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UNCOUNTABLE ADMISSIBLES I: FORCING

BY
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ABSTRACT. Assume $V = L$. Let κ be a regular cardinal and for $X \subseteq \kappa$ let $\alpha(X)$ denote the least ordinal α such that $L_\alpha[X]$ is admissible. In this paper we characterize those ordinals of the form $\alpha(X)$ using forcing and fine structure of L techniques. This generalizes a theorem of Sacks which deals with the case $\kappa = \omega$.

Forcing has proved to be an extremely valuable tool for the recursion-theorist. Generic sets are used to construct a minimal hyperdegree (Gandy and Sacks [**1967**]), establish the plus-one and plus-two theorems in higher types (Sacks [**1974**], Harrington [**1973**]), refute the (relativized) McLaughlin conjecture (Steel [**1978**]) and characterize the countable admissible ordinals (Sacks [**1976**]). This is because genericity provides a natural way to control the definability properties of sets.

In the countable case generic set existence is not a problem. (This suffices for most applications of forcing in set theory.) However the existence of generic sets in the uncountable case can be an obstacle. One way around it is to exploit strong closure properties of the ground model in question.

This last technique can be illustrated as follows: Suppose κ is regular and M is a transitive set of cardinality κ which is $< \kappa$-*closed*; i.e., any sequence from M of length $< \kappa$ belongs to M. Assume that $\mathcal{P}$ is a partial ordering of a subset of M which is also $< \kappa$-*closed*; i.e., for any sequence $p_0 \geqslant p_1 \geqslant p_2 \geqslant \ldots$ from $\mathcal{P}$ of length $< \kappa$ there is $p \in \mathcal{P}$ s.t. $p \leqslant p_\gamma$ for each γ. Then one can easily construct (in κ steps) sets which are $\mathcal{P}$-generic over M.

The main result of the present paper implies that the preceding paragraph describes the best existence theorem for generic sets in the uncountable case. We present a simple forcing problem for uncountable L_α's which can be solved positively only in cases where L_α possesses the strong closure property stated above.

The problem in question is a generalization of Sacks' characterization of countable admissible ordinals. Assume $V = L$ and let κ be an uncountable cardinal. For $X \subseteq \kappa$ we let $\alpha(X)$ denote the least ordinal $\alpha > \kappa$ s.t. $L_\alpha(X)$ is admissible.

Question ($V = L$). Which admissible ordinals are of the form $\alpha(X)$ for some $X \subseteq \kappa$?

We deal in this paper with the case where κ is regular. (The singular case will be treated in Friedman [**1981a**].) The answer to this Question is best phrased in terms of a strong form of $< \kappa$-closure which we call $< \kappa$-admissibility.

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DEFINITION. L_α is $<\kappa$-*admissible* if cofinality $(\alpha) \geqslant \kappa$ and L_α is both closed under and admissible relative to the function $y \mapsto [y]^{<\kappa}$, where $[y]^{<\kappa} = \{x \subseteq y \mid x$ has cardinality $< \kappa\}$.[2]

Notice that $<\kappa$-admissibility implies $<\kappa$-closure. It is natural to attempt to build X as above by forcing over L_α. If L_α is $<\kappa$-admissible then in §1 we construct a $<\kappa$-closed partial ordering $\mathcal{P}$ such that $\alpha = \alpha(X)$ whenever X is $\mathcal{P}$-generic over L. The $<\kappa$-closure of L_α implies the existence of such X and thus the Question is answered positively in this case.

The purpose of §2 is to provide a negative answer in all other cases.

THEOREM. $\alpha = \alpha(X)$ *for some* $X \subseteq \kappa$ *iff*
(i) $\kappa < \alpha < \kappa^+$,
(ii) L_α *is* $<\kappa$-*admissible.*

Thus even basic forcing results extend to the uncountable only in very special cases. In §2 we will also give examples where $\alpha < \kappa^+$ is admissible of cofinality κ but L_α is not $<\kappa$-closed and also where $\alpha < \kappa^+$ is admissible, L_α is $<\kappa$-closed but L_α is not $<\kappa$-admissible. The former example corrects an assertion made following Problem 48 in H. Friedman's list of problems in logic (H. Friedman [**1975**]). Our methods can be used to give a negative solution to that problem in ZF (see Example 3 in §2).

The "only if" direction of our Theorem (proved in §2) is established via a combination of techniques from β-recursion theory and the fine structure of L. We show that if $\alpha = \alpha(X)$ for some X then not only α but also Σ_n projectum (β) must have cofinality κ for various ordinals β closely related to α (for certain $n \in \omega$). This condition in turn implies the $<\kappa$-closure of L_α by an induction argument. The proof of $<\kappa$-admissibility uses related methods.

The "if" direction is proved by a forcing technique which extends the method of Jensen [**1972**] to the uncountable. The proof is in two parts. First a predicate $B \subseteq L_\alpha$ is constructed so that $\langle L_\alpha[B], B\rangle$ is admissible but $\langle L_\gamma[B \cap L_\gamma], B \cap L_\gamma\rangle$ is inadmissible for each $\gamma < \alpha$. This uses (and in fact necessitates) the $<\kappa$-admissibility of L_α. Second the predicate B is coded using almost disjoint set forcing by $X \subseteq \kappa$. Thus $\alpha = \alpha(X)$.

1. Killing admissibles. Our goal in this section is to establish

THEOREM 1 ($V = L$). *If α has cardinality ω_1, L_α is ω-admissible, then $\alpha = \alpha(X)$ for some $X \subseteq \omega_1$.*

Thus we consider only the case $\kappa = \omega_1$. We write "ω-admissible" for "$<\omega_1$-admissible". There are obvious modifications of what is described below when κ is any other regular L-cardinal.

Our proof is an extension to the uncountable of a forcing method developed by Jensen (see Jensen [**1972**]). He used a combination of unbounded Lévy forcing, a

[2]Lemma 4 below shows that if α is admissible then the words "and admissible relative to" can be deleted in this definition.

special forcing for destroying recursive Mahloness and almost disjoint set forcing to prove a strengthening of Sacks' theorem that any countable admissible $> \omega$ is the first admissible relative to some real $R \subseteq \omega$. There are several new problems which arise when attempting to adapt these techniques to the present context. To demonstrate that admissibility is preserved by unbounded Lévy forcing one must use the ω-admissibility of L_α. A Skolem hull argument is needed then to construct a predicate $B \subseteq L_\alpha$ such that $\langle L_\alpha[B], B\rangle$ is admissible but for each $\gamma < \alpha$ $\langle L_\gamma[B \cap L_\gamma], B \cap L_\gamma\rangle$ is inadmissible. Finally we make use of a technical lemma from Jensen [**1975**] to perform almost disjoint set coding in an admissibility-preserving way.

An ω_1-closed unbounded set of inadmissibles. If α is countable and admissible then a closed unbounded $C \subseteq \alpha$ is constructed in Jensen [**1972**] such that

(a) $L_\alpha[C] = L_\alpha$,
(b) $\langle L_\alpha, C\rangle$ is admissible, and
(c) $\gamma \in C \to \gamma$ is inadmissible.

Now suppose α is as in the hypothesis of Theorem 1. Then we construct a closed unbounded $C \subseteq \alpha$ such that

(a) $L_\alpha[C] = L_\alpha$,
(b) $\langle L_\alpha, C\rangle$ is admissible, and
(c) $\gamma \in C$, cofinality $(\gamma) > \omega \to \gamma$ is inadmissible.

The extra hypothesis in (c) is necessary: For, there may be a closed unbounded $D \subseteq \alpha$ consisting only of admissibles. But then $C \cap D \neq \varnothing$ since C, D are closed unbounded subsets of an ordinal of uncountable cofinality. We arrange that $C \cap D$ consists only of ordinals of cofinality ω.

A condition p is a closed subset of α such that

(i) p has a greatest element,
(ii) $\gamma \in p$, cofinality $(\gamma) > \omega \to \gamma$ inadmissible, and
(iii) $p \in L_\alpha$.

If p, q are conditions then p is stronger than q, $p \leqslant q$, if $p \cap (\max(q) + 1) = q$. Let $\mathcal{P}_J$ denote the collection of all conditions. $\mathcal{P}_J$ is countably closed. If $\phi(\underset{\sim}{G})$ is a ranked sentence of L_α, $\operatorname{rank}(\phi) = \beta$, then we define

$$p \Vdash \phi \leftrightarrow \beta < \max(p) \quad \text{and} \quad L_\beta[p] \vDash \phi(p).$$

So forcing for ranked sentences is a Δ_1 relation. Forcing for unranked sentences is defined in the usual way. Thus forcing for Σ_1 sentences is Δ_1 over L_α. The existence of sets $\mathcal{P}_J$-generic for sentences of L_α is guaranteed by the countable closure of L_α, the countable closure of $\mathcal{P}_J$ and the next lemma.

LEMMA 2. *If p is a condition and ϕ is ranked then $\exists q \leqslant p$ ($q \Vdash \phi$ or $q \Vdash \sim\phi$).*

PROOF. Choose $\beta < \alpha$ to be inadmissible and greater than $\operatorname{rank}(\phi)$, $\max(p)$. Then $p \cup \{\beta\} \Vdash \phi$ or $p \cup \{\beta\} \Vdash \sim \phi$. Also $p \cup \{\beta\} \leqslant p$. $\square$

If G is $\mathcal{P}_J$-generic for sentences of L_α then $C = \bigcup G$ is a closed unbounded subset of α such that $L_\alpha[C] = L_\alpha$. The next lemma guarantees that $\langle L_\alpha, C\rangle$ is admissible.

Lemma 3. *Suppose $\beta < \alpha$ and $f\colon \beta \to \alpha$ is $\Sigma_1\langle L_\alpha, C\rangle$. Then $f \in L_\alpha$.*

Proof. Let $\phi(x, y, z, \underline{C})$ be a Δ_0 formula ($\underline{C}$ is a name for C). Suppose $p \Vdash \forall y < \underline{\beta} \exists! \delta \exists x \phi(x, y, \delta, \underline{C})$ and $\max(p) > \beta$, $\operatorname{rank}(\phi)$. Define a sequence of conditions as follows: $p_0 = p$. $P_{\gamma+1} =$ least $p \leqslant p_\gamma$ s.t. $p \Vdash \exists x \exists \delta \phi(x, \gamma, \delta, \underline{C})$ and $p_\gamma \in L_{\max(p)}$. $p_\lambda = \bigcup_{\gamma<\lambda} p_\gamma \cup \{\sup(\bigcup_{\gamma<\lambda} p_\gamma)\}$. Note that $\langle p_\gamma \restriction \gamma < \lambda \rangle$ is $\Sigma_1(L_{\max(p_\lambda)})$ for each limit λ as forcing for Σ_1 sentences is Δ_1. So $\max(p_\lambda)$ is inadmissible for each limit $\lambda \leqslant \beta$. Thus p_β is an extension of p and $p_\beta \Vdash^*$ "$\exists x \phi(x, y, z, \underline{C})$ defines a function in L_α", where $\Vdash^*$ denotes weak forcing. If $f \in \Sigma_1\langle L_\alpha, C\rangle$ is defined by $\exists x \phi(x, y, z, \underline{C})$ then $\{p \mid p \Vdash$ "If $\exists x \phi(x, y, z, \underline{C})$ defines a function then this function is in L_α"$\}$ is dense and therefore $\exists p \in G\ p \Vdash$ "The function defined by $\exists x \phi(x, y, z, \underline{C})$ is in L_α". So $f \in L_\alpha$. $\square$

For the application below we will need a relativized version of the preceding. If $L_\alpha[S] = L_\alpha$ is countably closed and $\langle L_\alpha, S\rangle$ is admissible then there is a closed unbounded $C \subseteq \alpha$ such that

(a) $L_\alpha[S, C] = L_\alpha$,

(b) $\langle L_\alpha, S, C\rangle$ is admissible, and

(c) $\gamma \in C$, cofinality $(\gamma) > \omega \to \langle L_\gamma, S \cap L_\gamma\rangle$ is amenable and inadmissible.

This is proved exactly as above, replacing "admissible" by "S-admissible" and "Σ_1" by "Σ_1 in S" throughout. Thus one can define a countably closed forcing $\mathcal{P}_J^S$ which is Δ_1 over $\langle L_\alpha, S\rangle$ and such that any C which is $\mathcal{P}_J^S$-generic over $\langle L_\alpha, S\rangle$ satisfies (a), (b) and (c) above.

Unbounded Lévy forcing. Again let α be as in the hypothesis of Theorem 1. We will choose $C \subseteq \alpha$ to be $\mathcal{P}_J^S$-generic over L_α where S is defined as follows. $S = \{(x, y) \mid x, y \in L_\alpha \text{ and } y = [x]^\omega\}$. But first we establish some properties of S which will be of use to us in §2.

Note that L_α is ω-admissible if and only if cofinality $(\alpha) \geqslant \omega_1$, L_α is closed under $y \mapsto [y]^\omega$ and $\langle L_\alpha, S\rangle$ is admissible. We first show that this last condition is redundant.

Lemma 4. *Suppose L_α is admissible and closed under $y \mapsto [y]^\omega$. Then $\langle L_\alpha, S\rangle$ is admissible.*

Proof. There are two cases.

Case 1. There is a largest α-cardinal κ. Then κ has uncountable cofinality as otherwise the elements of $[\kappa]^\omega$ are constructed cofinally in α. But $[\kappa]^\omega \in L_\alpha$ by hypothesis.

We now show that S is actually Δ_1 over L_α: Given $y \in L_\alpha$ let c be the $<_L$-least injection of y into κ and let $z = c[y]$. Then $[y]^\omega = \{c^{-1}[s] \mid s \in [z]^\omega\}$ and $[z]^\omega = [z]^\omega \cap L_\kappa$. This gives a Σ_1 definition for $y \mapsto [y]^\omega$.

Case 2. There is no largest α-cardinal. We show that if κ is a regular α-cardinal then $\langle L_{\kappa^+}, S \cap L_{\kappa^+}\rangle$ is a Σ_1-elementary substructure of $\langle L_\alpha, S\rangle$. This establishes the admissibility of $\langle L_\alpha, S\rangle$.

Suppose κ is a regular α-cardinal. Cofinality $(\kappa) > \omega$ as L_α is countably closed. Now suppose $\langle L_\alpha, S\rangle \vDash \exists y \phi(x, y, S)$ where ϕ is Δ_0 and $x \in L_{\kappa^+}$. We must show that

$\langle L_{\kappa^+}, S \cap L_{\kappa^+}\rangle \models \exists y \phi(x, y, S \cap L_{\kappa^+})$. Choose $y \in L_\alpha$ so that $\langle L_\alpha, S\rangle \models \phi(x, y, S)$ and let λ be a regular α-cardinal so that $y \in L_\lambda$. Then $\langle L_\lambda, S \cap L_\lambda\rangle$ is amenable so we may choose a Σ_1 Skolem function h for this structure. As κ is a regular α-cardinal we may choose $\delta < \kappa^+$ so that L_δ is countably closed and $x \in L_\delta$. Then let $H = \Sigma_1$ Skolem hull of $L_\delta \cup \{y\}$ inside L_λ.

We claim that H is countably closed. For, if $x_1, x_2, \ldots \in H$ then $x_i = h(n_i, \gamma_i, y)$ for some $n_i \in \omega$, $\gamma_i < \delta$. Then $\langle (n_i, \gamma_i) \mid i \in \omega\rangle \in L_\delta$ and as the Σ_1 sentence $\exists z \forall i$ $(z(i) = h(n_i, \gamma_i, y))$ is true in L_λ (by the countable closure of L_λ), it is true in H. So $\langle x_i \mid i \in \omega\rangle \in H$.

Now collapse H isomorphically to L_γ, $\delta \leqslant \gamma < \kappa^+$. Then L_γ is countably closed and $S \cap H$ collapses to $S \cap L_\gamma$. So as $\langle L_\lambda, S \cap L_\lambda\rangle \models \phi(x, y, S \cap L_\lambda)$ we have $\langle L_\gamma, S \cap L_\gamma\rangle \models \phi(x, \bar{y}, S \cap L_\gamma)$ where $\bar{y} = \text{image}(y)$ under the collapse. Hence $\langle L_{\kappa^+}, S \cap L_{\kappa^+}\rangle \models \phi(x, \bar{y}, S \cap L_{\kappa^+})$ since ϕ is Δ_0. □

This now gives us an explicit characterization of ω-admissibility in terms of countable closure. Countable closure is in turn characterized by a "fine structure" condition in the next section.

Lemma 5. *L_α is ω-admissible iff L_α is countably closed and*
($*$) *either there is no largest α-cardinal or the largest α-cardinal has uncountable cofinality.*

Proof. The "only if" direction was covered in the proof of Lemma 4, Case 1. By Lemma 4, to prove the "if" direction it suffices to show that L_α is closed under $y \mapsto [y]^\omega$ if L_α is countably closed and obeys ($*$). But this is clear by Gödel unless y has cardinality = greatest α-cardinal. But then if $c \in L_\alpha$ is a bijection between y and gc α then $[y]^\omega = \{c^{-1}[s] \mid s \in [\text{gc}\,\alpha]^\omega\}$ and $[\text{gc}\,\alpha]^\omega = [\text{gc}\,\alpha]^\omega \in L_{\text{gc}\alpha}$ by ($*$). □

Now let C be $\mathcal{P}_J^S$-generic over L_α. Then $L_\alpha[S, C] = L_\alpha$ is admissible. We seek to build a class $A \subseteq \alpha$ such that $\langle L_\alpha[A], A, C\rangle$ is admissible and $L_\alpha[A] \models \omega_1$ is the largest cardinal. We construct A to be generic for a countably closed forcing for then the countable closure of L_α guarantees the existence of generic classes. In the countable case this type of forcing was done in Sacks [**1976**] with finite conditions. He exploits the fact that $\beta < \alpha$ implies $[\beta]^{<\omega} \in L_\alpha$ in order to help reduce this class forcing to set forcing. The analogous property in our present context, $\beta < \alpha$ implies $[\beta]^\omega \in L_\alpha$, is guaranteed by the ω-admissibility of L_α. Moreover by working relative to S we can assume that the operation $\beta \mapsto [\beta]^\omega$ is effective.

We now define the countably closed forcing $\mathcal{P}_L$ for making ω_1 the largest cardinal. A condition p is a countable partial function $\alpha \times \omega_1 \to \alpha$ such that $p(\beta, \gamma) < \beta$ for each $(\beta, \gamma) \in \text{Domain}(p)$. The condition p is *stronger* than the condition q, $p \leqslant q$, if p extends q as a partial function. Thus if A is $\mathcal{P}_L$-generic over L_α then $L_\alpha[A] \models \omega_1$ is the largest cardinal. We in fact want A to be $\mathcal{P}_L$-generic over $\langle L_\alpha, S, C\rangle$ in order to guarantee that $\langle L_\alpha[A], A, C\rangle$ is admissible. The countable closure of L_α guarantees the existence of such an A.

The proof of admissibility is much as in the countable case. ω-admissibility is used to "bound the forcing relation": For $\beta < \alpha$ and $p \in \mathcal{P}_L$ let $p^{<\beta} = p \cap (\beta \times \omega_1) \times \beta$. Then $\mathcal{P}_L^{<\beta} = \{p^{<\beta} \mid p \in \mathcal{P}_L\}$ is a member of L_α and the function $\beta \mapsto \mathcal{P}_L^{<\beta}$ is

$\Sigma_1\langle L_\alpha, S\rangle$. A simple induction shows

(*) if $p \Vdash \phi$, rank$(\phi) < \beta$ then $p^{<\beta} \Vdash \phi$.

Forcing is defined in the usual way by induction. All instances of this induction for ranked ϕ are clearly $\Sigma_1\langle L_\alpha, S, C\rangle$ except the negation case which is Σ_1 due to (*). $p \Vdash \sim\phi$ iff $\forall q \leqslant p(\sim q \Vdash \phi)$ iff $\exists\beta$ (rank $\phi < \beta$ and $\forall q \in \mathcal{P}_L^{<\beta}(q \leqslant p \to \sim q \Vdash \phi)$). Thus the relation $p \Vdash \phi$ of p, ϕ is Σ_1 (for ranked ϕ).

It is now easy to establish the admissibility of $L_\alpha[A]$. Suppose ϕ is ranked and $p_0 \Vdash \forall x \exists y \phi(x, y)$. Thus for each $p \leqslant p_0$, constant c there is $q(p, c) \leqslant p$, constant $d(p, c)$ such that $q(p, c) \Vdash \phi(c, d(p, c))$. The functions q, d can be assumed to be Σ_1 as the forcing relation is Σ_1 for ranked sentences. Let $\beta < \alpha$ be a *fixed point* so that L_β is countably closed; i.e.,

$$p_0 \in L_\beta,$$

$$p, c \in L_\beta \to q(p, c), \quad d(p, c) \in L_\beta,$$

$$[L_\beta]^\omega \subseteq L_\beta.$$

β exists by admissibility of $\langle L_\alpha, S, C\rangle$.

Claim. $p_0 \Vdash \forall x^\beta \exists y^\beta \phi(x, y)$.

PROOF OF CLAIM. Suppose $p \leqslant p_0$, $c \in L_\beta$. Then $p^{<\beta} \in L_\beta$ since L_β is countably closed. As $p^{<\beta} \leqslant p_0$ we see that $q = q(p^{<\beta}, c)$, $d = d(p^{<\beta}, c)$ are defined and belong to L_β. Thus $q \cup p \Vdash \phi(c, d)$. So we have shown

$$\forall p \leqslant p_0 \forall c \in L_\beta \exists r \leqslant p \exists d \in L_\beta r \Vdash \phi(c, d).$$

This is exactly the statement of the Claim.

Now if $\langle L_\alpha[A], A, S, C\rangle \vDash \forall x \exists y \phi(x, y)$, ϕ ranked then for some $p \in A$ $p \Vdash \forall x \exists y \phi(x, y)$. By the claim, for some $\beta < \alpha$ $p \Vdash \forall x^\beta \exists y^\beta \phi(x, y)$ so $L_\alpha[A] \vDash \forall x^\beta \exists y^\beta \phi(x, y)$. We have demonstrated Π_2-reflection for $\langle L_\alpha[A], A, S, C\rangle$.

Almost disjoint set forcing. We are now in the following situation: There are predicates $A, S, C \subseteq L_\alpha$ such that

(a) $\langle L_\alpha[A], A, S, C\rangle$ is admissible,

(b) $L_\alpha[A] \vDash \omega_1$ is the largest cardinal,

(c) C is closed unbounded in α, and

(d) $\gamma \in C \to \mathrm{cof}(\gamma) = \omega$ or $\langle L_\gamma, S \cap L_\gamma\rangle$ is inadmissible and amenable.

It is now fairly easy to construct a predicate $B \subseteq L_\alpha$ such that $\langle L_\gamma[B \cap L_\gamma], B \cap L_\gamma\rangle$ is inadmissible for each $\gamma < \alpha$ and B is $\Delta_1\langle L_\alpha[A], A, S, C\rangle$. Then the idea will be to code B by a subset X of ω_1 in a very efficient way, so that $B \cap L_\gamma$ is Δ_1 over $L_\gamma[X]$ for each $\gamma < \alpha$ and $L_\alpha[X]$ is admissible.

The construction of B is as follows: For each $\gamma \in C$ let w_γ be the $L_\alpha[A]$-least wellordering of ω_1 of ordertype γ', where γ' = least member of C greater than γ. We identify w_γ with a subset of ω_1. Let $B' = \{\gamma + \delta \mid \gamma \in C$ and $\delta \in w_\gamma\}$. Then $B = \mathrm{join}(S, B')$. (Thus $B = \{2\mu \mid \mu \in S\} \cup \{2\mu + 1 \mid \mu \in B'\}$.) Certainly $\langle L_\gamma[B \cap L_\gamma], B \cap L_\gamma\rangle$ is inadmissible when γ is not the limit of elements of C as this structure contains a wellordering of ω_1 of ordertype $\geqslant \gamma$ in this case. If $\gamma < \alpha$ is the limit of elements of C and has uncountable cofinality then again this structure is inadmissible since $\langle L_\gamma, S \cap L_\gamma\rangle$ is.

We now show that $\langle L_\gamma[B \cap L_\gamma], B \cap L_\gamma\rangle$ is inadmissible for every $\gamma < \alpha$. Otherwise let γ be the least ordinal such that $\langle L_\gamma[B \cap L_\gamma], B \cap L_\gamma\rangle$ is admissible. We show that γ has uncountable cofinality, contradicting the previous paragraph. Note that as γ is the limit point of elements of C, $L_\gamma[B \cap L_\gamma] \models \omega_1$ is the largest cardinal.

Let h be a Σ_1 Skolem function for $\langle L_\gamma[B \cap L_\gamma], B \cap L_\gamma\rangle$. Then for each $\delta < \gamma$ $h[\omega \times L_\delta]$ is transitive as $L_\gamma[B \cap L_\gamma] \models \omega_1$ is the largest cardinal. Thus $h[\omega \times L_{\omega_1+1}]$ is of the form $\langle L_{\gamma'}[B \cap L_{\gamma'}], B \cap L_{\gamma'}\rangle$ and is admissible. The leastness of γ implies that $\gamma' = \gamma$ and so there is a $\Sigma_1\langle L_\gamma[B \cap L_\gamma], B \cap L_\gamma\rangle$ injection g of γ into ω_1. For each $\delta < \omega_1$, $g^{-1}[\delta]$ is bounded in γ by the admissibility of $\langle L_\gamma[B \cap L_\gamma], B \cap L_\gamma\rangle$. The function $\delta \mapsto \sup g^{-1}[\delta]$ is a nondecreasing unbounded function from ω_1 into γ so cofinality $(\gamma) = \omega_1$.

LEMMA 6. *For each* $\gamma \geq \omega_1$, $L_{\gamma+1}[B \cap L_\gamma] \models \gamma$ *has cardinality* ω_1.

PROOF. Otherwise let γ be the least $\delta \geq \omega_1$ s.t. $L_{\delta+1}[B \cap L_\delta] \models \delta$ has cardinality $> \omega_1$. Then $L_{\gamma+1}[B \cap L_\gamma] \models \gamma = \omega_2$ and so certainly $L_\gamma[B \cap L_\gamma]$ is admissible. But this contradicts the above established property of B. $\square$

Now we must code B by a subset of ω_1. The standard way to do this is with almost disjoint set forcing, invented in Jensen and Solovay [**1970**] and perfected in Jensen [**1972**, **1975**]. This method can be described as follows: First choose a $\Delta_1\langle L_\alpha[B], B\rangle$ sequence $\langle X_\gamma \mid \gamma < \alpha\rangle$ of subsets of ω_1 such that $\gamma \neq \gamma' \to X_\gamma \cap X_{\gamma'}$ is countable (X_γ, $X_{\gamma'}$ are almost disjoint). This is easily done. Also define $B^* = \{\omega_1 + \gamma \mid$ the γth set in $<_{L_\alpha}$ belongs to $B\}$. Then consider the forcing $\mathcal{P}_S$ where a typical condition is of the form (s, Y), s a countable subset of ω_1 and Y a countable subset of $\{X_\gamma \mid \gamma \in B^*\}$. We write $(s, Y) \leq (t, Z)$ if

$$s \supseteq t, \quad Y \supseteq Z, \quad X_\gamma \in Z \to (s - t) \cap X_\gamma = \varnothing .$$

We identify a generic object G with $X = \bigcup\{s \mid \exists Y(s, Y) \in G\}$. Thus if X is $\mathcal{P}_S$-generic over $\langle L_\alpha[B], B\rangle$ then $\omega_1 + \gamma \in B^* \leftrightarrow X_\gamma$ is almost disjoint from X. So $B \cap L_\gamma$ is Δ_1 over $L_\gamma[X, \langle X_{\gamma'} \mid \gamma' < \gamma\rangle]$ provided $\gamma > \omega_1$ and is (say) primitive recursively closed. Conversely we arrange that $\langle X_{\gamma'} \mid \gamma' < \gamma\rangle$ is Δ_1 over $L_{\tilde{\gamma}}[B \cap L_\gamma]$ where $\tilde{\gamma}$ = least p.r. closed ordinal $> \gamma \cup \omega_1$. Thus knowing $B \cap L_\gamma$ allows one to determine $\langle X_{\gamma'} \mid \gamma' < \tilde{\gamma}\rangle$ which in turn allows one to recover $B \cap L_{\tilde{\gamma}}$. This "bootstrap" idea is key to the decoding process as the proof of the next lemma indicates.

LEMMA 7. $L_\gamma[X]$ *is inadmissible for each* $\gamma < \alpha$.

PROOF. It is enough to show by induction that $B \cap L_\gamma$ is Δ_1 over $L_\gamma[X]$ uniformly, for p.r. closed $\gamma > \omega_1$. If $\gamma = \tilde{\omega}_1$ then $\langle X_{\gamma'} \mid \gamma' < \gamma\rangle$ is Δ_1 over L_γ so $B \cap L_\gamma$ is Δ_1 over $L_\gamma[X]$. The uniformity renders trivial the case of γ being a limit of p.r. closed ordinals. Otherwise $\gamma = \tilde{\delta}$ where $\delta > \omega_1$ is p.r. closed and hence by induction $B \cap L_\delta$ is Δ_1 over $L_\delta[X]$. But then $\langle X_{\gamma'} \mid \gamma' < \gamma\rangle$ is Δ_1 over $L_\gamma[X]$ as it is Δ_1 over $L_\gamma[B \cap L_\delta]$. Thus $B \cap L_\gamma$ is Δ_1 over $L_\gamma[X]$ as it is Δ_1 over

$$L_\gamma[X, \langle X_{\gamma'} \mid \gamma' < \gamma\rangle]. \quad \square$$

The proof of Lemma 7 depends on the fact that the sequence of codes $\langle X_\gamma \mid \gamma < \alpha\rangle$ can be chosen so that $\langle X_{\gamma'} \mid \gamma' < \tilde{\gamma}\rangle$ is Δ_1 over $L_{\tilde{\gamma}}[B \cap L_\gamma]$ for each γ. This is possible thanks to Lemma 6.

Guaranteeing the admissibility of $L_\alpha[X]$ for $\mathcal{P}_S$-generic X requires a further restriction on the sequence $\langle X_\gamma \mid \gamma < \alpha\rangle$. As in Jensen [**1972**] one must choose the X_γ's to be generic. In Jensen [**1972**] it suffices to arrange that for each p.r. closed γ, X_γ is Cohen generic over $L_{\gamma+1}[B \cap L_\gamma]$ and belongs to $L_{\tilde{\gamma}}[B \cap L_\gamma]$. For our purposes we must use the more sophisticated technology of Jensen [**1975**, p. 13], where it is arranged that for each p.r. closed γ and each 1-1 function $f\colon \omega \to (\alpha - \gamma)$, the sequence $(X_{f(0)}, X_{f(1)}, \ldots)$ is Cohen generic (as an ω-sequence) over $L_{\gamma+1}[B \cap L_\gamma]$ and in addition $\langle X_\gamma \mid \gamma < \tilde{\gamma}\rangle$ is uniformly Σ_1 definable over $L_{\tilde{\gamma}}[B \cap L_\gamma]$. We now assume that the sequence $\langle X_\gamma \mid \gamma < \alpha\rangle$ has these properties and in addition for each p.r. closed γ and each Cohen condition p there is X_δ extending p, $\gamma < \delta < \tilde{\gamma}$.

LEMMA 8. *$L_\alpha[X]$ is admissible when X is $\mathcal{P}_S$-generic over $\langle L_\alpha[B], B\rangle$.*

PROOF. For each γ let $\mathcal{P}_S^\gamma = \{\langle s, Y\rangle \in \mathcal{P}_S \mid Y \subseteq \{X_\delta \mid \delta < \gamma\}\}$. We claim that if $\omega_1 < \gamma$ is p.r. closed and has uncountable cofinality, $M \subseteq \mathcal{P}_S^\gamma$ is a maximal antichain in $\mathcal{P}_S^\gamma$ and $M \in L_{\gamma+1}[B \cap L_\gamma]$ then M is a maximal antichain in $\mathcal{P}_S$. From this it will be fairly easy to establish the admissibility of $L_\alpha[X]$. But first we establish this claim.

Suppose not and let $\langle s, Y\rangle \in \mathcal{P}_S$ be a condition incompatible with each element of M. Write $Y = Y_1 \cup Y_2$ where $Y_1 \subseteq \{X_\delta \mid \delta < \gamma\}$ and $Y_2 \subseteq \{X_\delta \mid \delta \geqslant \gamma\}$. List the elements of Y_2 in an ω-sequence $X_{f(0)}, X_{f(1)}, \ldots$ and then since $\langle X_{f(0)}, X_{f(1)}, \ldots\rangle$ is $\mathcal{C}^\omega$-generic over $L_{\gamma+1}[B \cap L_\gamma]$ ($\mathcal{C} =$ Cohen forcing) there is a condition $(p_0, p_1, \ldots) \in \mathcal{C}^\omega$ such that $X_{f(i)}$ extends p_i for each i and

$$(p_0, p_1, \ldots) \Vdash \text{“}\langle s, Y_1 \cup \{\underset{\sim}{G}(n) \mid n \in \omega\}\rangle \text{ is incompatible with each element of } M.\text{”}$$

Now $\langle s, Y_1\rangle \in \mathcal{P}_S^\delta$ for some $\delta < \gamma$ since γ has uncountable cofinality. For each i choose $X_{g(i)}$, $\delta < g(i) < \tilde{\delta}$, such that $X_{g(i)}$ extends p_i (see the remark immediately before the statement of Lemma 8). As $\langle s, Y_1 \cup \{X_{g(i)} \mid i \in \omega\}\rangle$ is a condition in $\mathcal{P}_S^\gamma$ it is compatible with some $\langle t, Z\rangle \in M$. Now let $\beta = \sup(t)$ and consider the condition $(X_{g(0)} \restriction \beta, X_{g(1)} \restriction \beta, \ldots) \in \mathcal{C}^\omega$. This condition extends $(p_0, p_1, \ldots)$ and (weakly) forces "$\langle s, Y_1 \cup \{\underset{\sim}{G}(n) \mid n \in \omega\}\rangle$ is compatible with $\langle t, X\rangle$," contradicting the choice of $(p_0, p_1, \ldots)$.

It now follows that if $\omega_1 < \gamma$ is p.r. closed and of uncountable cofinality and G is $\mathcal{P}_S$-generic over $\langle L_\alpha[B], B\rangle$ then $G \cap \mathcal{P}_S^\gamma$ is $\mathcal{P}_S^\gamma$-generic over $L_{\gamma+1}[B \cap L_\gamma]$. For if $M \in L_{\gamma+1}[B \cap L_\gamma]$ is a maximal antichain in $\mathcal{P}_S^\gamma$ then M is maximal in $\mathcal{P}_S$ and thus $G \cap M = (G \cap \mathcal{P}_S^\gamma) \cap M \neq \varnothing$. As any dense open subset of $\mathcal{P}_S^\gamma$ contains a maximal antichain, this proves that $G \cap \mathcal{P}_S^\gamma$ is $\mathcal{P}_S^\gamma$-generic over $L_{\gamma+1}[B \cap L_\gamma]$. We now show by induction on $\phi \in L_\gamma[B \cap L_\gamma]$ that for all $\langle s, Y\rangle \in \mathcal{P}_S^\gamma$

$$\langle s, Y\rangle \underset{\mathcal{P}_S}{\Vdash} \phi \leftrightarrow \langle s, Y\rangle \underset{\mathcal{P}_S^\gamma}{\Vdash} \phi.$$

Each step of the induction is clear save the negation case. If $\langle s, Y\rangle \Vdash_{\mathcal{P}_S} \sim \phi$ then certainly $\langle s, Y\rangle \Vdash_{\mathcal{P}_S^\gamma} \sim \phi$ follows by induction. Conversely suppose $\langle s, Y\rangle \Vdash_{\mathcal{P}_S^\gamma} \sim \phi$. Then for evey $\mathcal{P}_S^\gamma$-generic H over $L_{\gamma+1}[B \cap L_\gamma]$ extending $\langle s, Y\rangle$ we have $\langle L_\gamma[B \cap L_\gamma, H], B \cap L_\gamma\rangle \models \sim \phi$ by the truth lemma. But then for every $\mathcal{P}_S$-generic G over $\langle L_\alpha[B], B\rangle$ extending $\langle s, Y\rangle$ we have $\langle L_\alpha[B, G], B\rangle \models \sim \phi$ since in this case $G \cap \mathcal{P}_S^\gamma$ is $\mathcal{P}_S^\gamma$-generic over $L_{\gamma+1}[B \cap L_\gamma]$. Thus $\langle s, Y\rangle \Vdash_{\mathcal{P}_S} \phi$ (since we are using weak forcing).

We have proved that the relation $\langle s, Y\rangle \Vdash \phi$ restricted to ranked ϕ is Δ_1 over $\langle L_\alpha[B], B\rangle$. We can now prove the lemma. Suppose $\langle s, Y\rangle \Vdash \forall x\, \exists y \phi(x, y)$ where ϕ is Δ_0. We shall produce an ordinal $\gamma < \alpha$ such that $\langle s, Y\rangle \Vdash \forall x_\gamma \exists y_\gamma \phi(x, y)$ (where x_γ, y_γ range over elements of $L_\gamma[B \cap L_\gamma]$). Our hypothesis and the Δ_1-ness of the forcing relation imply the existence of a $\Sigma_1\langle L_\alpha[B], B\rangle$ function f: $\alpha \to \alpha$ such that $\forall x\, \forall \langle t, Z\rangle \leq \langle s, Y\rangle[(x \in L_\delta[B \cap L_\delta]$ and $\langle t, Z\rangle \in \mathcal{P}_S^\delta) \to \exists \langle t', Z'\rangle \leq \langle t, Z\rangle \exists y (y \in L_{f(\delta)}[B \cap f(\delta)], \langle t', Z'\rangle \in \mathcal{P}_S^{f(\delta)}$ and $\langle t', Z'\rangle \Vdash \phi(x, y))]$. Let $\omega_1 < \gamma$ be a p.r. closed fixed point for f of uncountable cofinality. Then for each

$$x \in L_\gamma\big[B \cap L_\gamma\big], \quad D = \big\{\langle t, Z\rangle \in \mathcal{P}_S^\gamma \mid \exists y \in L_\gamma\big[B \cap L_\gamma\big]\langle t, Z\rangle \Vdash \phi(x, y)\big\}$$

is dense in $\mathcal{P}_S^\gamma$ below $\langle s, Y\rangle$ and belongs to $L_{\gamma+1}[B \cap L_\gamma]$. Thus if $\langle t, Z\rangle \leq \langle s, Y\rangle$ then $\langle t, Z\rangle$ is compatible with some element of D and hence

$$\exists \langle t', Z'\rangle \leq \langle t, Z\rangle \exists y \in L_\gamma\big[B \cap L_\gamma\big]\langle t', Z'\rangle \Vdash \phi(x, y).$$

We have just proved $\langle s, Y\rangle \Vdash \forall x_\gamma \exists y_\gamma \phi(x, y)$. Thus Π_2-reflection holds for $L_\alpha[X]$ and so $L_\alpha[X]$ is admissible. $\square$

2. Converse to Theorem 1. We make use of Jensen's theory of projecta (see Jensen [**1972A**]) to characterize ordinals of the form $\alpha(X)$ for $X \subseteq \omega_1$. In fact we show that unless L_α is ω-admissible there is no $X \subseteq \omega_1$ such that $L_\alpha[X]$ is admissible and $L_\alpha[X] \models \omega_1$ is the largest cardinal. Our proof is best motivated by considering the following example which provides an admissible α of cardinality and cofinality ω_1 such that $L_\alpha[X]$ is inadmissible whenever $L_\alpha[X] \models \omega_1$ is the largest cardinal.

EXAMPLE 1. Choose $\alpha < \omega_2$ to be admissible of cofinality ω_1 and such that $\sigma 2 p\alpha = \Sigma_2$ projectum of α has cofinality ω in L_α. Such an α is obtained by choosing $L_\alpha =$ transitive collapse (M), M is an elementary submodel of L_γ of cardinality ω_1, where $\gamma =$ the ω_1st stable past $\aleph_\omega$. Thus $\sigma 2 p\alpha = (\aleph_\omega)^{L_\alpha}$ and Σ_2 cofinality $(\alpha) = \omega_1$.

Now let $\beta = \sigma 2 p\alpha = (\aleph_\omega)^{L_\alpha}$. We can obtain a wellordering R of a subset of β of ordertype α which is $\Sigma_2(L_\alpha)$ as follows. Choose a $\Sigma_2(L_\alpha)$ injection f: $\alpha \to \beta$ and let $R(x, y) \leftrightarrow f^{-1}(x) < f^{-1}(y)$, $x, y \in \text{Range}(f)$. Jensen's fundamental theorem about Σ_n projecta states that any $\Sigma_n(L_\alpha)$ bounded subest of $\sigma n p\alpha = \Sigma_n$ projectum (α) is a member of $L_{\sigma n p\alpha}$. Thus if we let $\beta_n = (\aleph_n)^{L_\alpha}$ we see that $R \cap (\beta_n \times \beta_n) \in L_\beta$ for each n. We can thereby "code" R by the ω-sequence s: $\omega \to L_\beta$ defined by $s(n) = R \cap (\beta_n \times \beta_n)$.

Finally suppose $X \subseteq \omega_1$ and $L_\alpha[X] \models \omega_1$ is the largest cardinal. Then there exists an injection c: $L_\beta \to \omega_1$ such that $c \in L_\alpha[X]$. The composition $c \circ s$ belongs to L_{ω_1} as $c \circ s$ is a function from ω to ω_1 and we have assumed $V = L$. Thus $s = c^{-1} \circ (c \circ s)$ $\in L_\alpha[X]$. We now have $\bigcup \text{Range}(s) = R \in L_\alpha[X]$ and thus $L_\alpha[X]$ contains a wellordering of ordertype α. So $L_\alpha[X]$ is inadmissible.

The idea of this example can be used to show that not only Σ_n projectum (α) but also many other "projecta" associated to α must have uncountable cofinality as well. By establishing the uncountable cofinality of a sufficient number of these related projecta we are able to ultimately show that L_α is countably closed. Thus we have also obtained a "fine structure" characterization of countable closure.

For the sake of the definition below recall the S-hierarchy of Jensen, defined and discussed in Devlin [**1973**, p. 82]. This hierarchy is a more convenient way of generating L than the usual L-hierarchy. S_β has very nice closure properties for limit β. $S_\beta \cap OR = \beta$ for limit β. In what follows β always denotes a limit ordinal.

DEFINITION. Let $\alpha \leqslant \beta$ and n a positive integer. The (n, β) *projectum of* α = least γ s.t. there is a $\Sigma_n(S_\beta)$ injection of α into γ. We write $(n', \beta') < (n, \beta)$ if $\beta' < \beta$ or $(\beta' = \beta$ and $n' < n)$. Then (n, β) is an α-*critical* pair if (n', β') projectum $(\alpha) > (n, \beta)$ projectum (α) whenever $(n', \beta') < (n, \beta)$. Notice that there are only finitely many α-critical pairs, beginning with $(1, \alpha)$. Let $(1, \alpha) = (n_1, \beta_1) < (n_2, \beta_2) < \cdots < (n_k, \beta_k)$ be a list of all α-critical pairs and let $\rho_1 > \rho_2 > \cdots > \rho_k = \omega_1$ be a list of the corresponding projecta; i.e., $\rho_i = (n_i, \beta_i)$ projectum (α).

LEMMA 9. *If (n, β) is α-critical then (n, β) projectum $(\alpha) = \Sigma_n$ projectum (β).*

PROOF. Let $\eta_1 = (n, \beta)$ projectum (α) and $\eta_2 = \Sigma_n$ projectum(β). As $\alpha \leqslant \beta$ it follows that $\eta_1 \leqslant \eta_2$. Now choose a $\Sigma_n(S_\beta)$ injection f: $\alpha \to \eta_1$. Then $R = \{(x, y) \mid x, y \in \text{Range}(f) \text{ and } f^{-1}(x) < f^{-1}(y)\}$ is a $\Sigma_n(S_\beta)$ subset of $\eta_1 \times \eta_2$. But $R \notin S_\beta$ as otherwise (n', β') projectum$(\alpha) \leqslant \eta_1$ for some $(n', \beta') < (n, \beta)$, contradicting the hypothesis that (n, β) is α-critical. Now by Jensen's characterization of the Σ_n projectum we must have $\eta_2 \leqslant \eta_1$. □

The idea used in Example 1 also establishes the next result.

LEMMA 10. *If $\alpha = \alpha(X)$ for some X then ρ_i has uncountable cofinality for each i.*

PROOF. Suppose not. Choose t: $\omega \to \rho_i$ to be unbounded. As before choose a $\Sigma_{n_i}(S_{\beta_i})$ injection f: $\alpha \to \rho_i$ and let $R = \{(x, y) \mid x, y \in \text{Range}(f) \text{ and } f^{-1}(x) < f^{-1}(y)\}$. Then by Lemma 9 $R \cap (t(n) \times t(n)) \in L_{\rho_i}$ for each n. We let $s(n) = R \cap (t(n) \times t(n))$.

If $L_\alpha[X] \vDash \omega_1$ is the largest cardinal then as before $s \in L_\alpha[X]$. But then $L_\alpha[X]$ is inadmissible as $R = \bigcup \text{Range}(s) \in L_\alpha[X]$ and $L_\alpha[X]$ contains a wellordering of ordertype α. □

Establishing the countable closure of L_α (when $\alpha = \alpha(X)$ for some X) necessitates consideration of other projecta closely related to those above. For each i, $1 \leqslant i \leqslant k$, we define

$$\rho_i' = \Sigma_{n_i - 1} \text{ projectum } (\beta_i).$$

Recall that $\rho_i = \Sigma_{n_i}$ projectum (β_i). If $n_i = 1$ we define $\Sigma_{n_i - 1}$ projectum $(\beta_i) = \Sigma_0$ projectum $(\beta_i) = \beta_i$. The countable closure results will follow once we demonstrate that ρ_i' also has uncountable cofinality for each i. The proof uses a combination of ideas from the theory of master codes and β-recursion theory.

Suppose $n > 0$, $A \subseteq \Sigma_n$ projectum(β) is a Σ_n *Master Code* for β if A is $\Sigma_n(S_\beta)$ and for any $B \subseteq \Sigma_n$ projectum $(\beta) = \sigma np\beta$,

$$B \text{ is } \Sigma_1\left\langle L_{\sigma np\beta}, A\right\rangle \quad \text{iff } B \text{ is } \Sigma_{n+1}(S_\beta).$$

Jensen defined and proved the existence of Σ_n Master Codes (Jensen [**1972A**]). If $n = 0$ we define $\varnothing = \Sigma_0$ Master Code for β. Then for each i define

$$A_i = \text{a } \Sigma_{n_i - 1} \text{ Master Code for } \beta_i.$$

Thus $A_i \subseteq \rho_i'$ and the structure $\mathfrak{A}_i = \langle S_{\rho_i'}, A_i\rangle$ is amenable (that is, $A_i \cap \gamma \in S_{\rho_i'}$ for each $\gamma < \rho_i'$). We can also think of ρ_i as the Σ_1 projectum of the structure $\mathfrak{A}_i$. Thus the introduction of master codes allows us to deal with Σ_1 predicates (over an appropriate amenable structure) where the methods of β-recursion theory apply.

LEMMA 11. *Suppose $\rho_i' > \alpha$. Then ρ_{i-1} is a ρ_i'-cardinal.*

PROOF. Since $\rho_i' > \alpha = \rho_1'$ we see that $i > 1$. Let (n, β) be the least pair such that there is a $\Sigma_n(S_\beta)$ injection of ρ_{i-1} into a smaller ordinal. Then (n, β) is α-critical. Thus $(n, \beta) = (n_i, \beta_i)$ and ρ_{i-1} is a β_i-cardinal. We are done since $\rho_i' = \Sigma_{n_i - 1}$ projectum $(\beta_i) \leqslant \beta_i$. □

We are now prepared to prove the key lemma toward establishing countable closure.

LEMMA 12. *If $\alpha = \alpha(X)$ for some X then ρ_i' has uncountable cofinality for each i.*

PROOF. If $\rho_i' < \alpha$ then the result follows from Lemma 10. For in this case $\rho_i' = \rho_{i-1}$ as $\rho_i' = (n_i - 1, \beta_i)$ projectum (α) and (n_i, β_i) is α-critical.

We must deal with the case $\rho_i' > \alpha$. Suppose then that cofinality $(\rho_i') = \omega$. Choose $f\colon \omega \to \rho_i'$ to be cofinal and increasing. Again there is a $\Sigma_{n_i}(S_{\beta_i})$ wellordering R of ρ_i of ordertype α. R is $\Sigma_1(\mathfrak{A}_i)$ so we can choose a Σ_1 formula $\phi(x, y, A_i)$ which defines R over $\mathfrak{A}_i$. We let R_n be the $f(n)$th approximation to R; i.e., $(x, y) \in R_n \leftrightarrow \langle S_{f(n)}, A_i \cap f(n)\rangle \models \phi(x, y, A_i \cap f(n))$. Thus $R_n \in L_{\rho_{i-1}}$ for each n as $R_n \in S_{\rho_i'}$ and ρ_{i-1} is a ρ_i'-cardinal. ρ_{i-1} has uncountable cofinality. So there is a $\gamma < \alpha$ s.t. $R_n \in S_\gamma$ for all n.

Now if $L_\alpha[X] \models \omega_1$ is the largest cardinal then as before $\langle R_n \mid n \in \omega\rangle \in L_\alpha[X]$. So $R \in L_\alpha[X]$ and $L_\alpha[X]$ is inadmissible. □

THEOREM 13. *If $\alpha = \alpha(X)$ for some X then L_α is countably closed.*

PROOF. Define $\rho_0 = \alpha$. We show by induction on $k - i$ that if $f\colon \omega \to \rho_i$ then $f \in L_\alpha$. If $k - i = 0$ then $\rho_i = \rho_k = \omega_1$ so the result is clear.

Now assume the result for $k - (i + 1)$ and we demonstrate it for $k - i$. Thus we are given $f\colon \omega \to \rho_i$ and we know that $g\colon \omega \to \rho_{i+1}$ implies $g \in L_\alpha$. Note that $\rho_{i+1}' \geqslant \rho_i$ as either $\rho_{i+1}' \geqslant \alpha$ or $\rho_{i+1}' = \rho_i$ (see the proof of Lemma 12). Thus $f\colon \omega \to \rho_{i+1}'$. Also let h be a $\Sigma_1(\mathfrak{A}_{i+1})$ injection of ρ_{i+1}' into ρ_{i+1}.

By induction $h \circ f \in L_\alpha$. Moreover $h \circ f \in L_{\rho_{i+1}}$ as ρ_{i+1} is an α-cardinal of uncountable cofinality. Now $f = h^{-1} \circ (h \circ f)$ is therefore $\Sigma_1(\mathfrak{A}_{i+1})$. As cofinality $(\rho_{i+1}') > \omega$ we see that $f \in L_{\rho_{i+1}'}$. If $\rho_{i+1}' \leqslant \alpha$ we are done. Otherwise $f \in L_{\rho_i}$ as ρ_i is a ρ_{i+1}'-cardinal of uncountable cofinality. As $\rho_i \leqslant \alpha$ we have $f \in L_\alpha$. □

Now that we have dispensed with countable closure, ω-admissibility can be easily dealt with.

THEOREM 14. *If $\alpha = \alpha(X)$ for some $X \subseteq \omega_1$ then L_α is ω-admissible.*

PROOF. The theorem follows if we show that L_α is closed under $y \mapsto [y]^\omega$ (see Lemma 4). Fix $y \in L_\alpha$. As we know that L_α is countably closed we can define a $\Sigma_1(L_\alpha[X])$ function as follows: Given $z \in [y]^\omega$ let $h(z) =$ the ordinal at which z is constructed in L_α. Then Range(h) is bounded by some $\beta < \alpha$ and so $[y]^\omega = [y]^\omega \cap L_\beta \in L_\alpha$. □

Lastly we show that ω-admissibility may fail for L_α even when L_α is countably closed.

EXAMPLE 2. Let M be the Σ_2 Skolem Hull of ω_1 inside L_γ where $\gamma = \omega_1$st stable above $\aleph_\alpha$. Let L_α be the transitive collapse of M. Then α has cofinality ω_1, $L_\alpha \vDash (\aleph_\omega$ is the largest cardinal) and Σ_2 projectum $(\alpha) = \omega_1$. Also $\alpha^* = \alpha$. We show that L_α is countably closed in two steps: (a) If $f: \omega \to L_\alpha$ then f is $\Sigma_2(L_\alpha)$. For, let $p: L_\alpha \to \omega_1$ be a $\Sigma_2(L_\alpha)$ injection. Then $p \circ f \in L_{\omega_1}$ and thus $f = p^{-1} \circ (p \circ f)$ is $\Sigma_2(L_\alpha)$. (b) If $f: \omega \to L_\alpha$ is $\Sigma_2(L_\alpha)$ then $f \in L_\alpha$. For, if C is a Σ_1 Master Code for L_α then f is $\Sigma_1\langle L_\alpha, C\rangle$ (recall that $\alpha^* = \alpha$). But α has uncountable cofinality so $f \in L_\alpha$. Note that Lemma 5 implies that L_α is not ω-admissible.

We can sum up our results as follows.

THEOREM 15 ($V = L$). (a) *L_α is countably closed iff ρ_i, ρ_i' have uncountable cofinality for each i.*

(b) *L_α is ω-admissible iff L_α is countably closed and in addition if L_α has a largest cardinal, it has uncountable cofinality.*

(c) *$\alpha = \alpha(X)$ for some $X \subseteq \omega_1$ iff $\omega_1 < \alpha < \omega_2$ and L_α is ω-admissible.*

EXAMPLE 3 (ZF). There is an admissible α, $\omega_1 < \alpha < \omega_2$, such that cofinality $(\alpha) = \omega_1$ and $\alpha \neq \alpha(X)$ for any $X \subseteq \omega_1$.

PROOF. Let κ denote true ω_1. As in Example 1 let $\alpha < (\kappa^+)^L$ have L-cofinality κ and be such that $\sigma 2 p \alpha$ has cofinality ω in L_α. Then there is a wellordering R of $\lambda = \sigma 2 p \alpha$ of ordertype α which is $\Sigma_2(L_\alpha)$. Moreover $R = \bigcup_n R_n$ where $R_n \in L_\lambda$ for each n.

Suppose $\alpha = \alpha(X)$. Then $L_\alpha[X]$ contains an injection $g: L_\lambda \to \kappa$. But then $\langle g(R_n) \mid n \in \omega\rangle \in [\kappa]^\omega \cap L[X] \subseteq L_\kappa[X]$. So $R \in L_\alpha[X]$, contradicting the admissibility of $L_\alpha[X]$. □

3. Further results and open questions. (1) There is a version of our result for Σ_n admissibility, $n > 1$. Thus for regular κ, α is the least Σ_n admissible relative to a subset of κ iff $\kappa < \alpha < \kappa^+$, α has cofinality κ and L_α is closed under the Σ_n admissible relative to the function $y \mapsto [y]^{<\kappa}$. For example, let $n = 2$, $\kappa = \omega_1$. The necessity of this condition follows as in §2. For the sufficiency, one first adds a closed unbounded $C \subseteq \alpha$ so that $\gamma \in C$, cofinality $(\gamma) > \omega \to \langle L_\gamma, S \cap L_\gamma\rangle$ is Σ_2 inadmissible (where S is defined as before). Moreover $\langle L_\alpha, S, C\rangle$ is amenable and Σ_2 admissible. Then generically add $A \subseteq L_\alpha$ so that $\langle L_\alpha[A], S, C, A\rangle$ is Σ_2 admissible and $L_\alpha[A] \vDash \omega_1$ is the largest cardinal. There is a predicate $B \subseteq L_\alpha$ so that B is

$\Delta_1\langle L_\alpha[A], S, C, A\rangle$ and for each $\gamma < \alpha$ $\langle L_\gamma[B \cap L_\gamma], B \cap L_\gamma\rangle$ is Σ_2 inadmissible. Now code B by $X \subseteq \omega_1$ as before. Then $B \cap L_\gamma$ is Δ_1 over $L_\gamma[X]$ for p.r. closed $\gamma > \omega_1$ so $L_\gamma[X]$ is Σ_2 inadmissible for $\gamma < \alpha$. The proof that $L_\alpha[X]$ is Σ_2 admissible proceeds as before, using the fact that maximal antichains when the forcing is restricted to p.r. closed γ of uncountable cofinality are maximal in the whole partial-ordering, and the resulting fact that forcing for Σ_2 sentences is a Σ_2 relation.

(2) Sacks' pointed perfect forcing (Sacks [**1976**]) can be adapted to the present context to prove: If α has cardinality and cofinality κ, κ regular, and L_α is Σ_n admissible relative to $y \mapsto [y]^{<\kappa}$ then there is $X \subseteq \kappa$ s.t. α is the least Σ_n admissible relative to X and if $Y \in L_\alpha[X]$, $Y \subseteq \kappa$, then either $X \in L_\alpha[Y]$ or the least Σ_n admissible relative to Y is less than α. One must use here the full version of this forcing, including Sacks' "triple forcing," adapted to trees on κ.

(3) In the countable case there is a model-theoretic proof of Sacks' characterization of countable admissibles. Of course compactness fails at regular cardinals $> \omega$. But is there a model-theoretic proof of Theorem 1, say like that given of Sacks' theorem in Friedman [**1981**]? It would help to develop a countably closed version of Steel forcing.[3]

(4) R. David [**1981**] has shown that if α is countable and $L_\alpha \models \mathrm{ZF}$ then for some $R \subseteq \omega$, α is the least ordinal such that $L_\alpha[R] \models \mathrm{ZF}$. What is the uncountable version of this result?

(t) Is there a nice characterization of which sequences of admissibles $\langle \alpha_\gamma \mid \gamma < \lambda\rangle$, $\lambda < \kappa^+$, between κ and κ^+ are the first λ admissibles relative to some $X \subseteq \kappa$ (as in Jensen [**1972**] for the case $\kappa = \omega$)?

References

[1981] R. David, *Some applications of Jensen's coding theorem* (to appear).

[1973] Keith J. Devlin, *Aspects of constructibility*, Lecture Notes in Math., vol. 354, Springer-Verlag, Berlin and New York.

[1981] Sy D. Friedman, *Steel forcing and Barwise compactness*, Ann. Math. Logic (to appear).

[1981] ______, *Uncountable admissibles* II: *Compactness*, Israel J. Math. (to appear).

[1975] Harvey M. Friedman, *One hundred and two problems in mathematical logic*, J. Symbolic Logic **4**.

[1967] R. Gandy and G. Sacks, *A minimal hyperdegree*, Fund. Math. **61**.

[1973] Leo Harrington, *Contributions to recursion theory in higher types*, Ph.D. Thesis, MIT.

[1972] R. Jensen, *Forcing over admissible sets*, Notes by K. Devlin, unpublished.

[1972A] ______, *The fine structure of the constructible hierarchy*, Ann. Math. Logic **4**.

[1975] ______, *Coding the universe by a real*, unpublished manuscript.

[1970] R. Jensen and R. Solovay, *Some applications of almost disjoint sets*, Math. Logic and Foundations of Set Theory, (Bar-Hillel, Editor), North-Holland, Amsterdam.

[1974] G. Sacks, *The* 1-*section of a type n object*, Generalized Recursion Theory, (Fenstad-Hinman, Editors), North-Holland, Amsterdam.

[1976] ______, *Countable admissible ordinals and hyperdegrees*, Adv. in Math. **20**.

[1978] J. Steel, *Forcing with tagged trees*, Ann. Math. Logic **15**.

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[3]This has now been accomplished: See our forthcoming paper *Model theory for* $L_{\infty\omega_1}$.