

In particular, if $r \in \bigcap G$ and $\gamma = \sup\{|T| \mid T \in G\}$, then in $L[Y][G]$,

$$Y \cap \gamma = \{j < \gamma \mid \mathcal{B}_j[r] \models \mathsf{ZF}^-\}.$$

Fact 51. The following hold.

1. ([FF10, Lemma 4]) For any $j \in \omega_1$ and any $T \in C(Y)$ with $|T| \leq j$, there exists $T' \leq T$ such that $|T'| = j$.

(2) ([FF10, Lemma 7]) $C(Y)$ is proper.
(3) ([FF10, Lemma 6]) Let G be $C(Y)$-generic over V and let $r := \bigcap G$. Then in $V[G]$, r codes the set Y in the sense that for all $j < \omega_1$

$$j \in Y \Leftrightarrow \mathcal{B}_j[r] \models \mathsf{ZF}^-.$$

(4) ([FF10, Lemma 8]) $C(Y)$ is ω^ω-bounding.[5]
(5) ([BFB22, Lemma 4.3]) Countable support iterations of $C(Y)$ preserve the tightness of Π^1_1 tight mad families.

3.3. **A Δ^1_3 long wellorder.** With the ingredients provided by the previous sections we proceed with the proof of Theorem 63. The definition of the forcing construction requires the establishment of some preliminaries.

Recall that $\diamondsuit$ is the assertion that there exists a sequence $\vec{A} = \langle A_\alpha : \xi < \omega_1\rangle$ where $A_\xi \subseteq \xi$ for each $\xi < \omega_1$ and for any $X \subseteq \omega_1$, the set $\{\xi < \omega_1 \mid X \cap \xi = A_\xi\}$ is stationary. $\vec{A}$ is called a $\diamondsuit$-sequence, and such a sequence can be constructed in L so that the sequence is Σ_1-definable over L_{ω_1}; see, for example, [Dev17, Theorem 3.3].

Proposition 52. Assume $\diamondsuit$ holds. Then there exists a sequence

$$\vec{S} = \langle S_\alpha : \alpha < \omega_2\rangle$$

which is Σ_1-definable over L_{ω_2} consisting of sets $S_\alpha \subseteq \omega_1$ that are stationary costationary in L, and are almost disjoint in the sense that $|S_\alpha \cap S_\beta| < \omega_1$ for all distinct $\alpha, \beta < \omega_2$. Furthermore there exists a stationary $S_{-1} \subseteq \omega_1$ such that $S_{-1} \cap S = \emptyset$ for all $S \in \vec{S}$.

Moreover, if $\mathcal{M}$, $\mathcal{N}$ are suitable models such that $\omega_1^{\mathcal{M}} = \omega_1^{\mathcal{N}}$, then $\vec{S}^{\mathcal{M}}$ and $\vec{S}^{\mathcal{N}}$ coincide on $\omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}}$. If $\mathcal{M}$ is suitable with $\omega_1^{\mathcal{M}} = \omega_1$, then $\vec{S}^{\mathcal{M}} = \vec{S} \restriction \omega_2^{\mathcal{M}}$.

We assume $V = L$, and therefore fix $\vec{S}$ and S_{-1} as above. For our bookeeping function:

Lemma 53 ([FF10, Lemma 14])**.** There exists $F\colon \omega_2 \to L_{\omega_2}$ such that for all $a \in L_{\omega_2}$, $F^{-1}(a)$ is unbounded in ω_2, and F is Σ_1-definable over L_{ω_2}. Moreover, if $\mathcal{M}$, $\mathcal{N}$ are suitable models such that $\omega_1^{\mathcal{M}} = \omega_1^{\mathcal{N}}$, then $F^{\mathcal{M}}$ and $F^{\mathcal{N}}$ coincide on $\omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}}$. If $\mathcal{M}$ is suitable with $\omega_1^{\mathcal{M}} = \omega_1$, then $F^{\mathcal{M}} = F \restriction \omega_2^{\mathcal{M}}$.

Fix F as above.

We will also need to preserve a tight mad family to preserve and so will use the following lemma. It is important, for the preservation by countable support iterations of $C(Y)$, that the tight mad family has a projective definition and provably consists of only constructible reals.

Lemma 54 ([BFB22, Lemma 4.2])**.** If $V = L$ then there is a Π^1_1 tight mad family $\mathcal{A}$ such that ZFC proves A is a subset of L.

[5] A proper forcing notion $\mathbb{P}$ is said to be ω^ω*-bounding* if the ground model reals remain a dominating family in the extension, in other words, for any $f \in \omega^\omega$ in the $\mathbb{P}$-generic extension, there exists $g \in V \cap \omega^\omega$ such that $f \leq^* g$. Such forcings preserve the ground model ω^ω as a dominating family.

By recursion on $\alpha < \omega_2$, define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$, where $\mathbb{P}_0$ is taken to be the trivial poset. Suppose $\mathbb{P}_\alpha$ has been defined, and let G_α be $\mathbb{P}_\alpha$-generic over L.

The wellorder $<_\alpha$ on $L[G_\alpha]$ has a natural definition using the global wellorder $<_L$ of the universe L and the collection of $\mathbb{P}_\alpha$-names for reals which we can assume to be *nice*:

Definition 55. For a forcing notion $\mathbb{P}$ and G a $\mathbb{P}$-generic filter over V, for any real $x \in V[G]$, a *nice* $\mathbb{P}$*-name for* x is a $\mathbb{P}_\alpha$-name of the form $\dot{x} = \bigcup_{n \in \omega} \{\langle \langle n, m_p^n \rangle, p \rangle \mid p \in A_n(\dot{x})\}$, where $A_n(\dot{x})$ is a maximal antichain in $\mathbb{P}_\alpha$; note that $p \Vdash \dot{x}(\check{n}) = m_p^n$.

Since every name for a real has a nice names in the above sense, this allows us to suppose that if $\alpha < \beta < \omega_2$ and $\dot{x}$ is a $\mathbb{P}_\beta$-name which is not a $\mathbb{P}_\alpha$-name, then all $\mathbb{P}_\alpha$-names precede $\dot{x}$ with respect to $<_L$, as it takes longer to construct x. Whenever is a real in $L[G_\alpha]$, there exists $\gamma \leq \alpha$ such that x has a nice $\mathbb{P}_\gamma$-name; let γ_x be the minimal such γ. Define σ_x^α to be the L-minimal nice $\mathbb{P}_{\gamma_x}$-name for x. In this way we can understand the reals of $L[G_\alpha]$ by considering the set

$$N = \{\sigma_x^\alpha \mid x \in L[G_\alpha] \cap \omega^\omega\}.$$

Notice that as N is a subset of L, N is canonically wellordered by $<_L$; therefore define $<_\alpha$ by letting

$$x <_\alpha y \text{ if and only if } \sigma_x^\alpha <_L \sigma_y^\alpha,$$

whenever x, y are reals of $L[G_\alpha]$. Equivalently, $x <_\alpha y$ if and only if $\gamma_x < \gamma_y$ or $\gamma_x = \gamma_y$ and $\sigma_x^\alpha <_L \sigma_y^\alpha$. Since $\sigma_x^\alpha = \sigma_x^\beta = \sigma_x^{\gamma_x}$ for any $\beta < \alpha$, $<_\beta$ is an initial segment of $<_\alpha$. Let $\dot{<}_\alpha$ denote a $\mathbb{P}_\alpha$-name for $<_\alpha$.

Fix a recursive coding $\cdot * \cdot \colon \omega^\omega \times \omega^\omega \to \omega^\omega$ by letting

$$x * y = \{2n \mid n \in x\} \cup \{2n+1 \mid n \in y\}.$$

Note the pair (x, y) is constructible from $x * y$. For any real x define $\Delta(x) := x * (\omega \setminus x)$.

Lastly we fix an absolute way of coding the ordinals of ω_2^L.

Definition 56. Let $\beta < \omega_2^L$ and $X \subseteq \omega_1^L$. We say X is a *sufficiently absolute code for* β if there exists a formula $\psi(x, y)$ such that for any suitable model $\mathcal{M}$ containing $X \cap \omega_1^{\mathcal{M}}$, there exists a unique $\overline{\beta} \in \omega_2^{\mathcal{M}}$ such that $\psi(\overline{\beta}, X \cap \beta)$ holds in $\mathcal{M}$, and $\overline{\beta} = \beta$ in the case $\omega_1^{\mathcal{M}} = \omega_1^L$. Moreover, if $\mathcal{M}, \mathcal{N}$ are any suitable models such that $\omega_1^{\mathcal{M}} = \omega_1^{\mathcal{N}}$ and $X \cap \omega_1^{\mathcal{M}} \in \mathcal{M} \cap \mathcal{N}$, then it is the same $\overline{\alpha} \in \omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}}$ such that $(\psi(\overline{\beta}, X \cap \omega_1^{\mathcal{M}}))^{\mathcal{M}}$ and $(\psi(\overline{\beta}, X \cap \omega_1^{\mathcal{N}}))^{\mathcal{N}}$.

Fact 57. ([FZ10, Fact 5]) Sufficiently absolute codes exist for every $\beta < \omega_2^L$.

Proof. Let $G \colon \mathrm{Ord} \times \mathrm{Ord} \to \mathrm{Ord}$ be Gödel's pairing function, and let $\psi(x, y)$ hold if and only if x is an ordinal, $x \in \omega_2$ and y is the $\leq_L$-least subset of ω_1 such that $\langle x, \in \rangle$ is order-isomorphic to $\langle \omega_1, G^{-1}[y] \rangle$. The "moreover" of Definition 56 is satisfied because the $\leq_L$-rank of $G^{-1}[y \cap \omega_1^{\mathcal{M}}]$ is absolute for transitive models. □

Henceforth we fix the formula $\psi(x, y)$ above.

Working in $L[G_\alpha]$, let $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{Q}}_\alpha^0 * \dot{\mathbb{Q}}_\alpha^1$ be a $\mathbb{P}_\alpha$-name for a two-step iteration in which $\dot{\mathbb{Q}}_\alpha^0$ is a $\mathbb{P}_\alpha$-name for the creature forcing $\mathbb{Q}$ of Definition 18, and $\dot{\mathbb{Q}}_\alpha^1$ is a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha^0$-name for the trivial forcing, unless the following occurs: $F(\alpha) = \{\sigma^\alpha_x, \sigma^\alpha_y\}$ for some reals $x, y \in L[G_\alpha]$ such that $x <_\alpha y$. In this case set $x_\alpha = x$ and $y_\alpha = y$, and define $\dot{\mathbb{Q}}^1_\alpha$ to be a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha$-name for a three-step iteration $\dot{\mathbb{K}}^0_\alpha * \dot{\mathbb{K}}^1_\alpha * \dot{\mathbb{K}}^2_\alpha$, where:

(1) $\dot{\mathbb{K}}^0_\alpha$ is a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha$-name for the countable support iteration $\langle \mathbb{P}^0_{\alpha,\beta}, \mathbb{K}^0_{\alpha,n} : \beta \leq \omega, n \in \omega\rangle$, where $\mathbb{K}^0_{\alpha,m}$ is a $\mathbb{P}^0_{\alpha,m}$-name for $Q(S_{\alpha+m})$ for all $m \in \Delta(x_\alpha * y_\alpha)$.

(2) Let R_α be $\mathbb{Q}^0_\alpha$-generic over $L[G_\alpha]$, and let H_α be $\mathbb{K}^0_\alpha$-generic over $L[G_\alpha * R_\alpha]$. In $L[G_\alpha * R_\alpha * H_\alpha]$ fix:
   - Subsets $W_\alpha, W_\eta \subseteq \omega_1$ such that W_α is a sufficiently absolute code for the ordinal α and W_η is a sufficiently absolute code for an ordinal η such that $L_\eta \models |\alpha| \leq \omega_1$;
   - A real $x_\alpha \oplus y_\alpha$ recursively coding the pair (x_α, y_α);
   - A subset $Z_\alpha \subseteq \omega_1$ coding $G_\alpha * R_\alpha * H_\alpha$.

   Fix a computable bijection $\langle \cdot,\cdot,\cdot,\cdot\rangle \colon \mathcal{P}(\omega_1) \to \mathcal{P}(\omega_1)$, and for $X \subseteq \omega_1$ and $i < 4$ write $(X)_i$ for those elements of $\mathcal{P}(\omega_1)$ such that $X = \langle (X)_i :< 4\rangle$. Let $X_\alpha = \langle W_\alpha, x_\alpha \oplus y_\alpha, W_\eta, Z_\alpha\rangle$.

   Let $\varphi_\alpha = \varphi_\alpha(\omega_1, X_\alpha)$ be a sentence with parameters ω_1 and X_α such that φ_α holds if and only if:

   There exists an ordinal $\overline{\alpha} \in \omega_2$ such that $(X_\alpha)_0 = \overline{\alpha}$ and there exists a pair (x, y) such that $(X_\alpha)_2 = x \oplus y$ and for all $m \in x * y$, $S_{\overline{\alpha}+2_m}$ is nonstationary, and for all $m \notin x * y$, $S_{\overline{\alpha}+2m+1}$ is nonstationary.

   More formally, $\varphi_\alpha(\omega_1, X_\alpha)$ is the formula:

$$\exists \overline{\alpha}, (x, y) \in \Delta^1_1(X_\alpha)[\overline{\alpha} \in \omega_2 \wedge \forall m \in \Delta(x * y)\ S_{\overline{\alpha}+m} \in \mathrm{NS}_{\omega_1}],$$

   where NS_{ω_1} denotes the set of nonstationary subsets of ω_1. Then φ_α is a Σ_1-sentence with parameters ω_1, X_α, and if $\mathcal{M}$ is any suitable model containing ω_1 and X_α as elements, then $\varphi_\alpha(\omega_1, X_\alpha)$ holds in $\mathcal{M}$ (this uses that $\mathcal{M}$ has the code W_η). We can therefore define $\dot{\mathbb{K}}^1_\alpha$ be a $(\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha * \mathbb{K}^0_\alpha)$-name for $\mathcal{L}(\varphi_\alpha)$.

(3) Let Y_α be $\mathbb{K}^1_\alpha$-generic over $L[G_\alpha * R_\alpha * H_\alpha]$. Then as $\{n \in \omega \mid 2n \in Y_\alpha\} = X_\alpha$ and X_α codes $G_\alpha * R_\alpha * H_\alpha$, we have $L[G_\alpha * R_\alpha * H_\alpha * Y_\alpha] = L[Y_\alpha]$. In this model, let $\mathbb{K}^2_\alpha$ be a $(\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha * \dot{\mathbb{K}}^0_\alpha * \dot{\mathbb{K}}^1_\alpha)$-name for $C(Y_\alpha)$.

This completes the definition of $\mathbb{P} = \mathbb{P}_{\omega_2}$. Then $\mathbb{P}$ is a countable support iteration such that each $\mathbb{P}_\alpha$ forces that $\mathbb{Q}_\alpha$ is an S_{-1}-proper forcing notion of size $\leq \omega_1$, and by Proposition 29 and Fact 51, $\mathbb{Q}_\alpha$ is forced to strongly preserves the tightness of $\mathcal{A}$. Therefore:

Lemma 58. $\mathbb{P}$ is S_{-1}-proper, strongly preserves the tightness of $\mathcal{A}$, and has the $\aleph_2$-cc.

Proof. By Lemma 39, Lemma 6, and Lemma 40. □

Let G be $\mathbb{P}$-generic over L, and define $<_G = \bigcup <_\alpha$, where $<_\alpha = \dot{<}^G_\alpha$.

Lemma 59. Let G be $\mathbb{P}$-generic over L, and let x, y be reals in $L[G]$. Then $x <_G y$ if and only if:

(∗) there exists a real r such that for every countable suitable $\mathcal{M}$ containing r as an element, there exists $\overline{\alpha} < \omega_2^{\mathcal{M}}$ such that for all $m \in \Delta(x * y)$, $S^{\mathcal{M}}_{\overline{\alpha}+m}$ is nonstationary in $\mathcal{M}$.

Proof. Let x, y be reals in $L[G]$, and let $\gamma_x, \gamma_y < \omega_2$ be minimal such that x has a nice $\mathbb{P}_{\gamma_x}$-name, and y has a nice $\mathbb{P}_{\gamma_y}$-name, respectively. First suppose $x <_G y$. Then $\{\sigma_x^{\gamma_x}, \sigma_y^{\gamma_y}\} \in L_{\omega_2}$ and $F^{-1}(\{\sigma_x^{\gamma_x}, \sigma_y^{\gamma_y}\})$ is unbounded in ω_2, so there is $\alpha \geq \max(\gamma_x, \gamma_y)$ such that $F(\alpha) = \{\sigma_x^{\gamma_x}, \sigma_y^{\gamma_y}\} = \{\sigma_x^{\alpha}, \sigma_y^{\alpha}\}$. Then our definition of $\dot{\mathbb{Q}}_\alpha^1$ at this point was nontrivial, and was taken with respect to the reals $x_\alpha = x$ and $y_\alpha = y$. Let R_α be $\mathbb{Q}_\alpha^0$-generic over $L[G_\alpha]$ where $G_\alpha = \mathbb{P}_\alpha \cap G$, let H_α be $\mathbb{K}_\alpha^0$-generic over $L[G_\alpha * R_\alpha]$, let Y_α be $\mathbb{K}_\alpha^1$-generic over $L[G_\alpha * R_\alpha * H_\alpha]$, and let D_α be $\mathbb{K}_\alpha^2$-generic over $L[G_\alpha * R_\alpha * H_\alpha * Y_\alpha] = L[Y_\alpha]$.

As D_α is $C(Y_\alpha)$-generic, by Fact 51, item(3), $d_\alpha := \bigcap D_\alpha$ is a real coding Y_α. Since the even part of Y_α codes X_α, also X_α and hence the pair (x_α, y_α) are constructible from d_α.

Now let $\mathcal{M}$ be any countable suitable model containing d_α as an element. By the above observations and the fact $\omega_1^{\mathcal{M}} = \omega_1^{L^{\mathcal{M}}}$, the model $\mathcal{M}$ can construct $Y_\alpha \restriction \omega_1^{\mathcal{M}}$ and $X_\alpha \cap \omega_1^{\mathcal{M}}$.

Since Y_α is $\mathcal{L}(\varphi_\alpha)$ generic, by Lemma 45, $\mathcal{M} \models \varphi_\alpha(\omega_1^{\mathcal{M}}, X_\alpha \cap \omega_1^{\mathcal{M}})$, meaning in $\mathcal{M}$ it holds:

"there exists $\overline{\alpha} \in \omega_2$ such that $(X_\alpha \cap \omega_1^{\mathcal{M}})_0 = \overline{\alpha}$ and $(X_\alpha \cap \omega_1^{\mathcal{M}})_2$ is the code of a pair of reals (x', y') such that for all $m \in \Delta(x' * y')$ $(S^{\mathcal{M}}_{\overline{\alpha}+m} \in \mathrm{NS}_{\omega_1})$."

Since then $\Delta(x', y') = \Delta(x, y)$ in $\mathcal{M}$, this implies necessarily $(x', y') = (x, y)$.

To see the converse implication, fix a real r given by $(*)$. We need the following.

Claim 60. If $\mathcal{N}$ is a suitable model of *any* cardinality with $r \in \mathcal{N}$, then there exists $\overline{\alpha} \in \omega_2^{\mathcal{N}}$ such that $S_{\overline{\alpha}+m} \in \mathrm{NS}_{\omega_1}$ for all $m \in \Delta(x * y)$.

Proof. Suppose towards contradiction that $\mathcal{N}$ is a suitable model containing r as an element, with $|\mathcal{N}| \geq \aleph_1$, but

$$(\otimes) \qquad \mathcal{N} \models \forall \alpha \in \omega_2 (\Delta(x * y) \neq \{m \in \omega \mid S_{\alpha+m} \in \mathrm{NS}_{\omega_1}\}).$$

By the downwards Löwenheim–Skolem theorem, there exists a countable elementary submodel $\mathcal{N}_0 \prec \mathcal{N}$ with $r \in \mathcal{N}_0$. Note $\mathcal{N}_0$ is a model of ZF^-, and $\mathcal{N}_0$ models $(\otimes)$. Let $\overline{\mathcal{N}_0}$ be the transitive collapse of $\mathcal{N}_0$; as the Mostowski isomorphism is the identity on countable objects, we also have $r \in \overline{\mathcal{N}_0}$. Moreover, $\overline{\mathcal{N}_0}$ is a countable, transitive model of ZF^- plus "ω_2 exists", and $\omega_2^{\overline{\mathcal{N}_0}} = \omega_2^{\mathcal{N}} \cap \mathcal{N}_0 = (\omega_2^L)^{\mathcal{N}} \cap \mathcal{N}_0 = (\omega_2^L)^{\overline{\mathcal{N}_0}}$. Therefore $\overline{\mathcal{N}_0}$ is a countable suitable model containing r and by downwards absoluteness of the formula in $(\otimes)$, there is no $\alpha \in \omega_2^{\overline{\mathcal{N}_0}}$ such that for all $m \in \Delta(x * y)$, $S^{\overline{\mathcal{N}_0}}_{\alpha+m}$ is nonstationary in $\overline{\mathcal{N}_0}$. Then $\overline{\mathcal{N}_0}$ witnesses the failure of $(*)$, a contradiction. □

Therefore we can consider the suitable model $\mathcal{M} = L_{\omega_6}[r]$. By $(*)$ there exists $\alpha \in \omega_2^{\mathcal{M}} = \omega_2$ such that for all $m \in \Delta(x * y)$, $(S^{\mathcal{M}}_{\alpha+m} \in \mathrm{NS}_{\omega_1})^{\mathcal{M}}$. As $\vec{S}$ was Σ_1-definable over L_{ω_2} and the fact $\omega_2^{\mathcal{M}} = \omega_2^L = \omega_2^{L[G]}$, we have that $S^{\mathcal{M}}_{\alpha+m} = S_{\alpha+m}$, and $S_{\alpha+m}$ is nonstationary in $\mathcal{M}$. By upwards absoluteness, for all such m we also have $(S_{\alpha+m} \in \mathrm{NS})^{L[G]}$.

Claim 61. In $L[G]$, suppose $\beta = \alpha + m < \omega_2$, and $m \notin \Delta(x_\alpha * y_\alpha)$. Then S_β is stationary in $L[G]$.

Proof. Let $p \in G$ be such that $p \Vdash \beta \notin \{\alpha + m \mid m \in \Delta(x_\alpha * y_\alpha)\}$. Let $\mathbb{P} \restriction p = \{q \in \mathbb{P} \mid q \leq p\}$. Note that as G is $\mathbb{P}$-generic over L and $p \in G$, then G is also $\mathbb{P} \restriction p$-generic over L. We have that

$\mathbb{P} \restriction p$ is S_β-proper, since it is a countable support iteration of the S_β-proper forcing notions. To verify this it suffices to consider the iterand $\mathbb{Q}_\alpha \restriction p(\alpha)$. But indeed, any condition in $\mathbb{Q}_\alpha \restriction p(\alpha)$ adds no new countable ordinals to $C_\beta \subseteq \omega_1 \setminus S_\beta$, and therefore $\mathbb{P} \restriction p$ is S_β-proper. $\square$

Using the claim, the fact that $S_{\alpha+m}$ is nonstationary in $L[G]$ implies $m \in \Delta(x * y)$, where $(x, y) = (x_\alpha, y_\alpha) = (\sigma_x^{\alpha G}, \sigma_y^{\alpha G})$, and $x <_\alpha y$. Therefore $x <_G y$. $\square$

Lemma 62. In $L[G]$, $<_G$ is a Δ^1_3-definable wellorder of the reals.

Proof. By Lemma 59, we have

$$x <_G y \Leftrightarrow \Phi(x, y),$$

where $\Phi(x, y)$ is the formula

$$\underbrace{\exists r \in \mathcal{P}(\omega^\omega)}_{\exists}[\underbrace{\forall \mathcal{M} \text{ countable, suitable, } r \in \mathcal{M}}_{\forall}$$

$$\underbrace{(\exists \alpha < \omega_2^{\mathcal{M}}}_{\exists}\ (\underbrace{\mathcal{M} \models \forall m \in \Delta(x * y)\ S_{\alpha+m} \in \mathrm{NS}_{\omega_1})}_{\Delta^1_1})]$$

Thus, $\Phi(x, y)$ is a Σ^1_3 formula. However, $<_G$ is also a total wellorder, since if x, y are any reals in $L[G]$, there is $\alpha = \max(\gamma_x, \gamma_y) < \omega_2$ such that $x = (\sigma_x^\alpha)^G$ and $y = (\sigma_y^\alpha)^G$. Either $\sigma_x^\alpha <_L \sigma_y^\alpha$ or $\sigma_y^\alpha <_L \sigma_x^\alpha$; in the case $\neg(x <_G y)$ then we must have $y <_G x$. Therefore the complement of $<_G$ is Σ^1_3-definable, giving that $<_G$ is Δ^1_3-definable. $\square$

Thus, we proved the following:

Theorem 63. *It is consistent that $\mathfrak{a} = \aleph_1 < \mathfrak{s} = \aleph_2$ that there exists a Δ^1_3 definable wellorder of the reals, and a Π^1_1 tight mad family of size $\aleph_1$.*

Proof. Suppose $V = L$, and fix a Π^1_1 definable tight mad family $\mathcal{A}$ from Lemma 54. Let $\mathbb{P}$ be the ω_2-length countable support iteration constructed in this section, and let G be $\mathbb{P}$-generic over V. First note $\mathbb{P}$ is proper by 58. Define $<_G = \bigcup_{\alpha<\omega_2} \dot{<}_\alpha^G$; by Lemma 62, $<_G$ is a Δ^1_3 wellorder of the reals. Since any set $\mathcal{S} \subseteq [\omega]^\omega \cap V[G]$ such that $|\mathcal{S}| < \omega_2$ appears at some inital stage $\alpha < \omega_2$ of the iteration, by definition $\dot{\mathbb{Q}}^0_\alpha$ is a $\mathbb{P}_\alpha$-name for the forcing $\mathbb{Q}$ of Definition 18, so $\mathcal{S}$ is not splitting in $V[G_{\alpha+1}]$. Therefore $(\mathfrak{s} = \aleph_2)^{V[G]}$. Next notice that by Shoenfield absoluteness, $\mathcal{A}$ is defined by the same Π^1_1 formula in $V[G]$ as in V. Moreover $\mathcal{A}$ remains a tight mad family in $V[G]$ by Lemma 58, and therefore $\mathcal{A}$ provides a witness for $(\mathfrak{a} = \aleph_1)^{V[G]}$. This completes the proof. $\square$

4. Definable Spectra

In this section we consider projective mad families, with a definition of optimal complexity, which are of size $\kappa > \aleph_1$. We briefly give an account of the work in this direction thus far.

Friedman and Zdomskyy [FZ10] established that a tight mad family with optimal projective definition is consistent with $\mathfrak{b} = \mathfrak{c} = \aleph_2$ and this was extended by Fischer, Friedman and Zdomskyy [FFZ11] to $\mathfrak{b} = \mathfrak{c} = \aleph_3$. However, a previous result of Raghavan [Rag09] shows no tight mad family can contain a perfect subset. The Mansfield-Solovay theorem states that any Σ^1_2 set is either a subset of L, or contains a perfect set of nonconstructible reals. Therefore, no Σ^1_2 tight mad family can exist in a model of $\mathfrak{b} > \aleph_1$; the optimal definition of a tight mad family of size greater than $\aleph_2$ is thus Π^1_2. Dropping the tightness requirement, Brendle and Khomskii [BK13] constructed a model in which $\mathfrak{b} = \mathfrak{c} \geq \aleph_2$, and there exists a Π^1_1 mad family; this is shown to be consistent with a Δ^1_3 wellorder of the reals in Fischer, Friedman, and Khomskii [FFK13]. The first work on projective witnesses of size κ when $\aleph_1 < \kappa < \mathfrak{c}$ is done by Fischer, Friedman, Schrittesser, and Törnquist [FFST25]; again by the Mansfield-Solovay theorem, the best possible complexity of such an object is Π^1_2.

So far the attention has been on finding definable mad families witnessing the value of $\mathfrak{a}$— the minimal element of the *mad spectrum*, this being the set

$$\mathrm{spec}(\mathfrak{a}) = \{|\mathcal{A}| \mid \mathcal{A} \subseteq [\omega]^\omega \text{ is mad}\}.$$

Hechler [Hec72] pioneered the study of the mad spectrum, and in particular he introduces techniques with which one can include a set C as a subset of the spectrum, provided C satisfies a certain list of assumptions. His work was later pursed by Blass [Bla93], and Shelah and Spinas [SS15]. Similar considerations have then since been taken with regards to other cardinal characteristics, such as $\mathfrak{a}_T$ and $\mathfrak{i}$, where $\mathfrak{a}_T$ is the minimal size of a partition of ω^ω into compact sets, and $\mathfrak{i}$ is the minimal size of a maximal independent family (See [FS25], [Bri24] for the former, and [FS19], [FS22] for the latter). By now there is fairly substantial knowledge of the ways to control realizations of spec($\mathfrak{a}$) in various models of set theory, and likewise recent results provide methods for controlling the definability of mad families of size $\mathfrak{a}$.

One way to build upon these two lines of research is by asking that for each cardinal κ which is the possible size of a mad family, there exists a projective mad family of size κ with an optimal definition. The main result of this section is the following.

Theorem 64 (See Theorem 85)**.** *It is consistent with $\mathfrak{b} = \mathfrak{a} = \aleph_1 < \mathfrak{c} = \aleph_2$ that there exists a Π^1_1-definable tight mad family of size $\aleph_1$, and a Π^1_2-definable tight mad family of size $\aleph_2$.*

The strategy will be as follows. We begin in a model of $V = L$ and fix a Π^1_1 tight mad family $\mathcal{A}_1$. We recursively define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$, along the way constructing a Π^1_2 tight mad family $\mathcal{A}_2$ consisting of ω_2-many $\mathbb{Q}_\alpha$-generic reals a_α, and such that each $\mathbb{Q}_\alpha$ is forced by $\mathbb{P}_\alpha$ to be an S-proper poset strongly preserving the tightness of $\mathcal{A}_1$. The definition will be a slight modification of the iteration defined in the proof of [FZ10, Theorem 1], which showed the consistency of $\mathfrak{b} = \mathfrak{c} = \aleph_2$ with a Π^1_2-definable tight mad family. Specifically, they define a countable support iteration of S-proper forcing notions $\mathbb{Q}_\alpha$ such that:

(1) For cofinally many $\alpha < \omega_2$, $\mathbb{Q}_\alpha$ is a certain forcing $\mathbb{K}_\alpha$ (see Definition 70 below) which adds a generic real a_α as well as sequences $\vec{C}_\alpha$ and $\vec{Y}_\alpha$ so that:
   (a) $\mathcal{A}_\alpha \cup \{a_\alpha\}$ is almost disjoint, where $\mathcal{A}_\alpha$ is the union of elements constructed thus far;
   (b) For a $\mathbb{P}_\alpha$-generic filter G_α, $a_\alpha \cap b_i$ is infinite, where $\langle \dot{b}_i : i \in \omega \rangle$, given by a bookeeping function, is a sequence of $\mathbb{P}_\alpha$-names such that $\dot{b}_i^{G_\alpha} = b_i$ is an element of $\mathcal{I}(\mathcal{A}_\alpha)^+$ in $V[G_\alpha]$;

(c) $\vec{C}_\alpha = \langle C_{\alpha+m} : m \in \Delta(a_\alpha)\rangle$ is a countable sequence of generic club subsets of ω_1 and $\vec{Y}_\alpha = \langle Y_{\alpha+m} : m \in \Delta(a_\alpha)\rangle$ is a sequence of subsets of ω_1 localizing the addition of the generic clubs;
(d) Using almost disjoint coding, a_α codes the sequences $\vec{C}_\alpha$ and $\vec{Y}_\alpha$.

(2) For cofinally many $\alpha < \omega_2$, $\mathbb{Q}_\alpha = \mathbb{D}$, Hechler's forcing for adding a dominating real over $V[G_\alpha]$.

This ensures that when G is a $\mathbb{P}_{\omega_2}$-generic filter, in $V[G]$ we have that $\mathcal{A}_2 := \{a_\alpha \mid \alpha < \omega_2\}$ is a tight almost disjoint family; this is handled by items (1)(a) and (1)(b). Item (1)(c) renders $\mathcal{A}_2$ a Π^1_2-definable subset of $[\omega]^\omega$ in $V[G]$, similarly to how the Δ^1_3 wellorder was obtained in the previous section. Item (2) guarantees that there are no mad families of size $\aleph_1$ in $V[G]$ so that $\mathcal{A}_2$ is a witness to $\mathfrak{a} = \aleph_2)^{V[G]}$; this is because iterations of Hechler forcing $\mathbb{D}$ not only increase the size of $\mathfrak{d}$, but also increase $\mathfrak{b}$ (see, for example, [Bla10]), killing all mad families of size strictly less than the length of the iteration. This use of $\mathbb{D}$ in the consistency of $\mathfrak{b} = \mathfrak{c}$ with a Π^1_2 tight mad family was pointed out in [FZ10, Question 18], which asks about the consistency of $\mathfrak{b} < \mathfrak{a}$ with a Π^1_2 tight mad family, suggesting that it was unknown to the authors whether the iterand $\mathbb{K}_\alpha$ itself added a dominating real. The main result needed for Theorem 85 will show this is not the case.

Proposition 65 (See Proposition 80)**.** Let $\mathbb{K}$ be the Friedman-Zdomskyy forcing notion of Definition 70, and let $\mathcal{A}$ be a tight mad family in the ground model. Then $\mathbb{K}$ strongly preserves the tightness of $\mathcal{A}$.

With this, Theorem 85 can be achieved with a countable support iteration much as that of [FZ10, Theorem 1] outlined above, however we can modify item (2):

(2)' For all $\alpha < \omega_2$, unless explicitly stated otherwise, $\mathbb{Q}_\alpha$ is some S-proper poset which strongly preserves the tightness of $\mathcal{A}_1$.

This modification allows flexibility for further applications of the forcing,

Let us say more about how the Π^1_2-definability of $\mathcal{A}_2$ is achieved. Fixing $\langle S_\alpha : \alpha < \omega_2\rangle$ and S_{-1} as in Proposition 52 in a model of $V = L$, at stage α in defining the iteration we want a_α to uniquely determine a pattern of stationarity/nonstationarity in the sequence

$$\langle S_{\alpha+m} : m \in \omega\rangle,$$

namely by coding the sequence $\vec{C}_\alpha = \langle C_{\alpha+m} : m \in \Delta(a_\alpha)\rangle$, where $C_{\alpha+m}$ is a generic club disjoint from $S_{\alpha+m}$ for every $m \in \Delta(a_\alpha)$. This will give that in the final extension, $\mathcal{A}_2$ is an element of $L(\mathbb{R})$, since membership in $\mathcal{A}_2$ can be defined as:

$$(\star) \qquad a \in \mathcal{A}_2 \Leftrightarrow \exists \alpha \in \omega_2\ L[a_\alpha] \models \Delta(a_\alpha) = \{m \in \omega \mid S_{\alpha+m} \in \mathrm{NS}_{\omega_1}\}.$$

Therefore another job of the iterand $\mathbb{K}_\alpha$ is to add such generic clubs, and so conditions will consist of a *finite part*, taking care of approximations to the set a_α, and an *infinite part*, making countable approximations to a club in $\omega_1 \setminus S_{\alpha+m}$. It is important that for $m \notin \Delta(a_\alpha)$, $S_{\alpha+m}$ remains stationary, so this simultaneous construction must be carried out carefully. To make the definiton of $\mathcal{A}_2$ projective obtain a projective definition of $\mathcal{A}_2$, the right hand side of $(\star)$

is localized to the class of countable suitable models (see Definition 42), again relying on the localization techniques of René David appearing in Definition 43. For this purpose $\mathbb{K}_\alpha$ adds the sequence $\langle Y_{\alpha+m} : m \in \Delta(a_\alpha)\rangle$ such that for each $Y_{\alpha+m}$ the set $\{\beta < \omega_1 \mid Y^\alpha_m(2\beta) = 1\}$ is a sufficiently absolute code for the ordinal $\alpha < \omega_2$. As a consequence, the initial existential quantification over ω_2 in $(\star)$ will range over the $\omega_2^{\mathcal{M}}$ for countable suitable $\mathcal{M}$ containing the real a_α. Formally, in the final generic extension $V[G]$, for all $a \in V[G] \cap [\omega]^\omega$,

$$\begin{aligned} a \in \mathcal{A}_2 \Leftrightarrow & \forall \mathcal{M}(\mathcal{M} \text{ is a countable suitable model and } a \in \mathcal{M}) \\ & \exists \overline{\alpha} < \omega_2^{\mathcal{M}}\ \forall m \in \Delta(a)(\mathcal{M} \models S_{\overline{\alpha}+m} \text{ is nonstationary}). \end{aligned}$$

The right hand side of the above is in the form $\forall\exists$ and thus is a Π^1_2 formula.

For the proof of Theorem 85 we first establish preliminaries; throughout the rest of the section we work in a model of $V = L$ unless explicitly stated otherwise. Fix a coanalytic tight mad family $\mathcal{A}_1$, as well as $\vec{S} = \langle S_\alpha : \alpha < \omega_2\rangle$ a sequence of pairwise almost disjoint stationary subsets of ω_1, and a stationary subset $S_{-1} \subseteq \omega_1$ such that $S_{-1} \cap S_\alpha = \emptyset$ for all $S_\alpha \in \vec{S}$, given by Proposition 52. Let $F\colon \mathrm{Lim}(\omega_2) \to L_{\omega_2}$ be such that $F^{-1}(x)$ is unbounded in ω_2 for all $x \in L_{\omega_2}$. By [FF10, Lemma 14], we can take $\vec{S}$ and F to be Σ_1-definable over L_{ω_2}, and moreover that whenever $\mathcal{M}, \mathcal{N}$ are suitable models with $\omega_1^{\mathcal{M}} = \omega_1^{\mathcal{N}}$, then $\vec{S}^{\mathcal{M}}$ and $\vec{S}^{\mathcal{N}}$ agree on $\omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}}$.

Fix $\psi(x, y)$ given by Fact 57 and for all $\alpha < \omega_2^L$, let X_α denote a sufficiently absolute code for α.

Whereas the coding of the Δ^1_3 wellorder was achieved by the $C(Y)$-generic reals, the generic reals in the current context will result from almost disjoint coding, a method developed by Solovay and Jensen [SJ70], and for which we give a general definition.

Definition 66. Let $\mathcal{R}$ be an almost disjoint family in V, and let $X \in V$ be a subset of ω_1. The *almost disjoint coding of X with respect to $\mathcal{R}$* is the partial order $\mathbb{P}_{\mathcal{R}}(X)$ consisting of conditions (s, F) such that s is a finite subset of ω and F is a finite subset of $\{r_\xi \mid \xi \in X\} \subseteq \mathcal{R}$. The extension relation is defined by letting $(t, G) \le (s, F)$ if and only if

(1) t end-extends s, $G \supseteq F$;
(2) For all $\xi \in X$ and $r_\xi \in F$, $(t \setminus s) \cap r_\xi = \emptyset$.

Fact 67. The following hold.

- If $(s, F), (t, G)$ are compatible conditions in $\mathbb{P}_{\mathcal{R}}(X)$, then they admit a minimal lower bound $(s \cup t, F \cup G)$.
- $\mathbb{P}_{\mathcal{R}}(X)$ is σ-centered.
- If G is $\mathbb{P}_{\mathcal{R}}(X)$-generic over M, let
$$a := \bigcup\{s \mid \exists F(s, F) \in G\}.$$
Then G and a are mutually definable in the generic extension; that is, $M[G] = M[a]$.

Lemma 68. Let G be a $\mathbb{P}_{\mathcal{R}}(X)$-generic filter over V, and let a be the generic real defined by G as above. Then in $V[a]$, for all $\xi < \omega_1$:

$$\xi \in X \Leftrightarrow a \cap r_\xi \text{ is finite.}$$

For proofs of the above, see, for example, [Har77] or [Jec97, Example IV]. A proof similar to that of the last lemma will be given in Lemma 81.

For the present coding purposes we therefore fix an almost disjoint family

$$\mathcal{R} = \{R_{\langle\eta,\xi\rangle} \mid \eta \in \omega \cdot 2, \xi \in \omega_1\}$$

which is Σ_1-definable over L_{ω_1}, and such that for every suitable model $\mathcal{M}$, $\mathcal{R} \cap \mathcal{M} = \{R_{\eta,\xi} \mid \eta \in \omega \cdot 2, \xi \in \omega_1^{\mathcal{M}}\}$. For an example of such a family see [FZ10, Proposition 3].

Recall in the previous section we had defined a function $\Delta \colon \omega^\omega \to \omega^\omega$ such that $\Delta(x)$ was a real coding both x and $\omega \setminus x$. Because of technical reasons in the coding we will do, we will modify the definition of this function, and in particular extend its definition to the finite subsets. Specifically, for $s \subseteq \omega$, finite or infinite, let

$$\Delta(s) = \{2n + 1 \mid n \in s\} \cup \{2n + 2 \mid n \in (\sup s \setminus s)\},$$

and let $C(s) = \Delta(s) \cup (\omega \setminus \max(\Delta(s)))$. We think of $C(s)$ as the "coding area" associated with the finite subset s. Denote by $E(s), O(s)$ the sets $\{s(2n) \mid 2n < |s|\}$ and $\{s(2n+1) \mid 2n+1 < |s|\}$ respectively, where $s(\ell)$ denotes the nth element of s for every $\ell < |s|$. For a limit ordinal γ and a function $r \colon \gamma \to 2$, let $\mathrm{Even}(r) = \{\alpha < \gamma \mid r(2\alpha) = 1\}$ and $\mathrm{Odd}(r) = \{\alpha < \gamma \mid r(2\alpha + 1) = 1\}$.

Lastly, for ordinals $\alpha < \beta$, let $\beta - \alpha$ denote the ordinal γ such that $\alpha + \gamma = \beta$. If B is a set of ordinals, let $B - \alpha = \{\beta - \alpha \mid \beta \in \beta\}$. If δ is an indecomposable ordinal, i.e. $\delta = \omega^\gamma$ for some ordinal γ, it straightforward to check $(\alpha + B) \cap \delta - \alpha = B \cap \delta$.

Recall our goal is to define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2\rangle$, such that for all $\alpha < \omega_2$, $\dot{\mathbb{Q}}_\alpha$ is forced to be an S_{-1}-proper poset which strongly preserves the tightness of $\mathcal{A}_1$, and along the way we construct a Π^1_2 tight mad family $\mathcal{A}_2 = \{a_\alpha \mid \alpha < \omega_2\}$ consisting of $\mathbb{Q}_\alpha$-generic reals a_α.

Continuing with the recursive definition, suppose $\alpha < \omega_2$ and $\mathbb{P}_\alpha$ has been defined. Let G_α be a $\mathbb{P}_\alpha$-generic filter, and let $\dot{\mathcal{A}}^2_\alpha$ be a $\mathbb{P}_\alpha$-name for the set of elements of $\mathcal{A}_2$ constructed up to stage α. The density arguments of Lemma 81 require the following inductive assumption:

$$(*) \qquad \forall r \in \mathcal{R} \forall A' \in [\mathcal{A}^2_\alpha]^{<\omega} \; (|\mathrm{E}(r) \setminus \bigcup A'| = |\mathrm{O}(r) \setminus \bigcup A'| = \aleph_0),$$

Since $\mathcal{R}$ is an almost disjoint family, $(*)$ implies that for every $A' \in [\mathcal{A}^2_\alpha \cup \mathcal{R}]^{<\omega}$ and $r \in \mathcal{R} \setminus A'$, also $|E(r) \setminus \bigcup A'| = |O(r) \setminus \bigcup A'| = \aleph_0$, as otherwise $r \cap r'$ is infinite for some $r' \in A' \cap \mathcal{R}$.

If $\alpha < \omega_2$ is a successor ordinal, let $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{Q}}^0_\alpha$ be a $\mathbb{P}_\alpha$-name for a proper poset of cardinality $\aleph_1$ such that $\Vdash_{\mathbb{P}_\alpha}$"$\dot{\mathbb{Q}}^0_\alpha$ strongly preserves the tightness of $\mathcal{A}_1$". $\mathbb{Q}^0_\alpha$ is reserved so to yield shorter proofs of our results in the subsequent sections.

For limit $\alpha \in \omega_2$, unless explicitly mentioned otherwise, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for the trivial poset. Suppose $F(\alpha)$ is a sequence $\langle \dot{x}_i : i \in \omega\rangle$ of $\mathbb{P}_\alpha$-names such that $x_i = \dot{x}_i^{G_\alpha}$ is an infinite subset of ω such that for all $i \in \omega$, $x_i \in \mathcal{I}(\mathcal{A}^2_\alpha)^+$.

Claim 69. There exists a limit ordinal $\eta_\alpha \in \omega_1$ with the property that for there exist no finite subsets J, E of $\omega \cdot 2 \times (\omega_1 \setminus \eta_\alpha), \mathcal{A}^2_\alpha$, respectively, and $i \in \omega$ such that $x_i \subseteq \bigcup_{\langle\eta,\xi\rangle \in J} R_{\langle\eta,\xi\rangle} \cup \bigcup E$.

Proof. Let I denote the set of all $i \in \omega$ for which there exists $J_i \in [\omega \cdot 2 \times \omega_1]^{<\omega}$ and $E_i \in [\mathcal{A}^2_\alpha]^{<\omega}$ such that $x_i \subseteq \bigcup_{\langle\eta,\xi\rangle \in J_i} R_{\langle\eta,\xi\rangle} \cup \bigcup E_i$. Let $\eta_\alpha \in \omega_1$ be a limit ordinal such that for all $i \in I$, if

$\langle\gamma,\eta\rangle \in J_i$, then $\eta < \eta_\alpha$. Such a limit ordinal exists, since for each $i \in I$ there are only countably many choices for the value of the second coordinate of J_i, as J_i is finite.

We will show that this η_α satisfies the conclusion of the claim. Fix $i \in I$, $J \in [\omega\cdot 2\times(\omega_1\setminus\eta_\alpha)]^{<\omega}$, and $E \in [\mathcal{A}^2_\alpha]^{<\omega}$; we show that $x_i \not\subseteq \bigcup_{\langle\eta,\xi\rangle\in J} R_{\langle\eta,\xi\rangle} \cup \bigcup E$. Since E, E_i are finite subsets of $\mathcal{A}^2_\alpha$, by hypothesis on x_i we have that $c_i := x_i \setminus \bigcup E \cup \bigcup E_i$ is infinite. But since $i \in I$, this means that $c_i \subseteq^* \bigcup_{\langle\eta,\xi\rangle\in J_i} R_{\langle\eta,\xi\rangle}$. Notice that as $\mathcal{R}$ is an almost disjoint family and $J \cap J_i = \emptyset$, we have that $\bigcup_{\langle\eta,\xi\rangle\in J_i} R_{\langle\eta,\xi\rangle} \cap \bigcup_{\langle\eta,\xi\rangle\in J} R_{\langle\eta,\xi\rangle}$ is finite. Therefore $c_i \cap \bigcup_{\langle\eta,\xi\rangle\in J} R_{\langle\eta,\xi\rangle}$ is finite. In particular $c_i \not\subseteq \bigcup_{\langle\eta,\xi\rangle\in J} R_{\langle\eta,\xi\rangle}$, and thus $x_i \not\subseteq \bigcup_{\langle\eta,\xi\rangle\in J} R_{\langle\eta,\xi\rangle} \cup \bigcup E$ as desired. □

Fix $\eta_\alpha \in \omega_1$ as given by the above claim. Fix also $Z_\alpha \subset \omega$ coding a surjection of ω onto η_α. The following is the original definition from [FZ10, Section 3].

Definition 70. (Friedman, Zdomskyy; [FZ10]) The partial order $\mathbb{K}_\alpha$ consists of conditions of the form $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$, such that:

(1) $c_k \subseteq \omega_1 \setminus \eta_\alpha$ is a closed bounded subset such that $S_{\alpha+k} \cap c_k = \emptyset$;
(2) $y_k \colon |y_k| \to 2$ is a function from a countable limit ordinal $|y_k| \in \omega_1$ such that
   - $|y_k| > \eta_\alpha$, $y_k \restriction \eta_\alpha = 0$;
   - for all $\gamma < |y_k|$, $y_k(\eta_\alpha + 2\gamma) = 1$ if and only if $\gamma = \eta_\alpha$ or $\gamma > \eta_\alpha$ and $\mathrm{Even}(y_k) = \{\eta_\alpha\} \cup (\eta_\alpha + X_\alpha)$.[6]

(3) $s \in [\omega]^{<\omega}$ and s^* is a finite subset of the set
$$\{R_{\langle m,\xi\rangle} \mid m \in \Delta(s), \xi \in c_m\} \cup \{R_{\langle \omega+m,\xi\rangle} \mid m \in \Delta(s),\ y_m(\xi) = 1\} \cup \mathcal{A}^2_\alpha.$$
Additionally, for all $n \in \omega$ such that $2n < |s \cap R_{\langle 0,0\rangle}|$, $n \in Z_\alpha$ if and only if there exists $m \in \omega$ such that $(s \cap R_{\langle 0,0\rangle})(2n) = R_{\langle 0,0\rangle}(2m)$;
(4) For all $k \in C(s)$ and for all limit ordinals $\gamma \in \omega_1$ such that $\eta_\alpha < \gamma \le |y_k|$, if γ is a limit point of c_k and $\gamma = \omega_1^{\mathcal{M}}$ for some countable suitable model $\mathcal{M}$ containing both $y_k \restriction \gamma$ and $c_k \cap \gamma$ as elements, then the following holds in $\mathcal{M}$: " $[\mathrm{Even}(y_k) - \min(\mathrm{Even}(y_k))] \cap \gamma$ is the code of some limit ordinal $\overline{\alpha} \in \omega_2$ such that $S_{\overline{\alpha}+k}$ is nonstationary."

For $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ and $q = \langle\langle t, t^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$ conditions in $\mathbb{K}_\alpha$, define q to be an extension of p and write $q \le p$ if and only if:

(1) t end-extends s, $t^* \supseteq s^*$, and for all $x \in s^*$, $(t \setminus s) \cap x = \emptyset$;
(2) For all $k \in C(t)$, d_k end-extends c_k and $z_k \supseteq y_k$.

For $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha$, let $\mathrm{Fin}(p) = \langle s, s^*\rangle$ denote the finite part of p, and let $\mathrm{Inf}(p) = \langle c_k, y_k : k \in \omega\rangle$ denote the infinite part of p. When $p, q \in \mathbb{K}_\alpha$ and $q \le p$, we say q is a *pure extension of* p if $\mathrm{Fin}(p) = \mathrm{Fin}(q)$.

Remark 71. The notion $\mathbb{K}_\alpha$ can be seen as a hybrid of the forcing notions of almost disjoint coding (Definition 66), club shooting (Definition 34), and localization (Definition 43). This is made more explicit in the proof of Lemma 77 below.

[6] Recall X_α denotes the sufficiently absolute code for α given by Fact 57.

The following is our main result of this section; it will establish both properness of the iteration and preservation and of the tight mad family $\mathcal{A}_1$.

Proposition 72 (See Proposition 80)**.** For every $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha$, every θ sufficiently large and countable elementary submodel $M \prec H_\theta$ containing $p, \mathbb{K}_\alpha, \mathcal{A}_1$, and every $B \in \mathcal{I}(\mathcal{A}_1)$ such that $B \cap Y$ is infinite for all $Y \in \mathcal{I}(\mathcal{A}_1)^+ \cap M$, if $M \cap \omega_1 = j \notin \bigcup_{k \in C(s)} S_{\alpha+k}$, then there is an $(M, \mathbb{K}_\alpha, \mathcal{A}_1, B)$-generic condition $q \leq p$ such that $\mathrm{Fin}(q) = \mathrm{Fin}(p)$.

To prove Proposition 80 we need the following intermediary lemmas.

Lemma 73 ([FF10, Lemma 1])**.** For every $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ in $\mathbb{K}_\alpha$ and every $\gamma \in \omega_1$, there exists a pure extension $q \leq p$ with $\mathrm{Inf}(q) = \langle d_k, z_k : k \in \omega\rangle$, such that $|z_k| \geq \gamma$ and $\max(d_k) \geq \gamma$, for every $k \in \omega$.

Proof. Fix $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ in $\mathbb{K}_\alpha$ and $k \in \omega$, and suppose $\gamma \in \omega_1$ is such that $\gamma > |y_k|$ and $\gamma > \max(c_k)$.

First we extend c_k. Since $\omega_1 \setminus S_{\alpha+k}$ is stationary, and for any $\eta \in \omega_1$ the set

$$T_\eta = \{\xi \in \omega_1 \mid \xi \text{ is a limit ordinal, } \xi \geq \eta\}$$

is a club, there exists $\xi \in T_{\eta_\alpha} \cap T_\gamma \cap (\omega_1 \setminus S_{\alpha+k})$. Let $\langle \xi_n : n \in \omega\rangle$ be an increasing cofinal sequence with limit ξ such that $\xi_n \geq |y_k|$ and $\xi_n \notin S_{\alpha+k}$ for all $n \in \omega$. Define $d_k := c_k \cup \{\xi_n \mid n \in \omega\} \cup \{\xi\}$. Then d_k is a bounded subset of $\omega_1 \setminus \eta_\alpha$ with $d_k \cap S_{\alpha+k} = \emptyset$, and it is closed since any increasing sequence in $d_k \setminus c_k$ has limit $\xi \in d_k$.

Next we extend y_k. Let $W \subseteq \omega$ be any code for a bijection between ω and ξ. Define $z_k \colon \xi \to 2$ so that $z_k \restriction |y_k| = y_k$, and $\mathrm{Odd}(z_k \restriction [|y_k|, |y_k| + \omega)) = W$. For $\beta \in [|y_k| + \omega, \xi)$, let $z_k(\beta) = 0$.

Then $\langle\langle s, s^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$ is a condition in $\mathbb{K}_\alpha$; the only item to be checked is (4) of Definition 70, for limit ordinals η such that $|y_k| < \eta \leq |z_k|$. For such η, $\gamma \geq |y_k| + \omega$, so if $\mathcal{M}$ is a transitive model such that $\eta = \omega_1^{\mathcal{M}}$ and $z_k \restriction \eta \in \mathcal{M}$, then $\mathcal{M} \models$"$\eta = \omega_1^{\mathcal{M}}$ is countable", and hence $\mathcal{M}$ cannot be a model of ZF^-. Therefore (4) is vacuously true. $\square$

The following notion appears implicitly in [FZ10].

Definition 74. For a condition $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ in $\mathbb{K}_\alpha$ and open dense $D \subseteq \mathbb{K}_\alpha$, we say p is *preprocessed* for D if and only if for every extension $q = \langle\langle t, t^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle \leq p$, if $q \in D$, then already there is some t^*_2 such that $q' = \langle\langle t, t^*_2\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ is a condition in $\mathbb{K}_\alpha$ extending p, and $q' \in D$.

Lemma 75. Suppose $p \in \mathbb{K}_\alpha$ is preprocessed for a dense open set D, and let $r \leq p$. Then r is preprocessed for D.

Proof. Let $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ be preprocessed for D and let $r \leq p$, with $r = \langle\langle t, t^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$. If $q = \langle\langle t_1, t^*_1\rangle, \langle d'_k, z'_k : k \in \omega\rangle\rangle$ is any extension of r such that $q \in D$, then also $q \leq p$ and so by preprocessedness of p there is t^*_2 such that $\langle\langle t_1, t^*_2\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in D$. Let $t^*_3 = t^*_2 \cup t^*$. Then $\langle\langle t_1, t^*_3\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$ is an extension of r which is also an element of D, as D is open and $\langle\langle t_1, t^*_3\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle \leq \langle\langle t_1, t^*_1\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in D$.

$\square$

The properness of Shelah's forcing $\mathbb{Q}$ required the set of pure extensions which are preprocessed for a given open dense set to be dense in $\mathbb{Q}$. This will also be the case for establishing the S_{-1}-properness of $\mathbb{K}_\alpha$ and motivates the next lemma.

Lemma 76 ([FZ10, Claim 9])**.** For any $p \in \mathbb{K}_\alpha$ and open dense $D \subseteq \mathbb{K}_\alpha$, there exists a pure extension $q \leq p$ such that q is preprocessed for D.

Proof. Let $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha$ and let $D \subseteq \mathbb{K}_\alpha$ be an open dense set. Let $M \prec H_\theta$ be a countable elementary submodel, where θ is sufficiently large and M contains $p, \mathbb{K}_\alpha, X_\alpha$, and D as elements. Let $j = M \cap \omega_1$. Since $S_{\alpha+k}$ is costationary for all $k \in C(s)$, without loss of generality we may assume $j \notin \bigcup_{k\in C(s)} S_{\alpha+k}$.

Let $\{\langle r_n, s_n\rangle \mid n \in \omega\}$ enumerate all pairs $\langle r, s\rangle \in (\mathbb{K}_\alpha \cap M) \times [\omega]^{<\omega}$ such that $r \leq p$ and each pair appears infinitely often. Let also $\langle j_n : n \in \omega\rangle$ be an increasing cofinal sequence in j with $\{j_n \mid n \in \omega\} \subseteq M$. By induction on $n \in \omega$, construct sequences $\langle d^n_k, z^n_k : k \in \omega\rangle \in M$ such that:

(1) $d^0_k = c_k$, $z^0_k = y_k$ for all $k \in \omega$;

(2) <u>*If*</u> there exists some $\tilde{r} \in \mathbb{K}_\alpha \cap M$ such that:
   (a) $\tilde{r} \leq r_n$;
   (b) $\tilde{r} \leq \langle\langle s, s^*\rangle, \langle d^n_k, z^n_k : k \in \omega\rangle\rangle$;
   (c) $\tilde{r} \in D$;
   (d) $\mathrm{Fin}(\tilde{r}) = \langle s_n, t^*\rangle$ and $\mathrm{Inf}(\tilde{r}) = \langle d'_k, z'_k : k \in \omega\rangle$ for some t^* and some $\langle d'_k, z'_k : k \in \omega\rangle$;

   <u>*then*</u> let $\langle d^{n+1}_k, z^{n+1}_k : k \in \omega\rangle$ be an extension of $\mathrm{Inf}(\tilde{r})$, in the sense that d^{n+1}_k end-extends d'_k and $z^{n+1}_k \supseteq z'_k$, so that $\langle\langle s, s^*\rangle, \langle d^{n+1}_k, z^{n+1}_k : k \in \omega\rangle\rangle$ is a condition in $\mathbb{K}_\alpha$ with $\max(d^{n+1}_k) \geq j_n$ and $|z^{n+1}_k| \geq j_n$ for each $k \in C(s)$. To be precise, for all $k \in C(s_n)$, Lemma 73 gives the existence of d^{n+1}_k and z^{n+1}_k as desired, though to ensure $\langle\langle s, s^*\rangle, \langle d^{n+1}_k, z^{n+1}_k : k \in \omega\rangle\rangle$ is a condition in $\mathbb{K}_\alpha$ below $p_n := \langle\langle s, s^*\rangle, \langle d^n_k, z^n_k : k \in \omega\rangle\rangle$, we need to also take care of $k \in C(s) \setminus C(s_n)$; this is still taken care of by Lemma 73 applied to p_n.

   If no such $\tilde{r}$ exists, let $\langle d^{n+1}_k, z^{n+1}_k : k \in \omega\rangle$ be any extension of $\mathrm{Inf}(\tilde{r})$ such that $\langle\langle s, s^*\rangle, \langle d^{n+1}_k, z^{n+1}_k : k \in \omega\rangle\rangle$ is a condition in $\mathbb{K}_\alpha$ with $\max(d^{n+1}_k) \geq j_n$ and $|z^{n+1}_k| \geq j_n$ for each $k \in C(s)$.

Set $d_k = \bigcup_{n\in\omega} d^n_k \cup \{j\}$ and $z_k = \bigcup_n z^n_k$ for all $k \in C(s)$, and for $k \notin C(s)$ let $d_k = z_k = \emptyset$.

We verify that $q := \langle\langle s, s^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$ is a condition in $\mathbb{K}_\alpha$. That clauses (1) and (2) hold in the definition of $\mathbb{K}_\alpha$ follows from the fact $j = M \cap \omega_1$ is a countable limit ordinal. As $\mathrm{Fin}(q) = \mathrm{Fin}(p)$, (3) is immediately satisfied. It suffices to check item (4) for $k \in C(s)$ and the limit ordinal $\gamma = j$. Fix $k \in C(s)$ and let $\mathcal{M}$ be a countable suitable model with $\omega_1^{\mathcal{M}} = \gamma$ and $z_k \restriction \gamma, d_k \cap \gamma \in \mathcal{M}$. Since γ is an indecomposable ordinal, $[\mathrm{Even}(z_k) - \min(\mathrm{Even}(z_k))] \cap \gamma = X_\alpha \cap \gamma \in \mathcal{M}$. Since X_α is a sufficiently absolute code for α, in $\mathcal{M}$ it holds that there exists a unique $\tilde{\alpha} \in \omega_2^{\mathcal{M}}$ such that $\mathcal{M} \models$"$X_\alpha \cap \gamma$ codes $\tilde{\alpha}$".

Let $\pi\colon M \to \overline{M}$ be the Mostowski collapse isomorphism, and so note that $\omega_1^{\overline{M}} = \pi(M \cap \omega_1) = \gamma$, i.e. $\omega_1^{\mathcal{M}} = \omega_1^{\overline{M}}$. As M was an elementary submodel of H_θ and $X_\alpha \in M$, in M the ordinal α is uniquely coded by X_α, and so in $\overline{M}$ the ordinal $\pi(\alpha) = \overline{\alpha}$ is the unique solution to $\psi(x, \pi(X_\alpha))$, that is, $\overline{\alpha}$ is uniquely coded by $\pi(X_\alpha) = X_\alpha \cap \gamma$. Again since we had chosen X_α to be a sufficiently absolute code for α and $\overline{M}, \mathcal{M}$ are suitable models with $\omega_1^{\overline{M}} = \omega_1^{\mathcal{M}}$, we have that in both $\mathcal{M}$ and $\overline{M}$, the unique ordinal coded by $X_\alpha \cap \gamma$ is $\overline{\alpha} = \tilde{\alpha}$.

In M, $(d_k \cap S_{\alpha+k}) \cap \gamma = \emptyset$, so by elementarity, in $\overline{M}$, $\pi(d_k) = d_k \cap \gamma$ is disjoint from $\pi(S_{\alpha+k}) = S^{\overline{M}}_{\overline{\alpha}+k} = S_{\overline{\alpha}+k} \cap \gamma$. Again as $\omega_1^{\overline{M}} = \omega_1^{\mathcal{M}}$, by choice of the sequence $\vec{S}$, we have $S^{\overline{M}}_{\overline{\alpha}+k} = S^{\mathcal{M}}_{\overline{\alpha}+k}$, so also in $\mathcal{M}$ it holds that $d_k^{\mathcal{M}} = d_k \cap \gamma$ is disjoint from $S^{\mathcal{M}}_{\overline{\alpha}+k}$. Since $d_k \cap \gamma \in \mathcal{M}$ and $d_k \cap \gamma$ is a closed unbounded set in $\omega_1^{\mathcal{M}}$, $\mathcal{M} \models$"$S_{\overline{\alpha}+k}$ is nonstationary". Therefore $q \in \mathbb{K}_\alpha$.

Let us see that $q \in \mathbb{K}_\alpha$ is as desired. Take any $r \leq q$; without loss of generality $r \in D$. Write $r = \langle\langle t, t^*\rangle, \langle d'_k, z'_k : k \in \omega\rangle\rangle$. Since both t, t^* are finite objects and for all $n \in \omega$, $\langle d^n_k, z^n_k : k \in \omega\rangle \in M$, there exists $m \in \omega$ such that $r^M = \langle\langle t, t^* \cap M\rangle, \langle d^m_k, z^m_k : k \in \omega\rangle\rangle$ is a condition in $\mathbb{K}_\alpha \cap M$, where $t^* \cap M$ is a finite subset of

$$\{R_{\langle \ell,\xi\rangle} \mid \ell \in \Delta(t), \xi \in d'_k \cap j\} \cup \{R_{\langle \omega+\ell,\xi\rangle} \mid \ell \in \Delta(t), y'_k \restriction j(\xi) = 1\}$$
$$\cup \{a_\beta \mid \beta < \alpha \cap M\}.$$

Note that $r \leq r^M$, and also $r^M \leq p$. Therefore there exists $n \geq m$ such that $r^M = r_n$ and $t = s_n$. Note that $r \in \mathbb{K}_\alpha$ has properties (a)-(d) above: (a) and (b) follow from the fact $r \leq q, r^M$, and (c) is immediate by assumption. Lastly (d) follows from the fact $t = s_n$. Then in H_θ it holds:

$$\exists x (x \in \mathbb{K}_\alpha \wedge x \text{ satisfies (a)-(d)}),$$

so by elementarity M satisifes the same formula, i.e., there exists some condition $\tilde{r} \in \mathbb{K}_\alpha \cap M$, $\tilde{r}$ has properties (a)-(d). In particular there are t_2^* and $\langle \tilde{d}_k, \tilde{z}_k : k \in \omega\rangle$ in M with the properties that $t_2^* \supseteq s^* \cap (t^* \cap M)$ and $\tilde{d}_k, \tilde{z}_k$ end-extend d^n_k, z^n_k respectively for $k \in C(s_n)$, and

$$\tilde{r} = \langle\langle s_n = t, t_2^*\rangle, \langle \tilde{d}_k, \tilde{z}_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha \cap M.$$

Then at this stage of the inductive construction we chose d^{n+1}_k, z^{n+1}_k to be appropriate end-extensions of $\tilde{d}_k, \tilde{z}_k$ respectively for all $k \in C(s)$. One can verify that $\overline{r} = \langle\langle s_n, t_2^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$ is a condition in $\mathbb{K}_\alpha$ extending q, and since $\overline{r} \leq \tilde{r}$ and D is open, $\overline{r} \in D$. □

Lemma 77. Let $q \in \mathbb{K}_\alpha \cap M$, where $M \prec H_\theta$ is a countable elementary submodel containing $\mathbb{K}_\alpha$ and $\mathcal{A}_1$, and let $\dot{Z}$ be a $\mathbb{K}_\alpha$-name for an element of $\mathcal{I}(\mathcal{A}_1)^+$. Then

$$W = \{m \in \omega \mid \exists p \leq q \ (\mathrm{Fin}(p) = \mathrm{Fin}(q) \wedge p \Vdash m \ \in \dot{Z})\}$$

is an element of $\mathcal{I}(\mathcal{A}_1)^+ \cap M$.

Proof. Fix a finite $F \subseteq \mathcal{A}_1 \cap M$. We have that

$$\Vdash_{\mathbb{K}_\alpha} \dot{Z} \setminus \bigcup F \text{ is infinite},$$

so in particular $q \Vdash_{\mathbb{K}_\alpha} \dot{Z} \setminus \bigcup F \in [\omega]^\omega$.

Lemma 78. For all $q \in \mathbb{K}_\alpha$ and $\dot{X}$ a $\mathbb{K}_\alpha$-name for an infinite subset of ω, there exists $p \leq q$ with $\mathrm{Fin}(p) = \mathrm{Fin}(q)$ and there exists $m^p \in \omega$ such that $p \Vdash m^p = \dot{X}(j)$, where $\dot{X}(j)$ denotes the j-th element of $\dot{X}$.

Proof. Fix q and $\dot{X}$ as above, and write $q = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$. Consider the countable support product

$$\mathbb{P}_s := \prod_{k \in C(s)} Q^{\eta_\alpha}(S_{\alpha+k}) \times \mathcal{L}^{\eta_\alpha}(Y_{\alpha+k}),$$

where $Q^{\eta_\alpha}(S_{\alpha+k})$ consists of closed bounded subsets $c_k \subseteq \omega_1 \setminus \eta_\alpha$ such that $c_k \cap S_{\alpha+k} = \emptyset$ and is ordered by end extension; the partial order $\mathcal{L}^{\eta_\alpha}(Y_{\alpha+k})$ consists of functions $y_k \colon |y_k| \to 2$ with domain $|y_k|$, where $|y_k| \in \omega_1 \setminus \eta_\alpha$ is a countable limit ordinal, such that

- $|y_k| \in \omega_1 \setminus \eta_\alpha$ is a countable limit ordinal and $y_k \restriction \eta_\alpha = 0$;
- $\mathrm{Even}(y_k) = (\{\eta_\alpha\} \cup (\eta_\alpha + X_\alpha)) \cap |y_k|$;
- for all $\gamma \leq |y_k|$, if $\gamma = \omega_1^{\mathcal{M}}$ for some suitable model such that $y_k \restriction \gamma \in \mathcal{M}$ and γ is a limit point of c_k, then $\mathcal{M} \models$"$\mathrm{Even}(y_k)$ is the code for some $\overline{\alpha} \in \omega_1$ such that $S_{\overline{\alpha}+k}$ is nonstationary ".

$\mathcal{L}^{\eta_\alpha}(Y_{\alpha+k})$ is ordered by end-extension. For notational simplicity we will suppress the superscript η_α in what follows.

The ordering on $\mathbb{P}_s$ is defined by $\langle d_k, z_k : k \in \omega\rangle \leq \langle c_k, y_k : k \in \omega\rangle$ if and only if d_k is an end-extension of c_k and $z_k \supseteq y_k$, i.e., if and only if $(d_k, z_k) \leq_{Q(S_{\alpha+k}) \times \mathcal{L}(Y_{\alpha+k})} (c_k, y_k)$.

For all $k \in C(s)$, find c'_k, y'_k such that $c'_k \leq_{Q(S_{\alpha+k})} c_k$ and $y'_k \leq_{\mathcal{L}(Y_{\alpha+k})} y_k$, and

$$(c_k, y_k) \Vdash_{Q(S_{\alpha+k}) \times \mathcal{L}(Y_{\alpha+k})} \dot{X}(j) = \check{m}_j$$

for some $m_j \in \omega$.

Then $\langle c'_k, y'_k : k \in \omega\rangle \in \mathbb{P}_s$ is an extension of $\langle c_k, y_k : k \in \omega\rangle$ and forces $\dot{X}(j) = \check{m}_j$. Therefore $\langle \mathrm{Fin}(q), \langle c'_k, y'_k : k \in \omega\rangle\rangle \leq_{\mathbb{K}_\alpha} q$ and decides $\dot{X}(j)$. □

Therefore, for every $k \in \omega$ there exists $j > k$ and a pure extension $q_j \leq q$ in $\mathbb{K}_\alpha$, and there exists $m_j \in \omega$ with $q \Vdash \dot{X}(j) = \check{m}_j$. Then

$$Y_k = \{m_j \mid j > k, q_j \leq q \wedge q_j \Vdash \check{m}_j = \dot{X}(j)\}$$

is an infinite set such that for all $m_j \in Y$, $m_j \in W$ as witnessed by q_j, and moreover $m_j \notin \bigcup F$ since $q_j \Vdash m_j \in \dot{Z} \setminus \bigcup F$. This shows $W \setminus \bigcup F$ is infinite, proving the lemma. □

Remark 79. We can make the following observations about the forcings $\mathbb{P}_s$, for $s \in [\omega]^{<\omega}$, defined in the above proof.

- If $\langle c_k, y_k : k \in \omega\rangle \Vdash_{\mathbb{P}_s} \dot{X}(j) = m_j$, then $\langle\langle\emptyset, \emptyset\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \Vdash_{\mathbb{K}_\alpha} \dot{X}(j) = m_j$.
- $\mathbb{P}_s$ is a complete suborder of $\mathbb{P}_\emptyset$.
- For all $s \in [\omega]^{<\omega}$, $\mathbb{P}_s$ does not add new reals. This follows from the fact $Q(S_{\alpha+k})$ and $\mathcal{L}(Y_{\alpha+k})$ do not add new reals.

Proposition 80. For every $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha$, every θ sufficiently large and countable elementary submodel $M \prec H_\theta$ containing $p, \mathbb{K}_\alpha, \mathcal{A}_1$, and every $B \in \mathcal{I}(\mathcal{A}_1)$ such that $B \cap Y$ is infinite for all $Y \in \mathcal{I}(\mathcal{A}_1)^+ \cap M$, if $M \cap \omega_1 = j \notin \bigcup_{k \in C(s)} S_{\alpha+k}$, then there is an $(M, \mathbb{K}_\alpha, \mathcal{A}_1, B)$-generic condition $q \leq p$ such that $\mathrm{Fin}(q) = \mathrm{Fin}(p)$.

Proof. Let θ be a sufficiently large regular cardinal and let $M \prec H_\theta$ be a countable elementary submodel containing $p, \mathbb{K}_\alpha$ and $\mathcal{A}_1$, such that $j = M \cap \omega_1 \notin \bigcup_{k\in C(s)} S_{\alpha+k}$. Fix $B \in \mathcal{I}(\mathcal{A}_1)$ such that $B \cap Y$ is infinite for all $Y \in \mathcal{I}(\mathcal{A}_1)^+ \cap M$. Let $\{D_n \mid n \in \omega\}$ enumerate all open dense subsets of $\mathbb{K}_\alpha$ in M, and let $\{\dot{Z}_n \mid n \in \omega\}$ enumerate all $\mathbb{K}_\alpha$-names for subsets of ω in M which are forced to be in $\mathcal{I}(\mathcal{A}_1)^+$ such that each name appears infinitely often. Let $\langle j_n : n \in \omega\rangle$ be an increasing cofinal sequence of ordinals converging to j. We inductively define a descending sequence $\langle q_n : n \in \omega\rangle \subseteq M \cap \mathbb{K}_\alpha$ where $q_n = \langle\langle s, s^*\rangle, \langle d^n_k, c^n_k : k \in \omega\rangle\rangle$ and such that:

(1) $d^0_k = c_k$, $z^0_k = y_k$;
(2) For all $n \in \omega$ and $k \in C(s)$, d^{n+1}_k is an end-extension of d^n_k and $z^{n+1}_k \supseteq z^n_k$;
(3) $\max(d^{n+1}_k), |z^{n+1}_k| \geq j_n$;
(4) q_{n+1} is preprocessed for D_n;
(5) $q_{n+1} \Vdash (\dot{Z}_n \cap B) \setminus n \neq \emptyset$.

Assume q_n has been constructed. First extend q_n with a pure extension q'_n such that q_n is preprocessed for D_n. Next let

$$W_{n+1} = \{m \in \omega \mid \exists r \leq q'_n(\mathrm{Fin}(r) = \mathrm{Fin}(q) \wedge\ r \Vdash m \in \dot{Z}_n)\}.$$

Since $W_{n+1} \in \mathcal{I}(\mathcal{A}_1)^+$ by Lemma 77, fix $m_n \in \omega$ such that $m_n > n$ and $m_n \in W_{n+1} \cap B$. Let $r \leq q^0_n$ be given by $m_n \in W$, and let $q_{n+1} \leq r$ be a pure extension of r such that $\max(d^{n+1}_k) \geq j_n$ and $|z^{n+1}_k| \geq j_n$. Then q_{n+1} satisfies the above clauses, so this completes the inductive construction.

Set $d_k = \bigcup_{n\in\omega} d^n_k \cup \{j\}$ and $z_k = \bigcup_{n\in\omega} z^n_k$ for all $k \in C(s)$, and for $k \notin C(s)$, let $d_k = z_k = \emptyset$. Define $q := \langle\langle s, s^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$. Then q is a condition in $\mathbb{K}_\alpha$, as this can be verified as in the proof of Lemma 76. It remains to see that q is an $(M, \mathbb{K}_\alpha, \mathcal{A}_1, B)$-generic condition.

First we show q is $(M, \mathbb{Q}_\alpha)$-generic by showing that for all $n \in \omega$, $D_n \cap M$ is predense below q. Fix n and let $r = \langle\langle t, t^*\rangle, \langle d'_k, z'_k : k \in \omega\rangle\rangle \leq q$, and we can assume $r \in D_n$. Then as $r \leq q_{n+1}$ and q_{n+1} is preprocessed for D_n, there is $r' = \langle\langle t, t^*_2\rangle, \langle d^{n+1}_k, z^{n+1}_k : k \in \omega\rangle \leq q_{n+1}$ for some finite $t^*_2 \in M$, such that already $r' \in D_n$. Clearly $r' \in M$. Then r and r' are compatible, as witnessed by the condition $\langle\langle t, t^* \cup t^*_2\rangle, \langle d'_k, z'_k : k \in \omega\rangle\rangle$.

Lastly, for all $n \in \omega$ we have $q \Vdash |\dot{Z}_n \cap B| = \omega$. Let $\ell \in \omega$, and take any $r \leq q$. Find $i > \ell$ such that $\dot{Z}_i = \dot{Z}_n$; then $r \leq q_{i+1}$ and so since q_{i+1} satisifes property (5), we have

$$r \Vdash \emptyset \neq (\dot{Z}_i \cap B) \setminus i = (\dot{Z}_n \cap B) \setminus i \subseteq (\dot{Z}_n \cap B) \setminus \ell.$$

As ℓ was arbitrary this shows q forces $\dot{Z}_n \cap B$ is infinite, and therefore $(M, \mathbb{K}_\alpha, \mathcal{A}_1, B)$-generic condition.

□

Let H_α be $\mathbb{K}_\alpha$-generic over $V[G_\alpha]$, and set $Y^\alpha_k = \bigcup_{p\in H_\alpha} y_k$, $C^\alpha_k = \bigcup_{p\in H_\alpha} c_k$, $a_\alpha = \bigcup_{p\in H_\alpha} s$, $\mathcal{A}^2_{\alpha+1} = \mathcal{A}_2 \cup \{a_\alpha\}$. The following lemma gives consequences of forcing with $\mathbb{K}_\alpha$.

Lemma 81 ([FZ10, Claim 11])**.** The following hold.

(1) $a_\alpha \in [\omega]^\omega$ is almost disjoint from all elements of $\mathcal{A}^2_\alpha$;
(2) For all $i \in \omega$, $a_\alpha \cap x_i$ is infinite;

(3) For all $m \in \Delta(a_\alpha)$, C^α_m is a club in ω_1 such that $C^\alpha_m \cap S_{\alpha+m} = \emptyset$, and for all $\xi \in \omega_1$, $\xi \in C^\alpha_m$ if and only if $a_\alpha \cap R_{\langle m,\xi\rangle}$ is finite;
(4) For all $m \in \Delta(a_\alpha)$, $Y^\alpha_m\colon \omega_1 \to 2$ is a total function, and for all $\xi \in \omega_1$, $Y^\alpha_m(\xi) = 1$ if and only if $a_\alpha \cap R_{\langle \omega+m,\xi\rangle}$ is finite;
(5) For all $n \in \omega$, $n \in Z_\alpha$ if and only if there exists $m \in \omega$ such that $(a_\alpha \cap R_{\langle 0,0\rangle})(2n) = R_{\langle 0,0\rangle}(2m)$;
(6) For every $r \in \mathcal{R}$ and finite $A' \subseteq \mathcal{A}^2_{\alpha+1}$, $|E(r) \setminus \bigcup A'| = |O(r) \setminus \bigcup A'| = \omega$.

Proof. For item (1), to see a_α is infinite it suffices to show that for every $n \in \omega$ the set

$$D_n = \{p \in \mathbb{K}_\alpha \mid \mathrm{Fin}(p) = \langle s, s^*\rangle,\ \exists k > n(k \in s)\}$$

is dense in $\mathbb{K}_\alpha$ for each $n \in \omega$. Let $p = \langle\langle s, s^*\rangle, \langle c_k, d_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha$, and suppose $\max s \leq n$. Since $s^* \in [\mathcal{R} \cup \mathcal{A}^2_\alpha]^{<\omega}$, the assumption $(*)$ implies that $y := \omega \setminus (\bigcup s^* \cup R_{\langle 0,0\rangle} \cup n)$ is infinite (as it contains both $E(R), O(R)$ for any $R \in \mathcal{R} \setminus (s^* \cup \{R_{\langle 0,0\rangle}\})$). Therefore we may fix $m := \min y$ and define $p' = \langle\langle s \cup \{m\}, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha$; then $p' \leq p$ and $p' \in D_n$. So D_n is dense for every n, implying a_α is infinite. Next, for any $a \in \mathcal{A}^2_\alpha$, $a_\alpha \cap a$ is finite, since first of all

$$D_a = \{p \in \mathbb{K}_\alpha \mid \mathrm{Fin}(p) = \langle s, s^*\rangle,\ a \in s^*\}$$

is dense, and if $p \in H_\alpha \cap D_a$, then $a \cap a_\alpha \subseteq a \cap s$.

For the proof of (2), fix $i, n \in \omega$, and let

$$D_{i,n} = \{p \in \mathbb{K}_\alpha \mid \mathrm{Fin}(p) = \langle s, s^*\rangle, \exists k > n(k \in x_i \cap s)\}.$$

We show $D_{i,n}$ is dense in $\mathbb{K}_\alpha$. Let $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha$. Recall that $\eta_\alpha < \omega_1$ was a limit ordinal fixed so to ensure that $x_i \setminus \bigcup F$ is never a subset of $\bigcup_{\langle\gamma,\xi\rangle \in J} R_{\langle\gamma,\xi\rangle}$, for any $F \in [\mathcal{A}^2_\alpha]^{<\omega}$ and any $J \in [\omega \cdot 2 \times (\omega_1 \setminus \eta_\alpha)]^{<\omega}$. Since for any $\langle\gamma,\xi\rangle \in \omega \cdot 2 \times \omega_1$ with $R_{\langle\gamma,\xi\rangle} \in s^*$, items (1) and (2) of the definition of $\mathbb{K}_\alpha$ imply $\xi > \eta_\alpha$ and $\gamma > 0$, the set $y := x_i \setminus \bigcup s^*$ is infinite by the former. However by the latter we have two cases:

<u>Case I:</u> $y \setminus R_{\langle 0,0\rangle}$ is infinite. Then let $m := \min(y \setminus (\bigcup R_{\langle 0,0\rangle} \cup n \cup (\max s) + 1)$, and let $p' := \langle\langle s \cup \{m\}, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$. Then $p' \leq p$ and $p' \in D_{i,n}$.

<u>Case II:</u> $y \subseteq^* R_{\langle 0,0\rangle}$. In this case we need to take care of the additional assumption in item (3) of the definition of $\mathbb{K}_\alpha$, namely the coding of $Z_\alpha \subseteq \omega$. We may assume without loss of generality that $y \setminus R_{\langle 0,0\rangle} \subseteq n$. In the case $|s \cap R_{\langle 0,0\rangle}| = 2j$ for some $j \in \omega$, let $m := \min(y \cap R_{\langle 0,0\rangle}) \setminus (n \cup (\max s + 1))$, and define p' as in Case I with respect to this latter choice of integer m. Then $p' \leq p$ and $p' \in D_{i,n}$.

If $|s \cap R_{\langle 0,0\rangle}| = 2_j - 1$ for some $j \in \omega$, we consider whether or not $j \in Z_\alpha$. If $j \in Z_\alpha$, let $m_0 := \min(E(R_{\langle 0,0\rangle}) \setminus (n \cup (\max s + 1))$, and note this can be done as $E(R_{\langle 0,0\rangle}) \setminus \bigcup s^*$ is infinite by $(*)$. Let $m_1 := \min(y \setminus m_0)$, and define $p' := \langle\langle s \cup \{m_0, m_1\}, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$. Then $p' \in \mathbb{K}_\alpha$ and $p' \in D_{i,n}$

If $j \notin Z_\alpha$, take $m_0 := \min(O(R_{\langle 0,0\rangle})) \setminus (n \cup (\max s + 1))$ and $m_1 := \min(y \setminus m_0)$, and then define p' as above; in either case $p' \leq p$ and $p' \in D_{i,n}$.

Regarding item (3), if $m \in \Delta(a_\alpha)$ then $m \in \Delta(s)$ for some $p \in G$ with $\mathrm{Fin}(p) = \langle s, s^*\rangle$, so Lemma 73 implies C^α_m is a closed unbounded set of $\omega_1 \setminus S_{\alpha+m}$. To see a_α almost disjointly codes C^α_m using $\mathcal{R}_m = \{R_{\langle m,\xi\rangle} \mid \xi \in \omega_1\}$, first let $\xi \in \omega_1$ be an element of C^α_m. Then $\xi \in c_m$ for some

$p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in H_\alpha$. If $m \notin \Delta(s)$, $m \in \Delta(a_\alpha) = \bigcup_{q \in H_\alpha}\{\Delta(t) \mid \mathrm{Fin}(q) = \langle t, t^*\rangle\}$, so there $q \in H_\alpha$ with $\mathrm{Fin}(q) = \langle t, t^*\rangle$ and $m \in \Delta(t)$. The there is $r \in H_\alpha$ be a common extension of p, q. So without loss of generality $m \in \Delta(s)$. If $R_{\langle m,\xi\rangle} \in s^*$ then $C^\alpha_m \cap a_\alpha \subseteq s$, and if $R_{\langle m,\xi\rangle} \notin s^*$ then we use the density of the set

$$D_{m,\xi} = \{q \in \mathbb{K}_\alpha \mid \mathrm{Fin}(q) = \langle t, t^*\rangle,\ R_{\langle m,\xi\rangle} \in t^*\}$$

to find an extension $q \leq p$ with $q \in H_\alpha \cap D_{m,\xi}$. This q witnesses that $C^\alpha_m \cap a_\alpha$ is finite as in the previous case.

If $\xi \notin C^\alpha_m$, let $p \in H_\alpha$ be such that $m \in \Delta(s)$, where $\mathrm{Fin}(p) = \langle s, s^*\rangle$. The set

$$D'_n = \{p \in \mathbb{K}_\alpha \mid \exists k > n\ k \in R_{\langle m,\xi\rangle} \cap s\}$$

is dense; the proof is similar to the proof of item (1), and uses the fact that $R_{\langle m,\xi\rangle} \setminus \bigcup s^* \cup R_{\langle 0,0\rangle}$ is infinite, by $(*)$ and the fact $R_{\langle m,\xi\rangle}$ is never an element of t^* for any $\langle\langle t, t^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle \in H_\alpha$.

Item (4) is verified almost identically to item (3).

Next we check (5), which implies that a_α codes Z_α, using $R_{\langle 0,0\rangle}$. Fix $n \in \omega$ such that $n \in Z_\alpha$. The set

$$D_{R,2n} = \{p \in \mathbb{K}_\alpha \mid \mathrm{Fin}(p) = \langle s, s^*\rangle,\ |s \cap R_{\langle 0,0\rangle}| \geq 2n\}$$

is dense in $\mathbb{K}_\alpha$, using the assumption $(*)$. Indeed, if $p = \langle\langle s, s^*\rangle : \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha$ with $|s \cap R_{\langle 0,0\rangle}| = k < 2n$, we perform a finite induction of length $2n - k$, picking integers m_j to be the $(k+j)$-th element of $s \cap R_{\langle 0,0\rangle}$. If $k+j$ is odd, let $m_j = \min(R_{\langle 0,0\rangle} \setminus (\bigcup s^* \cup \max s \cup \{m_\ell \mid \ell < j\}))$. If $k+j$ is even and $\frac{k+j}{2} \in Z_\alpha$, let $m_j = \min(E(R_{\langle 0,0\rangle}) \setminus (\bigcup s^* \cup \max s \cup \{m_\ell \mid \ell < j\}))$. In the case $\frac{k+j}{2} \notin Z_\alpha$, replace O with E in the previous definition of m_j. That all this is possible is by assumption $(*)$. Then it can be verified $\langle\langle s \cup \{m_j \mid j < 2n-k\}, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ is a condition extending p, and is an element of $D_{R,2n}$.

Lastly, to prove item (6), fix $r \in \mathcal{R}$. Since $(*)$ holds, it suffices to show that $E(r) \setminus a_\alpha$ and $O(r) \setminus a_\alpha$ are infinite. For the latter suffices to show that the set

$$D^E_n = \{p \in \mathbb{K}_\alpha \mid \mathrm{Fin}(p) = \langle s, s^*\rangle \wedge \exists k > n(k < \max s \wedge k \in E(r) \setminus s)\}$$

is dense in $\mathbb{K}_\alpha$, and for the former it suffices that D^O_n is dense, where D^O_n is defined analagously for $O(t)$. Letting $p = \langle\langle s, s^*\rangle, \langle c_k, d_k : k \in \omega\rangle\rangle$ be any condition in $\mathbb{K}_\alpha$, let $m_e = \min E(r) \setminus (\bigcup s^* \cup R_{\langle 0,0\rangle} \cup (\max s + 1) \cup n)$, and let $m_o = \min O(r) \setminus (\bigcup s^* \cup R_{\langle 0,0\rangle} \cup (\max s + 1) \cup n)$.

Define $p' = \langle\langle s \cup \{\min(\omega \setminus (m_e + m_o + 1))\}, s^*\rangle, \langle c_k, d_k : k \in \omega\rangle\rangle$. Then $p' \in D^E_n \cap D^O_n$ and extends p, as desired.

□

Recall that S_{-1} was a stationary subset of ω_1 such that $S_{-1} \cap S = \emptyset$ for each $S \in \vec{S}$.

Lemma 82 ([FZ10, Corollary 12])**.** $\mathbb{K}_\alpha$ is S_{-1}-proper. Moreover, for every $p = \langle\langle s, s^*\rangle : \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha$, the subposet $\mathbb{K}_\alpha \restriction p = \{r \in \mathbb{K}_\alpha \mid r \leq p\}$ is $(\omega_1 \setminus \bigcup_{n \in C(s)} S_{\alpha+n})$- proper.

Proof. The first statement follows from the fact S_{-1} was a fixed stationary subset in L disjoint from each $S \in \vec{S}$. For the second statement, let $\beta = \alpha + m$ where $m \notin C(s)$, so $m < \max(\Delta(s))$ and $m \notin \Delta(s)$. Let $q = \langle\langle t, t^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha \restriction p$. Then t is an end extension of s, so $\Delta(t)$

is an end-extension of $\Delta(s)$, and therefore $m \notin C(t)$. Then define $q' = \langle\langle t, t^*\rangle, \langle d'_k, z'_k : k \in \omega\rangle\rangle$ such that $d'_k = d_k$ and $y'_k = y_k$ for all $k \in C(t)$, and let $d'_m = y'_m = \emptyset$. Then $q' \in \mathbb{K}_\alpha \restriction p$ and $q' \leq q$. Therefore it is dense in $\mathbb{K}_\alpha \restriction p$ to not be adding any ordinals to $\omega_1 \setminus S_{\alpha+m}$, and so $S_{\alpha+m}$ remains stationary in any $\mathbb{K}_\alpha \restriction p$ extension.

□

This completes the definition of $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2\rangle$.

Corollary 83. $\mathbb{P}_{\omega_2}$ is S_{-1}-proper and strongly preserves the tightness of $\mathcal{A}_1$. Moreover for all $m \in \omega \setminus \Delta(a_\alpha)$, $S_{\alpha+m}$ remains stationary in $L[G]$.

Proof. By Proposition 80, Lemmas 82 and 39. □

Lemma 84 ([FZ10, Lemma 13])**.** If G is $\mathbb{P}$-generic over L, then in $L[G]$, $\mathcal{A}_2$ is definable by the following Π^1_2 formula:

$$a \in \mathcal{A}_2 \Leftrightarrow \forall \mathcal{M}[(\mathcal{M} \text{ is a countable suitable model, } a \in \mathcal{M})$$
$$\exists \overline{\alpha} < \omega_2^{\mathcal{M}} \forall m \in \Delta(a)(\mathcal{M} \models \text{“}S_{\overline{\alpha}+m} \text{ is nonstationary”})]$$

Proof. Let $\varphi(a)$ denote the formula

$$\forall \mathcal{M}(\mathcal{M} \text{ is countable, suitable, } a \in \mathcal{M})$$
$$[\exists \overline{\alpha} \in \omega_2^{\mathcal{M}}(\mathcal{M} \models \forall m \in \Delta(a)(S_{\alpha+m} \text{ is nonstationary }))]$$

Then since any countable suitable model can be recursively coded by a real and the satisfaction relation $\models$ for such models is Δ^1_1, it is easy to see $\varphi(a)$ is of the form $\forall\exists$, i.e. is a Π^1_2 formula.

Now we show $\mathcal{A}_2 = \{a \in L[G] \mid \varphi(a)\}$. Suppose first $a \in \mathcal{A}_2$. Then there exists a limit ordinal $\alpha < \omega_2$ such that $a = a_\alpha$, the generic real added by $\mathbb{K}_\alpha$. Let $\mathcal{M}$ be a countable suitable model containing a_α. By absoluteness of the family $\mathcal{R}$ and items (3),(4),(5) of Lemma 12, $\langle C^\alpha_m \cap \omega_1^{\mathcal{M}} : m \in \Delta(a_\alpha)\rangle$, $\langle Y^\alpha_m \restriction \omega_1^{\mathcal{M}} : m \in \Delta(a_\alpha)\rangle, Z_\alpha \in \mathcal{M}$. Note that

$$\langle\langle\emptyset, \emptyset\rangle, \langle C^\alpha_k \cap (\omega_1^{\mathcal{M}} + 1), Y^\alpha_k \restriction \omega_1^{\mathcal{M}} : k \in \omega\rangle\rangle$$

is a condition in $\mathbb{K}_\alpha$. So by (4) of Definition 70, for each $m \in \Delta(a_\alpha)$, in $\mathcal{M}$ it holds that $\text{Even}(Y^\alpha_m \restriction \omega_1^{\mathcal{M}}) - \min(\text{Even}(Y^\alpha_m \restriction \omega_1^{\mathcal{M}}) =$ is the unique code of some limit ordinal $\overline{\alpha_m}$ such that $S_{\overline{\alpha_m}+m}$ is nonstationary. By item (2) of Definition 70, $\text{Even}(Y^\alpha_m \restriction \omega_1^{\mathcal{M}}) - \min(\text{Even}(Y^\alpha_m \restriction \omega_1^{\mathcal{M}}))$ for every $m \in \Delta(a_\alpha)$, so the unique ordinal $\overline{\alpha_m}$ is the same $\overline{\alpha} \in \omega_2^{\mathcal{M}}$ for all such m.

Conversesly, let $a \in L[G]$ be such that $\varphi(a)$ holds. As shown in Claim 60, $\varphi(a)$ holds in suitable models containing a of any infinite cardinality, in particular for the suitable model $\mathcal{N} = L_{\omega_5}[G]$. Since $\omega_2^{\mathcal{N}} = \omega_2^{V[G]} = \omega_2^V$ and $\vec{S}^{\mathcal{N}} = \vec{S}^L = \vec{S}$, if $a \in \mathcal{N}$ and there exists $\alpha < \omega_2^{\mathcal{N}}$ such that $S^{\mathcal{N}}_{\alpha+m} = S_{\alpha+m}$ is nonstationary in $\mathcal{N}$, by upwards absoluteness, $S_{\alpha+m}$ is no longer stationary in $L[G]$. By Lemma 83, it must be that $\mathbb{K}_\alpha$ was nontrivial and the set of m for which $S_{\alpha+m}$ is nonstationary in $L[G]$ were those $m \in \Delta(a_\alpha)$, where a_α was the generic real added by $\mathbb{K}_\alpha$ Therefore $\Delta(a) = \Delta(a_\alpha)$ and so $a = a_\alpha \in \mathcal{A}_2$. □

Theorem 85. *It is consistent that $\mathfrak{b} = \mathfrak{a} = \aleph_1 < \mathfrak{c} = \aleph_2$, there exists a Π^1_1-definable tight mad family of size $\aleph_1$, and a Π^1_2-definable tight mad family of size $\aleph_2$.*

Proof. Let $\mathbb{P}$ be the countable support iteration defined above, let G be $\mathbb{P}$-generic over L, and let $\mathcal{A}_2 = \{a_\alpha \mid \alpha < \omega_2\}$, where $a_\alpha = \dot{a}_\alpha^G$ where $\dot{a}_\alpha$ is the generic real added by $\mathbb{Q}_\alpha$. By Lemma 81, item (1), $\mathcal{A}_2$ is an almost disjoint family of infinite subsets of ω. To see it is tight, suppose that there is $\{x_i \mid i \in \omega\} \in L[G]$ such that $x_i \in \mathcal{I}(\mathcal{A})^+$ for every $i \in \omega$. Then there is $\alpha < \omega_2$ such that $\langle x_i : i \in \omega\rangle \in L[G_\alpha]$, where $G_\alpha = G \cap \mathbb{P}_\alpha$, so there is a sequence of $\mathbb{P}_\alpha$-names $\langle \dot{x}_i : i \in \omega\rangle \in L_{\omega_2}$ such that $x_i = \dot{x}_i^{G_\alpha}$. Since $F^{-1}(\langle \dot{x}_i : i \in \omega\rangle)$ is unbounded in ω_2, there exists $\beta \geq \alpha$ such that $F(\beta) = \langle \dot{x}_i : i \in \omega\rangle$. By definition of $\dot{\mathbb{Q}}_\beta$, and Lemma 81 item (2), $a_\beta \cap x_i$ is infinite in $L[G_\beta]$ for all $i \in \omega$, where a_β is the $\mathbb{Q}_\beta$-generic real. As $a_\beta \in \mathcal{A}_2$, we have $(\mathcal{A}_2 \text{ is tight})^{L[G]}$. That $\mathcal{A}_2$ is Π^1_2-definable in $L[G]$ is by Lemma 84.

As $\mathcal{A}_1 \in L$ and $\mathcal{A}_1$ is Π^1_1-definable in L, by Shoenfield absoluteness $\mathcal{A}_1$ remains Π^1_1-definable in $L[G]$. By Proposition 80, for every $\alpha < \omega_2$, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a proper forcing strongly preserving the tightness of $\mathcal{A}_1$. Therefore $\aleph_1^L = \aleph_1^{L[G]}$ and $\aleph_2^L = \aleph_2^{L[G]}$, and $(\mathfrak{a} = |\mathcal{A}_1| = \aleph_1 < \mathfrak{c} = \aleph_2 = |\mathcal{A}_2|)^{L[G]}$

□

5. Cardinal Characteristic Constellations and Optimal Projective Spectra

We show the compatibility of the conclusions of Theorem 63 and Theorem 85 above.

With the results of the previous section we can prove our main theorem.

Theorem 86. *It is consistent that $\aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_2$, there exists a Δ^1_3 wellorder of the reals, as well as tight mad families of cardinality $\aleph_1$ and $\aleph_2$, which are respectively Π^1_1 and Π^1_2 definable.*

Proof. We work in a model of $V = L$. The following can be obtained analagously as in Lemma 52, using Solovay's theorem on the existence of ω_1-many pairwise disjoint stationary subsets of ω_1.

Lemma 87. There exist pairwise disjoint stationary sets $T_0, T_1, T_2 \subseteq \omega_1$ such that for $i < 2$ there are sequences

$$\vec{S}^i = \langle S^i_\alpha \subseteq T_i : \alpha < \omega_2\rangle$$

where $S^i_\alpha \subseteq \omega_1$ is stationary/costationary, and for distinct $\alpha, \beta < \omega_2$, $S^i_\alpha \cap S^i_\beta$ is bounded. Moreover, whenever $\mathcal{M}, \mathcal{N}$ are suitable models such that $\omega_1^{\mathcal{M}} = \omega_1^{\mathcal{N}}$, then $\langle (S^i_\alpha)^{\mathcal{M}} : \alpha < \omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}}\rangle = \langle (S^i_\alpha)^{\mathcal{N}} : \alpha < \omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}}\rangle$.

Let $\mathcal{A}_1$ be a coanalytic tight mad family, and fix Σ_1-definable bookeeping function $F\colon \mathrm{Lim} \cap \omega_2 \to L_{\omega_2}$, and a Σ_1-definable almost disjoint family $\mathcal{R}$ such that $F, \mathcal{R}$ are as in the proof of Corollary 63 and Definition 70.

Define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2\rangle$, where for successor $\alpha < \omega_2$, $\dot{\mathbb{Q}}_\alpha$ is the creature forcing notion $\mathbb{Q}$ of Definition 18. For limit $\alpha < \omega_2$ we consider the following cases:

<u>Case I:</u> $F(\alpha) = \{\dot{b}_i \mid i \in \omega\}$ is a sequence of $\mathbb{P}_\alpha$-names such that in $V[G_\alpha]$, $\dot{b}_i^G$ is an element of $\mathcal{I}(\mathcal{A}^2_\alpha)^+$ for each $i \in \omega$. Then let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$ name for the forcing notion $\mathbb{K}_\alpha$ of Definition 70, with respect to the same countable limit ordinal $\eta_\alpha \in \omega_1$, and modifying item (1) by letting $c_k \subseteq \omega_{\eta_\alpha}$ be a closed bounded subset such that $S^0_{\alpha+k} \cap c_k = \emptyset$.

Case II: $F(\alpha) = \{\sigma_x^\alpha, \sigma_y^\alpha\}$ is a pair of $\mathbb{P}_\alpha$-names for reals in $L[G_\alpha]$ such that $\sigma_x^\alpha <_L \sigma_y^\alpha$ (i.e., $x = (\sigma_x^\alpha)^G <_\alpha y = (\sigma_y^\alpha)^G$). In this case define $\dot{\mathbb{Q}}_\alpha$ to be a $\mathbb{P}_\alpha$-name for $\mathbb{Q}_\alpha^1 = \mathbb{K}_\alpha^0 * \mathbb{K}_\alpha^1 * \mathbb{K}_\alpha^2$ as defined in the proof of Corollary 63, however modifying the definition of $\mathbb{K}_\alpha^0$ by taking closed bounded subsets of $\omega_1 \setminus S^1_{\alpha+k}$, for $k \in \Delta(x_\alpha * y_\alpha)$.

This completes the definition of $\mathbb{P} = \mathbb{P}_{\omega_2}$. Note that for all $\alpha < \omega_2$, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for either a proper, a $T_1 \cup T_2$-proper, or a $T_0 \cup T_2$-proper forcing notion, and in each case $\dot{\mathbb{Q}}_\alpha$ strongly preserves the tightness of $\mathcal{A}_1$. Therefore $\mathbb{P}$ is T_2-proper and so preserves ω_1 as well as the tightness of $\mathcal{A}_1$. Let G be $\mathbb{P}$-generic over V, first we have $(\mathfrak{a} = \aleph_1)^{L[G]}$ by the previous observation. For cofinally many $\alpha < \omega_2$, $\mathbb{Q}_\alpha = \mathbb{Q}$ adds a real not split by the ground model reals, so $(\mathfrak{s} = \aleph_2)^{L[G]}$. Lastly for cofinally many $\alpha < \omega_2$, $\mathbb{Q}_\alpha$ adds an infinite $a_\alpha \subseteq \omega$ such that $\mathcal{A}_2 = \{a_\alpha \mid \alpha < \omega_2\}$ is a Π^1_2-definable tight mad family. This shows altogether that $L[G]$ witnesses the conclusions of the theorem. □

6. Concluding remarks and Questions

6.1. **Weakly ω^ω-bounding iterations.** Proposition 80 about the iterand $\mathbb{K}_\alpha$ shows that the ω_2-length iteration of the in the proof of Theorem 85 is weakly ω^ω-bounding. However, this does not immediately imply $\mathbb{K}_\alpha$ is almost ω^ω-bounding. In general we may ask:

Question 88. If a countable support iteration of proper forcings is weakly ω^ω-bounding, is each iterand an almost ω^ω-bounding forcing notion?

In particular, it is still open if the Friedman-Zdomskyy poset is almost ω^ω-bounding. Given a positive answer to this, however, could yield a proof of the relative consistency of $\mathfrak{b} < \mathfrak{a} = \mathfrak{s} = \aleph_2$, with a Π^1_2 tight witness for $\mathfrak{a}$, in the following way.

In [She84], Shelah shows the consistency of both $\mathfrak{b} = \mathfrak{a} < \mathfrak{s}$ as well as $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$. The latter is achieved by a modification of the original forcing $\mathbb{Q}$ which we have studied in Section **??** (see Definition 18). This modification is to first add ω_1-many Cohen reals to a model V of CH, obtaining a model $V_1 = V[\langle r_i : i < \omega_1\rangle]$, and in this latter model there exists a partial order $\mathbb{Q}[I]$ which is proper, almost ω^ω-bounding, and adds a real almost disjoint from every element of a given mad family $\mathcal{A} \in V_1$. This partial order was integrated in the construction of a Δ^1_3 wellorder with countable support and produced Theorem 3 of [FF10], showing $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$ is consistent with a Δ^1_3 wellorder of the reals. That the initial segments of the iteration were almost ω^ω-bounding followed from the fact that Sacks coding $C(Y)$ is ω^ω-bounding.

Recall the definability of the Δ^1_3 wellorder and the definability of the Π^1_2 tight mad family are both achieved by similar coding methods (namely, coding a certain pattern of stationarity/nonstationarity into an ω-block of the fixed sequence $\vec{S}$), and the primary difference between the two constructions is the type of generic real added. Whereas in the former construction of [FF10] the generic reals were Sacks reals, in the latter the generic reals can be seen as a form of Mathias real. More specifically, when a_α is a $\mathbb{K}_\alpha$-generic real, a_α arises as a result of the forcing notion of almost disjoint coding; this forcing can be considered a form of restricted Mathias forcing.

This relative consistency result would answer Question 18 of [FZ10]:

Question 89. Is $\mathfrak{b} < \mathfrak{a}$ consistent with a Π^1_2 (tight) mad family of size $\mathfrak{a}$?

6.2. **Separating parallel notions of tightness.** While our constructions in Theorem 85 and Theorem 86 required active work to ensure both $\aleph_1, \mathfrak{c} \in \mathrm{spec}(\mathfrak{a})$ were witnessed by definable mad families with an definition of minimal complexity, in other cases, obtaining optimal projective witnesses for multiple values in the spectrum associated to a cardinal characteristic is almost immediate. For example, consider the cardinal

$$\mathfrak{a}_e = \min\{|\mathcal{F}| \mid \mathcal{F} \subseteq \omega^\omega,\ \mathcal{F} \text{ is maximal eventually different}\},$$

where a family $\mathcal{F} \subseteq \omega^\omega$ is *eventually different* if $f(n) = g(n)$ for all but finitely many $n \in \omega$, for every distinct $f, g \in \mathcal{F}$. Such a family is *maximal eventually different* (med) if it is maximal with respect to inclusion. Shelah and Horowitz [HS24] have shown that there always exists a Borel maximal eventually different of size $\mathfrak{c}$, and a model of $\aleph_1 = \mathfrak{a}_e = \mathfrak{d} < \mathfrak{c}$ with a coanalytic witness to $\mathfrak{a}_e$ is given in [FS21, Theorem 8]; therefore in this model both $\aleph_1, \mathfrak{c} \in \mathrm{spec}(\mathfrak{a}_e)$ have optimal projective witnesses.[7] Fischer and Switzer [FS21] introduce a parallel notion of tightness for eventually different families as well as a notion of strong preservation of tightness of eventually different families by proper forcings. It is shown in [FS21, Theorem 6.1] that Miller forcing strongly preserves the tightness of eventually different families; recall Miller forcing also strongly preserves tightness in the case of mad families. However it is not the case that the class of forcings strongly preserving tightness of eventually different families coincides with the class of forcings preserving tightness of almost disjoint families, and Shelah's forcing of Definition 18 witnesses this. Indeed, the analagous version of Theorem **??** for eventually different families cannot hold, by the following ZFC result:

Theorem 90. $\mathfrak{s} \leq \mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_\mathfrak{e}$

The latter inequality follows from the combinatorial characterization of $\mathrm{non}(\mathcal{M})$ given by Bartoszyński and Judah; see also [FS21, Fact 1.2]. The former inequality can be found in [Bla93].

A notion of tightness has also been introduced for another combinatorial family, maximal cofinitary groups, which give rise to the cardinal invariant $\mathfrak{a}_g$:

$$\mathfrak{a}_g = \min\{|\mathcal{G}| \mid \mathcal{G} \leq S_\infty,\ \mathcal{G} \text{ is a maximal cofinitary group}\}.$$

A subgroup $\mathcal{G}$ of the symmetric group $S_\infty = \{f \in \omega^\omega \mid f \text{ is a bijection}\}$ is called *cofinitary* if every $f \in \mathcal{G}$ which is not the identity function has only finitely many fixed points. A cofinitary group is *maximal (mcg)* if if it is maximal with respect to inclusion. The recent work of Fischer, Schrittesser, and Schembecker [FSS25] defines the notion of a tight cofinitary group, strengthening the notion of maximality, as well as a notion of preservation of tightness (see [FSS25, Definition 3]). They show that the Sacks coding satisfies this property in [FSS25, Theorem 17], obtaining the consistency of $\mathfrak{a}_g = \aleph_1 < \mathfrak{c} = \aleph_2$ with a Δ^1_3 wellorder and moreover a coanalytic tight cofinitary group of size $\aleph_1$; previously a coanalytic witness of the fact $\mathfrak{a}_g = \aleph_1 < \mathfrak{c}$ was constructed in [FST17]. Similar to the case of eventually different families, Horowitz and Shelah [HS25] have constructed a Borel maximal cofinitary group (under ZF), and thus there always exists an

[7] The definability here is optimal, since any Borel set is either countable or of size continuum.

optimally definable mcg of size $\mathfrak{c}$. Therefore in the model of [FSS25], $\mathrm{spec}(\mathfrak{a}_g) = \{\aleph_1, \aleph_2\}$ is realized with optimal projective witnesses in a model with a Δ^1_3 wellorder.

The ZFC theorem above again precludes Shelah's forcing $\mathbb{Q}$ from satisfying this preservation notion. Taking these facts into account we can see Theorem 63 as separating the different combinatorial strengthenings for almost disjoint families, eventually different families, and cofinitary groups.

6.3. **Further directions.** While a model of $\aleph_2 < \mathfrak{b} = \mathfrak{c}$ and each $\kappa \in \mathrm{spec}(\mathfrak{a})$ has a Π^1_1 witness is by Brendle and Khomskii [BK13], their construction relies heavily on the preservation of splitting families and increases the size of $\mathfrak{c}$ by using Hechler forcing. Therefore their construction cannot be used to answer the following.

Question 91. Is it consistent with $\mathfrak{a} < \mathfrak{c}$ or even with $\mathfrak{a} < \mathfrak{s} = \mathfrak{c}$ that there exist coanalytic mad families of sizes $\mathfrak{a}$ and $\mathfrak{c}$?

Because our above constructions used countable support iterations, our techniques can only yield models with $\mathfrak{c} = \aleph_2$. A natural question following our work is the following:

Question 92. Can we obtain a model in which $|\mathrm{spec}(\mathfrak{a})| \geq 3$ and each $\kappa \in \mathrm{spec}(\mathfrak{a})$ admits an optimal projective witness?

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