Classification theory and $0^{\#}$

Sy D. Friedman Tapani Hyttinen* Mika Rautila*

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Abstract

We characterize the classifiability of a countable first-order theory T in terms of the solvability (in the sense of [2]) of the potential-isomorphism problem for models of T.

1 Introduction

[2] studies the solvability of a number of problems concerning constructible sets under the assumption that $0^{\#}$ exists. As an example, consider the collection of constructible subsets S of a regular L-cardinal κ such that S has a cub subset in a cardinal-preserving extension of L. If κ is $(\aleph_1)^L$, then this set is constructible (as it is just the collection of constructible subsets of $(\aleph_1)^L$ which are stationary in L), and therefore we may say that this problem is "solvable". But if κ is a regular L-cardinal greater than $(\aleph_1)^L$, then this collection is not in L but equiconstructible with $0^{\#}$. Thus in the latter case we have an "unsolvable" problem. [2] studies a number of related problems in terms of their solvability in this sense.

In this article we relate the solvability of problems defined in terms of potential-isomorphism to stability theory. In particular we show (Theorem 3.5):

Assume that $0^{\#}$ exists and let T be a constructible first-order theory which is countable in L. Then the following are equivalent:

(i) The potential-isomorphism problem in cardinal- and real-preserving extension of L for constructible models of T of size $(\omega_2)^L$ is solvable. (More precisely: The collection $\{\langle \mathfrak{A}, \mathfrak{B}\rangle \in L \mid \mathfrak{A}$ and $\mathfrak{B}$ models T with universe $(\aleph_2)^L$ which are isomorphic in an extension of L with the same cardinals and reals as $L\}$ is constructible.)

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(ii) The theory T is classifiable (i.e., superstable with NDOP *and* NOTOP*).*

Potential isomorphism with respect to *set-generic* extension was studied in [6], [1], [5], and [3]. However the question raised in the present article require class-forcing, as the existence of an isomorphism between two constructible models in a *set-generic* extension of L is clearly an L-definable property.

2 Prerequisites

We assume throughout that $0^{\#}$ exists. The following two definitions and the theorem are from [2].

Definition 2.1 ([2]) *(i) By a cardinal preserving extension of L we mean a transitive model containing all the ordinals which satisfies* AC*, is contained in a set-generic extension of V and has the same cardinals as L. The notions of $\mathcal{P}(\lambda)$-preserving and real-preserving extensions of L are defined analogously.*

(ii) A subset X of L is Σ_1^{CP} if and only if X can be written in the form:

$$a \in X \text{ iff } \varphi(a) \textit{ holds in a cardinal-preserving extension of } L$$

for some Σ_1-formula φ with constructible parameters.

Definition 2.2 ([2]) *Suppose that $\langle X_0, X_1\rangle$ and $\langle Y_0, Y_1\rangle$ are pairs of disjoint subsets of L. Then we write*

$$\langle X_0, X_1\rangle \longrightarrow_L \langle Y_0, Y_1\rangle$$

if and only if there is a function $f \in L$ such that

$$\begin{aligned} a \in X_0 &\to f(a) \in Y_0, \\ a \in X_1 &\to f(a) \in Y_1. \end{aligned}$$

If X_1 is the complement of X_0 within some constructible set, we write X_0 instead of $\langle X_0, X_1\rangle$, and similarly for $\langle Y_0, Y_1\rangle$.

By S^κ_λ we denote the set of all ordinals $\alpha < \kappa$ such that $\mathrm{cf}(\alpha) = \lambda$.

Definition 2.3 *Let λ be an infinite cardinal in L and let $(\lambda^+ = \kappa)^L$.*

(i) Let $\mathcal{T}(\kappa)$ denote the set of trees $t \in L$ on κ of height κ.

(ii) Let $\mathcal{T}_{CP}(\kappa)$ denote the set of trees $t \in \mathcal{T}(\kappa)$ such that in a cardinal-preserving extension of L there is a κ-branch in t.

(iii) *Let* $\mathcal{T}^{\Delta}_{CP}(\kappa)$ *denote the collection of trees* $t \in \mathcal{T}_{CP}(\kappa)$ *such that* t *is* Δ_1*-definable over* L_κ *from the parameter* λ *and in a cardinal-preserving extension of* L*,* t *has a* κ*-branch,*

(iv) *Let* $\mathcal{T}^{\Delta}_{\mathcal{P}}(\kappa)$ *denote the collection of trees* $t \in \mathcal{T}_{CP}(\kappa)$ *such that* t *is* Δ_1*-definable over* L_κ *from the parameter* λ *and there is a cardinal- and* $\mathcal{P}(\lambda)$*-preserving extension of* L *in which* t *has a* κ*-branch,*

(v) *Let* $S(\kappa)$ *denote the collection of sets* $S \in L$ *such that* $S \subsetneq (S^{\kappa}_{\omega})^L$ *is stationary in* L *and in a cardinal-preserving extension of* L*,* $\kappa \backslash S$ *contains a cub.*

(vi) *Let* $\mathrm{Sr}(\kappa)$ *denote the collection of sets* $S \in L$ *such that* $S \subsetneq (S^{\kappa}_{\omega})^L$ *is stationary in* L *and in a cardinal- and real-preserving extension of* L*,* $\kappa \backslash S$ *contains a cub.*

Theorem 2.4 ([2]) (i) *If* X *is* Σ^{CP}_1*, then* X *is constructible from* $0^{\#}$*.*

(ii) *Suppose* λ *is an infinite cardinal in* L *and* $(\lambda^+ = \kappa)^L$*.*

(a) *The set* $\mathcal{T}_{CP}(\kappa)$ *is equiconstructible with* $0^{\#}$*.*

(b) $0^{\#} \longrightarrow_L \langle \mathcal{T}^{\Delta}_{\mathcal{P}}(\kappa), \mathcal{T}(\kappa) \backslash \mathcal{T}^{\Delta}_{CP}(\kappa) \rangle$*.*

(c) *If* λ *is regular in* L *and* $> \omega$*, then* $0^{\#} \longrightarrow_L S(\kappa)$*.*

(d) *If* λ *is regular in* L *and* $> \omega$*, then* $0^{\#} \longrightarrow_L \mathrm{Sr}(\kappa)$*.*

The rest of the section is from [4]. Let κ be a cardinal. By t^{κ}_{κ} we denote the tree whose universe ${}^{<\kappa}\kappa$ is ordered by end-extension. For the exact definitions of the tree operations supremum ($\oplus$), sum ($+$), and product ($\cdot$) we refer to [4, Section 2]. A (κ, α)-tree is a tree with the following properties: Every node of the tree has $< \kappa$ immediate successors and the tree does not contain branches of length α.

Definition 2.5 *A* ***lexically ordered*** (λ, α)***-tree*** t *is a tuple*

$$t = (U, <, <_1, <_2)$$

which satisfies the following conditions:

(i) *The pair* $\langle U, < \rangle$ *is a* (λ, α)*-tree.*

(ii) *For all* $x \in U$*,* $<_1 \restriction \mathrm{succ}(x)$ *is a linear order.*

(iii) *If* $x <_1 y$*, then* $x, y \in \mathrm{succ}(z)$ *for some* $z \in U$*.*

(iv) *For all* $x, y \in U$ *it holds that* $x <_2 y$ *if and only if* $x < y$ *or* $x_\gamma <_1 y_\gamma$ *where* γ *is the least ordinal with* $x \restriction \gamma \neq y \restriction \gamma$*, and* x_γ *(*y_γ*) is the node on level* γ *which is below* x *(*y*).*

Let M be a model of ZFC. Suppose that κ is a regular uncountable cardinal and t is a $(\kappa^+, \kappa+1)$-tree in M. In [4, proof of Theorem 3.4] two unary operations on trees are defined. We now shortly describe the operations without giving the exact definitions. The first operation, $c(t)$, gives a lexically ordered tree which is obtained by starting from the tree t and adding κ copies of t on every node of t. This adding of the tree t is done ω times and on round $n+1$ the tree t is added only on those nodes that were added to the resulting tree on round n, i.e., on the previous round. The nodes added to the tree on round n are called phase n nodes and the phase of a node $x \in c(t)$ is denoted by $p(x)$. We do not need the exact definition of the relation $<_1$ only that if N is a model of ZFC extending M and having the same cardinals as M, then in N there is no $x \in t$ such that the successors of x has an ascending $<_1$-sequence of length κ.

The other tree, $m(t)$, is obtained by multiplying every node on successor level in t by κ.

The lexically ordered trees and linear orders in the next definition are from the proofs of Theorem 3.4 and Corollary 3.6 in [4]. These are the basic ingredients used in building very equivalent but non-isomorphic models for unstable theories.

Definition 2.6 *Let κ be an uncountable regular cardinal, u a (κ^+, κ)-tree, and*

$$\begin{aligned}
\mathfrak{t}_\kappa &= c(t_\kappa^\kappa), \\
\mathfrak{t}_\kappa(u) &= c\big(m\big((\bigoplus_{\alpha<\kappa} \alpha) \cdot u + 1\big)\big), \\
\mathfrak{u}_\kappa(u) &= (\bigoplus_{n<\omega} n) \cdot u, \\
\eta_\kappa &= \mathbb{Q} \cdot (\mathrm{univ}(\mathfrak{t}_\kappa), <_2), \\
\eta_\kappa(u) &= \mathbb{Q} \cdot \big(\mathrm{univ}(\mathfrak{t}_\kappa(\mathfrak{u}_\kappa(u))), <_2\ \big).
\end{aligned}$$

The Ehrenfeucht-Mostowski model related to a model I, which is called the index model (often it is a linear order but not always), is denoted by $\mathrm{EM}(I, \Psi)$ where Ψ is the template. The set of sequences from $\mathrm{EM}(I, \Psi)$ indexed by I is called the skeleton. For the exact definitions see, e.g., [7] or [4, Section 8].

3 The results

For the sake of simplicity, we study only countable theories.

Lemma 3.1 *Let $M \subseteq N$ be models of* ZFC *with the same cardinals, λ an infinite cardinal in M, and $(\lambda^+ = \kappa)^M$. Suppose that $(\kappa^{<\kappa} = \kappa)^M$ and u is a (κ^+, κ)-tree in M.*

(i) In N there is a κ-branch in $\mathfrak{u}_\kappa(u)$ if and only if there is a κ-branch in u.

(ii) If in N there is a κ-branch in $\mathfrak{t}_\kappa(u)$, then there is also a κ-branch in u.

(iii) If in N there is an ascending $<_2$-sequence of length κ in $\mathfrak{t}_\kappa(u)$, then there is a κ-branch in $\mathfrak{t}_\kappa(u)$.

(iv) If in N there is a κ-branch in u, then $\mathfrak{t}_\kappa \cong \mathfrak{t}_\kappa(u)$.

(v) If in N there is an ascending κ-sequence in $\eta_\kappa(u)$, then there is an ascending $<_2$-sequence of length κ in $\mathfrak{t}_\kappa(u)$.

(vi) Let T be an unstable countable theory and the template Ψ as in [7]. If $(\mathcal{P}(\lambda))^M = (\mathcal{P}(\lambda))^N$ and in N it holds that

$$\mathrm{EM}(\eta_\kappa, \Psi) \cong \mathrm{EM}(\eta_\kappa(u), \Psi),$$

then in N there is an ascending κ-sequence in $\eta_\kappa(u)$.

Proof. (i) Follows from the definition of $\mathfrak{u}_\kappa(u)$.

(ii) Suppose $b \in N$ is a κ-branch in $\mathfrak{t}_\kappa(u)$. Let $t = (\bigoplus_{\alpha<\kappa} \alpha) \cdot u + 1$ and $t' = m(t)$. For $n \in \omega$, let $x_n \in b$ be the least node with $p(x_n) = n$ if there is such a node, and otherwise let x_n be the root of $\mathfrak{t}_\kappa(u)$. Now, if the branch b contains a node from every phase, the mapping $n \mapsto \mathrm{ht}(x_n)$ is unbounded in κ. Clearly this contradicts the assumption $(\lambda^+ = \kappa)^N$. Hence there is $r \in b$ and $n < \omega$ such that for each $x \in b$, if $x > r$ then $p(x) = n$. Let $b_x = \{y \in b \mid y > x\}$. Since t is a "projection" of t', b_x determines a κ-branch in t. Thus there is a κ-branch b' in $(\bigoplus_{\alpha<\kappa} \alpha) \cdot u$. Since κ is a regular cardinal in N, there is a κ-branch in u.

(iii) This follows from the fact that in N there is no node $x \in \mathfrak{t}_\kappa(u)$ such that the successors of x contains an ascending $<_1$-sequence of length κ and [4, Lemma 3.1(iii)].

(iv) By Theorem 3.4 in [4], there is, in M, a winning strategy w for player $\exists$ in $\mathrm{EF}^2_u(\mathfrak{t}_\kappa, \mathfrak{t}_\kappa(u))$. Without loss of generality we may assume that $\mathfrak{t}_\kappa$ and $\mathfrak{t}_\kappa(u)$ are disjoint. Since in M it holds that $\kappa^{<\kappa} = \kappa$, there is a bijection $f \in M$ from κ to $\mathfrak{t}_\kappa \cup \mathfrak{t}_\kappa(u)$. Work in N. Let $b = \langle b_i \mid i < \kappa \rangle$ be a κ-branch in u. Since every initial segment of

$$\langle \langle f(i), b_i \rangle \mid i < \kappa \rangle$$

is in M, the play

$$\langle\langle\langle f(i), b_i\rangle \mid i < \kappa\rangle, \langle w(\langle f(0), b_0\rangle, \ldots, \langle f(i), b_i\rangle) \mid i < \kappa\rangle\rangle$$

determines an isomorphism between $\mathfrak{t}_\kappa$ and $\mathfrak{t}_\kappa(u)$.

(v) Follows from the definition of $\eta_\kappa(u)$.

(vi) Let $\eta = \eta_\kappa$ and $\eta' = \eta_\kappa(u)$. First we note that $(\mathrm{EM}(\eta, \Psi))^M = (\mathrm{EM}(\eta, \Psi))^N$. If follows from $(\mathcal{P}(\lambda))^M = (\mathcal{P}(\lambda))^N$ and $(\lambda^+ = \kappa)^N$ that $(\mathcal{P}_\kappa(\kappa))^M = (\mathcal{P}_\kappa(\kappa))^N$. Since $\mathrm{EM}(\eta, \Psi) \cong \mathrm{EM}(\eta', \Psi)$ in N, some of the assumptions used in the proof of Theorem 4.9 in [4], which shows that the Ehrenfeucht-Mostowski models are non-isomorphic, must not hold in N. Since in N there are no new subsets of κ of cardinality $< \kappa$, the only assumption that can fail is the following assumed in Lemma 5.3 in [4]: η' does not contain ascending κ-sequences. Hence in N there is an ascending κ-sequence in $\eta' = \eta_\kappa(u)$. (3.1)■

Theorem 3.2 *Suppose λ is an infinite cardinal in L and $(\lambda^+ = \kappa)^L$. Then the following Σ_1^{CP} sets are equiconstructible with $0^\#$.*

- *(i) The collection of trees $t' \in L$ on κ such that there is a cardinal-preserving extension of L in which $\mathfrak{t}_\kappa \cong t'$. This collection of trees is denoted by $\mathcal{C}_{\mathfrak{t}_\kappa}$.*
- *(ii) The collection of pairs of (κ^+, κ)-trees on κ $\langle t, t'\rangle \in L$ such that in a cardinal-preserving extension of L it holds that $t \cong t'$. This collection is denoted by $\mathcal{C}_{\mathrm{isom}}$.*
- *(iii) The collection of pairs of models $\langle \mathfrak{A}, \mathfrak{B}\rangle \in L$ such that $\mathrm{univ}(\mathfrak{A}) = \mathrm{univ}(\mathfrak{B}) = \kappa$, the similarity type of $\mathfrak{A}$ and $\mathfrak{B}$ is a subset of τ, and in a cardinal-preserving extension of L there is an elementary embedding from $\mathfrak{A}$ to $\mathfrak{B}$ or from $\mathfrak{B}$ to $\mathfrak{A}$ where τ is a similarity type, which contains at least two constant symbols and one binary relation symbol. This collection of pairs of models is denoted by $\mathcal{C}_{\mathrm{elem}}$.*
- *(iv) If in (iii), elementary embedding is replaced by embedding, the claim still holds.*

Proof. By Theorem 2.4(i) it suffices to show that there is a reduction of $0^\#$ to each of the sets.

(i) Work in L. Let the function f with domain ω be defined by

$$f(n) = t_n$$

where t_n is as in [2, proof of Theorem 3(a)] for $n \in \omega$. So, we have $n \in 0^\# \leftrightarrow t_n \in \mathcal{T}_{CP}(\kappa)$. Define a mapping g by

$$t_n \mapsto \mathfrak{t}_\kappa(t_n).$$

The mapping $g \circ f$ demonstrates that $0^{\#} \longrightarrow_L \mathcal{C}_{\mathrm{t}_\kappa}$. To see this, suppose first that $t_n \in \mathcal{T}_{CP}(\kappa)$. Let N be a cardinal-preserving extension of L in which t_n has a κ-branch. By 3.1(iv), $\mathrm{t}_\kappa \cong \mathrm{t}_\kappa(t_n) = g(t_n)$. Suppose then that $t_n \in \mathcal{T}(\kappa) \backslash \mathcal{T}_{CP}(\kappa)$. For a contradiction assume that in a cardinal-preserving extension N of L it holds that $\mathrm{t}_\kappa \cong \mathrm{t}_\kappa(t_n)$. Then, in N, there is a κ-branch in $\mathrm{t}_\kappa(t_n)$. So, by 3.1(ii), there is a κ-branch in t_n contradicting $t_n \notin \mathcal{T}_{CP}(\kappa)$. Hence $g(t_n) \notin \mathcal{C}_{\mathrm{t}_\kappa}$.

(ii) Work in L. Let the function f be as in the previous case. Define a mapping g by

$$t_n \mapsto \langle \mathrm{t}_\kappa, \mathrm{t}_\kappa(t_n) \rangle$$

The fact that $g \circ f$ reduces $0^{\#}$ to $\mathcal{C}_{\mathrm{isom}}$ can be seen as in the previous case.

(iii) Let t and t' be $(\kappa^+, \kappa+1)$-trees. Without loss of generality, we may assume that t and t' are disjoint. Choose distinct objects x and y not in $t \cup t'$. Let $\mathcal{M}(t,t')$ be the structure $(U, <, c, d)$ satisfying the following conditions:

(1.1) $U = t \cup t' \cup \{x, y\}$.

(1.2) $(c)^{\mathcal{M}(t,t')} = x$ and $(d)^{\mathcal{M}(t,t')} = y$.

(1.3) For all $a, b \in U$, $a < b$ if and only if

$$\begin{array}{rcl} a = x \wedge b \in t & \vee & a = y \wedge b \in t' \quad \vee \\ a, b \in t \wedge a <_t b & \vee & a, b \in t' \wedge a <_{t'} b. \end{array}$$

Work in L. Let the function f be as in previous cases. Define a mapping g by

$$t_n \mapsto \langle \mathcal{M}(\mathrm{t}_\kappa, \mathrm{t}_\kappa(t_n)), \mathcal{M}(\mathrm{t}_\kappa(t_n), \mathrm{t}_\kappa) \rangle.$$

We show that $g \circ f$ reduces $0^{\#}$ to $\mathcal{C}_{\mathrm{elem}}$. First assume that $t_n \in \mathcal{T}_{CP}(\kappa)$. Let N be a cardinal-preserving extension of L which witnesses this. By 3.1(iv), $\mathrm{t}_\kappa \cong \mathrm{t}_\kappa(t_n)$ in N. It follows that

$$\mathcal{M}(\mathrm{t}_\kappa, \mathrm{t}_\kappa(t_n)) \cong \mathcal{M}(\mathrm{t}_\kappa(t_n), \mathrm{t}_\kappa).$$

Hence $g(t_n) \in \mathcal{C}_{\mathrm{elem}}$.

Suppose then that $t_n \in \mathcal{T}(\kappa) \backslash \mathcal{T}_{CP}(\kappa)$. For a contradiction assume that N is a cardinal-preserving extension of L such that

$$\mathcal{M}(\mathrm{t}_\kappa, \mathrm{t}_\kappa(t_n)) \preceq \mathcal{M}(\mathrm{t}_\kappa(t_n), \mathrm{t}_\kappa).$$

Hence there is a κ-branch in $\mathrm{t}_\kappa(t_n)$. By (ii) of Lemma 3.1, there is a κ-branch in t_n whence $t_n \notin \mathcal{T}(\kappa) \backslash \mathcal{T}_{CP}(\kappa)$. Hence $g(t_n) \notin \mathcal{C}_{\mathrm{elem}}$.

(iv) As above. (3.2)■

Theorem 3.3 *Suppose λ is an infinite L-cardinal, $(\lambda^+ = \kappa)^L$, $T \in L$ is a complete unstable theory with $(|T| = \omega)^L$, the template Ψ is as in [7, Lemma 1.2], and $\mathfrak{A} = \mathrm{EM}(\eta_\kappa, \Psi)$.*

(i) Let $\mathcal{C}^{\mathfrak{A}}_\kappa$ be the collection of models $\mathfrak{B} \in L$ of T with universe κ such that there is a cardinal- and $\mathcal{P}(\lambda)$-preserving extension of L in which $\mathfrak{A} \cong \mathfrak{B}$. Then $0^\# \longrightarrow_L \mathcal{C}^{\mathfrak{A}}_\kappa$.

(ii) Let $\mathcal{C}^{\mathfrak{A},\mathrm{ee}}_\kappa$ be the collection of models $\mathfrak{B} \in L$ of T with universe κ such that in a cardinal- and $\mathcal{P}(\lambda)$-preserving extension of L there is an elementary embedding from $\mathfrak{A}$ to $\mathfrak{B}$. Then $0^\# \longrightarrow_L \mathcal{C}^{\mathfrak{A},\mathrm{ee}}_\kappa$.

Proof. (i) Let $f : \omega \to \mathcal{T}(\kappa)$ be the function given by (ii-b) in Theorem 2.4, i.e., f reduces $0^\#$ to $(\mathcal{T}^\Delta_\mathcal{P}(\kappa), \mathcal{T}(\kappa)\backslash\mathcal{T}^\Delta_{CP}(\kappa))$. Let $t^*_n = f(n)$. Define a mapping g by

$$t^*_n \mapsto \mathrm{EM}(\eta_\kappa(t^*_n), \Psi).$$

Then $g \circ f$ reduces $0^\#$ to $\mathcal{C}^{\mathfrak{A}}_\kappa$. To see this assume first $t^*_n \in \mathcal{T}^\Delta_\mathcal{P}(\kappa)$. Let N be a cardinal- and $\mathcal{P}(\lambda)$-preserving extension of L in which t^*_n has a κ-branch. By 3.1(iv), $g(t_n) \in \mathcal{C}^{\mathfrak{A}}_\kappa$.

Suppose then that $t^*_n \in \mathcal{T}(\kappa)\backslash\mathcal{T}^\Delta_{CP}(\kappa)$. Towards a contradiction assume that in a cardinal- and $\mathcal{P}(\lambda)$-preserving extension N of L it holds that $\mathfrak{A} \cong g(t_n)$. Then, by (vi), (v), (iii), and (ii) of Lemma 3.1, t^*_n has a κ-branch. But then $t^*_n \notin \mathcal{T}(\kappa)\backslash\mathcal{T}^\Delta_{CP}(\kappa)$ and hence we have proved that $0^\# \longrightarrow_L \mathcal{C}^{\mathfrak{A}}_\kappa$.

(ii) The proof is based on the fact that by the proof of Theorem 4.9 in [4] there is no elementary embedding from $\mathrm{EM}(\eta_\kappa, \Psi) = \mathfrak{A}$ to $\mathrm{EM}(\eta_\kappa(t^*_n), \Psi)$. Otherwise the proof goes exactly as in the previous case. ■

As in the proof of Theorem 3.3, one can combine known model constructions with results from [2]. Below we give two more results but only sketch the proofs.

Theorem 3.4 *Suppose λ is a regular L-cardinal $> \omega$, $(\lambda^+ = \kappa)^L$, $T \in L$ is a complete unsuperstable theory with $(|T| = \omega)^L$. Let D^T_κ be the collection of pairs $(\mathfrak{A}, \mathfrak{B}) \in L$ of models of T with universe κ such that there is a cardinal-preserving extension of L in which $\mathfrak{A} \cong \mathfrak{B}$. Then D^T_κ is equiconstructible with $0^\#$.*

Proof. By (i) and (ii-c) in Theorem 2.4, it is enough to show the following: For all stationary $S \subseteq S^\kappa_\omega$ there are models $\mathfrak{A}, \mathfrak{B} \in L$ of T of power κ such that in any cardinal-preserving extension of L,

$$\mathfrak{A} \cong \mathfrak{B} \text{ iff } \kappa\backslash S \text{ contains a cub set.}$$

Let the trees $J_0 \in K^\omega_{\mathrm{tr}}$ and $J_1 \in K^\omega_{\mathrm{tr}}$ be as in [3, Lemma 7.29] and $\mathfrak{A} = \mathrm{EM}(J_0, \Psi)$ and $\mathfrak{B} = \mathrm{EM}(J_1, \Psi)$, see [3, Section 7] and notice that this model construction is originally due to S. Shelah.

If in a cardinal-preserving extension of L, $\kappa \setminus S$ contains a cub set, then in the extension $\mathfrak{A} \cong \mathfrak{B}$ holds by the proofs of [3, Lemmas 7.31 and 7.15] (notice that when the isomorphism is constructed along the cub, the partial isomorphisms may not be in L but this is not a problem).

On the other hand, if $\kappa \setminus S$ does not contain a cub set in a cardinal preserving extension of L, then the S-invariants of J_0 and J_1 are different in the extension, see [8] or [4, Lemmas 8.14 and 8.20]. Since the trees J_0 and J_1 are of power κ, it is easy to see that the properties $(<\kappa, \mathrm{bs})$-stable and locally $(\kappa, \mathrm{bs}, \mathrm{bs})$-nice are preserved in cardinal-preserving extensions of L (as well as the truth of first-order formulas and "being an Ehrenfeucht-Fraïssé model"). So $\mathfrak{A} \not\cong \mathfrak{B}$ by [8] (or [4, Theorem 8.13]). (3.4)■

Theorem 3.5 *Suppose λ is a regular L-cardinal $> \omega$, $(\lambda^+ = \kappa)^L$, $T \in L$ is a complete theory with $(|T| = \omega)^L$. Let P^T_κ be the collection of pairs $(\mathfrak{A}, \mathfrak{B}) \in L$ of models of T with universe κ such that there is a cardinal- and real-preserving extension of L in which $\mathfrak{A} \cong \mathfrak{B}$. Then the following are equivalent:*

(i) P^T_κ is constructible,

(ii) T is classifiable (i.e. superstable with NDOP *and* NOTOP*).*

Proof. If T is classifiable, then any two models of T are isomorphic in a cardinal- and real-preserving extension of L if and only if they are isomorphic in L, see [1]. Hence P^T_κ is constructible.

For the other direction, assume that T is not classifiable. By (i) and (ii-d) in Theorem 2.4, it is enough to show the following: For all stationary $S \subseteq S^\kappa_\omega$ there are models $\mathfrak{A}, \mathfrak{B}$ of T of power κ such that in any cardinal- and real-preserving extension of L it holds that

$$\mathfrak{A} \cong \mathfrak{B} \text{ iff } \kappa \setminus S \text{ contains a cub set.}$$

If T is unsuperstable, this can be seen exactly as in the proof of Theorem 3.4. If T is superstable with OTOP or DOP, then we use the following observation: Firstly, we have Ehrenfeucht-Fraïssé models over linearly ordered skeletons with the linear order definable by a formula which is absolute for cardinal- and real-preserving extensions of L. (In fact, in the OTOP case the formula is absolute for all extensions and in the DOP case, for model theoretic reasons, the formula is absolute in the models of T for cardinal-preserving extensions.) Secondly, the trees in K^ω_{tr} can be coded into linear orders so that the bs-type of a sequence of elements of the tree determines the bs-type of the corresponding sequence in the linear order and the tree order can be defined by a quantifier free formula in the linear order. Hence the proof can be completed analogously to the proof of Theorem 3.4. (3.5)■

Notice that by the proof of Theorem 3.5, Theorem 3.4 holds also for countable superstable theories with OTOP or DOP.

The model construction in the proof of the following theorem is a modification of a construction in [9].

Theorem 3.6 *Suppose λ is a regular L-cardinal $\geq \omega$ and $(\lambda^+ = \kappa)^L$. Let A_κ be the set of all models $\mathfrak{A} \in L$ with universe κ such that there is a cardinal-preserving extension of L in which $\mathfrak{A}$ has a non-trivial automorphism. Then A_κ is equiconstructible with $0^\#$.*

Proof. For every tree t on κ of height and cardinality κ, a model $\mathfrak{A}_t$ is defined as follows. First some some preliminary definitions are given.

For every $\alpha < \kappa$, let t_α denote the set of all elements of t of height α, and let G_α be the set of all finite subsets of t_α. Make G_α to an Abelian group by letting $a + b = a \triangle b$ (the symmetric difference). For all $\alpha < \beta < \kappa$ and $\eta \in t_\beta$, let $\pi_{\beta\alpha}(\eta)$ denote the unique element $\xi \in t_\alpha$ with $\xi < \eta$.

For all $\alpha < \beta < \kappa$, define function $F_{\beta\alpha} : G_\beta \to G_\alpha$ so that $F_{\beta\alpha}(a) = \Sigma_{\eta \in a}\{\pi_{\beta\alpha}(\eta)\}$. Note that $F_{\beta\alpha}$ is a homomorphism and that for $\gamma < \beta < \alpha < \kappa$, $F_{\alpha\gamma} = F_{\beta\gamma} \circ F_{\alpha\beta}$.

For all $\alpha < \kappa$, let $\overline{G}_\alpha$ denote the set of all functions g on α such that for all $\beta < \alpha$, $g(\beta) \in G_\beta$ and for all $\gamma < \beta < \alpha$, $g(\gamma) = F_{\beta\gamma}(g(\beta))$, and we make $\overline{G}_\alpha$ to an Abelian group by defining $+$ coordinate-wise.

Now we are ready to define $\mathfrak{A}_t$. Let the universe of $\mathfrak{A}_t$ be $\bigcup_{\alpha<\kappa} \overline{G}_\alpha$, and equip $\mathfrak{A}_t$ with the following relations and functions: For every $g \in \bigcup_{\alpha<\kappa} G_\alpha$ define the relation S_g by

$$S_g(g_0, g_1) \text{ iff } \exists \alpha < \kappa (g, g_0, g_1 \in \overline{G}_\alpha \wedge g_0 = g_1 + g)$$

and for every $\alpha < \kappa$ define the function F_α by

$$g' \mapsto g' \restriction \alpha.$$

(So if $g' \in \overline{G}_\beta$ and $\beta < \alpha$, $F_\alpha(g') = g'$.) There are infinitely many relations and functions but, if wanted, the relations and functions can be coded so that the similarity type becomes finite.

By (i) and (ii-a) of Theorem 2.4 it suffices to show that for all trees t on κ, if t has height κ, then

$$\mathfrak{A}_t \in A_k \text{ iff } t \in \mathcal{T}_{CP}(\kappa).$$

Assume first that $t \in \mathcal{T}_{CP}(\kappa)$. Fix a cardinal-preserving extension of L in which there exists a κ-branch b in t. Let $g : \kappa \to \bigcup_{\alpha<\kappa} G_\alpha$ be such that for all $\alpha < \kappa$, $g(\alpha) = b \cap t_\alpha$. Define function $f_g : \mathfrak{A}_t \to \mathfrak{A}_t$ so that if $g' \in \overline{G}_\alpha$, then $f_g(g') = g' + g\restriction\alpha$. It is easy to see that f_g is a non-trivial automorphism of $\mathfrak{A}_t$.

Assume then that $\mathfrak{A}_t \in A_\kappa$. Fix a cardinal-preserving extension of L in which there is a non-trivial automorphism f of $\mathfrak{A}_t$. Let the function $s : \kappa \to \bigcup_{\alpha<\kappa} G_\alpha$ be such that for all $\alpha < \kappa$, $s(\alpha) = \emptyset$, and let the function $g : \kappa \to \bigcup_{\alpha<\kappa} G_\alpha$ be such that for all $\alpha < \kappa$, $g\restriction\alpha = f(s\restriction\alpha)$. If $g = s$, then it is easy to see that f is the trivial automorphism. So $g \neq s$ and since κ is still a successor cardinal in the extension, there are non-zero $n < \omega$ and $\alpha < \kappa$ such that for all $\alpha < \beta < \kappa$, $|g(\beta)| = n$. With this it is easy to find a κ-branch from t (in fact, n many of them). (3.6)■

Question 3.7 *(i) Does Theorem 3.3 need preservation of subsets of λ? Does Theorem 3.5 need real-preservation?*

(ii) Are there other dividing lines within first-order theories which can be characterized along the lines of Theorem 3.5?

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Institute for Logic
University of Vienna
Vienna
AUSTRIA

Department of Mathematics
P.O.Box 4
00014 University of Helsinki
FINLAND

Department of Mathematics
P.O.Box 4
00014 University of Helsinki
FINLAND

Stonesoft Corp.
Itälahdenkatu 22 A
00210 Helsinki
FINLAND