Descriptive Set Theory for Finite Structures

KGRC: Infinitary Logic
St. Petersburg: Finitary Logic
Q. Can we connect the two?

Set Theory:
Forcing
Large Cardinals
Descriptive Set Theory

Forcing and the Finite?
Takeuti, Ajtai, Krajicek: Forcing in complexity theory

Large Cardinals and the Finite?
H.Friedman: Create finite combinatorial principles whose consistency requires (small) Large Cardinals

Descriptive Set Theory for Finite Structures

Descriptive Set Theory (DST) and the Finite?

Idea: Transfer ideas from the DST of countably infinite structures to create a DST for finite structures

The DST of countable structures

Fix a countable language $\mathcal{L}$
Mod $= \mathcal{L}$-structures with universe N
Goal: Compare interesting subclasses of Mod

Examples of interesting subclasses of Mod:

a. Linear orders, Groups, Graphs, Trees, Fields, BA's
These are described by first-order sentences

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b. Sometimes one needs first-order theories:

Infinite linear orders

$\forall x_0 \exists x(x \neq x_0),\ \forall x_0, x_1 \exists x(x \neq x_0 \wedge x \neq x_1),\ \cdots$

Torsion-free groups

$\forall x \neq 0(x + x \neq 0),\ \forall x \neq 0(x + x + x \neq 0),\ \cdots$

Fields of characteristic zero

$1 + 1 \neq 0,\ 1 + 1 + 1 \neq 0,\ \cdots$

c. Sometimes one needs sentences with infinite conjunctions and disjunctions:

Torsion groups

$\forall x \ \bigvee\{x = 0, x + x = 0, x + x + x = 0, \cdots\}$

Connected graphs

$\forall x, y \bigvee\{\exists x_1 \ xEx_1Ey, \exists x_1, x_2 \ xEx_1Ex_2Ey, \cdots\}$

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d. Sometimes one needs second-order sentences:

Wellorders

$\forall X\ (X \neq \emptyset \rightarrow X$ has a least element)

Non-Superatomic BA's

$\exists X\ (X$ is an atomless subalgebra)

In the standard topology on Mod:

Sentences with countable conjunctions, disjunctions define exactly the *Borel* subclasses of Mod which are *invariant* (closed under $\simeq$)

Wellorders: Π^1_1, not Borel (complicated)

Non-Superatomic BA's: Σ^1_1, not Borel (complicated)

Nice subclasses of Mod $=$ *Borel invariant* subclasses

CYCΛH's Theorem: Borel $= \Sigma^1_1 \cap \Pi^1_1 = \Delta^1_1$

(Preview: $\Sigma^1_1 \approx$ NP, $\Pi^1_1 \approx$ CoNP, $\Delta^1_1 \approx$ NP $\cap$ CoNP, Borel $\approx P$??)

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Compare Borel invariant classes $\mathcal{C}_0$, $\mathcal{C}_1$:

$\mathcal{C}_0 \leq \mathcal{C}_1$ ($\mathcal{C}_0$ is *reducible* to $\mathcal{C}_1$) iff there is a Borel function $F : \mathcal{C}_0 \rightarrow \mathcal{C}_1$ such that $M_0 \simeq M_1$ iff $F(M_0) \simeq F(M_1)$

Borel function $=$ function with Borel graph

$\mathcal{C}$ is *complete* if every Borel invariant class reduces to it

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Examples

1. At most countably many $\simeq$ classes
2. Orders of type ω with a unary relation ($2^{\aleph_0}$ classes)
3. Subgroups of $(\mathbb{Q}, +)$ (equivalently, torsion-free Abelian groups where any two nonzero elements are linearly dependent).
4. Finitely generated groups
5. Locally finite graphs
6. Graphs, trees, fields, groups, linear orders, BA's

Theorem

Examples 1-6 are strictly increasing under reducibility.
Example 6 is complete.

Examples 1-5: $\simeq$ is Borel

There are many inequivalent nice classes with Borel $\simeq$ relations

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The above examples are analysed as follows:

1. At most countably many $\simeq$ classes

Equivalent to $=_n$ $(n < \omega)$, $=_\omega$

2. Orders of type ω with a unary relation

Equivalent to $=_{\mathcal{P}(\omega)}$

3. Subgroups of $(\mathbb{Q}, +)$

Equivalent to $(\mathcal{P}(\omega), E_0)$: xE_0y iff $x \triangle y$ is finite

4. Finitely generated groups

Equivalent to E_∞ (shift action of FG_2, the free group on two generators, on 2^{FG_2}; complete for Borel equivalence relations with countable equivalence classes).

5. Locally finite graphs

Equivalent to $F_2 =$ (countable sets of reals, $=$)

6. Graphs, trees, fields, groups, linear orders, BA's

Complete

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So we have:
$=_1 < =_2 < \cdots < =_\omega < =_{\mathcal{P}(\omega)} < E_0 < E_\infty < F_2 <$ Complete

For Borel invariant classes we have:

Silver: Nothing between $=_\omega$ and $=_{\mathcal{P}(\omega)}$
Vaught's Conjecture: Nothing incomparable with $=_{\mathcal{P}(\omega)}$

Harrington-Kechris-Louveau: Nothing between $=_{\mathcal{P}(\omega)}$ and E_0
Abelian torsion groups is incomparable with E_0

Key Question. Is there an analogous theory for finite structures?

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Reducibility of isomorphism relations on finite structures

Fix a finite language $\mathcal{L}$
Identify n with $n = \{0, 1, \ldots, n-1\}$ for finite n
Finmod $=$ $\mathcal{L}$-structures with universe n for some finite n

Goal: Compare *nice* subclasses of Finmod

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Examples:

1. Finite Linear orders
2. Finite vector spaces over a fixed finite field.
3. Finite fields
4. Finite linear orders with a unary relation
5. Finite Abelian groups
6. Finite cyclic groups
7. Finite groups with a fixed number of generators
8. Finite connected graphs with a fixed bound on the degree
9. Finite graphs with a fixed bound on the degree
10. Finite groups
11. Finite graphs

Except for 6,7,8: Above examples are first-order

Examples 6,7,8 belong to P (recognisable in polynomial time)

Nice subclass of *Finmod* $=$ *Invariant P-time subclass*

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If $\mathcal{C}_0$, $\mathcal{C}_1$ are invariant P-time classes then $\mathcal{C}_0$ is *reducible* to $\mathcal{C}_1$ iff there is a P-time function F such that $M_0 \simeq N_0$ iff $F(M_0) \simeq F(N_0)$.

$\mathcal{C}$ is *complete* iff all invariant P-time classes are reducible to it

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Analogies

Nice (invariant Borel) subclasses of Mod $\approx$
Nice (invariant P-time) subclasses of Finmod

$\simeq$ on a nice subclass of Mod is Σ^1_1
$\simeq$ on a nice subclass of Finmod is NP

$\simeq$ on a nice subclass of Mod need not be Borel
$\simeq$ on a nice subclass of Finmod need not be in P ???

There are many inequivalent nice subclasses of Mod
There are indeed many inequivalent nice subclasses of Finmod !!!

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$\mathcal{C}$ a nice subclass of Finmod.
$\mathcal{C}(n)$ = the set of models in $\mathcal{C}$ with universe m for some $m \leq n$
$\#_{\mathcal{C}}$ is defined by:
$\#_{\mathcal{C}}(n) = \#$ of isomorphism classes of models in $\mathcal{C}(n)$

Observation 1: Suppose that $\mathcal{C}_0$, $\mathcal{C}_1$ are nice subclasses of Finmod and $\mathcal{C}_0$ is reducible to $\mathcal{C}_1$. Then $\#_{\mathcal{C}_0}$ is bounded by $\#_{\mathcal{C}_1} \circ p$ for some polynomial p

Proof: Suppose that $F : \mathcal{C}_0 \to \mathcal{C}_1$ is in P-time, $M_0 \simeq N_0$ iff $F(M_0) \simeq F(M_1)$. Let p be a polynomial such that if $M \in \mathcal{C}_0$ has size at most n then $F(M)$ has size at most $p(n)$. Then $\#_{\mathcal{C}_0}(n)$ is at most $\#_{\mathcal{C}_1}(p(n))$.

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Observation 2: There are nice subclasses $\mathcal{C}_0$, $\mathcal{C}_1$ of Finmod such that for no polynomial p is $\#_{\mathcal{C}_0}$ bounded by $\#_{\mathcal{C}_1} \circ p$ or vice-versa.

Proof: Let $\mathcal{C}_i$ consist of all linear orders of size $f(2n+i)$ with one distinguished element, where f has its graph in P but grows very fast. Then $\#_{\mathcal{C}_0}(f(2n))$ is $\sum_{k \leq n} f(2k)$,
$\#_{\mathcal{C}_1}(f(2n)) = \sum_{k<n} f(2k+1)$ and for any polynomial p,
$\sum_{k \leq n} f(2k)$ is greater than $p(\sum_{k<n} f(2k+1))$ for large n.

It therefore follows that none of the following is complete under reducibility:
Finite Linear orders
Finite vector spaces over a fixed finite field.
Finite fields
Finite cyclic groups

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(Question: Can the above argument be applied to any of these?
Finite Abelian groups
Finite groups with a fixed number of generators
Finite connected graphs with a fixed bound on the degree
Finite graphs with a fixed bound on the degree
Finite groups)

It is not hard to see that Finite graphs is complete

Interesting questions are:
Q1. Is Finite Graphs reducible to Finite Linear orders with a Unary relation (FLU)?
Q2. (Silver analogue) Suppose that $\#_{\mathcal{C}}$ is exponential ($2^n \leq \#_{\mathcal{C}}(p(n))$ for large n, p polynomial). Is FLU reducible to $\mathcal{C}$?
Q3. Is there an analogue of the Harrington-Kechris-Louveau theorem in this context?