Trapping Cofinality

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We begin with a simple observation about the inner model $L[\text{Card}]$, where Card denotes the class of infinite cardinals. If m is a mouse built from measures which is *active* (i.e. has a measure at the top) then a *good Ord-iteration* of m is an iteration of m of length Ord using total measures whose iteration map sends the top measurable of m to Ord in the iterate. A *good Ord-iterate* of m is the result of a good Ord-iteration.

Proposition 1 *Let $m = m_1^\#$ be the least mouse with a measurable limit of measurables.*
(a) There is a good Ord-iterate m^+ of m with $\aleph_1^V$ as its least measurable whose truncation to Ord contains $L[Card]$ as an inner model.
(b) There is a good Ord-iterate m^- of m with $\aleph_\omega^V$ as its least measurable whose truncation to Ord is contained in $L[Card]$ as an inner model.

The iterate m^+ is obtained by iterating measurables of $m_1^\#$ and its iterates which are below the top measurable to cardinals of V. The measurables of m^+ below Ord are the infinite successor cardinals of V and therefore Card is definable over $m^+|\text{Ord}$. The iterate m^- is obtained by iterating the measurables of $m_1^\#$ and its iterates which are below the top measurable to limit cardinals of V. The measurable cardinals of m^- below Ord are the successor limit cardinals of V (i.e. the limit cardinals of V which are not limits of limit cardinals of V), and the measure in m^- on such a cardinal is determined by the tail filter on the cardinals below it. It follows that $m^-|\text{Ord}$ is definable over $L[\text{Card}]$.

We say that the mouse $m_1^\#$ *traps* the inner model $L[\text{Card}]$.

Corollary 2 *The subsets of $\aleph_1^V$ in $L[Card]$ are those of m^- (= those of m^+). Thus $\aleph_1^V$ is weakly compact in $L[Card]$ and the GCH holds in $L[Card]$*

up to and including $\aleph_1^V$*. The reals of* $L[Card]$ *are those of* $m_1^{\#}$*.* [1]

The Corollary follows because $m^-|\text{Ord}$ is a simple iterate of $m^+|\text{Ord}$ (i.e. an iterate using total measures) and $\aleph_1^V$ is the critical point of this iteration. Thus $L[\text{Card}]$ has the same subsets of $\aleph_1^V$ as these models and as $\aleph_1^V$ is measurable in $m^+|\text{Ord}$ it is weakly compact in $m^-|\text{Ord}$ and hence in $L[\text{Card}]$. Also note that if a mouse m has a measure at the top with critical point κ then m and $m|\kappa$ have the same bounded subsets of κ[2].

The aim of this paper is to establish a similar result for the model $L[\text{Cof}]$, where Cof denotes the proper class function that assigns to each limit ordinal its cofinality. For this result we make use of the least mouse with a measurable κ of Mitchell order $\kappa+1$, which we denote by m_{Cof}. [3]

Our result requires that we be clear about what we mean by "mouse" in this paper. The modern notion of mouse (see [8]) allows the use of partial extenders (in the present setting, partial measures), facilitating good condensation properties. The classical notion (see [6]) made use only of total extenders. The difference is often just cosmetic as the hierarchy of total extenders of a classical mouse can be reorganised into a hierarchy with partial extenders, producing a modern mouse with the same sets. However the distinction is important in thisi paper[4].

To produce our desired smaller iterate of m_{Cof} we need to assume that the universe is rich in inaccessible cardinals. This is guaranteed by assuming that *Ord is* Δ_2*-Mahlo*, i.e. that every club of ordinals which is Δ_2-definable in V contains an inaccessible.

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[1] Actually, as discussed later in this paper, [7] characterises $L[\text{Card}]$ as a hyperclass-generic extension of the model $(m^-|\text{Ord}, \mathcal{C})$ of class theory, where $\mathcal{C}$ consists of the subclasses of $m^-|\text{Ord}$ belonging to m^-, via a cardinal- and GCH-preserving Prikry Product forcing. This yields the stronger result that all infinite successor V-cardinals are weakly compact in $L[\text{Card}]$ and the GCH holds everywhere in $L[\text{Card}]$.

[2] This is true not only for "modern" mice built from partial measures, but also for the "classical" mice used in this paper which are built from only total measures.

[3] Normal measures on a cardinal κ are ordered by $U_0 < U_1$ iff U_0 belongs to the ultrapower of the universe by U_1. In a mouse this is a wellorder. The *Mitchell order of a measure* is the ordertype of the measures below it in this order and the *Mitchell order of* κ is the ordertype in this order of all meaures with critical point κ. Thus if there is just one normal measure on κ, this measure has Mitchell order 0 and κ has Mitchell order 1.

[4] The reason is that we will only recover the total measures of the truncation to Ord of an Ord-iterate of m_{Cof} using Cof, and it is therefore important to know that these are the only measures of that truncation.

Theorem 3 *Regard m_{Cof} as a classical mouse (presented with only total measures).*
(a) There is an Ord-iterate m^+ of m_{Cof} with $\aleph_1^V$ as its least measurable such that $m^+|Ord$ contains $L[Cof]$ as an inner model. If Ord is Δ_2-Mahlo then m^+ is a good Ord-iterate of m_{Cof}.
(b) Assume that Ord is Δ_2-Mahlo. Then there is a good Ord-iterate m^- of m_{Cof} with $\aleph_\omega$ as its least measurable such that $L[Cof]$ contains $m^-|Ord$ as an inner model.

Corollary 4 *Assume that Ord is Δ_2-Mahlo. Then letting m^+ and m^- be as in Theorem 3, the subsets of $\aleph_1^V$ in $L[Cof]$ are those of m^- (= those of m^+). Thus $\aleph_1^V$ is weakly compact in $L[Cof]$ and the GCH holds in $L[Cof]$ up to and including $\aleph_1^V$. The reals of $L[Cof]$ are those of m_{Cof}.*

Again the Corollary follows from the Theorem because L[Cof] is trapped by m_{Cof}: The weasels $m^-|$Ord and $m^+|$Ord compare simply (i.e. with total measures) below Ord and $\aleph_1^V$ is the critical point of the comparison. It follows that L[Cof] has the same subsets of $\aleph_1^V$ as these models. The reals of m_{Cof} are the same as the reals of m^+ and m^-; as these mice have measures at the top with critical point Ord (thanks to the goodness of the Ord-iterations that yield m^+ and m^-), all of these reals appear in $m^+|$Ord and $m^-|$Ord.

An iterate containing $L[Cof]$

We prove Theorem 3 (a).

Let $(\aleph_\alpha^* \mid \alpha \in \text{Ord})$ be the increasing enumeration of the infinite regular cardinals of V. (Thus for example $\aleph_n^* = \aleph_n$ for finite n, $\aleph_\omega^* = \aleph_{\omega+1}$ and for weakly inaccessible α, $\aleph_\alpha^* = \alpha$.) Suppose m is a mouse (presented with only total measures) and U is a measure in m with critical point κ. We say that U is *satisfied* iff

U has (Mitchell-) order α in m and κ has cofinality greater than $\aleph_\alpha^*$ (in V).

Our iteration is defined as follows. Let m_0 be m_{Cof}, the least mouse (presented with only total measures) containing a measurable κ of order $\kappa + 1$. For limit ordinals α we take m_α to be the direct limit of the m_β, $\beta < \alpha$.

Now suppose that m_α is defined and we wish to define $m_{\alpha+1}$. Let U_α be the least measure of m_α which is not satisfied (where measures are ordered by their critical points and for a fixed critical point by their orders). Note that U_α exists as the top measure of m_α has order $\kappa =$ the top measurable of m_α and therefore is not satisfied. Then we take $m_{\alpha+1}$ to be the ultrapower of m_α via the measure U_α. Let κ_α denote the critical point of the measure U_α.

Note that the critical points κ_α used in the iteration are strictly increasing: When the measure U_α on κ_α is applied, all smaller measures have been satisfied and will continue to be satisfied as measures in the ultrapower of m_α by U_α. Moreover the only measures in $m_{\alpha+1}$ with critical point at most κ_α are smaller than U_α in m_α. So κ_α cannot be re-used in the iteration. It follows that for each α, κ_α is at least α and therefore m_α has ordinal height greater than α. And as taking an ultrapower does not increase cardinality, m_α has the same cardinality as α for infinite α (m_n is countable for finite n).

Lemma 5 *Let κ_α^- denote the sup of the κ_β, $\beta < \alpha$ ($\kappa_0^- = 0$). If κ is greater than κ_α^- and is regular in m_α then κ has V-cofinality $\aleph_0$.*

Corollary 6 *If κ_α^- (defined as in the previous Lemma) is less than κ_α then κ_α is the least measurable of m_α greater than κ_α^-.*

The Corollary follows from the Lemma since $\kappa =$ the least measurable of m_α greater than κ_α^- is regular in m_α and therefore by the Lemma of V-cofinality $\aleph_0$; so the (order 0) measure on κ in m_κ is not satisfied.

Proof of Lemma 5. Note that as $m_{Cof} = m_0$ is sound and Σ_1-projectible to ω, every element of m_0 is Σ_1-definable in m_0. It follows that every element of m_α is Σ_1-definable in m_α from parameters in κ_α^-. Also note that m_0 is closed under taking transitive closures. But then if for each transitive x in m_0 and Σ_1 formula φ we let $a_{x,\varphi}$ consist of those elements of κ which are Σ_1-definable in $\pi_{0\alpha}(x)$ from parameters in κ_α^- via the formula φ, we see that the $a_{x,\varphi}$'s are bounded in κ, using the regularity of κ in m_α. And their union is all of κ as the union of the $\pi_{0\alpha}(x)$'s is all of m_α. As there are only V-countably many $a_{x,\varphi}$'s it follows that κ has V-cofinality $\aleph_0$. $\square$

To satisfy the measures on a κ of order κ we need κ to be weakly inaccessible (in V) and V might not have any weak inaccessibles. However we can satisfy every measure below the top if we assume that Ord is Δ_2-Mahlo:

Lemma 7 *Assume that Ord is Δ_2-Mahlo. Then the class of α for which all measures with critical point below the top measurable of m_α are satisfied forms a closed unbounded class.*

Proof. Say that the pair (α, γ) is *good* if γ is below the top measurable of m_α and there is a β at least α such that $\pi_{\alpha\beta}(\gamma)$ is less than κ_β. We show that all pairs (α, γ) with γ below the top measurable of m_α are good. If not, choose a bad pair (α_0, γ_0). If there is an α_1 greater than α_0 and γ_1 less than $\pi_{\alpha_0\alpha_1}(\gamma_0)$ such that (α_1, γ_1) is bad, choose such a bad pair (α_1, γ_1). And if there is an α_2 greater than α_1 and γ_2 less than $\pi_{\alpha_1\alpha_2}(\gamma_1)$ such that (α_2, γ_2) is bad, then choose such a bad pair (α_2, γ_2). Continue this, choosing bad pairs (α_n, γ_n). This sequence cannot be infinite, as that would contradict the wellfoundness of the iterate m_{α_ω} where α_ω is the sup of the α_n's. So there is some $\alpha = \alpha_n$ and $\gamma = \gamma_n$ such that (α, γ) is a bad pair and for all β at least α, $\pi_{\alpha\beta}(\gamma)$ is the least δ so that the pair (β, δ) is bad.

We may choose β_0 large enough so that $\pi_{\alpha\beta_0}(\delta)$ is less than κ_{β_0} for all δ less than γ, using the fact that (α, δ) is good for such δ. Then we may choose β_1 greater than β_0 so that $\pi_{\beta_0\beta_1}(\delta)$ is less than κ_{β_1} for all δ less than $\gamma_1 = \pi_{\alpha\beta_0}(\gamma)$, using the fact that (β_0, δ) is good for such δ. Note that for δ less than γ, $\pi_{\alpha\beta_1}(\delta)$ is equal to $\pi_{\alpha\beta_0}(\delta)$, as κ_{β_0} is the critical point of $\pi_{\beta_0\beta_1}$. If we continue this for ω steps we reach a β greater than α such that for all n, $\pi_{\beta_n\beta}(\delta)$ is less than κ_β for all δ less than $\pi_{\alpha\beta_n}(\gamma)$. It follows that $\gamma_\omega = \pi_{\alpha\beta}(\gamma)$ is at most κ_β. By the badness of the pair (α, γ), we have that γ_ω in fact equals κ_β. By iterating further, we can create a club C of such β's. Choose some β in C. If κ_β is not the top measurable of m_β then let δ be the order of κ_β. If δ is less than κ_β then choose β^* in C of V-cofinality $\aleph_\delta^*$. But then $\pi_{\beta\beta^*}(\kappa_\beta) = \pi_{\beta\beta^*}(\pi_{\alpha\beta}(\gamma)) = \pi_{\alpha\beta^*}(\gamma) = \kappa_{\beta^*}$, which is a contradiction since $\delta =$ the order of κ_β is also the order of κ_{β^*} by elementarity and therefore all measures with critical point κ_{β^*} are satisfied. If δ equals κ_β then choose β^* in C to be V-regular, using the Δ_2-Mahloness of Ord. Then we again reach a contradiction as all meaures with critical point κ_{β^*} are satisfied. So it must be that κ_β is the top measurable of m_β and therefore we have found a β such that all measures in m_β with critical point below the top measurable are satisfied.

This proves that the class of ordinals in the statement of the Lemma is unbounded. It is closed because if β is the limit of ordinals α such that κ_α is the top measurable of m_α, then κ_β is also the top measurable of m_β. □

The iteration continues for Ord steps, resulting in an iterate $m_{Ord} = m^+$ of height greater than Ord. It follows from Lemma 7 that if Ord is Δ_2-Mahlo then the top measurable of m_0 is sent to Ord under the iteration map from m_0 to m^+. So Ord is the top measurable of m^+ in that case. We denote $m^+|\text{Ord}$ by m_0^+.

Next we characterise the measurables of m_0^+ together with their orders. Let Meas(α) denote the class of measurables of order α. And recall that $\aleph_\alpha^*$ denotes the α-th regular cardinal.

Lemma 8 *Let α be greater than 0. If κ is a measurable of order at least α in m_0^+ then κ has V-cofinality at least $\aleph_\alpha^*$. Conversely, if κ has V-cofinality at least $\aleph_\alpha^*$ and is regular in m_0^+ then κ is measurable of order at least α in m_0^+.*

Proof. We prove that the Lemma holds up to κ_γ (i.e. for κ less than or equal to κ_γ) by induction on γ. For $\gamma = 0$ this is vacuous as the least measurable of m_0^+ is greater than κ_0 and, as κ_0 is countable in V, the least ordinal of uncountable V-cofinality is also greater than κ_0.

Suppose that $\gamma > 0$, the Lemma holds up to κ_δ for $\delta < \gamma$ and we want to verify it up to κ_γ. If $\gamma = \delta + 1$ is a successor then by Corollary 6 κ_γ is the least measurable of m_γ greater than κ_δ and therefore there are no measurables of m_0^+ in the interval $(\kappa_\delta, \kappa_\gamma]$. Also by Lemma 5 the m_γ-regulars and therefore also the m_0^+-regulars in this interval have V-cofinality $\aleph_0$. So the Lemma holds vacuously on this interval and therefore by induction up to and including κ_γ.

So assume that γ is a limit. Again by Lemma 5 there are no instances of the Lemma to verify in the interval $(\kappa_\gamma^-, \kappa_\gamma]$ so it will suffice to verify the Lemma at κ_γ^-. First suppose that κ_γ is greater than κ_γ^- and therefore all measures in m_γ on κ_γ^-, if any, are satisfied.

We first verify the Lemma at κ_γ^- in the case $\alpha = 1$; the proof in the general csae will be similar. Let κ denote κ_γ^-. If κ is measurable in m_0^+ then it is measurable in m_γ and therefore the order 0 measure on κ in m_γ is satisfied; it follows from the definition of satisfaction that κ has uncountable V-cofinality. Conversely, suppose that κ has uncountable V-cofinality and is regular in m_0^+ and therefore also in m_γ. We use the following variant of Lemma 5.

Lemma 9 *For all α, if κ is regular in m_α then κ is either measurable in m_α or has V-cofinality $\aleph_0$.*

Proof. By induction on α. The claim is vacuous for $\alpha = 0$ as m_0 is countable.

Suppose the result holds for m_α and we want to verify it for $m_{\alpha+1}$. If κ is less than κ_α then the result follows by induction. If κ equals κ_α then either κ is measurable in $m_{\alpha+1}$ (because the measure U_α which was applied to obtain $m_{\alpha+1}$ has order > 0) or κ has cofinality $\aleph_0$ in V (because U_α has order 0 and is not satisfied). If κ is greater than κ_α then we can apply Lemma 5 to infer that κ has V-cofinality $\aleph_0$.

So assume that α is a limit and the result holds for all β less than α. As before let κ_α^- denote the sup of the κ_β, $\beta < \alpha$. If κ is greater than κ_α^- then again by Lemma 5 it has V-cofinality $\aleph_0$ and if κ is less than κ_α^- then the result follows by induction. So suppose that κ equals κ_α^- and assume that κ has uncountable V-cofinality. Choose $\beta < \alpha$ and $\bar{\kappa}$ in m_β so that $\pi_{\beta\alpha}(\bar{\kappa})$ equals κ. If $\bar{\kappa}$ has uncountable V-cofinality then by induction $\bar{\kappa}$ is measurable in m_β and therefore by elementarity, κ is measurable in m_α. So suppose that for all $\beta < \alpha$, if $\pi_{\beta\alpha}(\bar{\kappa})$ equals κ then $\bar{\kappa}$ has V-cofinality $\aleph_0$ and therefore the map $\pi_{\beta\alpha}$ restricted to $\bar{\kappa}$ is not cofinal into κ. But then using the uncountable V-cofinality of $\kappa = \kappa_\alpha^-$, we can choose $\beta < \alpha$ and $\bar{\kappa}$ so that $\pi_{\beta\alpha}(\bar{\kappa})$ equals κ and $\bar{\kappa}$ is the critical point κ_β of $\pi_{\beta\alpha}$. It follows that $\bar{\kappa}$ is measurable in m_β and by elementarity, κ is measurable in m_α. $\square$ (Lemma 9)

Now returning to our verification of Lemma 8 at κ_γ^- when $\alpha = 1$, if $\kappa = \kappa_\gamma^-$ has uncountable V-cofinality and is regular in m_0^+ we can apply Lemma 9 to conclude that κ is measurable in m_γ and therefore also in m_0^+ (as κ is less than κ_γ).

The case $\alpha > 1$ is a natural generalisation of the case $\alpha = 1$: If $\kappa = \kappa_\gamma^-$ has order at least α in m_0^+ then as all measures in m_γ^- with critical point κ are satisfied (recall our assumption that $\kappa = \kappa_\gamma^-$ is less than κ_γ) it follows that κ has V-cofinality greater than $\aleph_\beta^*$ for each $\beta < \alpha$ and therefore has V-cofinality at least $\aleph_\alpha^*$. Conversely, assume that κ has V-cofinality at least $\aleph_\alpha^*$ and is regular in m_0^+. We want to show that κ has order at least α in m_0^+. We prove this using the following analogue of Lemma 9:

Lemma 10 *For all β, if κ is regular in m_β then κ either has order at least α in m_β or has V-cofinality less than $\aleph^*_\alpha$.*

Proof. By induction on β. The claim is vacuous for $\beta = 0$ as m_0 is countable.

Suppose the result holds for m_β and we want to verify it for $m_{\beta+1}$. If κ is less than κ_β then the result follows by induction. If κ equals κ_β then either κ has order at least α in $m_{\beta+1}$ (because the measure U_β which was applied to obtain $m_{\beta+1}$ has order at least α) or κ has V-cofinality less than $\aleph^*_\alpha$ (because U_β has order less than α and is not satisfied). If κ is greater than κ_β then by Lemma 5, κ has V-cofinality $\aleph_0$.

Assume that β is a limit and the result holds for all γ less than β. Let κ^-_β denote the sup of the κ_γ, $\gamma < \beta$. If κ is greater than κ^-_β then it has V-cofinality $\aleph_0$ by Lemma 5. If κ is less than κ^-_β then the result follows by induction. So suppose that κ equals κ^-_β and assume that κ has V-cofinality at least $\aleph^*_\alpha$. Choose $\gamma < \beta$ and $\bar{\kappa}$ in m_γ so that $\pi_{\gamma\beta}(\bar{\kappa})$ equals κ. If $\bar{\kappa}$ has V-cofinality at least $\aleph^*_\alpha$ then by induction $\bar{\kappa}$ has order at least α in m_γ and therefore by elementarity κ has order at least α in m_β. So suppose that for all $\gamma < \beta$, if $\pi_{\gamma\beta}(\bar{\kappa})$ equals κ then $\bar{\kappa}$ has V-cofinality less than $\aleph^*_\alpha$ and therefore the map $\pi_{\gamma\beta}$ restricted to $\bar{\kappa}$ is not cofinal into κ. Recall that $\kappa = \kappa^-_\beta$ has V-cofinality at least $\aleph^*_\alpha$. For each $\alpha_0 < \alpha$ we can choose $\gamma < \beta$ and $\bar{\kappa}$ of V-cofinality at least $\aleph^*_{\alpha_0}$ so that $\pi_{\gamma\beta}(\bar{\kappa})$ equals κ and $\bar{\kappa}$ is the critical point κ_γ of $\pi_{\gamma\beta}$. The measure U_γ must have order at least α_0 in m_γ, else it would not have been applied, so $\bar{\kappa}$ has order greater than α_0 in m_γ. By elementarity, κ has order greater than α_0 in m_β, and since $\alpha_0 < \alpha$ was arbitrary, κ has order at least α in m_β. $\square$ (Lemma 10)

Returning to our verificatiion of Lemma 8 at κ^-_γ for $\alpha > 1$, if $\kappa = \kappa^-_\gamma$ is a limit cardinal of V-cofinality at least $\aleph^*_\alpha$ which is regular in m^+_0 we can apply Lemma 10 to conclude that κ has order at least α in m_γ and therefore also in m^+_0 (as by assumption κ is less than κ_γ).

Finally, we must treat the case where γ is a limit and $\kappa_\gamma = \kappa^-_\gamma =$ the sup of the κ_β, $\beta < \gamma$. So some measure on κ_γ in m_γ is not satisfied. For simplicity of notation denote κ_γ simply by κ.

First suppose that α equals 1. Suppose $\kappa = \kappa_\gamma$ is measurable in m^+_0 and therefore in m_γ. If U_γ is the order 0 measure on κ in m_γ then κ would not be measurable in m_0^+, contrary to hypothesis. So the order 0 measure on κ in m_γ is satisfied and therefore κ has uncountable V-cofinality. Conversely, if κ has uncountable V-cofinality then the measure U_γ is not the order 0 measure on κ and therefore κ remains measurable in m_0^+.

For $\alpha > 1$: If $\kappa = \kappa_\gamma$ has order at least α in m_0^+ then the measure U_γ has order at least α and therefore κ has V-cofinality greater than $\aleph_\beta^*$ for each $\beta < \alpha$. It follows that κ has V-cofinality at least $\aleph_\alpha^*$. Conversely, if κ has V-cofinality at least $\aleph_\alpha^*$ then the measure U_γ must have order at least α in m_γ; it follows that κ has order at least α in m_0^+ as well.

This completes the proof of Lemma 8 in all cases. □

Using Lemmas 9 and 8 we can now show:

Lemma 11 *Cof is definable over m_0^+.*

Proof. We define the V-cofinality of β by induction on β in the model m_0^+. This is trivial for β finite, so assume that β is infinite. If β is not regular in m_0^+ then the V-cofinality of β is the same as the V-cofinality of the m_0^+-cofinality of β, so we can apply induction to compute the V-cofinality of β. If β is m_0^+-regular and not measurable in m_0^+ then by Lemma 9, β has V-cofinality $\aleph_0$. If β is measurable of order α in m_0^+ then by Lemma 8, β has V-cofinality $\aleph_\alpha^*$. Note that the sequence $(\aleph_\alpha^* \mid \alpha \in \mathrm{Ord})$ is definable in m_0^+ as for $\alpha > 0$, $\aleph_\alpha^*$ is the least measurable of order α in m_0^+. □

Thus $m^+|\mathrm{Ord} = m_0^+$ contains $L[\mathrm{Cof}]$ as a definable inner model, completing the proof of Theorem 3(b).

An iterate contained in $L[Cof]$

For this second iteration, which yields the smaller iterate m^- of m_{Cof}, we assume that Ord is Δ_2-Mahlo. We use a different notion of "satisfied measure". Let $((m_\beta, U_\beta) \mid \alpha \in \mathrm{Ord})$ be an iteration (with total measures) of m_{Cof} and let κ_β denote the critical point of U_β. Let U be a measure on κ of order $\alpha < \kappa$ in some m_β. A *critical predecessor* of κ is an ordinal $\bar\kappa < \kappa$ such that for some $\bar\beta < \beta$, $\bar\kappa = \kappa_{\bar\beta}$ and $\pi_{\bar\beta\beta}(\bar\kappa) = \kappa$. Then U is *satisfied* if the set of V-regular critical predecessors of κ is cofinal in κ and has ordertype at least $\aleph_\alpha$.

Our iteration is defined analogously to the previous iteration: Let m_0 be m_{Cof}. For limit ordinals β we take m_β to be the direct limit of the m_γ, $\gamma < \beta$. If m_β is defined and we wish to define $m_{\beta+1}$, we let U_β be the lesat measure of m_β which is not satisfied. If all measures of m_β below the top measure of m_β are satisfied then we let U_β be the top measure of m_β. Then we take $m_{\beta+1}$ to be the ultrapower of m_β via the measure U_β. Let κ_β denote the critical point of the measure U_β.

As before the critical points κ_β used in the iteration are strictly increasing, so κ_β is at least β and m_β has ordinal height greater than β for each β. And $\kappa_{\beta+1}$, the least critical point of an unsatisfied measure in $m_{\beta+1}$, is the least measurable of $m_{\beta+1}$ greater than κ_β.

Using the Δ_2-Mahloness of Ord we can again satisfy every measure below the top.

Lemma 12 *The class of β for which all measures with critical point below the top measurable of m_β are satisfied forms a closed unbounded class.*

Proof. Follow the proof of Lemma 7. The only difference is in the choice of β^* in C: We now choose it to be a limit of V-regular elements of C which has V-cofinality at least $\aleph_\delta$. This is possible by the Δ_2-Mahloness of Ord and yields the desired contradiction. $\square$

The iteration continues for Ord steps, resulting in an iterate $m_{Ord} = m^-$ of height greater than Ord. It follows from Lemma 12 that the top measurable of m_0 is sent to Ord under the iteration map π_0 from m_0 to m^-. So Ord is the top measurable of m^-. We denote $m^-|\text{Ord}$ by m_0^-.

We show that using the predicate Cof we can identify the measurables of m_0^- as well as the measures attached to them. Let M be the unary relation defined by $M(\beta)$ iff κ_β has no critical predecessors and let R be the binary relation defined by: $R(\beta, \gamma)$ iff $M(\beta)$, γ is V-regular and κ_β is a critical predecessor of γ.

Given the relation R we can determine the measures of m_0^-: κ is measurable in m_0^- if for some β there are cofinally-many V-regular $\gamma < \kappa$ such that $R(\beta, \gamma)$ holds. For such a κ and β, the V-regular critical predecessors of κ are those V-regular $\gamma < \kappa$ such that $R(\beta, \gamma)$ holds. The order α measure on κ is generated by the tail filter on the V-regular critical predecessors of κ

which have order α in m_0^- (the latter can be determined inductively). So to show that m_0^- is an inner model of $L[\text{Cof}]$ it suffices to define the relation R in $(L[\text{Cof}], \text{Cof})$.

Lemma 13 *The relation R is definable over $(L[Cof], Cof)$.*

Proof. By induction on the infinite cardinal δ we define the relations M and R up to δ (i.e. we define M on $\beta \leq \delta$ and R on pairs (β, γ) where $\beta < \gamma \leq \delta$). The case $\delta = \aleph_0$ is vacuous. If $\delta = \aleph_1$ then the only instances of M and R up to δ are $M(0)$ and $R(0, \delta)$. Given that we have defined M and R up to the uncountable cardinal δ, we can define it up to δ^+ by declaring: $M(\delta+1)$ to hold if δ is a limit cardinal (this is because $\kappa_{\delta+1}$, the least measurable of $m_{\delta+1}$ greater than δ, is the least critical predecessor of δ^+ if δ is a limit V-cardinal), $R(\beta, \delta^+)$ holds iff $R(\beta, \delta)$ holds if δ is a successor V-cardinal and $R(\beta, \delta^+)$ holds iff $\beta = \delta + 1$ if δ is a limit V-cardinal.

Suppose that δ is a limit V-cardinal. We say that $\bar{\delta} < \delta$ is *initial* if $M(\bar{\delta})$ holds. An initial $\bar{\delta}$ is *convergent below* δ if $\pi_{\bar{\delta}\delta}(\kappa_{\bar{\delta}})$ is less than κ_δ. The latter is definable from the relation R below δ as it is equivalent to the statement that the ordertype of the V-regular $\gamma < \delta$ such that $R(\bar{\delta}, \gamma)$ is at least $\aleph_\alpha$, where α is the order of $\kappa_{\bar{\delta}}$ in m_0^- (which can be computed inductively). And $\bar{\delta}$ is *divergent below* δ if it is not convergent below δ. Now $M(\delta)$ holds iff all initial $\bar{\delta} < \delta$ are convergent below δ. To define $R(\bar{\delta}, \delta)$ for $\bar{\delta} < \delta$ we use the following Claim.

Claim. There are only finitely many initial $\bar{\delta} < \delta$ which are divergent below δ.

Proof of Claim. Suppose that $\bar{\delta}_0 < \bar{\delta}_1 < \ldots$ were initial and divergent below δ. For some n, $\pi_{\bar{\delta}_n\bar{\delta}_{n+1}}(\kappa_{\bar{\delta}_n})$ must be at most $\kappa_{\bar{\delta}_{n+1}}$, else the sup of the $\kappa_{\bar{\delta}_n}$'s would not be wellfounded. If $\pi_{\bar{\delta}_n\bar{\delta}_{n+1}}(\kappa_{\bar{\delta}_n})$ were less than $\kappa_{\bar{\delta}_{n+1}}$ then it would be convergent below δ as $\kappa_{\bar{\delta}_{n+1}}$ is the critical point of $\pi_{\bar{\delta}_{n+1}\delta}$. So $\pi_{\bar{\delta}_n\bar{\delta}_{n+1}}(\kappa_{\bar{\delta}_n})$ equals $\kappa_{\bar{\delta}_{n+1}}$, contradicting the assumption that the latter is initial. $\square$ (*Claim*)

Now suppose that $R(\bar{\delta}, \delta)$ holds. Then $\bar{\delta}$ is initial and divergent below δ. Let $\bar{\delta}_0$ be the largest initial ordinal less than δ which is divergent below δ; $\bar{\delta}_0$ exists by the Claim. We argue that $\bar{\delta}_0$ equals $\bar{\delta}$: Otherwise, $\pi_{\bar{\delta}\bar{\delta}_0}(\kappa_{\bar{\delta}})$ cannot equal $\kappa_{\bar{\delta}_0}$ as $\bar{\delta}_0$ is initial, it cannot be less than $\kappa_{\bar{\delta}_0}$ as the latter is the critical point of $\pi_{\bar{\delta}_0\delta}$ and therefore $\bar{\delta}$ would be convergent below δ and it cannot be greater than $\kappa_{\bar{\delta}_0}$ else $\bar{\delta}$ would not be a critical predecessor of δ in light of the fact that $\pi_{\bar{\delta}_0\delta}(\kappa_{\bar{\delta}_0})$ is at least δ. In conclusion: $R(\beta, \delta)$ holds iff δ is V-regular and β is the largest initial ordinal which is divergent below δ. This completes the proof of Lemma 13. $\square$

This also completes the proof of Theorem 3(b). $\square$

Some Inner models of $L[Cof]$

We have seen that we can trap L[Cof] with iterates of the mouse m_{Cof}. With smaller mice we can trap smaller inner models of L[Cof]. Let Reg denote the class of V-regular cardinals.

Theorem 14 *Regard* 0*-sword, the least mouse with a measure of order* 1*, as a classical mouse (presented with only total measures).*
(a) There is an Ord-iterate m^+ of 0*-sword with $\aleph_1^V$ as its least measurable such that $m^+|Ord$ contains $L[Reg]$ as an inner model.*
(b) There is an Ord-iterate m^- of 0*-sword with $\aleph_\omega^V$ as its least measurable such that $L[Reg]$ contains $m^-|Ord$ as an inner model and $L[Reg]$ is hyperclass-generic over $(m^-|Ord, \mathcal{C})$, where $\mathcal{C}$ consists of the subclasses of $m^-|Ord$ belonging to m^-, via a Magidor iteration of Prikry forcings.*
(c) If Ord is Δ_2-Mahlo then m^+ and m^- are good Ord-iterates of 0*-sword.*

The Ord-iterate m^+ is obtained by iterating measurable cardinals below the top measurable of 0-sword to V-regular cardinals. Every V-regular will be measurable in the final iterate: Suppose that κ is V-regular and choose $\beta < \kappa$ and $\bar{\kappa}$ so that $\pi_{\beta\kappa}(\bar{\kappa}) = \kappa$. As κ is V-regular we may further suppose that $\bar{\kappa} = \kappa_\beta$ is the critical point of $\pi_{\beta\kappa}$ and therefore by elementarity κ is measurable in m_κ. But then again as κ is V-regular, the critical point of π_κ, the iteration map from m_κ to m^+, must be greater than κ and therefore κ is measurable in m^+. It follows that Reg is definable over $m^+|$Ord as the class of measurables of $m^+|$Ord.

The iteration that produces m^- moves measurables below the top measurable through ω-many V-regular cardinals, generating a Prikry sequence for their supremum. All but finitely–many V-regulars will appear in one of the Prikry sequences generated: If κ is V-regular then κ has an initial critical predecessor $\bar{\kappa}$; if $\bar{\kappa}$ converges below Ord then κ belongs to the Prikry sequence generated by it. By the analogue of the Claim in Lemma 13 there are only finitely-many initial $\bar{\kappa}$ which do not converge below Ord and for each such $\bar{\kappa}$ there are only finitely-many V-regulars having it as a critical predecessor. It follows that $L[\text{Reg}]$ is contained in $m^-|\text{Ord}[\vec{P}]$ where $\vec{P}$ is the sequence of Prikry sequences generated. The fact that $\vec{P}$ (and therefore the entire model $m^-|\text{Ord}[\vec{P}]$) is definable over $(L[\text{Reg}], \text{Reg})$ follows again from the analogue of the Claim in the proof of Lemma 13 (saying that there are only finitely-many initial, divergent ordinals below any V-regular cardinal), as we can use this to inductively recover the Prikry sequences and the measures they generate from the predicate Reg.

The above iterations need not be good because it could be that not all measurables below the top measurable of 0-sword and its iterates get used in the iteration (for example if there are no weakly inccessibles in V). But if we assume that Ord is Δ_2-Mahlo, then as in Lemmas 7 and 12, all measures below the top measurable get used.

In an Appendix below we discuss the Magidor iteration of Prikry forcings. For the following Corollary to Theorem 14 it suffices to note that this forcing is GCH- and cardinal-preserving.

Corollary 15 *GCH holds in $L[Reg]$. The reals of $L[Reg]$ belong to 0-sword and if Ord is Δ_2-Mahlo then they equal the reals of 0-sword. There are no measurables in $L[Reg]$.*

The Corollary follows because $L[\text{Reg}]$ is a hyperclass-generic extension of $m^-|\text{Ord}$ via a GCH- and cardinal-preserving forcing which does not add reals. If Ord is Δ_2-Mahlo then the iteration to produce m^- is good and therefore Ord is the top measurable in m^-; it follows that all reals in m^- (i.e. all reals in 0-sword) belong to $m^-|\text{Ord}$. Every measurable that appears in the iteration either gets iterated to the supremum of an ω-limit of V-regulars, and therefore is singular in $L[\text{Reg}]$, or gets iterated past the ordinals. So there are no measurables in $L[\text{Reg}]$.

Between $L[\text{Reg}]$ and $L[\text{Cof}]$ is the model $L[\text{Reg}, \text{Cof}_\omega]$, where Cof_ω denotes the class of ordinals of V-cofinality ω. We can trap this model using m_2, the least mouse with a measure of order 2 (presented with only total measures):

Theorem 16 *Assume that Ord is Δ_2-Mahlo. (a) There is a good Ord-iterate m^+ of m_2 with least measurable $\aleph_1^V$ such that $m^+|Ord$ contains $L[Reg, Cof_\omega]$ as an inner model.*

(b) There is a good Ord-iterate m^- of m_2 with least measurable $\aleph_\omega^V$ such that $m^-|Ord$ is contained in $L[Reg, Cof_\omega]$ as an inner model.

Corollary 17 *Assume that Ord is Δ_2-Mahlo.*
GCH holds in $L[Reg, Cof_\omega]$ up to and including $\aleph_1^V$ and $\aleph_1^V$ is weakly compact in $L[Reg, Cof_\omega]$. The reals of $L[Reg, Cof_\omega]$ are those of m_2 and there are no measurables in this model.

The iteration that produces m^+ of Theorem 16 moves the order 0 measures to ordinals of uncountable V-cofinality and moves the order 1 meaures to V-regulars. The predicate Cof_ω is definable over $m^+|\mathrm{Ord}$ using its order 0 measures, as in the proof of Theorem 3(a). The predicate Reg is definable over $m^+|\mathrm{Ord}$ using its order 1 measures, whose critical points are the V-regulars greater than $\aleph_1^V$.

The iteration that produces m^- of Theorem 16 moves all measures to ω-limits of regular cardinals. As in the proof of Theorem 3(b), the predicates Cof_ω, Reg can recover the Prikry sequences that are generated by the order 0, order 1 measures, respectively, and therefore can recover the enrire truncated iterate $m^-|\mathrm{Ord}$.

Similarly, for any finite n, the inner model $L[\mathrm{Reg}, \mathrm{Cof}_\omega, \mathrm{Cof}_{\omega_1}, \cdots, \mathrm{Cof}_{\omega_{n-1}}]$ can be trapped by the mouse m_{n+1} = the least mouse with a measure of order $n+1$, where Cof_{ω_i} is the class of ordinals of V-cofinality ω_i. To produce m^+ the measures of order $i < n$ move to ordinals of V-cofinality $\aleph_{i+1}$ and the measures of order n move to V-regulars. To produce m^- the measures of order $i < n$ move to ω_i-limits of V-regulars and the measures of order n to ω-limits of V-regulars.

Open questions

As in [7] (for the model $L[\mathrm{Card}]$) and Theorem 14(b) (for the model $L[\mathrm{Reg}]$) it is sometimes possible to show that an inner model is a hyperclass-generic extension (via a known forcing) of the truncation to Ord of an iterate of a mouse that traps it. Can this be done for $L[\mathrm{Cof}]$?

Question 1. Is $L[\mathrm{Cof}]$ hyperclass-generic over $m^-|\mathrm{Ord}$ (with its subclasses in m^- as classes) for some Ord-iterate m^- of m_{Cof}?

Conjecture 2. The GCH holds in $L[\mathrm{Cof}]$, the reals of $L[\mathrm{Cof}]$ are those of m_{Cof} and there are no measurables in $L[\mathrm{Cof}]$.

The methods of this paper appear to depend heavily on the predicate Reg and therefore only apply to inner models at least as large as $L[\mathrm{Reg}]$.

Question 3. Is there a mouse that traps $L[\mathrm{Cof}_\omega]$? Does $L[\mathrm{Cof}_\omega]$ satisfy GCH?

A natural inner model, far larger than $L[\mathrm{Cof}]$ is the Stable Core of [3]. In [4] it is shown that the Stable Core is contained in an iterate of Mighty Mouse (the least mouse with a definably-Woodin measurable).

Question 4. Is the Stable Core trapped by Mighty Mouse, i.e. does it lie between the truncations to Ord of two Ord-iterates of Mighty Mouse?

Appendix: The Magidor iteration of Prikry forcings

Suppose that to each measurable cardinal α is associated a normal measure U_α of order 0 (i.e. concentrating on non-measurables). The *Magidor iteration P_β of length β* is defined by induction on β. Conditions in P_β are β-sequences p where for $\alpha < \beta$, $p(\alpha)$ is 0 if α is not measurable and is of the form $(s_\alpha, \dot{A}_\alpha)$ otherwise. For limit β, P_β consists of all β-saequences p such that $p|\alpha$ belongs to P_α for all $\alpha < \beta$ and for only finitely-many measurable $\alpha < \beta$ is $p(\alpha)$ of the form $(s_\alpha, \dot{A}_\alpha)$ with s_α nonempty. If β is not measurable then $P_{\beta+1}$ consists of all $\beta + 1$-sequences p such that $p|\beta$ belongs to P_β and $p(\beta)$ is 0. If β is measurable then $P_{\beta+1}$ consists of all $\beta + 1$-sequences p such that $p|\beta$ belongs to P_β and forces in P_β that $p(\beta) = (s_\beta, \dot{A}_\beta)$ belongs to the Prikry forcing for the measure $\dot{U}_\beta$ on β in $V[\dot{G}_\beta]$ extending U_β which is defined as follows (where $\dot{G}_\beta$ denotes the P_β-generic):

A belongs to $\dot{U}_\beta$ iff $A = \dot{A}^{\dot{G}_\beta}$ for some P_β-name $\dot{A}$ such that in the forcing $j_\beta(P_\beta)$, β is forced into $j_\beta(\dot{A})$ by some condition q in $j_\beta(P_\beta)$ with $q|\beta$ in $\dot{G}_\beta$ and $q(\alpha)$ not of the form $(s_\alpha, \dot{A}_\alpha)$ with s_α nonempty for any α at least β.

Lemma 18 *For each measurable β, $\dot{U}_\beta$ as defined above is forced by the trivial condition of P_β to be a normal measure on β in $V[\dot{G}_\beta]$ which extends U_β.*

If p belongs to P_β then a *direct* extension of p is a condition q extending p such that $p(\alpha)$ and $q(\alpha)$ have the same first component for each measurable α less than β.

Lemma 19 *For each β, the Magidor iteration P_β of length β satisfies the Prikry property: For any sentence φ of the forcing language, any condition has a direct extension which decides φ.*

Lemma 20 *If G_β is P_β-generic then G_β adds ω-sequences $\vec{C} = (C_\alpha \mid \alpha < \beta,\ \alpha$ measurable) where each C_α is Prikry over $V[\vec{C}|\alpha]$ for the measure $\dot{U}^{\dot{G}_\beta}$ extending U_α. Moreover G_β is definable from $\vec{C}$ in $V[\vec{C}]$.*

We say that $\vec{C}$ is P_β-generic if it arises as in the previous lemma. [2] establishes the following "Mathias-style" characterisation of P_β-genericity, generalising the result of [5] for the case of discrete sequences of measures.

Lemma 21 *A sequence of ω-sequences $\vec{C} = (C_\alpha \mid \alpha < \beta,\ \alpha$ measurable) is P_β-generic iff $\vec{C}|\beta_0$ is P_{β_0}-generic for each $\beta_0 < \beta$ and whenever $\vec{\dot{A}} = (\dot{A}_\alpha \mid \alpha < \beta,\ \alpha$ measurable) is a sequence where for each α the trivial condition of P_α forces that $\dot{A}_\alpha$ is of measure one for the measure $\dot{U}^{\dot{G}_\alpha}$, then the union of the C_α's is contained in the union of the $\dot{A}_\alpha^{G_\alpha}$'s with only finitely-many exceptions (where G_α is the P_α-generic with associated Prikry sequences $\vec{C}|\alpha$).*

Mouse iteration can be used to produce a generic for the Magidor iteration of the iterate. To accomplish this we must use an iteration in which each order 0 measure is "used cofinality ω times". We call such iterations *Prikry iterations.*

Definition 1 *Let m be a mouse with exactly one order 0 measure at each measurable cardinal (and possibly neasures of higher order). An iteration $(m_\alpha \mid \alpha \leq \beta)$ hitting order 0 measures of m with final iterate $m_\beta = m^*$ is a* Prikry iteration of critical length λ *iff it is normal (i.e. critical points are increasing) and for measurable κ of m^* less than λ, the set of $\alpha < \beta$ such that $\pi_{\alpha\beta}(\kappa_\alpha) = \kappa$ (where the iteration map $\pi_{\alpha\beta}$ from m_α to m_β has critical point κ_α) has ordertype a limit ordinal of cofinality ω. In the latter case we refer to these κ_α's as the* critical predecessors of κ.

Prikry iterations are easy to create: At each stage of the iteration hit the order 0 measure at the least measurable κ whose set of critical predecessors is finite, until there are no such κ. The result will be an iteration where the set of critical predecessors of each measurable of the final iterate has ordertype ω. In this case we can take the critical length λ to be the ordinal height of the final iterate. In the final iterate of the Prikry iteration we consider below, the ordertypes of the set of critical predecessors will be greater than ω for measurables below the critical length.

Lemma 22 *Suppose that $(m_\alpha \mid \alpha \leq \beta)$ is a Prikry iteration of critical length λ. For each measurable κ of the final iterate m^* less than λ choose an ω-sequence C_κ cofinal in the set of critical predecessors of κ. Then $\vec{C} = (C_\kappa \mid \kappa$ measaurable in $m^*)$ is generic for the Magidor iteration of m^* using the order 0 measures at its measurables less than λ.*

The previous Lemma ensures that the sequence $\vec{P}$ of Prikry sequences resulting from the iteration of 0-sword as in Theorem 14(b) forms a generic for the Magidor iteration of Prikry forcings over the model of class theory $(m^-|\mathrm{Ord}, \mathcal{C})$ where $\mathcal{C}$ consists of the subclasses of $m^-|\mathrm{Ord}$ in m^-, and therefore $L[\mathrm{Reg}] = m^-|\mathrm{Ord}[\vec{P}]$ is a hyperclass-generic extension of $m^-|\mathrm{Ord}$ via this forcing.

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