PATRAS LOGIC SYMPOSION
G. Metakides (ed.)

THE TURING DEGREES AND THE METADEGREES HAVE ISOMORPHIC CONES*

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In this paper we study the global structure of the β-degrees. A cone of β-degrees is a set of β-degrees of the form $\{\underline{d} \mid \underline{d} \geq \underline{d}_o\}$ for some fixed β-degree $\underline{d}_o$, partially ordered by $\leq_\beta$. The base of this cone is the β-degree $\underline{d}_o$. Our main result is that if β is countable then the Turing degrees and the β-degrees have isomorphic cones. If $\beta = \omega_1^{ck}$ then the β-degrees are the metadegrees and in this case the cone of metadegrees with base $0'$ = complete meta-RE degree is isomorphic to the cone of Turing degrees with base the Turing degree of Kleene's $\mathcal{O}$.

If $V = L$ we also construct some isomorphisms in the uncountable case (sections two and three). If β has regular cardinality κ and S_β is $< \kappa$-closed then the β-degrees and the κ-degrees have isomorphic cones. If β has singular cardinality κ then some cone of κ-degrees is often isomorphic to a cone in the regular β-degrees. The hypothesis in this case is that there exist a tame injection $S_\beta \longrightarrow \kappa$; i.e., an injection g with the property that $g^{-1} \restriction \gamma \in S_\beta$ for each $\gamma < \kappa$.

Much of this work is based on techniques of Maass from Maass [1978] and Maass [1979]. The uncountable case draws heavily on Friedman [1981 a,b,c] which provide "fine structure" characterizations of the hypotheses used.

These results are in fact an application of ideas from inadmissible recursion theory to degree theory on admissible (and inadmissible) ordinals.

*The preparation of this paper was supported by NSF Grant #MCS 7906084

The isomorphism mentioned between a cone of Turing degrees and a cone of meta-degrees converts the Turing jump into the weak metajump, a notion which first arose out of the study of inadmissible ordinals.

SECTION ONE COUNTABLE CARDINALITY

Suppose $\mathcal{B} = \langle S_\beta, B\rangle$ is amenable; i.e., $B \cap S_\gamma \in S_\beta$ for each $\gamma < \beta$. Then we define:

$$\Sigma_1\text{cf}(\mathcal{B}) = \text{least } \gamma \text{ s.t. there is a } \Sigma_1(\mathcal{B}) \text{ unbounded } f:\gamma \to \beta$$

$$\Sigma_1\text{proj}(\mathcal{B}) = \text{least } \gamma \text{ s.t. there is a } \Sigma_1(\mathcal{B})\text{injection } f:\beta \to \gamma.$$

If $\Sigma_1\text{cf}(\mathcal{B}) = \Sigma_1\text{proj}(\mathcal{B}) = \omega$ then there is a $\Sigma_1(\mathcal{B})$ bijection $g:\beta \to \omega$ and in this case Maass [1978] constructs $A \subseteq \omega$ s.t. for $C \subseteq \omega$, C is $\Sigma_1(\mathcal{B})$ if and only if C is $\Sigma_1\langle S_\omega, A\rangle$. The structure $\mathcal{A} = \langle s_\omega, A\rangle$ is called the Maass collapse of $\mathcal{B}$.

THEOREM 1. Suppose $= \langle S_\beta, B\rangle$ is amenable and $\Sigma_1\text{cf}(\mathcal{B}) = \Sigma_1 \text{ proj}(\mathcal{B}) = \omega$. Let $\mathcal{A} = \langle S_\omega, A\rangle$ be the Maass collapse of $\mathcal{B}$. Then there is an isomorphism j between the Turing degrees $\geq$ Turing degree (A) and the β-degrees $\geq$ β-degree (B).

In the proof it will be convenient to refer to the notion of $\mathcal{B}$-degree: For $C, D \subseteq S_\beta$ write $C \leq_{\mathcal{B}} D$ iff $C \leq_\beta D \vee B$. Then $\mathcal{B}$-degree (C) = $\{D \mid C \leq_{\mathcal{B}} D, D \leq_{\mathcal{B}} C\}$. The $\mathcal{A}$-degrees are defined similarly. In this terminology Theorem 1 asserts that the $\mathcal{A}$-degrees and the $\mathcal{B}$-degrees are isomorphic.

Define $C \subseteq \omega$ to be β-bi-immune if neither C nor ω-C contains an infinite β-finite set. The following lemma is proved on pages 154-5 of Maass [1978].

LEMMA 2. Each $\mathcal{A}$-degree contains a β-bi-immune representative.

The significance of β-bi-immunity is this: The reducibilities $\leq_{\mathcal{A}}, \leq_{\mathcal{B}}$ agree on β-bi-immune sets. For, by the key property of the Maass collapse the reduction procedures coincide for these reducibilities and β-bi-immunity insures that the neighborhood conditions are also the same.

Proof of Theorem 1. Define j: $\mathcal{A}$-degrees $\longrightarrow$ $\mathcal{B}$-degrees by $j(\underline{d}) = \mathcal{B}$-degree (D) for any β-bi-immune $D \in \underline{d}$. By the above remarks this is a well-defined monomorphism. We show that j is onto. To do so we will make use of the reducibilities:

$$C \le_{f\mathcal{B}} D \longleftrightarrow C \le_{f\beta} D \vee B$$

$$C \le_{w\mathcal{B}} D \longleftrightarrow C \le_{w\beta} D \vee B.$$

Choose a $\mathcal{B}$-recursive (that is, $\sum_1(\mathcal{B})$) bijection $g:S_\beta \longleftrightarrow \omega$. Given $E \subseteq S_\beta$ we construct a β-bi-immune $\hat{E} \subseteq \omega$ such that $E,\hat{E}$ have the same $\mathcal{B}$-degree. Let $E^* = \{z \in S_\beta | z \subseteq E\} \vee \{z \in S_\beta | z \cap E \neq \emptyset\}$. Then let $E^{**} = g[E^*]$ and finally choose $\hat{E}$ to be any β-bi-immune set of the same A-degree as E^{**}.

Then $\{z \in S_\beta | z \subseteq E\} \vee \{z \in S_\beta | z \cap E = \emptyset\} \le_{f\mathcal{B}} E^*$, $E^* \le_{f\mathcal{B}} E^{**}$ and $E^{**} \le_{f\mathcal{B}} \hat{E}$ (since for any $C,D \subseteq \omega$, $C \le_A D$ iff $C \le_{f\mathcal{B}} D$). Putting this chain of reducibilities together and using the definition of $\le_{\mathcal{B}}$ we see that $E \le_{\mathcal{B}} \hat{E}$.

Moreover $\hat{E} \le_{f\mathcal{B}} E^{**}$, $E^{**} \le_{f\mathcal{B}} E^*$ and $E^* \le_{w\mathcal{B}} E$. This yields $\hat{E} \le_{f\mathcal{B}} E$. Since $\hat{E}$ is β-bi-immune we have $\hat{E} \le_{\mathcal{B}} E$. ——|

The applicability of Theorem 1 depends on the possibility, given a countable limit ordinal β, of choosing $B \subseteq S_\beta$ such that $\mathcal{B} = \langle S_\beta, B\rangle$ is amenable and $\sum_1 cf(\mathcal{B}) = \sum_1 proj(\mathcal{B}) = \omega$. Our next result says that this is always possible.

<u>THEOREM 3.</u> Suppose $f: \beta \longleftrightarrow \omega$ is a bijection. There is $B \subseteq S_\beta$, $B \le_{f\beta} f$ such that $\mathcal{B} = \langle S_\beta, B\rangle$ is amenable and $\sum_1 cf(\mathcal{B}) = \sum_1 proj(\mathcal{B}) = \omega$.

<u>Proof</u>. Define $B = \{\langle\beta_0,\ldots,\beta_n\rangle | f(\beta_i) = i$ for all $i\}$. Then $B \cap S_\gamma$ is the finite for all $\gamma < \beta$. Moreover f is $\sum_1\langle S_\beta, B\rangle$ and $B \le_{f\beta} f$. ——|

In case $\beta = \omega_1^{CK}$ we have:

<u>COROLLARY 4</u>. The Turing degrees $\ge$ Turing degree $(\mathcal{O})$ are isomorphic to the metadegrees $\ge 0'$ = the complete meta-RE degree.

<u>Proof</u>. There is a $\Delta_2(L_{\omega_1^{CK}})$ bijection between ω_1^{CK} and ω. By Theorem 3 we can choose a $\Delta_2(L_{\omega_1^{CK}})$ set $B \subseteq L_{\omega_1^{CK}}$ such that $\langle L_{\omega_1^{CK}}, B\rangle$ obeys the hypothesis of Theorem 1. But $\mathcal{O}$ is a <u>Δ_2 master code</u> for $L_{\omega_1^{CK}}$; i.e., $A \subseteq \omega$ is $\Delta_2(L_{\omega_1^{CK}})$ iff $A \le_T \mathcal{O}$ (for a proof see Jockusch-Simpson [1976]). If $\langle L_\omega, A\rangle$ is the Maass collapse of $\langle L_{\omega_1^{CK}}, B \vee C\rangle$ where C = a regular complete meta-RE set, then $A \le_T \mathcal{O}$ since A is $\Delta_2(L_{\omega_1^{CK}})$ and $\mathcal{O} \le_T A$ since $\mathcal{O}$ is $\Delta_1\langle L_{\omega_1^{CK}}, C\rangle$. So by Theorem 1 the Turing degrees $\ge$ Turing degree $(\mathcal{O})$ are isomorphic to the metadegrees $\ge$ metadegree $(B \vee C) = 0'$. ——|

For other ordinals $\beta < \omega_1^L$ we can choose the least pair (γ,n) so that there is a $\Delta_n(S_\gamma)$ bijection $\beta \longleftrightarrow \omega$. Then the Turing degrees $\geq$ Turing degree(A) are isomorphic to the β-degrees $\geq$ β-degree (A) where A is a β-bi-immune Δ_n master code for S_γ. The β-degree of A is the largest β-degree of a set $D \subseteq S_\beta$ such that both $\{z \in S_\beta | z \subseteq D\}$ and $\{z \in S_\beta | z \cap D = \emptyset\}$ are $\sum_n(S_\gamma)$. For example, if λ is a recursive limit ordinal then the λ-degrees are isomorphic to the Turing degrees $\geq 0^{(\beta)} = \beta^{th}$ jump of 0, where $\lambda = \omega+\beta$. (The degree $0^{(\beta)}$ is just the Turing degree of H_a where $a \in \mathcal{O}$ is a notation for β and H_a is the associated Kleene H-set.) If $\lambda = \omega_n^{CK} = n^{th}$ admissible greater than ω then the λ-degrees $\geq$ λ-degree (complete λ-RE set) are isomorphic to the Turing degrees $\geq \mathcal{O}^{(n)} = n^{th}$ hyperjump of 0. If $\lambda = \bigcup_n \omega_n^{CK}$ then the λ-degrees are isomorphic to the Turing degrees $\geq$ Turing degree$(\mathcal{O}^{(\omega)})$ where $\mathcal{O}^{(\omega)}$ = recursive join of $\langle \mathcal{O}, \mathcal{O}^{(2)}, \mathcal{O}^{(3)}, \ldots \rangle$.

We now discuss what effect the isomorphism j of Theorem 1 has on relative RE-ness. If $C, D \subseteq S_\beta$ then C is _tamely_ β-RE relative to D if $\{z \in S_\beta | z \subseteq C\}$ is β-RE relative to D. If $\mathcal{B} = \langle S_\beta, B\rangle$ is amenable then C is _tamely_ $\mathcal{B}$-RE if C is tamely β-RE relative to B. The following result describes the range of j on the A-RE degrees.

THEOREM 5. If j is the isomorphism of Theorem 1 then an A-degree $\underline{d}$ contains an A-RE set iff $j(\underline{d})$ contains a tamely $\mathcal{B}$-RE set.

Proof. If $D \in \underline{d}$ is A-RE then so is $\hat{D}$, the canonical β-bi-immune set of the same A-degree as D obtained from Maass [1978]. But then $\hat{D}$ is tamely $\mathcal{B}$-RE since $\hat{D}$ is $\mathcal{B}$-RE and $\{z \in S_\beta | z \subseteq \hat{D}\} = \{z \in S_\beta | z$ is finite, $z \subseteq \hat{D}\}$. So $j(\underline{d})$ contains the tamely $\mathcal{B}$-RE set $\hat{D}$.

Conversely if $E \subseteq S_\beta$ is tamely $\mathcal{B}$-RE then we show that E has the same $\mathcal{B}$-degree as a β-bi-immune A-RE set. Consider the sets $E^*, E^{**}, \hat{E}$ from the proof of Theorem 1. As E is tamely $\mathcal{B}$-RE we see that E^* is $\mathcal{B}$-RE. Hence E^{**} is $\mathcal{B}$-RE and therefore A-RE. Once again, $\hat{E}$ is also A-RE as it is obtained as the canonical β-bi-immune set of the same A-degree as E^{**}. The proof of Theorem 1 showed that $E, \hat{E}$ have the same $\mathcal{B}$-degree. ——|

Note that if $\mathcal{B} = \langle S_\beta, B\rangle$ satisfies the hypothesis of Theorem 1 then every $\mathcal{B}$-degree is _regular_; that is, every $\mathcal{B}$-degree contains a set C such that $\langle S_\beta, C\rangle$

is amenable. For, if $f:\omega \longrightarrow \beta$ is an order-preserving unbounded $\Sigma_1(\mathcal{B})$ function then any $E \subseteq S_\beta$ has the same $\mathcal{B}$-degree as $f[\hat{E}]$, where $\hat{E}$ comes from the proof of Theorem 1. As a result Theorem 5 can be applied to yield:

COROLLARY 6. If j is the isomorphism of Theorem 1 then for A-degrees d_1, d_2: d_1 is A-RE relative to $d_2 \longleftrightarrow j(d_1)$ is tamely $\mathcal{B}$-RE relative to $j(d_2)$.

Proof. Just note that if C has $\mathcal{B}$-degree $j(d_2)$, $\langle S_\beta, C\rangle$ amenable, then $\langle S_\beta, C\rangle$ obeys the hypothesis of Theorem 1 and the resulting isomorphism is $j \restriction$ (A-degrees $\geq d_2$). Thus by Theorem 5 d_1 is A-RE relative to $d_2 \longleftrightarrow j(d_1)$ is tamely $\langle S_\beta, C\rangle$ - RE $\longleftrightarrow j(d_1)$ is tamely $\mathcal{B}$-RE relative to $j(d_2)$. —|

We end this section by determining the inverse image under j of the $\mathcal{B}$-RE degrees. The answer comes from work of Maass [1979]. Theorem 1 of that paper implies that in case $\mathcal{B} = \langle S_\beta, \emptyset\rangle$ obeys the hypothesis of our Theorem 1 then a $\mathcal{B}$-degree contains a regular $\mathcal{B}$-RE set (i.e., a $\mathcal{B}$-RE set C such that $\langle S_\beta, C\rangle$ is amenable) if and only if it contains a β-bi-immune set S such that: D is A-RE relative to E for some A-RE set $E \leq_A D$. Our remark after Theorem 5 implies that each $\mathcal{B}$-RE degree contains a regular $\mathcal{B}$-RE set. Thus the $\mathcal{B}$-RE degrees in this case correspond under j to the A-degrees which are A-RE relative to an A-RE predecessor. As Maass remarks in statement 3) after Theorem 1 of Maass [1979], this also works for any structure $\mathcal{B}$ satisfying the conditions of our Theorem 1. We give Maass' proof specialized to the present context.

THEOREM 7. If j is the isomorphism of Theorem 1 then for any A-degree $\underline{d}$, $j(\underline{d})$ is $\mathcal{B}$-RE iff $\underline{d}$ is A-RE relative to some A-RE degree $\underline{e} \leq \underline{d}$.

Proof. Let $g: S_\beta \longleftrightarrow \omega$ be a $\Sigma_1(\mathcal{B})$ bijection and for $D \subseteq \omega$ let $\hat{D}$ be the canonical β-bi-immune set of the same A-degree as D.

If $C \subseteq S_\beta$ is $\mathcal{B}$-RE let $C^* = \{z \in S_\beta | z \cap C \neq \emptyset\} \vee \{z \in S_\beta | z \cap (S_\beta - C) \neq \emptyset\}$ and let $D = \widehat{g[C^*]}$. Then $C =_{\mathcal{B}} D$ and D is β-bi-immune. But D is A-RE relative to the A-RE set $g[C]$: To see this, it suffices to show that $g[C^*]$ is A-RE relative to $g[C]$. But $w \in g[C^*] \longleftrightarrow [(g^{-1}(w) = \langle 0,z\rangle,\ z \cap C \neq \emptyset)$ or $(g^{-1}(w) = \langle 1,z\rangle,\ z \cap (S_\beta - C) \neq \emptyset] \longleftrightarrow [\exists x \in g[C](g^{-1}(w) = \langle 0,z\rangle$ and $g^{-1}(x) \in z)$ or $\exists x \notin g[C](g^{-1}(w) = \langle 1,z\rangle$ and $g^{-1}(x) \in z)]$. So $g[C^*]$ is $\mathcal{B}$-RE relative to $g[C]$ using only finite neighborhood conditions on $g[C]$. Thus

$g[C^*]$ is A-RE relative to $g[C]$. Also note that $g[C] \leq_{fB} g[C^*]$ and hence $g[C] \leq_A g[C^*] \leq_A D$.

Conversely, suppose $D, E \subseteq \omega$ are β-bi-immune and E is A-RE, $E \leq_A D$, D is A-RE relative to E. Let $\{D^s | s \in \omega\}$ be a $\sum_1\langle A,E\rangle$ enumeration of D and $\{E^s | s \in \omega\}$ a $\sum_1(A)$ enumeration of E. Choose a $\sum_1(B)$ increasing, unbounded $f: \omega \to B$. Then define:

$$C = \{\langle f(n), s, \kappa, t\rangle | n \in D^{s+1} - D^s, \kappa = E \cap s, t = \text{least } t \geq s \text{ s.t. } E^t \cap s = F \cap s\}.$$

C is co-B-RE as D^{s+1} only depends on $E \cap s$ and the condition on t is co-A-RE (hence co-B-RE). Then $D \leq_{fB} C$ as $n \in D \longleftrightarrow \exists s, \kappa, t \in L_\omega(\langle f(n) s, \kappa, t\rangle \in C)$ and $n \notin D \longleftrightarrow \{f(n)\} \times L_\omega \times L_\omega \times L_\omega \subseteq S_\beta - C$. As D is β-bi-immune this gives $D \leq_B C$. To show that $C \leq_B D$: To determine $C \cap S_{f(n)}$ one need only know a stage s where $D \cap n = D^s \cap n$. This can be determined from $D \cap n$ given a large enough t s.t. $E^t \cap s = E \cap s$. Thus $C \leq_B D \vee E$ and we are done since $E \leq_A D$ and E is β-bi-immune. ——|

A consequence of Corollary 6 is that for any B-degree $\underline{b}$ there is a largest B-degree which is tamely B-RE relative to $\underline{b}$. This B-degree is called the weak B-jump of $\underline{b}$. Theorem 7 implies that weak B-jump (weak B-jump ($\underline{b}$)) = $\underline{b}'$ = B-jump($\underline{b}$) = largest B-degree which is B-RE relative to $\underline{b}$. Thus we have:

<u>COROLLARY 8</u>. For any A-degree $\underline{d}$, $j(\underline{d}') =$ weak B-jump$(j(\underline{d}))$ and $j(\underline{d}'') = j(\underline{d})'$

<u>Some Remarks</u> 1) Theorem 7 establishes the density and splitting theorems for the B-RE degrees whenever $\sum_1 \text{cf}(B) = \sum_1\text{proj}(B) = \omega$ (for example $B = S_\beta$, β a recursive ordinal). For it is not difficult to show that for any $A \subseteq \omega$ these theorems hold in $\{$Turing degree $(B) | A \leq_T B, B$ is RE in some $C \leq_T B$ such that C is RE in $A\}$.

2) Shore pointed out to me that another corollary of this work is: The Turing degrees with jump are <u>not</u> isomorphic to the metadegrees with metajump. For, if this were the case then the Turing degrees between $0'$ and $0''$ would be isomorphic to the metadegrees between metajump (0) and metajump (metajump (0)). By the above results the latter is isomorphic to the Turing degrees between $\mathcal{O}$ and $\mathcal{O}''$. But as in Feiner [1970] any arithmetically presented linear ordering can be

embedded as an initial segment of the Turing degrees between 0 and $0''$, but for some large N any topped initial segment of the degrees between $0'$ and $0''$ is $0^{(N)}$ presentable. There are arithmetically-presentable linear orderings which are not $0^{(N)}$-presentable.

SECTION TWO UNCOUNTABLE REGULAR CARDINALITY

We now investigate how the work of the preceding section can be adapted to ordinals β of cardinality κ,κ regular. Assume $\mathcal{B} = \langle S_\beta, B\rangle$ is amenable and $\sum_1 cf(\mathcal{B}) = \sum_1 proj(\mathcal{B}) = \kappa$. As before there is a $A \subseteq \kappa$ such that for $C \subseteq \kappa$, C is $\sum_1(\mathcal{B})$ if and only if C is $\sum_1\langle L_\kappa, A\rangle$. Moreover Maass' proof of Lemma 2' also covers the present situation: Thus, if $A = \langle L_\kappa, A\rangle$ then each A-degree contains a β-bi-immune set. We must now examine the proof of Theorem 1.

The function j: A-degrees $\longrightarrow$ $\mathcal{B}$-degrees can be defined as before. It is clearly a monomorphism but a further hypothesis is now needed to show that it is onto. Given $E \subseteq S_\beta$ define E^*, E^{**}, $\hat{E}$ as in the proof of Theorem 1. Also, for $C_1, C_2 \subseteq S_\beta$ define $C_1 \leq_{\kappa\mathcal{B}} C_2$ iff $\{(z_1,z_2) | z_1 \subseteq C_1, z_2 \cap C_1 = \emptyset, \beta\text{-card}(z_1 \cup z_2) < \kappa\} \leq_{f\mathcal{B}} \{z_1,z_2) | z_1 \subseteq C_2, z_2 \cap C_2 = \emptyset, \beta\text{-card}(z_1 \cup z_2) < \kappa\}$. Then as before one can show that $\{(z_1,z_2) | z_1, z_2 \in S_\beta, z_1 \subseteq E, z_2 \cap E = \emptyset\} \leq_{f\mathcal{B}} E^* \leq_{f\mathcal{B}} E^{**} \leq_{\kappa\mathcal{B}} \hat{E}$ and hence $E \leq_{\mathcal{B}} \hat{E}$. The problem is in showing $\hat{E} \leq_{\mathcal{B}} E$. We have $\hat{E} \leq_{\kappa\mathcal{B}} E^{**}$ but to have $E^{**} \leq_{\kappa\mathcal{B}} E^*$ we need the property: $g^{-1}[x] \in S_\beta$ whenever $x \in L_\kappa$(g is tame). More seriously, to get $E^* \leq_{\kappa\mathcal{B}} E$ we need to have: If $z \in S_\beta$ has β-cardinality $< \kappa$ and $\forall w \in z(w \cap E \neq \emptyset)$ then there is $z' \in S_\beta$ s.t. $z' \subseteq E$ and $w \cap z' \neq \emptyset$ for all $w \in z$ (similarly for $S_\beta - E$ as well). Of course both of these properties are guaranteed if we assume that S_β is $\langle\kappa$-closed; i.e., if we assume that any $f: \gamma \to S_\beta$ $\gamma < \kappa$ belongs to S_β. Thus we have:

THEOREM 9. Suppose $\mathcal{B} = \langle S_\beta, B\rangle$ is amenable and $\sum_1 cf(\mathcal{B}) = \sum_1 proj(\mathcal{B}) = \kappa,\kappa$ regular. Let $A = \langle L_\kappa, A\rangle$ be the Maass collapse of $\mathcal{B}$. Further assume that S_β is $\langle\kappa$-closed. Then there is an isomorphism j between the A-degrees and the $\mathcal{B}$-degrees.

Theorem 3 goes over in this situation if one assumes the $\langle\kappa$-closure of S_β.

For, if $f:\beta \longleftrightarrow \kappa$ is a bijection then the $\langle\kappa$-closure of S_β guarantees the amenability of $\langle S_\beta, B\rangle$ where $B = \{\langle\beta_0,\beta_1,\ldots,\beta_\gamma\rangle | \gamma < \kappa,\ f(\beta_\delta) = \delta$ for all $\delta \le \gamma\}$. And $B \le_{\kappa B} f$, f is $\sum_1\langle S_\beta, B\rangle$. The remaining results of section 1 also carry over when S_β is $\langle\kappa$-closed, provided uses of $\le_{fB}$ are replaced by $\le_{\kappa B}$. To summarize:

<u>THEOREM 10</u>. Suppose β has regular cardinality κ, S_β is $\langle\kappa$-closed. Then there is $A \subseteq \kappa$ and isomorphism j between the κ-degrees $\ge$ κ-degree(A) and the β-degrees $\ge$ β-degree(A). If $\underline{d}_1$, $\underline{d}_2$ are κ-degrees $\ge$ κ-degree(A) then $\underline{d}_1$ is κ-RE relative to $\underline{d}_2$ iff $j(\underline{d}_1)$ is tamely β-RE relative to $j(\underline{d}_2)$. Moreover $j(\underline{d}_1)$ is β-RE relative to $j(\underline{d}_2)$ iff $\underline{d}_1$ is κ-RE relative to some $\underline{d}_3 \le \underline{d}_1$ such that $\underline{d}_3$ is κ-RE relative to $\underline{d}_2$. Thus $j(\underline{d}') =$ weak β-jump$(j(\underline{d}))$ and $j(\underline{d}'') = (j(\underline{d}))'$.

In case $\beta < (\kappa^+)^L$ = least L-cardinal greater than κ then A can be chosen to be a Δ_n master code for S_γ where γ is the least ordinal $\ge \beta$ such that there is a $\Delta_n(S_\gamma)$ bijection $\beta \longleftrightarrow \kappa$.

If we assume $V = L$, Friedman [1981b] provides a characterization of those β such that S_β is $\langle\kappa$-closed in terms of the <u>critical projecta</u> of β. In the terminology of that paper, S_β is $\langle\kappa$-closed iff the critical projecta of β all have cofinality κ. We shall have great use for the critical projecta in the next section.

<u>*SECTION THREE*</u> <u>*SINGULAR CARDINALITY*</u>

In this case we do not obtain isomorphisms, only monomorphisms. (We know of cases where an isomorphism does not exist.) Thus let β have singular cardinality κ and assume $\mathcal{B} = \langle S_\beta, B\rangle$ is amenable, $\sum_1\text{proj}(\mathcal{B}) = \kappa$. Our first obstacle is the fact that the Maass collapse does not apply in this context unless $\sum_1\text{cf}(\mathcal{B}) = \kappa$. This last condition is a lot to hope for when κ is singular as κ may very well be singular inside S_β. However we show that if $\sum_1\text{cf}(\mathcal{B}) = \sum_1(\mathcal{B}) - \text{cof}(\kappa)$ then nevertheless there does exist a <u>Δ_1 master code</u> for $\mathcal{B}$; i.e., a set $A \subseteq \kappa$, such that for $C \subseteq \kappa$, C is $\sum_1 \langle L_\kappa, A\rangle$ iff C is $\sum_1(\mathcal{B})$. We can then use the structure $\mathcal{A} = \langle L_\kappa, A\rangle$ instead of the Maass collapse of $\mathcal{B}$.

<u>LEMMA 11</u>. If $\sum_1\text{cf}(\mathcal{B}) = \sum_1(\mathcal{B}) - \text{cof}(\kappa)$, $\sum_1\text{proj}(\mathcal{B}) = \kappa$ then there is a

$\sum_1(\mathcal{B})$bijection $g: S_\beta \longleftrightarrow \kappa$ which is <u>tame</u>; i.e., $g^{-1}[z] \in S_\beta$ for each $z \in L_\kappa$ and the function $z \longmapsto g^{-1}[z]$ is $\sum_1(\mathcal{B})$.

Proof. Much like Lemma 4 of Friedman [1981c]. Let $\gamma_0 = \sum_1 cf(\mathcal{B})$. Fix a $\sum_1(\mathcal{B})$ injection $f: S_\beta \longrightarrow \kappa$ and a $\sum_1(\mathcal{B})$ partition $\langle P_\gamma | \gamma < \gamma_0\rangle$ of κ into γ_0 - many subsets of ordertype κ. We assume that $h(\gamma + 1) - h(\gamma) \subseteq P_\gamma$, where $h: \gamma_0 \longrightarrow \kappa$ is a $\sum_1(\mathcal{B})$ orderpreserving, unbounded function and $h(0) = 0$. Also choose an orderpreserving, unbounded $\sum_1(\mathcal{B})$-function $k: \gamma_0 \longrightarrow \beta$ and for each $\gamma < \gamma_0$ let f^γ be the γ^{th} approximation to f (thus if graph(f) = $\{(x,y) | \mathcal{B} \models \phi(x,y)\}, \phi\ \sum_1$ then graph $(f^\gamma) = \{(x,y) | \langle S_{k(\gamma)}, B \cap S_{k(\gamma)}\rangle \models \phi(x,y)\}$). We can assume that Range($f^{\gamma+1}$)-Range(f^γ) has orderype κ for each $\gamma < \gamma_0$. Thus we can choose a $\sum_1(\mathcal{B})$ sequence $\langle \ell_\gamma | \gamma < \gamma_0\rangle$ such that for each $\gamma < \gamma_0$, ℓ_γ is a bijection between Range($f^{\gamma+1}$)-Range(f^γ) and P_γ. We are finally prepared to define:

$$g(x) = \ell_\gamma \circ f^{\gamma+1}(x), \text{ where } \gamma = \text{least } \gamma \text{ s.t. } x \in \text{Dom}(f^{\gamma+1}).$$

Then g is a $\sum_1(\mathcal{B})$ bijection between S_β and κ. The tameness of g follows from the fact that $z \in L_\kappa \longrightarrow z \cap \kappa \subseteq h(\gamma+1)$ for some $\gamma < \gamma_0 \longrightarrow z \cap \kappa \subseteq P_\gamma$ $\longrightarrow g^{-1}[z]$ can be computed from $f^{\gamma+1}$. ——|

<u>COROLLARY 12</u>. Under the hypothesis of Lemma 11 there is a Δ_1 master code for $\mathcal{B}$.

Proof. Let $\phi(e,x)$ be a universal $\sum_1$ formula and set $T = \{(e,x,\sigma) | \langle S_\sigma, B \cap S_\sigma\rangle | = \phi(e,x)\}$. By Lemma 11 choose a tame $g: S_\beta \longleftrightarrow \kappa$ and let $A = g[T]$. Then $\langle A \cap \gamma | \gamma < \kappa\rangle$ is $\sum_1(\mathcal{B})$ since by the tameness of g the sequence $\langle T \cap g^{-1}[\gamma] | \gamma < \kappa\rangle$ is $\sum_1(\mathcal{B})$. From this it follows that any C which is $\sum_1\langle L_\kappa, A\rangle$ is also $\sum_1(\mathcal{B})$. But if $C \subseteq \kappa$ is $\sum_1(\mathcal{B})$ then for some e $x \in C \longleftrightarrow \exists\sigma'((g(e),g(x),\sigma') \in A)$. As we can assume $g \restriction \kappa$ is $\sum_1\langle L_\kappa, A\rangle$ (for example, require $g(\gamma) = \gamma + \gamma$) this shows that C is $\sum_1\langle L_\kappa, A\rangle$. ——|

Our second obstacle to overcome is that Lemma 2 may not apply in the singular case. (In fact if κ is singular in S_β and $V = L$ then no subset of κ is β-bi-immune.) Instead we need a different way of choosing representatives of the A-degrees so that $\leq_A$ and $\leq_\mathcal{B}$ agree on these sets. This method is obtained as an extension of Lemma 11 of Friedman [1981a].

<u>Definition</u>. $C \subseteq \kappa$ is <u>$\mathcal{B}$-reduced</u> if

$\{(x,y) \in S_\beta | x \subseteq C, y \cap C = \emptyset\} \le_{f\mathcal{B}} \{(x,y) \in L_\kappa | x \subseteq C, y \cap C = \emptyset\}$

It is clear that if $C_1, C_2 \subseteq \kappa$ are $\mathcal{B}$-reduced then $C_1 \le_{\mathcal{B}} C_2 \longleftrightarrow C_1 \le_{\mathcal{A}} C_2$.

<u>Lemma 13</u>. (V = L) Suppose κ is singular with respect to $\sum_1(\mathcal{B})$ functions. Then any $\mathcal{A}$-degree contains a $\mathcal{B}$-reduced set.

<u>Proof</u>. Let $\gamma_0 = \sum_1 cf(\mathcal{B}) = \sum_1(\mathcal{B}) - cof(\kappa) < \kappa$. Choose order-preserving, unbounded $\sum_1(\mathcal{B})$ functions $f: \gamma_0 \longrightarrow \beta$ and $g: \gamma_0 \longrightarrow \kappa$ such that $g(\gamma)$ is a cardinal for each $\gamma < \gamma_0$. Choose a parameter $p \in S_\beta$ such that f,g are $\sum_1(\mathcal{B})$ with parameter p and for each $\gamma < \gamma_0$ let $H_\gamma = \sum_1$ Skolem hull of $g(\gamma) \cup \{p\}$ inside $\mathcal{B}_\gamma = \langle S_{f(\gamma)}, B \cap S_{f(\gamma)}\rangle$. For each $\gamma < \gamma_0$, $s_\gamma = H_\gamma \cap g(\gamma)^+$ = least ordinal not in H_γ.

Given $C \subseteq \kappa$ we let $g_C(\gamma)$ = rank $(C \cap g(\gamma))$ in the canonical wellordering of L_κ. Thus $g_C(\gamma) < g(\gamma)^+$ for each $\gamma < \gamma_0$. Finally, $C^* = \{\delta < \kappa \mid$ For some $\gamma < \gamma_0$, $g(\gamma) \le \delta$ and $(\delta < s_\gamma + g_C(\gamma)$ or $H_\gamma \cap (\delta, \delta^+) \neq \emptyset)\}$. The sets C and C^* have the same $\mathcal{A}$-degree as the sequences $\langle g(\gamma) | \gamma < \gamma_0\rangle$, $\langle s_\gamma | \gamma < \gamma_0\rangle$ are both $\sum_1(\mathcal{A})$ and g_C and C code the same information.

The point of C^* is that if $x \in H_\gamma$, $x \in S_\beta$ then whether or not x is a subset of C^* or is disjoint from C^* only depends on $C^* \cap g(\gamma)$. For, if $x \in H_\gamma$ then for all δ, $\cup(x \cap \delta^+) \in H_\gamma$. So, $x \subseteq C^*$ iff $|x \cap \delta^+| < \delta^+$ for each δ and $x \cap g(\gamma) \subseteq C^*$. Moreover, if $x \in H_\gamma$, $x \cap [\delta, \delta^+) \neq \emptyset$ for some $\delta \ge g(\gamma)$ then $x \cap C^* \neq \emptyset$. Thus $x \cap C^* = \emptyset$, $x \in H_\gamma \longrightarrow x \in L_\kappa$. As any $x \in S_\beta$ belongs to H_γ for some $\gamma < \gamma_0$ we see that C^* is $\mathcal{B}$-reduced. ——|

We now see how to define the monomorphism j. Assume V = L. Suppose $\mathcal{B} = \langle S_\beta, B\rangle$ is amenable and $\sum_1 proj(\mathcal{B}) = \kappa$, $\sum_1 cf(\mathcal{B}) = \sum_1(\mathcal{B}) - cof(\kappa)$. Then if $\sum_1 cf(\mathcal{B}) = \kappa$ define $\mathcal{A}$ and j as in section 2. Otherwise let $\mathcal{A} = \langle L_\kappa, A\rangle$ where A is a Δ_1 master code for $\mathcal{B}$ and for any $\mathcal{A}$-degree $\underline{d}$ define $j(\underline{d})$ = $\mathcal{B}$-degree(D) where $D \in \underline{d}$ is $\mathcal{B}$-reduced. Then j is a well-defined monomorphism.

<u>THEOREM 14</u>. If j is defined as above then j is an isomorphism between the $\mathcal{A}$-degrees and the regular $\mathcal{B}$-degrees.

<u>Proof</u>. Notice that if $f: \gamma_0 \longrightarrow \beta$, $g: \gamma_0 \longrightarrow \kappa$ are $\Sigma_1(\mathcal{B})$, orderpreserving and cofinal ($\gamma_0 = \Sigma_1 cf(\mathcal{B})$) then any $\mathcal{B}$-reduced set $D \subseteq \kappa$ has the same $\mathcal{B}$-degree as $\{f(\gamma), D \cap g(\gamma)) | \gamma < \gamma_0\}$ and this set is regular. Thus Range(j) $\subseteq$ regular $\mathcal{B}$-degrees.

Conversely, suppose $E \subseteq S_\beta$ is regular and (similarly to the proof of Theorem 1) define $E^* = \{z \in S_\beta | z \subseteq E\} \vee \{z \in S_\beta | z \cap E \neq \emptyset\}$, $E^{**} = g[E^*]$ where g comes from Lemma 11, $\hat{E}$ a $\mathcal{B}$-reduced set of the same A-degree as E^{**}. As before it is easy to show that $E \leq_{\mathcal{B}} \hat{E}$. But conversely $\hat{E} \leq_A E^{**}$, $\{(x,y) \in L_\kappa | x \subseteq E^{**}, y \cap E^{**} = \emptyset\} \leq_{w\mathcal{B}} E^*$ by the tameness of g, $E^* \leq_{\mathcal{B}} E$ by the regularity of E. ——|

In case $\kappa = \aleph_{\omega_1}$ (and $V = L$) the results of Friedman [1981a] show that the A-degrees are well-ordered. Thus the regular $\mathcal{B}$-degrees are well-ordered. However it is known that the $\mathcal{B}$-degrees are <u>not</u> well-ordered in this case and thus the monomorphism j of Theorem 14 is not onto all the $\mathcal{B}$-degrees. It is not known at present if j is onto when $\kappa = \aleph_\omega$. (In this case the A-degrees are not well-ordered.)

As in earlier sections one can determine the Range of j on the A-RE degrees and the inverse image under j of the $\mathcal{B}$-RE degrees. Thus j[A-RE degrees] = Tamely $\mathcal{B}$-RE degrees and j^{-1}[$\mathcal{B}$-RE degrees] = $\{\underline{d} \in$ A-degrees$|\underline{d}$ is A-RE relative to an A-RE $\underline{e} \leq \underline{d}\}$. The proof uses the tameness of g and a slightly modified construction of $\mathcal{B}$-reduced sets (so as to produce an A-RE $\mathcal{B}$-reduced set in a given A-RE degree).

Finally we determine under what conditions given an ordinal β of cardinality κ, there exists $B \subseteq S_\beta$ such that $\mathcal{B} = \langle S_\beta, B \rangle$ is amenable and satisfies our above hypotheses; i.e., $\Sigma_1 proj(\mathcal{B}) = \kappa$, $\Sigma_1 cf(\mathcal{B}) = \Sigma(\mathcal{B})$-cof$(\kappa)$. We have already seen that this implies the existence of an injection $g: S_\beta \longrightarrow \kappa$ such that $g^{-1} \upharpoonright \gamma \in S_\beta$ for each $\gamma < \kappa$ (if $\Sigma_1 cf(\mathcal{B}) = \kappa$ then let g be any $\Sigma_1(\mathcal{B})$ injection $S_\beta \longrightarrow \kappa$; otherwise use Lemma 11). But if g is any such injection we can define $B \subseteq S_\beta$ such that our conditions are satisfied: $B = \{g^{-1} \upharpoonright \gamma | \gamma < \kappa\}$. The regularity of B follows from the key property of g. Clearly $\Sigma_1 proj(\mathcal{B}) = \kappa$

as g is $\Sigma_1(\beta)$. And, $\Sigma_1 cf(\beta) = \Sigma_1(\beta)$-$cof(\kappa)$ as there is a $\Sigma_1(\beta)$ cofinal function $\kappa \longrightarrow \beta$: Let $f(\gamma)$ = least δ s.t. $g^{-1} \upharpoonright \gamma \in S_\delta$.

Now the existence of g can be completely analyzed using the techniques of Frienman [1981c] (see the proof of Theorem 9 of that paper). This yields the next result.

THEOREM 15. (V = L) Suppose β has cardinality κ and the critical projecta of β all have cofinality = cofinality(κ). Then there is an injection $g: S_\beta \longrightarrow \kappa$ such that $g^{-1} \upharpoonright \gamma \in S_{\beta'}$ for all $\gamma < \kappa$. Moreover this last condition implies that there is a set $A \subseteq \kappa$ such that the κ-degrees $\geq$ κ-degree(A) are isomorphic to the regular β-degrees $\geq$ β-degree(A). If $\underline{d_1}$, $\underline{d_2}$ are κ-degrees $\geq$ κ-degree(A) then $\underline{d_1}$ is κ-RE relative to $\underline{d_2}$ iff $j(\underline{d_1})$ is tamely β-RE relative to $j(\underline{d_2})$ and $j(\underline{d_1})$ is β-RE relative to $j(\underline{d_2})$ iff $\underline{d_1}$ is κ-RE relative to some $\underline{d_3} \leq \underline{d_1}$ s.t. $\underline{d_3}$ is κ-RE relative to $\underline{d_2}$. Thus $j(\underline{d}')$ = weak β-jump $(j(\underline{d}))$ and $j(\underline{d}'')$ = $(j(\underline{d}))'$.

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