ISRAEL JOURNAL OF MATHEMATICS, Vol. 40, No. 2, 1981

UNCOUNTABLE ADMISSIBLES II: COMPACTNESS

BY

SY D. FRIEDMAN

ABSTRACT

Assume $V = L$. Let κ be a cardinal and for $X \subseteq \kappa$, $n < \omega$ let $\alpha_n(X)$ denote the least ordinal α such that $L_\alpha[X]$ is Σ_n admissible. In our earlier paper *Uncountable admissibles I: forcing*, we characterized those ordinals of the form $\alpha_n(X)$ when κ is regular. This paper treats the singular case using Barwise compactness, an effective version of Jensen's covering lemma and β-recursion theory.

We begin by reviewing some terminology from our preceding paper, *Uncountable admissibles I.* Assume $V = L$ and let κ be a cardinal. For any $X \subseteq \kappa$ let $\alpha(X)$ be the least ordinal α greater than κ such that $L_\alpha[X]$ is admissible. In case κ is regular this class of ordinals was characterized in *Uncountable admissibles I.* The present paper treats the singular case. As expected, the cases where κ has countable versus uncountable cofinality are quite different; the latter makes heavy use of β-recursion theory (mainly the results of Friedman [4]) while the former relies on generalizations of the Barwise Compactness Theorem. A by-product of this work is a characterization of those L_α's which are Barwise compact with "ordinal omitting" in L (assuming α is not a limit L-cardinal).

If A is an admissible set then let $\mathrm{O}(A) = A \cap \mathrm{ORD}$. Suppose $T(<, \cdots)$ is a theory in the fragment $\mathscr{L}_A$ of $\mathscr{L}_{\infty,\omega}$ determined by A (as in Barwise [1]). We say that T *realizes* the ordinal β if for every model M of T, $<^M$ is a linear ordering whose well-founded part has order-type greater than β. If A is countable then $\mathrm{O}(A)$ is the least ordinal not realized by an A-RE theory (i.e., a theory Σ_1 over A). Then A is *Barwise compact with ordinal omitting* if A is Barwise compact (an A-RE theory has a model if every subtheory which is a member of A has a

Received September 21, 1980 and in revised form December 20, 1980

model) and in addition $O(A)$ is the least ordinal not realized by an A-RE theory.

The relevance of these concepts to the present problem is this: If L_α is Barwise compact with ordinal omitting then $\alpha = \alpha(X)$ for some $X \subseteq \kappa$ (where $\kappa = L$-Cardinality (α)). For consider the L_α-RE theory $T(<, \in, a)_{a \in L_\alpha}$ with the following axioms:

(a) $KP +$ "$V = L_{\alpha(x)}[X]$ for some $X \subseteq \kappa$",
(b) $\forall x\, (x \in \boldsymbol{a} \Leftrightarrow \bigvee_{b \in a} x = \boldsymbol{b})$, for each $a \in L_\alpha$,
(c) $<$ is the well-ordering of the ordinals.

Then $T_0 \subseteq T$ has a model whenever T_0 is α-finite (i.e., whenever $T_0 \in L_\alpha$). Thus by hypothesis T has a model M where the well founded part of $<^M$ has order type α. But then $L_\alpha[X]$ is admissible for each $X \in M \cap 2^\kappa$. Thus if we choose X so that $M \models$ "$V = L_{\alpha(X)}[X]$" then $\alpha = \alpha(X)$.

Assume $V = L$. In §1 we show that L_α is Barwise compact with ordinal omitting provided the following conditions are met:

(i) $\kappa =$ cardinality (α) has cofinality ω,

(ii) there is a 1-1 function $f\colon L_\alpha \to L_\kappa$ which is *tame*; i.e., $f^{-1}[x] \in L_\alpha$ for each $x \in L_\kappa$,

(iii) if there is a largest α-cardinal then it has cofinality ω.

The proof is a Henkin construction modelled after Green [6]. Special care must be taken to deal with the facts that there might be no cofinal function from ω into κ inside L_α and κ might not be the greatest α-cardinal. Conversely, if $\kappa =$ cardinality (α) has cofinality ω and $\alpha = \alpha(X)$ for some $X \subseteq \kappa$, we show that conditions (i), (ii), (iii) are met. The proof heavily involves the fine structure of L as in section two of *Uncountable admissibles I.* In the terminology of that paper, condition (ii) in this case is equivalent to: ρ_i, ρ'_i both have cofinality ω for each i. As it is easily shown that L_α is not Barwise compact with ordinal omitting when $\kappa =$ (greatest L-cardinal $< \alpha$) has uncountable cofinality, this gives a complete characterization of this property (assuming α is not a limit L-cardinal).

The case where $\kappa =$ cardinality (α) is singular of uncountable cofinality is treated in §2. Here we must use the ideas developed in Friedman [4]. We recall some terminology from that paper. For each ordinal β, S_β is the βth level of Jensen's S-hierarchy for L (see Jensen [7]). For limit β, $S_\beta \cap \mathrm{ORD} = \beta$ and $S_{\beta+\omega} \cap 2^{S_\beta} =$ all subsets of S_β first-order definable over $\langle S_\beta, \in \rangle$. Then for any limit β the Δ_n projectum of β, $\delta np\beta$, is the least γ such that there is a $\Delta_n \langle S_\beta, \in \rangle$ injection of S_β *onto* γ. Jensen showed that if A is a bounded subset of $\delta np\beta$ and A is $\Delta_n \langle S_\beta, \in \rangle$ then $A \in S_\beta$ (see Devlin [2]).

The following definition comes from Friedman [4]: $D \subseteq \delta np\beta$ is a Δ_n *Master Code for* β if D is $\Delta_n \langle S_\beta, \in \rangle$ and for any $A \subseteq \delta np\beta$:

$$A \text{ is } \Sigma_1 \langle S_{\delta np\beta}, D \rangle \leftrightarrow A \text{ is } \Sigma_n \langle S_\beta, \in \rangle.$$

There exists a Δ_n Master Code for β if and only if $\delta np\beta$, $\sigma(n-1)p\beta$ have the same cofinality relative to $\Sigma_n \langle S_\beta, \in \rangle$ functions ($\sigma mp\beta = \Sigma_m$ projectum of β).

The relevance of Δ_n Master Codes to the problem at hand is this: If $D \subseteq \kappa$ is a Δ_n Master Code for β then $\alpha(D) > \beta$. Moreover it is shown in Friedman [4] that if $\kappa = \aleph_{\omega_1}$ then for any $X \subseteq \kappa$, $X \vee C$ has the same κ-degree as some Δ_n Master Code $D \subseteq \kappa$, where C = complete Σ_1 set for $\langle L_\kappa, \in \rangle$. Thus we see that if α is admissible and has cardinality $\aleph_{\omega_1}$ then for any $X \subseteq \aleph_{\omega_1}$ either $\alpha(X) > \alpha$ or X is a member of L_α. So $\alpha = \alpha(X)$ for some $X \subseteq \aleph_{\omega_1}$ if and only if $\alpha = \beta^+$ for some β such that $L_\alpha \models \beta$ has cardinality $\aleph_{\omega_1}$.

In this paper we extend the above result from Friedman [4] to other singular cardinals of uncountable cofinality. Say that $X \subseteq \kappa$ is *hyperregular* if $\langle L_\kappa, X \rangle$ is admissible. Then we show in Theorem 5 that any nonhyperregular $X \subseteq \kappa$ has the same κ-degree as some Δ_n Master Code. Thus if α is admissible of cardinality κ and κ is singular of uncountable cofinality in L_α then we come to the same conclusion as we did in the case $\kappa = \aleph_{\omega_1}$: $\alpha = \alpha(X)$ for some X if and only if $\alpha = \beta^+$ for some β such that $L_\alpha \models \beta$ has cardinality κ.

Finally we must treat the case where κ is singular of uncountable cofinality but $L_\alpha \models \kappa$ is regular. If there is a *tame* function from L_α 1–1 into L_κ we show that the methods of section one in *Uncountable admissibles I* apply. Thus if in addition there is no largest α-cardinal or the largest α-cardinal has α-cofinality κ then there is an $X \subseteq \kappa$, $\alpha(X) = \alpha$. The converse follows much as in section two of that paper using the fact that $\alpha(X) > \alpha$ whenever $X \subseteq \kappa$ is nonhyperregular. Moreover the existence of such a tame function is equivalent to: ρ_i, ρ'_i have cofinality equal to cofinality (κ) for each i.

Our full characterization of ordinals of the form $\alpha(X)$ in L is described by the following:

THEOREM. *Suppose α is admissible of cardinality κ and $\kappa < \alpha$.*

(a) *$\alpha = \alpha(X)$ for some $X \subseteq \kappa$ if and only if there is a tame function $f: L_\alpha \xrightarrow{1\text{-}1} L_\kappa$ and in addition if λ = greatest α-cardinal (should it exist):*

κ regular $\rightarrow$ cofinality $(\lambda) = \kappa$,
κ singular of cofinality ω $\rightarrow$ cofinality $(\lambda) = \omega$,
κ singular of cofinality $> \omega$, κ regular in L_α $\rightarrow$ L_α-cofinality $(\lambda) \geqq \kappa$,
κ singular of cofinality $> \omega$ in $L_\alpha \rightarrow \lambda = \kappa$ and α is a successor admissible.

(b) *There is a tame function* $f\colon L_\alpha \xrightarrow{1\text{-}1} \kappa$ *if and only if* ρ_i, ρ'_i *have cofinality equal to* cofinality (κ) *for each* i.

In §3 we consider Σ_n admissibility, $n > 1$. When κ is either regular or singular of uncountable cofinality then the above results generalize in a straightforward manner. However an unexpected phenomenon occurs when κ is singular of cofinality ω in L_α. Then we use Silver's proof of Jensen's Covering Lemma (Silver [9]) to show: If α is Σ_n admissible relative to $X \subseteq \kappa$, $n > 1$, then $X \in L_\alpha$. Thus the situation for Σ_n admissibility, $n > 1$, when κ has cofinality ω in L_α is more like the case where κ is singular of uncountable cofinality in L_α. This result also implies the existence of an L_α which is Σ_2 compact with ordinal omitting and a theory T Σ_2 over L_α such that $T \supseteq KP + \text{Diagram}(L_\alpha)$, T has a model but the standard part of each model of T is Σ_2 inadmissible.

§1. Countable cofinality

Throughout this section assume $V = L$ and that κ is a singular cardinal of cofinality ω. Also let α be an admissible ordinal of cardinality κ. We begin by developing a sufficient condition for L_α to be Barwise compact with ordinal omitting. This condition permits one to perform a Henkin construction in the style of Green [6]. This construction takes place in ω steps; at each step decisions are made about fewer than κ sentences from L_α. The fact that $\lambda < \kappa$ implies $2^\lambda \in L_\alpha$ is of crucial importance. Moreover the collection of sentences considered at each stage should be α-finite (i.e., a member of L_α). Thus we arrive at the first additional condition that we wish to impose on L_α:

(ii) there is a function $f\colon L_\alpha \xrightarrow{1\text{-}1} L_\kappa$ such that $f^{-1}[x] \in L_\alpha$ for each $x \in L_\kappa$ (f is tame).

The other condition that we need will become apparent after a more detailed description of the construction is given.

Now let T be a Σ_1 theory over L_α. Thus T is a collection of sentences in the fragment of $\mathcal{L}_{\infty,\omega}$ determined by L_α, where $\mathcal{L}$ is some α-recursive language. (A complete discussion of fragments of $\mathcal{L}_{\infty,\omega}$ can be found in Barwise [1].) Suppose that every $T_0 \subseteq T$ such that $T_0 \in L_\alpha$ has a model. We wish to show that T has a model, thus establishing the Barwise compactness of L_α.

A set of L_α-sentences S is *consistent* if no contradiction can be derived from S via an α-finite proof in the usual system for logic on L_α (which can be found in Barwise [1]). Our given theory T is consistent as otherwise some α-finite $T_0 \subseteq T$ is inconsistent and hence has no model. In fact we know that T remains consistent upon the addition of any number of valid sentences. This fact is of use in this context as there exist valid sentences which are not derivable from the usual axioms and rules for logic on L_α.

To prove the existence of a model of T it suffices to construct $S \supseteq T$ where S is a collection of sentences in a possibly larger α-recursive language $\mathscr{L}$ with the following properties:

(a) for each $\phi \in \mathscr{L}$, $\phi \in S \leftrightarrow {}^\sim\phi \notin S$,

(b) for each $\vee\Phi \in \mathscr{L}$, $\vee\Phi \in S \leftrightarrow \phi \in S$ for some $\phi \in \Phi$,

(c) for each $\exists x\phi(x) \in \mathscr{L}$, $\exists x\phi(x) \in S \leftrightarrow \phi(c) \in S$ for some constant c.

For then one can construct a (canonical) model M of S out of equivalence classes of terms. Clauses (a), (b), (c) facilitate the inductive proof that for $\phi \in \mathscr{L}$, $\phi \in S \leftrightarrow M \models \phi$.

$\mathscr{L}$ = the language of S is obtained as follows: Let L_1 = language of T augmented by a new constant c^0_ϕ for each ϕ in the language of T. Then $\mathscr{L}_{n+1} = \mathscr{L}_n$ augmented by a new constant c^n_ϕ for each ϕ in $\mathscr{L}_n$. $\mathscr{L} = \bigcup_n \mathscr{L}_n$. The constants c^n_ϕ are used in satisfying (c).

As κ has cofinality ω and there is a tame injection of L_α into κ, we can write the collection of all L_α-sentences (in the language $\mathscr{L}$) as the union of an ω-sequence $\Phi_0 \subseteq \Phi_1 \subseteq \cdots$ of α-finite collections of sentences of α-cardinality $< \kappa$. (Thus $\Phi_n = f^{-1}[\kappa_n]$ where $\kappa_0 < \kappa_1 < \cdots$ is cofinal in κ.) Also arrange that no constant c^n_ϕ occurs in any element of Φ_n. We build S as the union of an ω-sequence $T = T_0 \subseteq T_1 \subseteq \cdots$ of consistent sets of sentences. We arrange that for each $\phi \in \Phi_n$ either $\phi \in T_{n+1}$ or ${}^\sim\phi \in T_{n+1}$. Moreover for each n if $\exists x\phi(x) \in T_n$ then $\phi(c^n_\phi) \in T_{n+1}$. Thus (a), (c) will be satisfied. Dealing with condition (b) is the major problem. The difficulty is this: We wish to choose a disjunct to put into T_{n+1} for each disjunction $\vee\Phi \in T_n \cap \Phi_n$. But this must be done so that T_{n+1} is consistent. There are a large number of possible ways of choosing these disjuncts. An argument is needed to show that one of these ways is both α-finite and consistent with T_n.

Instead we do not choose a unique disjunct for each disjunction in $T_n \cap \Phi_n$ but instead argue that for an appropriate ordinal λ there is a partition of $T_n \cap \Phi_n$ into ω pieces $P_0, P_1, \cdots$ such that T_n remains consistent upon the addition of each of the following sentences:

$$\bigvee_{\substack{x_m \in L_\lambda \\ \mathrm{card}(x_m) < \kappa}} \ \bigwedge_{\vee\Phi \in P_m} \vee(\Phi \cap x_m).$$

This asserts that we have narrowed down the possible choices for a disjunct from the disjunctions in P_n to a collection of possibilities in L_λ of size less than κ. For this to be conceivable L_α must obey the following condition:

(*) For each $X \in L_\alpha$ there is $\lambda < \alpha$ such that $X = X_0 \cup X_1 \cup \cdots$ where X_m has cardinality less than κ and $X_m \in L_\lambda$ for each m.

We later show that this can be obtained as a consequence of conditions (i), (ii), and:

(iii) If there is a largest α-cardinal then it has cofinality ω.

Now assume that (*) holds. Then for any α-finite collection of sentences $\{\phi_{i,j} \mid i \in I,\ j \in J_i\}$, $\operatorname{card}(I) < \kappa$, the following sentence is valid:

$$\bigwedge_i \bigvee_j \phi_{i,j} \to \bigvee_{\substack{\text{Partition}\\(I_0, I_1, \cdots) \text{ of } I}} \bigwedge_m \bigvee_{\substack{X_m \in L_\lambda\\ \operatorname{card}(X_m) < \kappa}} \bigwedge_{i \in I_m} \bigvee_{j \in J_i \cap X_m} \phi_{i,j}.$$

Here, λ is obtained by applying (*) to $\bigcup_i J_i$. (Note that the above is an L_α-sentence because $2^I \in L_\alpha$.) Now as every α-finite $T_0 \subseteq T$ has a model and the above sentence is valid we know that T is consistent with all sentences of that form. We would like to assume that all sentences of the above form belong to T. Unfortunately this assumption is unjustified as we are considering a possibly *non* α-RE collection of sentences.

If there exists an α-recursive function g such that for all $X \in L_\alpha$, $g(X) = \lambda$ satisfies (*) for X then the above collection can be made α-RE by letting $\lambda = g(\bigcup_i J_i)$. We show in Lemma 2 below that there does always exist $g\colon L_\alpha \to \alpha$ such that for all X, $g(X) = \lambda$ satisfies (*) for X and while g is not α-recursive, nevertheless $\langle L_\alpha, g\rangle$ is admissible. Then by working relative to g we can assume that the above sentences belong to T. (Thus T will be α-RE relative to g, though possibly not α-RE.) So for now we assume that the desired valid sentences all belong to T. We now use this fact to obtain the desired partition $P_0, P_1, \cdots$ of $T_n \cap \Phi_n$.

Define $I = T_n \cap \Phi_n$, $J_i = \Phi$ if $i = \bigvee\Phi$, $J_i = \{i \vee \sim i\}$ if i is not a disjunction. Then T_n proves $\bigwedge_i \bigvee_{j \in J_i} j$. Thus T_n also proves:

$$\bigvee_{\substack{\text{Partition}\\(P_0, P_1, \cdots) \text{ of } T_n \cap \Phi_n}} \left[\bigwedge_m \bigvee_{\substack{X_m \in L_\lambda\\ \operatorname{card}(X_m) < K}} \bigwedge_{\vee\Phi \in P_m} \vee(\Phi \cap X_m) \right].$$

So for some partition $(P_0, P_1, \cdots)$ of $T_n \cap \Phi_n$ the part in brackets is consistent with T_n. Now we can also choose $X_0, X_1, \cdots, X_n \in L_\lambda$ so that $\operatorname{card}(X_i) < \kappa$ and consistency is still preserved by addition of the sentences $\bigwedge_{\vee\Phi \in P_i} \vee(\Phi \cap X_i)$ for each $i \leqq n$. For, there are only α-finitely many choices for the X_i's, $i \leqq n$, and thus one such choice must preserve consistency. Given such a choice of $X_0, X_1, \cdots, X_n$ one can now choose a unique disjunct for each $\bigvee\Phi \in P_0 \cup \cdots \cup P_n$ as there are now only at most $\operatorname{card}(X_0 \cup \cdots \cup X_n)^{P_0 \cup \cdots \cup P_n} < \kappa$ and hence α-finitely many possible choices. Thus if we apply this same procedure at stages $m > n$ we will guarantee that for each $\bigvee\Phi \in T_n \cap \Phi_n$ there is a stage m and $\phi \in \Phi$ such that $\phi \in T_m$. Thus we have a procedure for satisfying (b).

Condition (c) is handled easily. One simply must know that if R is a consistent set of sentences not involving c_ϕ^n for any ϕ then $R \cup \{\phi(c_\phi^n) \mid \exists x\phi(x) \in R\}$ is still consistent. But otherwise $R \vdash \bigvee_{\exists x\phi(x)\in R} \sim \phi(c_\phi^n)$ and this implies $R \vdash \bigvee_{\exists x\phi(x)\in R} \forall x \sim \phi(x)$ by the next lemma.

LEMMA 1. *If* $R \vdash \bigvee \phi(c_\phi)$ *where* $\phi \neq \phi' \to c_\phi$ *does not occur in* R *or* ϕ' *then* $R \vdash \bigvee \forall x\phi(x)$.

PROOF. By absoluteness it suffices to check this for countable admissible fragments. But then use the completeness theorem and the fact that $R \models \bigvee \forall x\phi(x)$ ($\models$ is the semantic consequence relation). ⊣

For condition (a) we consider the valid sentence:

$$\bigwedge_{i\in I}\bigvee_{j\in J}\phi_{i,j} \to \bigvee_{f\in J^I}\bigwedge_{i\in I}\phi_{i,f(i)},$$

where I, J, $\{\phi_{i,j} \mid i \in I, j \in J\}$ are α-finite and I, J have cardinality $< \kappa$. This is an α-recursive collection of sentences. Thus we can assume that all of these sentences belong to T. Clearly $T \vdash \bigwedge_{\phi\in\Phi_n}(\phi \vee \sim\phi)$. So by the above $T \cup \{f(\phi) \mid \phi \in \Phi_n\}$ is consistent for some $f, f(\phi) = \phi$ or $\sim\phi$. Thus all sentences in Φ_n can be simultaneously decided without damaging consistency. We of course are strongly using the fact that $2^{\Phi_n} \in L_\alpha$.

It should now be clear how to perform the desired Henkin construction. $T_0 = T$. To define T_{n+1} first decide each sentence in Φ_n. This yields a consistent theory T_n'. Now let $T_n'' = \{\phi(c_\phi^n) \mid \exists x\phi(x) \in T_n'\}$. Then T_n'' is consistent. Lastly follow the procedure outlined above to satisfy (b). This yields T_{n+1}. Then $S = \bigcup_n T_n$ obeys (a)–(c) and hence has a model. We have established the Barwise compactness of L_α given cofinality $\kappa = \omega$, $\exists$ tame $f\colon L_\alpha \xrightarrow{1\text{-}1} \kappa$ and (*).

We show that (*) follows from the further hypothesis that the greatest α-cardinal, if it exists, has cofinality ω. First note that each successor α-cardinal must have cofinality ω. For if $f: L_\alpha \xrightarrow{1\text{-}1} L_\kappa$ is tame, $\kappa_0 < \kappa_1 < \cdots$ is cofinal in κ, λ a successor α-cardinal then $f^{-1}[L_{\kappa_n}] \cap \lambda$ must be bounded in λ for each n. But then $\langle \sup(f^{-1}[L_{\kappa_n}] \cap \lambda) \mid n \in \omega\rangle$ is a cofinal ω-sequence through λ. Thus (*) holds provided there is no largest α-cardinal. Otherwise it suffices to verify (*) when $X = \mu$ = largest α-cardinal. By hypothesis $\mu = \mu_0 \cup \mu_1 \cup \cdots$ where each μ_i is less than μ. Then

$$\mu\,(\mu_0 \cap f^{-1}[L_{\kappa_0}]) \cup (\mu_1 \cap f^{-1}[L_{\kappa_1}]) \cup \cdots$$

and each $\mu_i \cap f^{-1}[L_{\kappa_i}]$ is an α-finite bounded subset of μ. Thus $\mu_i \cap f^{-1}[L_{\kappa_i}] \in L_\mu$ and we may take $\lambda = \mu$.

The proof of ordinal omitting for L_α simply requires the addition of one further step in the above construction. We are given a $\Sigma_1(L_\alpha)$ theory T in a language containing a binary relation symbol $<$ and we suppose $T \vdash <$ is a linear ordering. We are interested in constructing a model M of T such that the well-founded part of $<^M$ has ordertype $\leqq \alpha$. Thus we must arrange that for $c \in \text{Field}(<^M)$, $P_<(c) = <$-predecessors of c has ordertype $< \alpha$ or is not well-ordered.

For each $\beta < \alpha$ let $\phi_\beta^<(x)$ be a formula of L_α which says that $P_<(x)$ has ordertype β. Such a formula is easily constructed by induction on $\beta < \alpha$. Now perform a new Henkin construction identical to the previous at even stages but proceed as follows at stage $2n+1$: Consider $C = \{c_\phi^i \mid i < n,\ \phi \in \Phi_n\}$. There must be a reflexive, transitive binary relation R on C such that T_{2n} remains consistent upon addition of the sentences $\psi_1 = \bigwedge_{c_1 R c_2}(c_1 = c_2 \vee c_1 < c_2)$, $\psi_2 = \bigwedge_{\sim c_1 R c_2}(c_2 < c_1)$. Suppose there is an R-minimal $c \in C$ such that $T_{2n} \cup \{\psi_1, \psi_2\} \cup \{\sim \phi_\beta^<(c) \mid \beta < \alpha\}$ is consistent. Then choose some (new) constant of the form c_ϕ^n and let $T_{2n+1} = T_{2n} \cup \{\psi_1, \psi_2\} \cup \{\sim \phi_\beta^<(c_\phi^n) \mid \beta < \alpha\} \cup \{c_\phi^n < c\}$. If there is no R-minimal such c then let $T_{2n+1} = T_{2n} \cup \{\psi_1, \psi_2\}$. T_{2n+1} is consistent in the former case as otherwise $T_{2n} \cup \{\psi_1, \psi_2\} \cup \{\sim \phi_\beta^<(c_\phi^n) \mid \beta < \beta_0\} \cup \{c_\phi^n < c\}$ is inconsistent for some $\beta_0 < \alpha$. But this contradicts the consistency of $T_{2n} \cup \{\psi_1, \psi_2\} \cup \{\sim \phi_\beta^<(c) \mid \beta < \beta_0 + 1\}$.

To see that α is not realized by $S = \bigcup_n T_n$ (and hence is not realized by T) argue as follows: Suppose that the $<^M$-predecessors of c have ordertype α where M is a model of S. Then c is of the form c_ϕ^i for some i. But then c is R-minimal with the property that $T_{2i+2} \cup \{\psi_1, \psi_2\} \cup \{\sim \phi_\beta^<(c) \mid \beta < \alpha\}$ is consistent, where ψ_1, ψ_2, R are defined at stage $2i+3$ (for otherwise $<^M$ is non-wellfounded below c). Thus some constant c_γ^{i+1} denotes a $<^M$-predecessor of c whose $<^M$-predecessors do not have ordertype β for any $\beta < \alpha$. Therefore the $<^M$-predecessors of c cannot have ordertype α.

It remains to show that if (*) holds then there is a function $g: L_\alpha \to \alpha$ such that $\langle L_\alpha, g\rangle$ is admissible and $\forall x \in L_\alpha\ \exists x_0, x_1, \cdots \in L_{g(x)}\ (x = x_0 \cup x_1 \cup \cdots$ and for all i $\text{card}(x_i) < \kappa)$. If there is a largest α-cardinal λ_0 then define $g(x) = \text{least } \lambda > \lambda_0$ such that L_λ is p.r. closed and $L_\lambda \models \text{card}(x) \leqq \lambda_0$. Then as $\lambda_0 = x_0 \cup x_1 \cup \cdots$ with $x_i \in L_{\lambda_0}$, $\text{card}(x_i) < \kappa$ we see that $x = h^{-1}[x_0] \cup h^{-1}[x_1] \cup \cdots$ where $h \in L_{g(x)}$ is an injection of x into λ_0. Thus g works and is α-recursive in this case.

If there is no largest α-cardinal then we use:

LEMMA 2. *If there is no largest α-cardinal and* cardinality$(\alpha)=\kappa$ *then* $\langle L_\alpha, g\rangle$ *is admissible where* $g(x)=([x]^{<\kappa})^{L_\alpha}=\{y\in L_\alpha \mid y\subseteq x,\ \text{card}(y)<\kappa\}$.

PROOF. We show that if λ is a regular α-cardinal greater than κ then $\langle L_{\lambda^+}, g\restriction L_{\lambda^+}\rangle$ is a Σ_1 elementary substructure of $\langle L_\alpha, g\rangle$. This suffices to establish the admissibility of $\langle L_\alpha, g\rangle$ as there are unboundedly many regular α-cardinals.

Let $\phi(x,y,g)$ be a Δ_0 formula and $x\in L_{\lambda^+}$. Suppose $\langle L_\alpha, g\rangle\models\phi(x,y,g)$ and we will show that $\langle L_{\lambda^+}, g\restriction L_{\lambda^+}\rangle\models\exists y\phi(x,y,g)$.

Choose a regular α-cardinal μ such that $y\in L_{\mu^+}$. Now let H be a Σ_1 elementary substructure of $\langle L_{\mu^+}, g\restriction L_{\mu^+}\rangle$ of α-cardinality λ containing $\lambda\cup\{x\}\cup\{y\}$ and closed under: $Z\subseteq H$, card$(Z)<\kappa\rightarrow Z\in H$. This is possible because λ is a regular α-cardinal. Now let L_β be the transitive collapse of H; $\pi: H\xrightarrow{\sim} L_\beta$. Then $\beta<\lambda^+$ and $\langle L_\beta, g'\rangle\models\phi(x,\pi(y),g')$ where $g'=\pi(g\restriction H)$. But $([L_\beta]^{<\kappa})^{L_\alpha}\subseteq L_\beta$ since $([H]^{<\kappa})^{L_\alpha}\subseteq H$. So $g'=g\restriction L_\beta$ and we are done. ⊣

To summarize, we have proved (d)→(e)→(a)→(b) in the following:

THEOREM 3. *Suppose $\alpha>\kappa$ is admissible and has cardinality κ,* cofinality$(\kappa)=\omega$. *Then the following are equivalent:*

(a) *L_α is Barwise compact with ordinal omitting,*

(b) *$\alpha=\alpha(X)$ for some $X\subseteq\kappa$,*

(c) *ρ_i,ρ'_i have cofinality ω for each i and if the greatest α-cardinal exists then its cofinality is ω,*

(d) *there is a tame injection $L_\alpha\rightarrow\kappa$ and if the greatest α-cardinal exists then its cofinality is ω,*

(e) cofinality $(\alpha)=\omega$ *and for all $x\in L_\alpha$ there is $\lambda<\alpha$ and $x_0,x_1,\cdots\in L_\lambda$ such that $x=x_0\cup x_1\cup\cdots$ and* card$(x_i)<\kappa$ *for all i.*

The definition of ρ_i,ρ'_i can be found in *Uncountable admissibles I.* The proof of (b)→(c) is entirely similar to the argument in section two of that paper. Thus the idea is to show that if ρ_i or ρ'_i has cofinality $\neq\omega$ for some i then $L_\alpha[X]$ contains a well-ordering of length α whenever $L_\alpha[X]\models\kappa$ is the largest cardinal. To prove (c)→(d) one constructs a series of tame injections $L_\alpha\rightarrow L_{\rho_1}, L_{\rho_1}\rightarrow L_{\rho_2},\cdots, L_{\rho_{k-1}}\rightarrow L_{\rho_k}=L_\kappa$. The ith such is constructed using the fact that $\rho_i,\rho'_{i+1}\ \rho_{i+1}$ have cofinality ω. Then the composition of these tame injections is tame.

PROOF OF THEOREM 3. As we remarked earlier we have already established (d)→(e)→(a)→(b).

(b)→(c): Suppose cofinality $(\rho_i)>\omega$ and let $\langle\gamma_j \mid j<\lambda\rangle$ be a cofinal increasing sequence through ρ_i. There exists a $\Sigma_{n_i}(S_{\beta_i})$ well-ordering W of ρ_i of ordertype α. Then $W\cap\gamma_j\in S_{\rho_i}$ for each j.

Suppose $\alpha = \alpha(X)$, $X \subseteq \kappa$. Then either $\rho_i = \alpha$ or $L_\alpha[X] \models \rho_i$ has cardinality κ. In the former case ρ_i has cofinality ω as $\alpha(X)$ has cofinality ω for any $X \subseteq \kappa$. In the latter case $L_\alpha[X]$ contains an injection $f\colon S_{\rho_i} \to \kappa$ and thus $f[\{W \cap \gamma_j \mid j \in Z\}]$ is bounded for some unbounded $Z \subseteq \lambda$ since $\mathrm{cof}(\lambda) \neq \mathrm{cof}(\kappa)$. But then $\langle W \cap \gamma_j \mid j \in Z\rangle$ and hence W belongs to $L_\alpha[X]$, contradicting its admissibility.

Suppose cofinality $(\rho_i') > \omega$. Let $\mathfrak{A}_i = \langle S_{\rho_i'}, A_i\rangle$ where A_i is a Σ_{n_i-1} master code for β_i (as in *Uncountable admissibles I*). Then there is a $\Sigma_1(\mathfrak{A}_i)$ well-ordering W of ρ_i of ordertype α. Let $\langle \gamma_j \mid j < \lambda\rangle$ be a cofinal increasing sequence through ρ_i' and for each $j < \lambda$, W^j = the amount of W enumerated by stage γ_j, in a $\Sigma_1(\mathfrak{A}_i)$ enumeration of W. Then $W^j \in L_\alpha$ for each j as W^j is ρ_i'-finite and hence constructed before the next ρ_i'-cardinal above ρ_i (if $\rho_i' > \alpha$). This next ρ_i'-cardinal is $\leqq \alpha$.

As cofinality $(\alpha) = \omega$ there is an unbounded $Z \subseteq \lambda$ such that for some $\beta < \alpha$, $W^j \in S_\beta$ for all $j \in Z$. If $L_\alpha[X] \models \kappa$ is the largest cardinal and $f\colon S_\beta \xrightarrow{1\text{-}1} \kappa$ belongs to $L_\alpha[X]$ then as before $\langle W^j \mid j \in Z'\rangle \in L_\alpha[X]$ for some unbounded $Z' \subseteq Z$. But then $W \in L_\alpha[X]$ so $L_\alpha[X]$ is inadmissible.

Finally we must show that if λ = greatest α-cardinal exists then cofinality$(\lambda) = \omega$. There is a $\Sigma_1(L_\alpha)$ function from $\{Y \subseteq \lambda \mid Y \in L_\alpha\} = \mathcal{S}$ onto an unbounded subset of α. Namely, send any $Y \in \mathcal{S}$ to the least ordinal β such that $Y \in L_\beta$. We now show that if cofinality $(\lambda) > \omega$ and $L_\alpha[X] \models \kappa$ is the largest cardinal, then there is a function in $L_\alpha[X]$ mapping L_κ onto $\mathcal{S}$. This shows that $L_\alpha[X]$ is inadmissible.

So suppose $\langle \gamma_j \mid j < \lambda'\rangle$ is a cofinal sequence through λ and that $f\colon L_\lambda \xrightarrow{1\text{-}1} \kappa$ belongs to $L_\alpha[X]$. Then for any $Y \in \mathcal{S}$ there is an unbounded $Z \subseteq \lambda'$ such that $f[\{Y \cap \gamma_j \mid j \in Z\}]$ is bounded in κ (and hence belongs to L_κ). The desired function from L_κ onto $\mathcal{S}$ is defined by: If $y = \bigcup f^{-1}[x]$ then send x to y.

(c)$\to$(d): We claim that it suffices to prove the following:

Lemma 4. *Suppose $\langle S_\beta, A\rangle$ is amenable, ρ is a β-cardinal and $\gamma > \rho$ is either a β-cardinal or $\gamma = \beta$. If there is a $\Sigma_1\langle S_\beta, A\rangle$ injection of L_γ into L_ρ and γ, ρ, β have cofinality ω then there is a* tame *injection of L_γ into L_ρ.*

The lemma can be applied as follows: Let $\langle S_\beta, A\rangle = \langle S_{\rho_i'}, A_i\rangle$, $\rho = \rho_i$ and $\gamma = \rho_{i-1}$. Then the lemma provides a tame injection f_i from $S_{\rho_{i-1}}$ into L_{ρ_i}. But then $f_i^{-1}(x) \in S_{\rho_{i-1}}$ for any $x \in S_{\rho_i}$ by tameness. So $f_k \circ f_{k-1} \circ \cdots \circ f_1$ is a tame injection of L_α into L_κ.

Proof of Lemma 4. Choose $\beta_1 < \beta_2 < \cdots$, $\rho_1 < \rho_2 < \cdots$ and $\gamma_1 < \gamma_2 < \cdots$ converging to β, ρ, γ respectively and let $f\colon L_\gamma \to L_\rho$ be a $\Sigma_1\langle S_\beta, A\rangle$ injection. For each n let f^n be the β-finite part of f enumerated by stage β_n in a $\Sigma_1\langle S_\beta, A\rangle$ enumeration of graph (f). Then define $g\colon L_\gamma \to L_\rho$ by:

$$g(x) = \langle f(x), \rho_n\rangle \qquad \text{where } n = \text{least } m \text{ s.t. } x \in L_{\gamma_m},\quad x \in \mathrm{Dom}(f^m).$$

For each n and each $y \in L_{\rho_n}$, $g^{-1}[y]$ is a β-finite bounded subset of L_γ. Then $g^{-1}[y] \in L_\gamma$ for each n. So g is tame. ⊣

Remarks. (1) There are admissible ordinals α of cardinality $\aleph_\omega$ such that L_α is Barwise compact but does not obey ordinal omitting. For, as Barwise compactness is equivalent to a reflection principle (see the discussion of strict Π^1_1-reflection in Barwise [1]) there is a closed unbounded collection of ordinals $\alpha < \aleph_{\omega+1}$ such that L_α is Barwise compact. But then there is such an α of cofinality ω_1. Then L_α does not obey ordinal omitting by Theorem 3(b).

(2) There do exist admissible α of cardinality $\aleph_\omega$ such that there is a tame injection $L_\alpha \to \aleph_\omega$ but L_α is not Barwise compact. Thus the second condition in (d) of Theorem 3 is necessary. An example is the least nonprojectible $\alpha > \aleph_\omega$ such that $L_\alpha \models \aleph_{\omega_1}$ exists. Then the greatest α-cardinal has uncountable cofinality yet $\rho_1 = \alpha$, $\rho_2 = \aleph_\omega$, $\rho'_1 = \alpha$, $\rho'_2 = \alpha$ all have cofinality ω. Thus there exists a tame injection $L_\alpha \to \aleph_\omega$. If L_α were Barwise compact then there would be a model of "Diagram $(L_\alpha) + X \subseteq \aleph_\omega + V = L_{\alpha(X)}[X] + \sim\exists\beta$ (β is nonprojectible and $L_\beta \models \aleph_{\omega_1}$ exists)". But the well-founded part of such a model would have ordinal height α, proving $\alpha = \alpha(X)$ for some $X \subseteq \aleph_\omega$. This contradicts Theorem 3.

(3) If α obeys the conditions of Theorem 3 then our proof of Barwise compactness from condition (d) yields an explicit axiomatization of the valid sentences in L_α. This set of axioms consists of the usual ones embellished by the following:

$$\bigwedge_{i\in I}\bigvee_{j\in J}\phi_{i,j} \to \bigvee_{f\in J^I}\bigwedge_{i\in I}\phi_{i,f(i)},$$

$$\bigwedge_{i}\bigvee_{j\in J_i}\phi_{i,j} \to \bigvee_{\substack{\text{Partition}\\ (I_0,I_1,\cdots)\text{ of } I}}\ \bigwedge_{m}\ \bigvee_{\substack{x_m\in L_\lambda\\ \mathrm{card}(x_m)<\kappa}}\ \bigwedge_{i\in I_m}\ \bigvee_{j\in J_i\cap x_m}\phi_{i,j},$$

where $\lambda = g(\bigcup_i J_i)$ and g is as constructed earlier. This gives a $\Sigma_1\langle L_\alpha, g\rangle$ complete axiomatization of the valid L_α-sentences. Magidor–Shelah–Stavi have shown that if there is no largest α-cardinal then any L_α obeying Theorem 3 is *not* Σ_1-complete. (Otherwise g is α-recursive.) See their forthcoming paper *Countably decomposable admissible sets.* Note that in Theorem 3 compactness has actually been established for any theory $\Sigma_1\langle L_\alpha, g\rangle$.

(4) If κ = largest (L-) cardinal $< \alpha$ has cofinality $> \omega$ then L_α cannot be

Barwise compact with ordinal omitting. For otherwise there is a non-well-founded model M of $KP + \text{Diagram}(L_\alpha) +$ "κ is the largest cardinal". If $c \in M$ is a non-standard ordinal then M contains a descending sequence below c as M contains a bijection $c \leftrightarrow \kappa$ and L_κ is countably closed. Thus Theorem 3 completely characterizes Barwise compactness with ordinal omitting unless α is a limit cardinal of L.

(5) The equivalence of (a), (b), (e) in Theorem 3 was obtained independently by Magidor–Shelah–Stavi. See their upcoming paper *Countably decomposable admissible sets.* They also obtained a characterization of which L_α's are Σ_1-complete.

§2. Uncountable cofinality

This case draws heavily on the methods of Friedman [4]. Assume $V = L$ throughout and that κ is a singular cardinal of uncountable cofinality. Our key lemma provides a characterization of the nonhyperregular κ-degrees:

THEOREM 5. *Any nonhyperregular $X \subseteq \kappa$ has the same κ-degree as a Δ_n master code.*

PROOF. In Friedman [4], section two this result is established when $\kappa = \aleph_{\omega_1}$. That proof shows: If $M \subseteq \kappa$ is nonhyperregular and $M \leqq_\kappa X$ then X has the same κ-degree as a $\Delta_n(M)$ *master code.* $X \subseteq \kappa$ is a $\Delta_n(M)$ master code for β if for any $Y \subseteq \kappa$, Y is $\Sigma_1\langle L_\kappa, X\rangle$ iff Y is $\Sigma_n\langle S_\beta[M], M\rangle$. Now if M is a Δ_n master code then the $\Delta_m(M)$ master codes $\subseteq \kappa$ are just the Δ_p master codes $Y \subseteq \kappa$ such that $M \leqq_\kappa Y$. Thus to prove the theorem it suffices to show that there is a nonhyperregular Δ_n master code $M \subseteq \kappa$ such that $M \leqq_\kappa X$ for every nonhyperregular $X \subseteq \kappa$.

Let β be the least ordinal such that for some n there is a $\Delta_n(S_\beta)$ function from an ordinal $< \kappa$ onto an unbounded subset of κ. Also choose n to be least.

LEMMA 6. *There is a Δ_n master code for β.*

PROOF. Note that Δ_n projectum $(\beta) = \kappa$. Also let $\mathfrak{A} = \langle S_\rho, A\rangle$ where $\rho = \Sigma_{n-1}$ projectum (β) and A is a Σ_{n-1} master code for β. Then as in Friedman [4], theorem 9 it suffices to show that Σ_1 cf $\mathfrak{A} =$ cofinality κ.

Let $\lambda =$ cofinality (κ) and $f: \lambda \to \kappa$ be increasing, continuous, cofinal and $\Sigma_1(\mathfrak{A})$. If $\Sigma_1 \operatorname{cf} \mathfrak{A} \neq \lambda$ then $f \restriction Z \in S_\rho$ for some unbounded $Z \subseteq \lambda$. But then κ is singular via a function in S_β, contradicting the leastness of β. ⊣

Let $X \subseteq \kappa$ be nonhyperregular. Choose a Σ_1 Skolem function h for $\mathfrak{A}$ and let

$g: \lambda \to \rho$ be increasing, cofinal, continuous and $\Sigma_1(\mathfrak{A})$ with parameter p. Also for each $\lambda' < \lambda$, $h^{\lambda'}$ is the λ' th approximation to h, obtained by restricting its $\Sigma_1(\mathfrak{A})$ definition to $\langle S_{g(\lambda')}, A \cap g(\lambda')\rangle$.

It suffices to show that for some closed unbounded $C \subseteq \text{Range}(f)$:

$$(*) \qquad \gamma = \lambda'\text{th member of } C \to \gamma \notin h^{\lambda'}[\omega \times (\gamma \cup \{p\})].$$

For, as in the proof of theorem 9 (a) in Friedman [4], a Δ_n master code for β is obtained by considering the sequence $\langle t_{\lambda'} \mid \lambda' < \lambda \rangle$ where $L_{t_{\lambda'}}$ = transitive collapse of $h^{\lambda'}[\omega \times (\gamma \cup \{p\})]$, γ as in (*). But as γ is regular in $L_{t_{\lambda'}}$ (κ is regular in $S_{g(\lambda')}$) we see that there is a function $k \leqq_{\omega\kappa} X$ such that $k: \lambda \to \kappa$, $k(\lambda')$ has the same cardinality as $t_{\lambda'}$ for all λ', and k dominates $\lambda' \mapsto t_{\lambda'}$ everywhere. But then $\langle t_{\lambda'} \mid \lambda' < \lambda \rangle \leqq_\kappa X$ and so κ-degree$(X) \geqq \kappa$-degree of some Δ_n master code for β.

To construct $C \subseteq \text{Range}(f)$ satisfying (*) begin with some closed unbounded $C_1 \subseteq \kappa$ of ordertype λ such that $C_1 \leqq_\kappa X$ (by the nonhyperregularity of X). But then for some c.u.b. $Z \subseteq \lambda$, $\{\gamma \mid \gamma$ is the λ'th element of C_1, $\gamma' \in Z\} = C_2$ must satisfy (*). For otherwise Fodor's Theorem implies that for some fixed $\gamma_0 < \kappa$, $\lambda_0 < \lambda$, $h^{\lambda_0}[\omega \times (\gamma_0 \cup \{p\})] \cap \kappa$ is unbounded in κ. This contradicts the fact that κ is regular with respect to β-finite functions. Finally let $C = C_2 \cap \text{Range}(f)$. ⊣

Corollary 7. *If $X \subseteq \kappa$ is nonhyperregular then $X \in L_{\alpha(X)}$.*

This immediately yields:

Theorem 8. *Suppose κ is singular of uncountable cofinality in L_α. Then $\alpha = \alpha(X)$ for some $X \subseteq \kappa$ if and only if $\alpha = \beta^+$ where $L_\alpha \models \text{cardinality}(\beta) = \kappa$.*

Note. Here we use β^+ to denote the least admissible $> \beta$. If κ is singular in L_α then L_α-cofinality(κ) = cofinality(κ).

Proof. Choose $Y \subseteq \kappa$, $Y \in L_\alpha$ so that Y is nonhyperregular. If $\alpha = \alpha(X)$ then Corollary 7 implies that $X \vee Y \in L_\alpha$. Thus $\alpha = \beta^+$ where $X \vee Y \in L_\beta$. Also $L_\alpha \models \text{card}(\beta) = \kappa$ as otherwise $X \vee Y \in L_{\text{gc}\,\alpha}$ and $\alpha(X) < \text{gc}\,\alpha$ (where $\text{gc}\,\alpha$ = the greatest α-cardinal).

Also note that if $\alpha = \beta^+$, $L_\alpha \models \text{card}(\beta) = \kappa$ then $\alpha = \alpha(X)$ where $X \in L_\alpha$ is a well-ordering of κ of ordertype β. ⊣

Finally it remains to consider the case where κ is singular of uncountable cofinality yet $L_\alpha \models \kappa$ is regular. First note by Corollary 7 that if $X \subseteq \kappa$ is nonhyperregular then $\alpha(X) > \alpha$ (as otherwise $X \in L_\alpha$, contradicting the regularity of κ in L_α). This fact is key to proving the "only if" direction of the following:

THEOREM 9. *Suppose κ is a singular cardinal of uncountable cofinality and $L_\alpha \models \kappa$ is regular. Then $\alpha = \alpha(X)$ for some $X \subseteq \kappa$ if and only if*:

(a) *there is a tame injection $L_\alpha \to L_\kappa$ and*

(b) *either there is no largest α-cardinal or if λ = largest α-cardinal then L_α-cof$(\lambda) \geqq \kappa$.*

PROOF. We first prove that conditions (a), (b) are necessary. The key fact to establish is: ρ_i, ρ'_i have cofinality = cofinality(κ) for each i. This will imply both the $<\mathrm{cof}(\kappa)$-closure of L_α and the existence of a tame injection $L_\alpha \to L_\kappa$.

Let λ = cofinality(κ). Suppose $\mathrm{cof}(\rho_i) = \lambda' \neq \lambda$. There is a well-ordering W of ρ_i of ordertype α s.t. $W \cap S_\gamma \in S_{\rho_i}$ for each $\gamma < \rho_i$. Choose $f\colon \lambda' \to \rho_i$ to be cofinal. If $L_\alpha[X] \models \kappa$ is the largest cardinal then there is $g\colon S_{\rho_i} \to \kappa$, $g \in L_\alpha[X]$. So since $\lambda' \neq \lambda$, $g[\{W \cap S_{f(\delta)} \mid \delta \in Z\}]$ is bounded in κ for some unbounded $Z \subseteq \lambda'$. But then $\{W \cap S_{f(\delta)} \mid \delta \in Z\rangle$ and hence W belongs to $L_\alpha[X]$ so $L_\alpha[X]$ is inadmissible. Thus $\alpha \neq \alpha(X)$ for any X, contrary to hypothesis.

Now suppose $\mathrm{cof}(\rho'_i) = \lambda' \neq \lambda$. Recall that $\rho'_i = \Sigma_{n_i-1}$ projectum (β_i). Choose a cofinal $f\colon \lambda' \to \rho'_i$. There is a $\Sigma_1(\mathfrak{A}_i)$ well-ordering W of ρ_i of ordertype α, where $\mathfrak{A}_i = \langle S_{\rho'_i}, A_i\rangle$ (A_i a Σ_{n_i-1} master code for β_i). Either $\rho'_i = \rho_{i-1}$ or ρ_{i-1} is a ρ'_i-cardinal. In either case there is an unbounded $Z \subseteq \lambda'$ and $\beta < \alpha$ such that W^δ = amount of W enumerated by stage $f(\delta)$ (in some $\Sigma_1(\mathfrak{A}_i)$ enumeration of W) belongs to S_β for all $\delta \in Z$. But then as before if $L_\alpha[X] \models \kappa$ is the largest cardinal, then $\langle W^\delta \mid \delta \in Z'\rangle \in L_\alpha[X]$ for some unbounded $Z' \subseteq Z$. So $W \in L_\alpha[X]$ and $L_\alpha[X]$ is inadmissible, contrary to hypothesis.

We now argue as in the proof of Theorem 13 in *Uncountable admissibles I* that L_α is $<\lambda$-closed. Recall that $\alpha \geqq \rho_1 > \rho_2 > \cdots > \rho_k = \kappa$. Also define $\rho_0 = \alpha$. We show by induction on $k - i$ that for $\lambda' < \lambda$, $f\colon \lambda' \to \rho_i$ implies $f \in S_{\rho_i}$. This is clear if $k - i = 0$. Given $\lambda' < \lambda$ and $f\colon \lambda' \to \rho_i$ let $g\colon \alpha \to \rho_{i+1}$ be $\Sigma_1(\mathfrak{A}_{i+1})$. Then $g \circ f\colon \lambda' \to \rho_{i+1}$ and hence $g \circ f \in S_{\rho_{i+1}}$ by induction. But then $f = g^{-1} \circ (g \circ f)$ is $\Sigma_1(\mathfrak{A}_{i+1})$ and since $\mathrm{cof}(\rho'_{i+1}) = \lambda > \lambda'$, $f \in S_{\rho'_{i+1}}$. But either $\rho'_{i+1} = \rho_i$ or ρ_i is a ρ'_{i+1}-cardinal. In the former case we are done; in the latter case $f \in S_{\rho_i}$ as f is bounded in ρ_i.

We use the $<\lambda$-closure of L_α to define a tame injection $L_\alpha \to L_\kappa$. Note that S_{ρ_i} is also $<\lambda$-closed for each i as ρ_i is an α-cardinal and $\mathrm{cof}(\rho_i) = \lambda$.

LEMMA 10. *Suppose $\langle S_\beta, A\rangle$ is amenable and $\rho < \gamma$ are β-cardinals, S_γ $<\lambda$-closed. If there is a $\Sigma_1\langle S_\beta, A\rangle$ injection of γ into ρ and γ, ρ, β have cofinality λ then there is a tame injection of L_γ into L_ρ.*

PROOF. Much like the proof of Lemma 4. Choose $\langle \beta_{\lambda'} \mid \lambda' < \lambda\rangle$, $\langle \rho_{\lambda'} \mid \lambda' < \lambda\rangle$

and $\langle \gamma_{\lambda'} \mid \lambda' < \lambda \rangle$ converging to β, ρ, γ respectively and let $f: \gamma \to \rho$ be a $\Sigma_1\langle S_\beta, A\rangle$ injection. For each λ' let $f^{\lambda'}$ be the β-finite part of f enumerated by stage $\beta_{\lambda'}$ in a $\Sigma_1\langle S_\beta, A\rangle$ enumeration of graph(f). Then define $g: L_\gamma \to L_\rho$ by:

$$g(x) = \langle f(x), \rho_{\lambda'}\rangle \qquad \text{where } \lambda' = \text{least } \lambda'' \text{ s.t. } x \in L_{\gamma_{\lambda''}}, \quad x \in \text{Dom}(f^{\lambda''}).$$

For each $y \in L_\rho$, $g^{-1}[y] \in L_\gamma$ using the fact that γ is a β-cardinal and L_γ is $<\lambda$-closed. ⊣

Now let $\langle S_\beta, A\rangle = \langle S_{\rho_i}, A_i\rangle$, $\rho = \rho_i$, $\gamma = \rho_{i-1}$. Notice that Lemma 10 also holds even if $\gamma = \beta$. Thus there is a tame injection f_i from $L_{\rho_{i-1}}$ into L_{ρ_i}. So $f_k \circ f_{k-1} \circ \cdots \circ f_1$ is a tame injection of L_α into L_κ. From this it is easy to get a tame injection of L_α into L_κ.

Thus we have demonstrated the necessity of condition (a). For condition (b) suppose that there is a largest α-cardinal λ. Note that if $\gamma = L_\alpha$-cofinality of λ then there is a $\Sigma_1(L_\alpha)$ function from $[L_\lambda]^\gamma \cap L_\alpha$ onto a cofinal subset of α: Namely, if $x \in [L_\lambda]^\gamma \cap L_\alpha$ then send x to the least β s.t. $\bigcup x \in S_\beta$. As λ is the largest α-cardinal there are subsets of λ constructed cofinally in α. As L_α-cofinality$(\lambda) = \gamma$ each of these sets is of the form $\bigcup x$ for some $x \in [L_\lambda]^\gamma \cap L_\alpha$.

We have the hypothesis $\alpha = \alpha(X)$ for some $X \subseteq \kappa$. By Corollary 7, $L_\alpha[X] \models \kappa$ is regular as otherwise for some $Y \in L_\alpha[X] \cap 2^\kappa$, Y is nonhyperregular and thus $X \vee Y \in L_{\alpha(X \vee Y)}$. But $\alpha(X \vee Y) = \alpha(X) = \alpha$, contradicting the fact that κ is regular in L_α. Also $L_\alpha[X] \models \kappa$ is the largest cardinal, so choose a bijection $f \in L_\alpha[X]$ between L_λ and κ. But then f induces a bijection between $[\kappa]^\gamma \cap L_\alpha$ & $[L_\lambda]^\gamma \cap L_\alpha$ & if $\gamma < \kappa$ then $[\kappa]^\gamma \cap L_\alpha \in L_\alpha[X]$ since $L_\alpha[X] \models \kappa$ is regular. So $\gamma \geqq \kappa$ as otherwise $L_\alpha[X]$ contains a Σ_1 function from $[\kappa]^\gamma \cap L_\alpha \in L_\alpha[X]$ onto a cofinal subset of α. This completes the proof of the necessity of (a), (b).

Now under the hypothesis of (a), (b) we show that the forcing construction of *Uncountable admissibles I*, section one can be carried out to yield $X \subseteq \kappa$ such that $\alpha(X) = \alpha$. The difficulty is that L_α is not $<\kappa$-closed (as κ is not regular). So there might be a problem in proving the existence of generic sets. However we show that L_α has enough $<\kappa$-closure to enable the construction of sufficiently generic sets. The degree of genericity required is captured by the next definition.

Definition. A sentence ϕ is $\Pi_0\Sigma$ if it is of the form $\forall x \in y\, \exists z \psi$ where ψ is Δ_0.

There are three steps to the forcing argument in section one of *Uncountable admissibles I*. In each case one has an admissible structure $\langle L_\alpha[Y], Y\rangle$ and a generic set is required for a notion of forcing such that the forcing relation is Σ_1

when restricted to Σ_1 sentences (over $\langle L_\alpha[Y], Y\rangle$). Moreover the proofs in that paper that admissibility is preserved really show that for any $\Delta_0\psi$:

$$(*)\qquad \rho \Vdash \forall x \in y\, \exists z \psi(x,z) \rightarrow \exists w\, \rho \Vdash \forall x \in y\, \exists z \in w \psi(x,z).$$

Thus forcing for $\Pi_0\Sigma$ sentences is actually Σ_1. Moreover genericity with respect to $\Pi_0\Sigma$ suffices to demonstrate that admissibility is preserved (by (*) and the Truth Lemma). So we have only to establish the existence of $\Pi_0\Sigma$-generic sets for the three forcing notions in question. Note that the hypothesis that κ is a regular α-cardinal implies that each of these forcings is $<\kappa$-closed with respect to sequences in $L_\alpha[Y]$. The unbounded Lévy collapse forcing requires in addition that L_α be admissible relative to the function $x \mapsto [x]^{<\kappa} \cap L_\alpha$. This function maps L_α to L_α by virture of property (b). Moreover this function is α-recursive if there is a largest α-cardinal. The admissibility of L_α relative to this function when there is no largest α-cardinal is established just as in Lemma 2.

Finally we use the existence of a tame injection $L_\alpha \rightarrow L_\kappa$ to establish the existence of $\Pi_0\Sigma$-generic sets over $\langle L_\alpha[Y], Y\rangle$ whenever this structure is admissible and the forcing partial ordering $\mathcal{P}$ satisfies:

(i) if $\langle \rho_\gamma \mid \gamma < \gamma_0\rangle \in L_\alpha[Y]$, $\gamma_0 < \kappa$ and $\gamma < \gamma' \rightarrow \rho_\gamma \geqq \rho_{\gamma'}$ then for some $\rho, \rho \leqq \rho_\gamma$ for all $\gamma < \gamma_0$,

(ii) the relation $\rho \Vdash \phi$ for $\Pi_0\Sigma\phi$, $\rho \in \mathcal{P}$ is $\Sigma_1\langle L_\alpha[Y], Y\rangle$.

The idea is to build a sequence $\langle \rho_\gamma \mid \gamma < \kappa\rangle$ of conditions in $\mathcal{P}$ such that $\gamma < \gamma' \rightarrow \rho_{\gamma'} \leqq \rho_\gamma$ and ρ_γ decides the γth $\Pi_0\Sigma$ sentence of $\langle L_\alpha[Y], Y\rangle$ in some tame listing of the $\Pi_0\Sigma$ sentences of length κ. Thus for each $\gamma < \kappa$ the sequence $\langle \phi_{\gamma'} \mid \gamma' < \gamma\rangle$ of the first γ $\Pi_0\Sigma$ sentences is α-finite; now define:

$$\rho_0 = \varnothing,$$

$$\rho_{\gamma+1} = \begin{cases} \text{least } \rho \leqq \rho_\gamma \text{ such that } \rho \Vdash \phi_\gamma, & \text{if } \rho \text{ exists,} \\ \rho_\gamma, & \text{otherwise,} \end{cases}$$

$$\rho_\lambda = \text{least } \rho \leqq \rho_\gamma \text{ for all } \gamma < \lambda \quad (\lambda \text{ limit}).$$

Then for each $\gamma < \kappa$ the sequence $\langle \rho_{\gamma'} \mid \gamma' < \gamma\rangle$ is a member of $L_\alpha[Y]$ as $y = \{\gamma' < \gamma \mid \rho_{\gamma'+1} = \rho_{\gamma'}\} \in L_\kappa$ (and $\langle \rho_{\gamma'} \mid \gamma' < \gamma\rangle$ is $\Sigma_1\langle L_\alpha[Y], Y\rangle$ using y as a parameter, by (ii) above). Thus by (i) ρ_γ is well-defined for all $\gamma < \kappa$. Also $G = \{\rho \mid \rho \geqq \rho_\gamma \text{ for some } \gamma < \kappa\}$ is generic for $\Pi_0\Sigma$ sentences as for any γ either $\rho_{\gamma+1} \Vdash \phi_\gamma$ or $\forall \rho \leqq \rho_{\gamma+1}(\sim \rho \Vdash \phi_\gamma)$; thus either $\rho_{\gamma+1} \Vdash \phi_\gamma$ or $\rho_{\gamma+1} \Vdash \sim \phi_\gamma$. This completes the proof of Theorem 9.

§3. Σ_n admissibility; $n > 1$

Let κ be a cardinal and assume $V = L$. For $X \subseteq \kappa$ and $n > 1$ we let $\alpha_n(X)$ denote the least ordinal $\alpha > \kappa$ such that $L_\alpha[X]$ is Σ_n admissible. As was remarked in section three of *Uncountable admissibles I*, if κ is regular then α is of the form $\alpha_n(X)$ for some $X \subseteq \kappa$ if and only if $\kappa < \alpha < \kappa^+$, α has cofinality κ and L_α is both closed under and Σ_n admissible relative to the function $y \mapsto [y]^{<\kappa}$. The purpose of this section is to characterize ordinals of the form $\alpha_n(X)$ when κ is singular.

If κ has uncountable cofinality then the methods of §2 generalize in a straightforward way. However when cofinality$(\kappa) = \omega$ the compactness techniques of §1 do not apply; the key reason is that the standard part of a model of Σ_n admissibility need not be Σ_n admissible when $n > 1$. Instead we establish an effective version of Jensen's Covering Lemma (Devlin–Jensen [3]) and use it to show: If $X \subseteq \kappa$ is nonhyperregular then $X \in L_{\alpha_n}(X)$. Thus the countable cofinality case becomes like the (singular of) uncountable cofinality case when $n > 1$.

THEOREM 11. (Effective Covering Lemma) *Suppose κ is a cardinal, $X \subseteq \kappa$ and $L_\alpha[X]$ is Σ_2 admissible. If $L_\alpha[X] \models \kappa$ is singular then $X \in L_\alpha$.*

COROLLARY 12. *Suppose κ is a singular cardinal, α is Σ_n admissible, $n > 1$. Then $\alpha = \alpha_n(X)$ for some $X \subseteq \kappa$ iff there is a tame injection $L_\alpha \to \kappa$ and in addition if λ = greatest α-cardinal (should it exist):*

$$\kappa \text{ regular in } L_\alpha \rightarrow L_\alpha\text{-cofinality}(\lambda) \geqq \kappa,$$

$$\kappa \text{ singular in } L_\alpha \rightarrow \lambda = \kappa \text{ and } \alpha \text{ is a successor } \Sigma_n \text{ admissible.}$$

PROOF OF COROLLARY FROM THEOREM. If κ is singular in L_α then the Theorem implies that $X \in L_\alpha$ whenever $\alpha = \alpha_n(X)$. Thus in this case $\alpha = \alpha_n(X)$ implies that α is a successor Σ_n admissible and $\lambda = \kappa$. The converse is easy. If κ is regular in L_α and $\alpha = \alpha_n(X)$ then κ is still regular in $L_\alpha[X]$ by the Theorem. Then just as in the proof of Theorem 9 we see that L_α-cofinality$(\lambda) \geqq \kappa$: Otherwise if $f \in L_\alpha[X]$ maps L_λ 1-1 onto κ then $y \mapsto \bigcup f^{-1}[y]$ defines a $\Sigma_1(L_\alpha[X])$ function from L_κ onto a set containing $2^\lambda \cap L_\alpha$. This contradicts the admissibility of $L_\alpha[X]$.

It remains to show that if α is Σ_n admissible, there is a tame injection $L_\alpha \to \kappa$, κ regular in L_α, there is a greatest α-cardinal λ implies L_α-cofinality$(\lambda) \geqq \kappa$, then $\alpha = \alpha_n(X)$ for some $X \subseteq \kappa$. This is proved much as was the "if" direction of Theorem 9. In this case one seeks to build sets which are generic with respect to

$\Pi_0\Sigma_n$ sentences for a forcing relation which is Σ_n when restricted to Σ_n sentences (a $\Pi_0\Sigma_n$ sentence is one of the form $\forall x \in y\psi$, where ψ is Σ_n). As in (*) of the proof of Theorem 9 forcing for $\Pi_0\Sigma_n$ sentences is actually Σ_n. Thus we can use the tame injection $L_\alpha \to \kappa$ to decide each of the κ-many $\Pi_0\Sigma_n$ sentences by a κ-sequence $\rho_0 \geqq \rho_1 \geqq \cdots$ of conditions; all that is needed is the closure of the partial-ordering relative to $< \kappa$-sequences inside the ground model. The unbounded Lévy collapse forcing requires in addition that L_α is Σ_n admissible relative to $g\colon y \mapsto [y]^{<\kappa} \cap L_\alpha$. The assumption on λ = greatest α-cardinal implies that L_α is closed under this function. The proof of Lemma 2 shows that $\langle L_{\mu^+}, g \restriction L_{\mu^+}\rangle$ is a Σ_1 elementary substructure of $\langle L_\alpha, g\rangle$ whenever μ is a regular α-cardinal $> \kappa$. Thus if g' = the complete Σ_1 set for $\langle L_\alpha, g\rangle$ then g' has the same α-degree as $0'$, as $\{\mu \mid \mu$ is a regular α-cardinal$\}$ is α-recursive in $0'$. So for $n > 1$, Σ_n relative to $g = \Sigma_n$ and the Σ_n admissibility of L_α implies that of $\langle L_\alpha, g\rangle$. ⊣

Theorem 11 is obtained by examining Silver's proof via machines of Jensen's covering lemma and verifying that it can be carried out inside a model of Σ_2 admissibility. Notice that we can assume that X has cardinality $< \kappa$ as κ is singular in $L_\alpha[X]$. To prove $X \in L_\alpha$ it suffices to obtain $Y \supseteq X$, $\operatorname{card}(Y) < \kappa$ where $Y \in L_\alpha$. (This is why we refer to Theorem 11 as a "covering lemma".)

We assume some familiarity with machines and with Silver's proof of Jensen's covering lemma (Silver [9]). Let M be an L-machine. We first recall some definitions concerning direct limit systems. A *neat* triple is one of the form $\langle \delta, \alpha, P\rangle$ where $\alpha \leqq \delta$, $P \in [\delta]^{<\omega}$ and $\delta = M^\delta(\alpha \cup P)$. A *neat* map $\pi\colon \langle \bar\delta, \bar\alpha, \bar P\rangle \to \langle \delta, \alpha, P\rangle$ between neat triples is actually a medium M-map $\pi\colon \bar\delta \to \delta$ so that $\pi \restriction \bar\alpha = \text{identity}$, $\pi[\bar\delta - \bar\alpha] \subseteq \delta - \alpha$ and $\pi[\bar P] \subseteq P$. Then a *κ-direct system* is a collection of neat triples and maps $\Pi = \{\langle \delta_i, \alpha_i, P_i\rangle, \pi_{ij} \mid i \leqq j$ in $I\}$ where I is a directed class such that

(a) each $\delta_i < \kappa$ and $\{\alpha_i \mid i \in I\}$ is cofinal in κ,

(b) if $i \leqq j \leqq k$ then $\pi_{ik} = \pi_{jk} \circ \pi_{ij}$,

(c) Range π_{ij} is not cofinal in δ_j.

We can describe how κ-direct systems arise as follows: Suppose η is a limit ordinal and for each $\alpha < \kappa$, finite $Q \subseteq \eta$, $M''(\alpha \cup Q)$ has ordertype less than κ. Then η can be realized as the limit of a κ-direct limit system. For, let $\{\langle \eta_i, \alpha_i, Q_i\rangle \mid i \in I\}$ enumerate all triples $\langle \eta', \alpha', Q'\rangle$ such that $\eta' < \eta$, $\alpha' < \kappa$ and $Q' \subseteq \eta'$ is finite. Define $i \leqq j \leftrightarrow \eta_i \leqq \eta_j$, $\alpha_i \leqq \alpha_j$, $Q_i \subseteq Q_j$ and $\eta_i \in Q_j$. Let $X_i = M^{\eta_i}(\alpha_i \cup Q_i)$ and let $\sigma_i\colon \delta_i \to X_i$ be the ascending enumeration. By hypothesis $\delta_i < \kappa$. If $P_i = \sigma_i^{-1}[Q_i]$ then $(\delta_i, \alpha_i, P_i)$ is a neat triple. If $i \leqq j$ then define $\pi_{ij} = \sigma_j^{-1} \circ \sigma_i$. Then $\{(\sigma_i, \alpha_i, P_i), \pi_{ij} \mid i \leqq j$ in $I\}$ is a κ-direct limit system with direct limit η.

Given this notion of a κ-direct limit system we can now outline the proof of Jensen's covering lemma. As $X \subseteq \kappa$ has cardinality less than κ we can obtain an elementary embedding $j\colon L_{\bar\kappa} \to L_\kappa$ where $\bar\kappa < \kappa$ and $X \subseteq \mathrm{Range}(j)$. Now if $\bar\kappa$ is an L-cardinal then $M^\eta(\alpha \cup Q)$ has ordertype $< \bar\kappa$ for all η, $\alpha < \bar\kappa$, finite $Q \subseteq \eta$. Thus OR is the limit of a κ-direct limit system $\Pi = \{\langle \delta_i, \alpha_i, P_i\rangle, \pi_{ij} \mid i \leqq j \text{ in } I\}$. Now $j(\Pi) = \{\langle j(\delta_i), j(\alpha_i), j[P_i]\rangle, j \circ \pi_{ij} \circ j^{-1} \mid i \leqq j \text{ in } I\}$ is a κ-direct limit system. If $j(\Pi)$ is well-founded (with direct limit OR) then j can be extended to an elementary embedding $L \to L$. Thus $0^\#$ exists.

Suppose that $\bar\kappa$ is not an L-cardinal and therefore $M^\eta(\alpha \cup Q)$ has ordertype $\geqq \kappa$ for some $\eta, \alpha < \bar\kappa$, finite $Q \subseteq \eta$. Assume that η is least; we can then also assume that $\eta = M^\eta(\alpha \cup Q)$. As before we can write η as the direct limit of some $\bar\kappa$-direct limit system Π. If $j(\Pi)$ has a well-founded direct limit μ then $j \restriction \bar\kappa$ can be extended to a medium M-map $j^*\colon \eta \to \mu$. Then we have $X \subseteq \mathrm{Range}(j^*) \subseteq M^\mu(j(\alpha) \cup j^*[Q]) = Y$. Y has cardinality $= \mathrm{card}(j(\alpha)) < \kappa$ and $Y \in L$.

Now assuming that $j\colon L_{\bar\kappa} \to L_\kappa$ can be chosen so that $j(\Pi)$ is a well-founded κ-direct limit system whenever Π is a well-founded $\bar\kappa$-direct limit system, the above argument easily takes place inside any Σ_2 admissible set. In fact all one needs is Σ_1 admissibility to construct an L-machine and to compute the ordinal limit of a (truly) well-founded direct limit system. Now consider the argument in $L_\alpha[X]$. In the first case we get $L_\alpha[X] \models 0^\#$ exists, which is a contradiction as $\alpha > \omega_1$ and therefore $L_\alpha[X]$ would then produce the true $0^\#$. This conflicts with our hypothesis $V = L$. Therefore the second case must hold and we get $X \subseteq Y \in L_\alpha$, $\mathrm{card}(Y) < \kappa$. But then $X \in L_\alpha$ as $2^{<\kappa} \cap L_\alpha[X] \subseteq L_\alpha$.

It remains only to justify the assumption on $j\colon L_{\bar\kappa} \to L_\kappa$ that $j(\Pi)$ is a well-founded κ-direct limit system if Π is a well-founded $\bar\kappa$-direct limit system. This is where Σ_2 admissibility is needed. (Note that well-founded $\leftrightarrow$ embeddable into OR follows from Σ_2 admissibility.)

If $\Pi = \{\langle \delta_i, \alpha_i, P_i\rangle, \pi_{ij} \mid i \leqq j \text{ in } I\}$ is a direct limit system then we write $\pi \subseteq Y$ if each $\langle \delta_i, \alpha_i, P_i\rangle$, $\pi_{ij} \in Y$. A *descending code* for Π is a sequence $\{\beta_n \mid n \in \omega\}$ so that for some $i_0 < i_1 < \cdots$ in I, $\pi_{i_n i_{n+1}}(\beta_n) > \beta_{n+1}$. Thus Π is ill-founded iff Π has a descending code. $\Pi \subseteq Y$ is *Y-well-founded* if Π has no descending code $\{\beta_n \mid n \in \omega\} \subseteq Y$. To justify our assumption on j it suffices to get $X \subseteq Z < L_\kappa$ so that $\mathrm{card}(Z) < \kappa$ and whenever $\Pi \subseteq Z$ is a Z-well-founded κ-direct limit system then Π is really well-founded. For then let $j\colon L_{\bar\kappa} \to L_\kappa$ be the inverse to the transitive collapse of Z.

To get Z it suffices to prove that there is a Σ_2 function $Y \mapsto \bar Y$ so that if $Y \subseteq L_\kappa$, $\mathrm{card}(Y) < \kappa$ then $Y \subseteq \bar Y < L_\kappa$, $\mathrm{card}(\bar Y) = \omega_1 \cup \mathrm{card}(Y)$ and any ill-founded

$\Pi \subseteq Y$ is $\bar{Y}$-ill-founded. For then by starting with X and iterating this function ω_1 times we get $Z \supseteq X$, $Z < L_\kappa$, $\mathrm{card}(Z) < \kappa$ so that any countable ill-founded $\Pi \subseteq Z$ is Z-ill-founded. But any ill-founded direct limit system contains a countable ill-founded direct limit system. Note that the Σ_2 admissibility of $L_\alpha[X]$ allows one to perform the above iteration inside $L_\alpha[X]$.

Finally we must describe the Σ_2 function $Y \mapsto \bar{Y}$. We can assume that $Y < L_\kappa$; let $k: L_{\bar{\kappa}} \to L_\kappa$ be the inverse to the transitive collapse of Y. Let α be the least ordinal so that there is a κ-direct limit system Π with direct limit α, yet $k[\Pi]$ is not well-founded. Let Π_0 be one such Π and $\{\beta_n \mid n \in \omega\}$ a descending code for $k[\Pi_0]$. Define $\bar{Y} < L_\kappa$ to be the $L_\alpha[X]$-least $\bar{Y} \supseteq Y \cup \{\beta_n \mid n \in \omega\}$, $\mathrm{card}(\bar{Y}) = \omega_1 \cup \mathrm{card}(Y)$.

This is certainly a Σ_2 definition. We must show that any ill-founded κ-direct limit system $\Pi \subseteq Y$ is $\bar{Y}$-ill-founded. Let $\bar{\Pi}$ be the pullback of Π via k (so $k(\bar{\Pi}) = \Pi$). If Π is $\bar{Y}$-well-founded then $\bar{\Pi}$ is well-founded, say with direct limit β. Now $\alpha \leqq \beta$ by definition of α. It is not difficult to construct a κ-direct limit system Γ such that $\Pi_0, \bar{\Pi}$ are both subsystems of Γ and $\bar{\Pi}$ is cofinal in Γ. Suppose $i_0 < i_1 < \cdots$ belong to the index set for Π_0 and $k(\pi_{i_n i_{n+1}})(\beta_n) > \beta_{n+1}$. Then choose $j_0 < j_1 < \cdots$ from the index set for $\bar{\Pi}$ so that $i_n < j_n$ for each n. The sequence $k(\pi_{i_0 j_0})(\beta_0)$, $k(\pi_{i_1 j_1})(\beta_1), \cdots$ is a descending code for Π and shows that Π is not $\bar{Y}$-well-founded. This completes the proof of Theorem 11.

§4. Further results and open questions

(1) The argument at the end of the proof of Corollary 12 shows: If α is Σ_n admissible of cardinality κ and either there is no greatest α-cardinal or the greatest α-cardinal has L_α-cofinality $\geqq \kappa$, then α is Σ_n admissible relative to the function $y \mapsto [y]^{<\kappa} \cap L_\alpha$. Thus if in addition there is a tame injection $L_\alpha \to L_\kappa$ then $\alpha = \alpha_n(X)$ for some $X \subseteq \kappa$. For many admissible α this function also gives a natural solution to Post's Problem in α-recursion theory (see Friedman [5]).

(2) Suppose $\alpha = \alpha(X)$ for some $X \subseteq \kappa$ and κ has cofinality ω. Then is there $X \subseteq \kappa$ such that $\alpha = \alpha(X)$ and for all $Y \in 2^\kappa \cap L_\alpha[X]$ either $\alpha(Y) < \alpha$ or $X \in L_\alpha[Y]$? (In other words: Is there a "minimal" solution to $\alpha = \alpha(X)$?)

(3) Suppose $L_\alpha[X]$ is admissible, $L_\alpha[X] \models \kappa =$ greatest cardinal is singular of uncountable cofinality, $X \subseteq \kappa$. Then does the Jensen covering lemma hold in $L_\alpha[X]$? If κ is a cardinal in L, $X \in L$ then this is established much as was the Effective Covering Lemma. This fact can be used to give an alternative proof of Theorem 8.

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Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139 USA