CO-STATIONARITY OF THE GROUND MODEL

NATASHA DOBRINEN AND SY-DAVID FRIEDMAN

Abstract. This paper investigates when it is possible for a partial ordering $\mathbb{P}$ to force $\mathscr{P}_\kappa(\lambda) \setminus V$ to be stationary in $V^{\mathbb{P}}$. It follows from a result of Gitik that whenever $\mathbb{P}$ adds a new real, then $\mathscr{P}_\kappa(\lambda) \setminus V$ is stationary in $V^{\mathbb{P}}$ for each regular uncountable cardinal κ in $V^{\mathbb{P}}$ and all cardinals $\lambda > \kappa$ in $V^{\mathbb{P}}$ [4]. However, a covering theorem of Magidor implies that when no new ω-sequences are added, large cardinals become necessary [7]. The following is equiconsistent with a proper class of ω_1-Erdős cardinals: If $\mathbb{P}$ is $\aleph_1$-Cohen forcing, then $\mathscr{P}_\kappa(\lambda) \setminus V$ is stationary in $V^{\mathbb{P}}$, for all regular $\kappa \geq \aleph_2$ and all $\lambda > \kappa$. The following is equiconsistent with an ω_1-Erdős cardinal: If $\mathbb{P}$ is $\aleph_1$-Cohen forcing, then $\mathscr{P}_{\aleph_2}(\aleph_3) \setminus V$ is stationary in $V^{\mathbb{P}}$. The following is equiconsistent with κ measurable cardinals: If $\mathbb{P}$ is κ-Cohen forcing, then $\mathscr{P}_{\kappa^+}(\aleph_\kappa) \setminus V$ is stationary in $V^{\mathbb{P}}$.

1. Introduction

Suppose $V \subseteq W$ are models of ZFC with the same ordinals, κ is regular and uncountable in W, and λ is a cardinal $> \kappa$ in W. We say that the *ground model V is stationary* or that *$(\mathscr{P}_\kappa(\lambda))^V$ is stationary in W* if $(\mathscr{P}_\kappa(\lambda))^V$ is a stationary subset of $(\mathscr{P}_\kappa(\lambda))^W$. We say that *the ground model is co-stationary* or that *$(\mathscr{P}_\kappa(\lambda))^V$ is co-stationary in W* if $(\mathscr{P}_\kappa(\lambda))^W \setminus (\mathscr{P}_\kappa(\lambda))^V$ is stationary in $(\mathscr{P}_\kappa(\lambda))^W$. Note that $(\mathscr{P}_\kappa(\lambda))^V = (\mathscr{P}_\kappa(\lambda))^W \cap V$; hence, $(\mathscr{P}_\kappa(\lambda))^W \setminus (\mathscr{P}_\kappa(\lambda))^V = (\mathscr{P}_\kappa(\lambda))^W \setminus V$.

The problem of preserving the stationarity of the ground model has been extensively studied. It is well-known that any κ-c.c. forcing preserves all stationary subsets of $\mathscr{P}_\kappa(\lambda)$ for all cardinals $\lambda > \kappa$, hence preserves the stationarity of $(\mathscr{P}_\kappa(\lambda))^V$ in $(\mathscr{P}_\kappa(\lambda))^{V^{\mathbb{P}}}$. Shelah has proved the following general theorem.

Theorem 1.1 (Shelah's Strong Covering Lemma [12]). *Suppose $V \subseteq W$ are models of ZFC, κ is uncountable regular in W, $(\kappa^+)^V = (\kappa^+)^W$, and Jensen covering holds between V and W. Then for all $\lambda \geq \kappa^+$, $(\mathscr{P}_\kappa(\lambda))^V$ is stationary in $(\mathscr{P}_\kappa(\lambda))^W$.*

The problem of making the ground model *co-stationary* in the $\mathscr{P}_\kappa(\lambda)$ of the larger model has received considerably less attention. The purpose of this paper is to investigate the following general problem.

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Main Problem. Given a partial ordering $\mathbb{P}$ and uncountable cardinals $\kappa < \lambda$ in $V^{\mathbb{P}}$ with κ regular in $V^{\mathbb{P}}$, when is $(\mathscr{P}_\kappa(\lambda))^V$ co-stationary in $V^{\mathbb{P}}$?

The following theorem of Gitik shows that any $\mathbb{P}$ which adds a new real makes $(\mathscr{P}_\kappa(\lambda))^V$ co-stationary in $V^{\mathbb{P}}$, for all cardinals $\aleph_1 \leq \kappa < \lambda$ in $V^{\mathbb{P}}$ with κ regular in $V^{\mathbb{P}}$.

Theorem 1.2 (Gitik [4]). *Let $V \subseteq W$ be two models of ZFC with the same ordinals, κ a regular uncountable cardinal in W, and $\lambda \geq (\kappa^+)^W$. Suppose that there is a real in $W \setminus V$. Then $(\mathscr{P}_\kappa(\lambda))^V$ is co-stationary in W.*

Naturally, one wonders what happens if no new reals are added, but a new ω-sequence is added.

Question 1.3. Suppose $\mathbb{P}$ adds no new reals but does add a new ω-sequence. Let λ_0 be least such that $\mathbb{P}$ adds a new function $r : \omega \to \lambda_0$. Does it follow that $(\mathscr{P}_\kappa(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$, for all cardinals $\aleph_1 < \kappa < \lambda$ in $V^{\mathbb{P}}$ with κ regular in $V^{\mathbb{P}}$ and $\lambda \geq \lambda_0$?

We do not currently know the answer to this, although we do have some partial results (see Fact 2.3, Theorems 3.2, 5.6, and 5.7, and Example 5.8).

Question 1.4. Suppose $\mathbb{P}$ adds no new ω-sequences but does add a new subset of $\aleph_1$. Does it follow that for all regular $\kappa > \aleph_1$ in $V^{\mathbb{P}}$ and all $\lambda \geq \kappa^+$ in $V^{\mathbb{P}}$ that $(\mathscr{P}_\kappa(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$?

We have obtained the equiconsistency of a positive answer to Question 1.4:

Theorem 3.8 (Consistency of Global Gitik). *The following are equiconsistent:*

1. *There is a proper class of ω_1-Erdős cardinals.*
2. *If $\mathbb{P}$ is $\aleph_1$-Cohen forcing, then $(\mathscr{P}_\kappa(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$ for all regular $\kappa \geq \aleph_2$ and all $\lambda > \kappa$.*
3. *If $\mathbb{P}$ adds a new subset of $\aleph_1$ and is $\aleph_2$-c.c. (or just satisfies the $(\kappa^+, \kappa^+, < \kappa)$-distributive law for all successor cardinals $\kappa \geq \aleph_2$ and is κ-c.c. for all strongly inaccessible κ), then $(\mathscr{P}_\kappa(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$ for all regular $\kappa \geq \aleph_2$ and all $\lambda > \kappa$.*

Why are ω_1-Erdős cardinals required for Global Gitik? Consider the following special case of Question 1.4.

Question 1.5. Suppose $\mathbb{P}$ adds no new ω-sequences but does add a new subset of $\aleph_1$. Does it follow that $(\mathscr{P}_{\aleph_2}(\aleph_3))^V$ is co-stationary in $V^{\mathbb{P}}$?

Magidor's Covering Theorem shows that at least one ω_1-Erdős cardinal is necessary.

Theorem 1.6 (Magidor [7]). *Assume there is no ω_1-Erdős cardinal in K_{DJ}, where K_{DJ} is the Dodd-Jensen core model. Then for every ordinal β one can define in K_{DJ} a countable collection of functions $\mathscr{C}$ on β such that every subset of β closed under $\mathscr{C}$ is a countable union of sets in K_{DJ}.*

It follows from Theorem 1.6 that if K_{DJ} has no ω_1-Erdős cardinal and $\mathbb{P}$ is a $(\omega, \lambda^{<\kappa})$-distributive partial ordering in V (for example, if $\mathbb{P}$ adds no new ω-sequences), where $\lambda > \kappa \geq \aleph_2$ in $V^{\mathbb{P}}$ and κ is regular in $V^{\mathbb{P}}$, then there is a club $C \subseteq \mathscr{P}_\kappa(\lambda)$ in $V^{\mathbb{P}}$ (namely, the one generated by the functions from the theorem) such that $C \subseteq V$. Hence, $(\mathscr{P}_\kappa(\lambda))^V$ is not co-stationary in $V^{\mathbb{P}}$. It follows that if K_{DJ} has no ω_1-Erdős cardinal and $\mathbb{P}$ adds no new ω-sequences, then for all $\lambda > \kappa \geq \aleph_1$ with κ regular in $V^{\mathbb{P}}$, $(\mathscr{P}_\kappa(\lambda))^V$ is not co-stationary in $V^{\mathbb{P}}$. We will show that an ω_1-Erdős cardinal is the exact consistency strength of a positive answer to Question 1.5 for $(\aleph_3, \aleph_3, \aleph_1)$-distributive partial orderings. (These include all $\aleph_2$-c.c. partial orderings.)

Theorem 3.6. *The following are equiconsistent:*

(1) *There is an ω_1-Erdős cardinal.*
(2) *If $\mathbb{P}$ is $\aleph_1$-Cohen forcing, then $(\mathscr{P}_{\aleph_2}(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$ for all $\lambda \geq \aleph_3$.*
(3) *If $\mathbb{P}$ adds a new subset of $\aleph_1$ and is $(\aleph_3, \aleph_3, \aleph_1)$-distributive, then $(\mathscr{P}_{\aleph_2}(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$ for all $\lambda \geq \aleph_3$.*

Now consider the following generalisation of Question 1.5.

Quest(κ, λ). Suppose κ is regular, $\mathbb{P}$ adds a new subset of κ but adds no new $< \kappa$-sequences, and $\lambda > \kappa^+$ in $V^{\mathbb{P}}$. Does it follow that $(\mathscr{P}_{\kappa^+}(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$?

Using indiscernibles, we have shown the following.

Theorem 3.2. *Suppose that in V, $\lambda > \kappa$, κ is regular, and λ is κ-Erdős. Let $\mathbb{P}$ be κ-Cohen forcing (or any partial ordering which adds a new subset of κ and is $(\lambda, \lambda, \kappa)$-distributive). Then $(\mathscr{P}_{\kappa^+}(\mu))^V$ is co-stationary in $V^{\mathbb{P}}$ for all $\mu \geq \lambda$.*

The next theorem shows the necessity of a κ-Erdős cardinal.

Theorem 1.7 (Magidor [7]). *If there is no κ-Erdős cardinal in K_{DJ}, then for every ordinal β, there exists a countable collection $\mathscr{C}$ of functions on β such that every subset of β closed under $\mathscr{C}$ is the union of $< \kappa$ sets in K_{DJ}.*

Hence, if K_{DJ} has no κ-Erdős cardinal, and $\mathbb{P}$ is a partial ordering in V which is $(\rho, \lambda^{<\mu})$-distributive for all $\rho < \kappa$ (which is weaker than saying that no new $< \kappa$-length sequences are added), where $\lambda > \mu \geq \kappa^+$ and μ is regular in $V^{\mathbb{P}}$, then $(\mathscr{P}_\mu(\lambda))^V$ is not co-stationary in $V^{\mathbb{P}}$. Thus, we have the following equiconsistency.

Theorem 3.3. *The following are equiconsistent:*

(1) *κ is regular and there is a κ-Erdős cardinal.*
(2) *κ is regular and there is a $\lambda > \kappa^+$ such that if $\mathbb{P}$ is κ-Cohen forcing, then $(\mathscr{P}_{\kappa^+}(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$.*
(3) *κ is regular and there is a $\lambda > \kappa^+$ such that if $\mathbb{P}$ is any partial ordering which adds a new subset of κ and is $(\lambda, \lambda, \kappa)$-distributive, then $(\mathscr{P}_{\kappa^+}(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$.*

When $\lambda < \aleph_{\omega_2}$, the following theorem implies that at least a measurable cardinal is required in order to make the ground model co-stationary in $\mathscr{P}_{\aleph_3}(\lambda)$.

Theorem 1.8 (Magidor [7]). *Assume there is no inner model with a measurable cardinal. Let $\beta < \aleph_{\omega_2}$. Then in V one can define a countable set of functions such that every subset of β closed under these functions is a union of $\aleph_1$ sets in K_{DJ}.*

It follows that if there is no inner model with a measurable cardinal, then the answer to **Quest**$(\aleph_2, \lambda)$ is negative for all $\aleph_3 < \lambda < \aleph_{\omega_2}$. The next theorem is a strengthening of Theorem 1.8 which implies that for any $\kappa \geq \aleph_2$, if there is no inner model with κ measurable cardinals, then the answer to **Quest**(κ, λ) is negative for all $\kappa^+ < \lambda \leq \aleph_\kappa$ (in fact, strongly negative in that every $(< \kappa, \lambda^\kappa)$-distributive partial ordering forces that the ground model is not co-stationary).

Theorem 4.3. *Let $\kappa \geq \aleph_2$ be regular and assume that there is no inner model with κ measurable cardinals. Then there is a countable collection $\mathscr{C}$ of functions on $\aleph_\kappa$ such that every subset of $\aleph_\kappa$ closed under $\mathscr{C}$ is the union of $< \kappa$ sets in K_M, Mitchell's core model for sequences of measures.*

Whenever the Free Subset Property $\mathrm{Fr}_\kappa(\lambda, \kappa)$ holds (see Definition 4.4), then **Quest**(κ, λ) has a positive answer for all partial orderings which are $(\lambda, \lambda, \kappa)$-distributive.

Proposition 4.5. *Suppose in V that $Fr_\kappa(\lambda, \kappa)$ holds. Suppose $\mathbb{P}$ adds a new subset of κ and is $(\lambda, \lambda, \kappa)$-distributive. Then $(\mathscr{P}_{\kappa^+}(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$.*

Shelah has shown that from κ measurable cardinals, one can obtain a model of ZFC in which $\mathrm{Fr}_\kappa(\aleph_\kappa, \kappa)$ holds [11]. Using this, we obtain the following equiconsistency.

Theorem 4.7. *The following are equiconsistent.*

(1) *$\aleph_\kappa > \kappa$ and there are κ measurable cardinals.*
(2) *$\aleph_\kappa > \kappa$, and if $\mathbb{P}$ is the κ-Cohen forcing, then $(\mathscr{P}_{\kappa^+}(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$ for all $\lambda \geq \aleph_\kappa$.*
(3) *$\aleph_\kappa > \kappa$, and if $\mathbb{P}$ adds a new subset of κ and is $(\aleph_\kappa, \aleph_\kappa, \kappa)$-distributive, then $(\mathscr{P}_{\kappa^+}(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$ for all $\lambda \geq \aleph_\kappa$.*

2. Basic Definitions and Facts

Throughout this paper, standard set-theoretic notation is used. α, β, γ are used to denote ordinals, while $\kappa, \lambda, \mu, \nu, \rho$ are used to denote cardinals. $\mathscr{P}_\kappa(X) = [X]^{<\kappa} = \{x \subseteq X : |x| < \kappa\}$. Usually we use $[X]^{<\omega}$ instead of $\mathscr{P}_\omega(X)$ to denote the collection of finite subsets of X. $(X)^{<\kappa}$ denotes the collection of all functions from an ordinal less than κ into X; i.e. the collection of all sequences of length less than κ of elements of X. We will hold to the convention that if $V \subseteq W$ are

models of ZFC with the same ordinals and $\kappa < \lambda$ are cardinals in W, then $\mathscr{P}_\kappa(\lambda)$ denotes $(\mathscr{P}_\kappa(\lambda))^W$.

Certain distributive laws imply preservation of the stationarity of the $\mathscr{P}_\kappa(\lambda)$ of the ground model. In addition, they will aid us in obtaining extension models in which the ground model is co-stationary. We present the forcing-equivalent definitions of distributivity, refering the reader to [6] for the Boolean algebraic versions.

Definition 2.1. Let κ, λ, μ be cardinals with $\mu \leq \lambda$. A partial ordering $\mathbb{P}$ is (κ, λ)-*distributive* if forcing with $\mathbb{P}$ adds no new functions from $\check{\kappa}$ into $\check{\lambda}$. (This implies all cardinals $\leq \kappa^+$ are preserved.) $\mathbb{P}$ is $(\kappa, \lambda, < \mu)$-*distributive* if for any function $\dot{f} : \check{\kappa} \to \check{\lambda}$, there is a function $g : \kappa \to [\lambda]^{<\mu}$ in V such that for each $\alpha < \kappa$, $\dot{f}(\alpha) \in g(\alpha)$ in $V^{\mathbb{P}}$. We will say that $\mathbb{P}$ is (κ, λ, μ)-*distributive* if it is $(\kappa, \lambda, < \mu^+)$-distributive.

Fact 2.2. (1) *If $\mathbb{P}$ is κ^+-c.c., then $\mathbb{P}$ is (ρ, λ, κ)-distributive for all ρ and for all $\lambda > \kappa$.*

(2) *The $(\kappa, \lambda, \kappa)$-d.l. holds iff every subset of λ of size κ in $V^{\mathbb{P}}$ can be covered by a subset of λ of size κ in V.*

(3) *If $\lambda > \kappa$ and $\mathbb{P}$ is $(\lambda, \lambda, \kappa)$-distributive, then $\mathbb{P}$ perserves all cardinals ρ with $\kappa^+ \leq \rho \leq \lambda$. Moreover, every stationary subset of $(\mathscr{P}_{\kappa^+}(\lambda))^V$ in V is a stationary subset of $(\mathscr{P}_{\kappa^+}(\lambda))^{V^{\mathbb{P}}}$ in $V^{\mathbb{P}}$. Hence, $(\mathscr{P}_{\kappa^+}(\lambda))^V$ is stationary in $V^{\mathbb{P}}$.*

(4) *If $\mathbb{P}$ is $(< \kappa, \kappa)$-distributive and $(\lambda, \lambda, \kappa)$-distributive, then $\mathbb{P}$ perserves all cardinals $\leq \lambda$.*

Suppose κ is a regular cardinal and $\mu > \kappa^+$. The question of whether $(\mathscr{P}_{\kappa^+}(\mu))^V$ is co-stationary in $V^{\mathbb{P}}$ is completely solved if $\mathbb{P}$ adds no new subsets of κ. To see this, let ν be least such that $\mathbb{P}$ adds a new function $r : \kappa \to \nu$. Note that $\nu > 2$ implies $\nu > 2^\kappa$. If $\nu = 2$, then $\mathbb{P}$ adds a new subset of κ. (Of course if $\mathbb{P}$ adds no new subsets of λ of size $\leq \kappa$, then $(\mathscr{P}_{\kappa^+}(\lambda))^V = (\mathscr{P}_{\kappa^+}(\lambda))^{V^{\mathbb{P}}}$, so the ground model cannot be co-stationary.) If $\nu > 2$, we have the following.

Fact 2.3. *Let $V \subseteq W$ be models of ZFC with the same ordinals. If κ is a cardinal in W and $\nu > \kappa$ is the least cardinal in V such that $W \setminus V$ has a new function from κ into ν, then $\forall \lambda \geq \nu$, $\mathscr{P}_{\kappa^+}(\lambda) \setminus V$ contains a cone. Moreover, for all cardinals ρ, λ in W with ρ regular in W, $\kappa < \rho \leq \nu \leq \lambda$, and $cf(\nu) \geq \rho$ in V, then $\mathscr{P}_\rho(\lambda) \setminus V$ contains a cone.*

Proof. Let $r : \kappa \to \nu$ be in $W \setminus V$. Let $z \in (\mathscr{P}_{\kappa^+}(\lambda))^V$. There is an injection $b : z \to \kappa$ in V. If $z \supseteq \mathrm{ran}(r)$, then $r = b^{-1} \circ (b \circ r) \in V$, since $b \circ r : \kappa \to \kappa$ and hence must be in V. Contradiction. Therefore, the cone $\{x \in \mathscr{P}_{\kappa^+}(\lambda) : x \supseteq \mathrm{ran}(r)\} \cap V = \emptyset$.

Now assume $\mathrm{cf}(\nu) \geq \rho$. Any $x \in (\mathscr{P}_\rho(\lambda))^V$ cannot contain $\mathrm{ran}(r)$, since $\mathrm{ran}(r)$ is unbounded in ν. $\square$

Remark. However, this tells us nothing about the co-stationarity of $(\mathscr{P}_\rho(\lambda))^V$ for $\lambda > \rho > \nu$. Theorem 5.6 will give sufficient conditions for making $(\mathscr{P}_\rho(\lambda))^V$ co-stationary for $\lambda > \rho > \nu$ when $\mathbb{P}$ adds a new function from ω_1 into ν.

Next we state a well-known result of Menas.

Theorem 2.4 (Menas [8]). *Let $A \subseteq B$ with $|A| \geq \kappa$. For $X \subseteq \mathscr{P}_\kappa(A)$, let $X^* = \{x \in \mathscr{P}_\kappa(B) : x \cap A \in X\}$. If $C \subseteq \mathscr{P}_\kappa(A)$ is club then C^* is club in $\mathscr{P}_\kappa(B)$. For $Y \subseteq \mathscr{P}_\kappa(B)$, let $Y \restriction A = \{y \cap A : y \in Y\}$. If $C \subseteq \mathscr{P}_\kappa(B)$ is club, then $C \restriction A$ contains a club set in $\mathscr{P}_\kappa(A)$.*

Two special facts follow from this theorem.

Fact 2.5. *Let $V \subseteq W$ be models of ZFC with the same ordinals and κ be regular and $\lambda > \kappa$ in W.*

(1) *If $(\mathscr{P}_\kappa(\lambda))^V$ is co-stationary in W, then for all $\mu \geq \lambda$, $(\mathscr{P}_\kappa(\mu))^V$ is also co-stationary in W.*
(2) *If $(\mathscr{P}_\kappa(\lambda))^V$ is stationary in W and $\kappa \leq \mu < \lambda$, then $(\mathscr{P}_\kappa(\mu))^V$ is also stationary in W.*

Proof. Suppose C is club in $\mathscr{P}_\kappa(\mu)$ and $\mu \geq \lambda$. Then $C \restriction \lambda$ contains a club in $\mathscr{P}_\kappa(\lambda)$, so there is a $y \in (C \restriction \lambda) \cap (\mathscr{P}_\kappa(\lambda) \setminus V)$. $y = x \cap \lambda$ for some $x \in C$, and $y \notin V \implies x \notin V$. If C is club in $\mathscr{P}_\kappa(\mu)$, then C^* is club in $\mathscr{P}_\kappa(\lambda)$, so there is an $x \in C^* \cap V$. Then $x \cap \mu \in C$ and $x \cap \mu$ must also be in V. □

Thus, to show that $(\mathscr{P}_\kappa(\lambda))^V$ is co-stationary in W for all $\lambda \geq \kappa^+$, it suffices to show that $(\mathscr{P}_\kappa(\kappa^+))^V$ is co-stationary in W.

3. α-Erdős cardinals and Global Gitik

In this section, we first look at Erdős cardinals and how they can be used to force co-stationarity at a single cardinal of the ground model. After this, we concentrate on ω_1-Erdős cardinals, culminating in the equiconsistency of a global Gitik-type result for partial orderings which add a new subset of $\aleph_1$.

Definition 3.1. [2] Let $\alpha \leq \lambda$, α a limit ordinal. λ is *α-Erdős* if whenever C is club in λ and $f : [C]^{<\omega} \to \lambda$ is regressive ($f(a) < \min(a)$), then f has a homogeneous set of order type α.

The following is a model-theoretic equivalent of being α-Erdős: λ is α-Erdős iff for any structure $\mathfrak{A}$ with universe λ (for a countable language) endowed with Skolem functions, for any club $C \subseteq \lambda$, there is an $I \subseteq C$ of order type α such that I is a set of indiscernibles for $\mathfrak{A}$ and in addition I is *remarkable*; i.e. whenever $\iota_0, \ldots, \iota_n$ and $\eta_0, \ldots, \eta_n$ are increasing sequences from I with $\iota_{i-1} < \eta_i$, τ is a term and $\tau^{\mathfrak{A}}(\iota_0, \ldots, \iota_n) < \iota_i$, then $\tau^{\mathfrak{A}}(\iota_0, \ldots, \iota_n) = \tau^{\mathfrak{A}}(\iota_0, \ldots, \iota_{i-1}, \eta_i, \ldots, \eta_n)$. (See [1].)

Theorem 3.2. *Suppose that in V, $\lambda > \kappa$, κ is regular, and λ is κ-Erdős. Let $\mathbb{P}$ be κ-Cohen forcing (or any partial ordering which adds a new subset of κ and is $(\lambda, \lambda, \kappa)$-distributive). Then $(\mathscr{P}_{\kappa^+}(\mu))^V$ is co-stationary in $V^{\mathbb{P}}$ for all $\mu \geq \lambda$.*

Proof. Let G be $\mathbb{P}$-generic over V. Let $C \subseteq \mathscr{P}_{\kappa^+}(\lambda)$ be club in $V[G]$. In $V[G]$, there is a function $g : \kappa \times [\lambda]^{<\omega} \to \lambda$ such that $C_g \subseteq C$, where $C_g = \{x \in \mathscr{P}_{\kappa^+}(\lambda) : \forall (\alpha, y) \in \kappa \times [x]^{<\omega},\ g(\alpha, y) \in x\}$. Using the $(\lambda, \lambda, \kappa)$-distributivity of $\mathbb{P}$, we can obtain a set $\mathcal{F} = \{f_\alpha : \alpha < \kappa\}$ of functions such that $\mathcal{F} \in V$, each $f_\alpha : [\lambda]^{<\omega} \to \lambda$, $\mathcal{F}$ is closed under compositions (we identify a finite subset of λ with its strictly increasing enumeration), and $C_{\mathcal{F}} \subseteq C$, where $C_{\mathcal{F}} = \{x \in \mathscr{P}_{\kappa^+}(\lambda) : \forall \alpha < \kappa,\ \forall y \in [x]^{<\omega},\ f_\alpha(y) \in x\}$.

Let $\mathfrak{A}$ be the structure $\langle \lambda, \in, f_\alpha (\alpha < \kappa)\rangle$. Let $I \subseteq \lambda$ be a set of indiscernibles for $\mathfrak{A}$ with o.t.$(I) = \kappa$. Define $\mathrm{cl}(x) = x \cup \{f_\alpha(y) : \alpha < \kappa,\ y \in [x]^{<\omega}\}$. Note that for each $x \in \mathscr{P}_{\kappa^+}(\lambda)$, $\mathrm{cl}(x) \in C_{\mathcal{F}}$, since $\mathcal{F}$ is closed under finite compositions. Let $\langle \iota_\alpha : \alpha < \kappa\rangle$ enumerate I in increasing order. Let $J = \{\iota_\alpha : \alpha < \kappa$ and α is a limit ordinal$\}$. Note that if $\iota_{\alpha_1} < \cdots < \iota_{\alpha_n}$ are in J, $\beta < \kappa$, and $f_\beta(\iota_{\alpha_1}, \ldots, \iota_{\alpha_n}) \in I$, then $f_\beta(\iota_{\alpha_1}, \ldots, \iota_{\alpha_n}) \in \{\iota_{\alpha_1}, \ldots, \iota_{\alpha_n}\}$. Thus, for all $x \subseteq J$, $\mathrm{cl}(x) \cap I = x$.

Now let r be a new subset of κ in $V[G]$. Let $\langle s_\alpha : \alpha < \kappa\rangle$ enumerate J in increasing order. Define $x = \{s_\alpha : \alpha \in r\}$. Let $u = \mathrm{cl}(x)$. Then $u \in C_{\mathcal{F}}$. From u we can read off r; hence, $u \notin V$. The result for $\mathscr{P}_{\kappa^+}(\mu) \setminus V$ for all $\mu \geq \lambda$ follows from Fact 2.4. □

The preceding theorem along with Theorem 1.7 yield the following equiconsistency.

Theorem 3.3. *The following are equiconsistent:*

(1) *There is a κ-Erdős cardinal.*
(2) *There is a $\lambda > \kappa^+$ such that if $\mathbb{P}$ is κ-Cohen forcing, then $(\mathscr{P}_{\kappa^+}(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$.*
(3) *There is a $\lambda > \kappa^+$ such that if $\mathbb{P}$ is any partial ordering which adds a new subset of κ and is $(\lambda, \lambda, \kappa)$-distributive, then $(\mathscr{P}_{\kappa^+}(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$.*

The proof of Theorem 3.2 works for $\mathscr{P}_{\kappa^+}(\lambda)$ only when λ is κ-Erdős. If we wish to make λ smaller, we need a different method. The rest of this paper is devoted to shrinking λ below a κ-Erdős cardinal. In this section we concentrate on those partial orderings which add a new subset of $\aleph_1$. The next lemma is a generalization of an argument due to Baumgartner, which he used to construct a model in which every club subset of $\mathscr{P}_{\aleph_2}(\aleph_3)$ has maximal size [1]. The first part of this lemma will enable us to obtain co-stationarity of $(\mathscr{P}_{\aleph_2}(\lambda))^V$ when λ is smaller than the least ω_1-Erdős cardinal and a new subset of $\aleph_1$ is added. The second part will be used in Section 5 to obtain a partial result when no new subsets of $\aleph_1$ are added but a new ω_1-sequence is added (see Theorem 5.6).

Lemma 3.4. (1) *Suppose that in V, $|2^\omega| < \kappa < \lambda$, κ is regular, and λ is ω_1-Erdős. Let $\mathbb{Q} = Col(\kappa, < \lambda)$ and G be $\mathbb{Q}$-generic over V. Then in $V[G]$, given a function $g : [\kappa^+]^{<\omega} \to [\kappa^+]^{<\kappa}$, there is a tree T isomorphic to $2^{<\omega_1}$ such that for any two branches b, c in T, $b \cap \bigcup g''[c]^{<\omega} \subseteq b \cap c$.*

(2) *Suppose that in V, $\nu \geq \aleph_1$, $|\nu^\omega| < \kappa < \lambda$, κ is regular, λ is ν-Erdős, and $\mathbb{Q} = Col(\kappa, < \lambda)$. Then in $V[G]$, given a function $g : [\kappa^+]^{<\omega} \to [\kappa^+]^{<\kappa}$, there is a tree T isomorphic to $\nu^{<\omega_1}$ such that for any branches b, c in T, $b \cap \bigcup g''[c]^{<\omega} \subseteq b \cap c$.*

Proof. (1) We closely follow the argument of Baumgartner in Theorem 5.9 of [1], using $\mathrm{Col}(\kappa, < \lambda)$ in place of $\mathrm{Col}(\aleph_2, < \lambda)$ and checking that everything goes through as before while supplying more details. Let $\dot{g} : [\check{\lambda}]^{<\omega} \to [\check{\lambda}]^{<\kappa}$. Let $\dot{R}_n$ be an $(n+1)$-ary relation on $\check{\lambda}$ such that $\dot{R}_n(\beta, \alpha_0, \ldots, \alpha_{n-1})$ holds iff $\beta \in \dot{g}(\{\alpha_0, \ldots, \alpha_{n-1}\})$. Let $\dot{\mathfrak{A}}$ be a $\mathbb{Q}$-name for the structure $\langle \lambda, \dot{R}_n \rangle_{n<\omega}$. Let $\mathfrak{B} = \langle V_\lambda, \in, <, \mathbb{Q}, \Vdash_\phi \rangle$, where ϕ ranges over the formulas of $\dot{\mathfrak{A}}$ and $\Vdash_\phi$ is the relation $\{(p, \alpha_0, \ldots, \alpha_{n-1}) : p \Vdash$ "$\dot{\mathfrak{A}} \models \phi(\alpha_0, \ldots, \alpha_{n-1})$"$\}$.

Let $C \subseteq \lambda$ be a club such that whenever τ is a term of $\mathfrak{B}$ and $\alpha_0, \ldots, \alpha_n$ are increasing from C, then $\tau(\alpha_0, \ldots, \alpha_{n-1}) < \alpha_n$ (provided $\tau(\alpha_0, \ldots, \alpha_{n-1})$ is an ordinal). We let $T' \subseteq C$ be a set of remarkable indiscernibles for $\mathfrak{B}$ of order type ω_1, with $\min(T') > \kappa^+$. By standard arguments, we can assume each indiscernible is Mahlo. Enumerate T' as $\langle \iota_\alpha : \alpha < \omega_1 \rangle$ and let $T = \{\iota_\alpha : \alpha < \omega_1$ and α is a limit ordinal$\}$. Put a tree ordering $<_T$ on T so that $(T, <_T)$ is isomorphic to $2^{<\omega_1}$ and $\alpha <_T \beta \to \alpha < \beta$ for all $\alpha, \beta \in T$. Unless otherwise specified, by a *branch* of T, we mean an ω_1-branch through T.

Fix a branch $b_0 \subseteq T$. Using Baumgartner's Lemmas 5.4 - 5.6 of [1], there is a $G(b_0) \subseteq H^{\mathfrak{B}}(b_0)$ which is $\mathbb{Q}$-generic over $H^{\mathfrak{B}}(b_0)$; meaning, for each maximal incompatible $D \subseteq \mathbb{Q}$ such that $D \in H^{\mathfrak{B}}(b_0)$, $D \cap G^{\mathfrak{B}}(b_0) \neq \emptyset$.

Now, for any branch $c \subseteq T$, let $\pi_c : b_0 \to c$ be an order-preserving bijection. Note that if $\tau(\alpha_0, \ldots, \alpha_{n-1}) \in G(b_0)$, then $H^{\mathfrak{B}}(c) \models \tau(\pi_c(\alpha_0), \ldots, \pi_c(\alpha_{n-1})) \in \mathbb{Q}$. Let $G(c) = \{\tau(\pi_c(\alpha_0, \ldots, \alpha_{n-1})) : \tau$ is a term, $\alpha_0, \ldots, \alpha_{n-1} \in b_0$, and $\tau(\alpha_0, \ldots, \alpha_{n-1}) \in G(b_0)\}$. Then $G(c)$ is $\mathbb{Q}$-generic over $H^{\mathfrak{B}}(c)$, by indiscernibility. Let $G(T)$ denote $\bigcup\{G(c) : c$ is a branch through $T\}$.

Claim 1. Any two elements in $G(T)$ are compatible.

Let $p, q \in G(T)$. Let b, c be branches of T such that $p \in G(b)$ and $q \in G(c)$. We will show that $p \| q$. Let σ, τ be terms such that $p = \sigma(\pi_b(\alpha_0), \ldots, \pi_b(\alpha_{k-1}))$, where $\alpha_0, \ldots, \alpha_{k-1} \in b_0$; and $q = \tau(\pi_c(\beta_0), \ldots, \pi_c(\beta_{m-1}))$, where $\beta_0, \ldots, \beta_{m-1} \in b_0$. Let $p_{b_0} = \sigma(\alpha, \ldots, \alpha_{k-1})$ and $q_{b_0} = \tau(\beta_0, \ldots, \beta_{m-1})$. Then $p_{b_0} \| q_{b_0}$, since they are both in $G(b_0)$. Let $r = \psi(\gamma_0, \ldots, \gamma_{n-1}) \in G(b_0)$ such that $r \leq p_{b_0}, q_{b_0}$. Let $r_b = \psi(\pi_b(\gamma_0), \ldots, \pi_b(\gamma_{n-1}))$ and $r_c = \psi(\pi_c(\gamma_0), \ldots, \pi_c(\gamma_{n-1}))$. Then $r_b \leq p$ and $r_c \leq q$, by indiscernibility.

We will show that $r_b \| r_c$. $|\mathrm{dom}(r_b)| \leq \kappa$, so every $\alpha \in \mathrm{dom}(r_b)$ is definable in $\mathfrak{B}$ from parameters in κ and $\{\pi_b(\gamma_0), \ldots, \pi_b(\gamma_{n-1})\}$. Say $\alpha = \phi(\xi_0, \ldots, \xi_{l-1}, \pi_b(\gamma_0), \ldots, \pi_b(\gamma_{n-1}))$, where $\xi_0, \ldots, \xi_{l-1} \in \kappa$. Let i be such that $\forall j < i$, $\pi_c(\gamma_j) = \pi_b(\gamma_j)$, and $\forall j \geq i$, $\pi_c(\gamma_j) \neq \pi_b(\gamma_j)$. Suppose that also $\alpha \in \mathrm{dom}(r_c)$. It can be shown using indiscernibility and remarkability that $\alpha < \pi_b(\gamma_i)$ and $\alpha < \pi_c(\gamma_i)$. Hence,

$\alpha = \phi(\xi_0, \ldots, \xi_{l-1}, \pi_c(\gamma_0), \ldots, \pi_c(\gamma_{n-1}))$, by remarkability. Again by remarkability, $r_c(\alpha) = r_b(\alpha)$. Thus, Claim 1 holds.

Since $|G(T)| \leq |2^\omega| < \kappa$, we have that $p = \bigwedge G(T) \in \mathbb{Q}$. Let G be $\mathbb{Q}$-generic with $p \in G$. Let $W = V[G]$ and work in W.

Claim 2. If c is any branch in T and $\bar{x} \in [c]^{<\omega}$, then $g(\bar{x}) \in H^{\mathfrak{B}}(c)$.

We will show that there is a $q \in H^{\mathfrak{B}}(c) \cap G(c)$ which decides $\dot{g}(\bar{x})$. If we can find such a q, then $W \models$ "$\beta \in g(\bar{x})$" iff $q \Vdash$ "$\dot{\mathfrak{A}} \models \dot{R}_n(\beta, \bar{x})$" iff $(q, \beta, \bar{x}) \in \Vdash_\phi$, where $\phi(\beta, \bar{x}) = \dot{R}_n(\beta, \bar{x})$. Then $\mathfrak{B} \models \exists z \forall \beta(\beta \in z \leftrightarrow (q, \beta, \bar{x}) \in \Vdash_\phi)$. Hence, by a Skolem function, such a z is in $H^{\mathfrak{B}}(c)$.

Define $q \in D$ iff $\mathfrak{B} \models \exists z((q, z, \bar{x}) \in \Vdash_\psi)$, where ψ is $\forall \beta(\dot{R}_n(\beta, \bar{x}) \leftrightarrow \beta \in z)$. D is dense since $\mathbb{Q}$ is κ-closed. Each maximal incompatible subset of D is in $\mathfrak{B}$, since $\mathbb{Q}$ is λ-c.c. By elementarity, there is a maximal incompatible $M \subseteq D$ such that $M \in H^{\mathfrak{B}}(c)$. Thus, $M \cap G(c) \neq \emptyset$, which proves Claim 2.

Claim 3. For all branches $b \neq c$ of T, $b \cap \bigcup g''[c]^{<\omega} \subseteq b \cap c$.

Let $\bar{x} \in [c]^{<\omega}$. All elements of T are above κ^+, and we can use parameters in κ^+ and still enjoy remarkability and indiscernibility. Let τ be a term and $\gamma_0, \ldots, \gamma_k \in c$ such that $g(\bar{x}) = \tau(\gamma_0, \ldots, \gamma_k)$. $\mathfrak{B} \models$ "There is an ordinal η and a bijection $f : \eta \to g(\bar{x})$". By elementarity, this is true for $H^{\mathfrak{B}}(c)$, since $g(\bar{x}) \in H^{\mathfrak{B}}(c)$. Take such η, f in $H^{\mathfrak{B}}(c)$. Suppose $\iota \in b \cap g(\bar{x})$. Then $\eta < \kappa$ and $\iota = f(\alpha)$ for some $\alpha < \eta$. But then ι is definable from α and parameters in c; hence, $\iota \in c$. Thus, we have proved Claim 3, and (1) of the Lemma follows.

The proof of (2) is analogous, giving the set of indiscernibles of size ν an appropriate tree ordering isomorphic to $\nu^{<\omega_1}$ and making the necessary changes in cardinals. □

The next theorem will be used to obtain an equiconsistency for $\mathbf{Quest}(\aleph_1, \aleph_3)$ (Corollary 3.6) and to obtain a result akin to Gitik's when a new subset of $\aleph_1$ is added (Theorem 3.8).

Theorem 3.5. *Suppose $V \models$ "$|2^\omega| < \kappa < \lambda$, κ is regular, and λ is ω_1-Erdős". Let $\mathbb{Q} = Col(\kappa, < \lambda)$, G be $\mathbb{Q}$-generic over V, and $W = V[G]$. In W, let $\mathbb{P}$ be $\aleph_1$-Cohen forcing (or any partial ordering which adds a new subset of $\aleph_1$ and satisfies the $(\kappa^+, \kappa^+, < \kappa)$-d.l. if κ is a successor cardinal, or the κ-c.c. if κ is inaccessible). Then $(\mathscr{P}_\kappa(\mu))^W$ is co-stationary in $W^{\mathbb{P}}$ for all $\mu \geq \kappa^+$.*

Proof. We assume $\mathbb{P}$ is κ-c.c., noting that if κ is a successor cardinal, then we can weaken this assumption to the $(\kappa^+, \kappa^+, < \kappa)$-d.l. Assume $\mathbb{P}$ adds no new reals (since otherwise Theorem 1.2 of Gitik suffices). In W, let $\dot{C}$ be a $\mathbb{P}$-name for a club subset of $\mathscr{P}_\kappa(\kappa^+)$. There is a function $\dot{f} : [\kappa^+]^{<\omega} \to [\kappa^+]^{<\kappa}$ such that $C_{\dot{f}} \subseteq \dot{C}$, where $C_{\dot{f}} = \{x \in \mathscr{P}_\kappa(\kappa^+) : \forall y \in [x]^{<\omega}\ \dot{f}(y) \in x\}$. Since $\mathbb{P}$ is κ-c.c., there is a function $h : [\kappa^+]^{<\omega} \to [\kappa^+]^{<\kappa}$ in W such that $\forall x \in [\kappa^+]^{<\omega}$, $\dot{f}(x) \in h(x)$. Let

$C_h^W = \{x \in (\mathscr{P}_\kappa(\kappa^+)^W : \forall y \in [x]^{<\omega},\ h(y) \subseteq x\}$. Then C_h^W is club in $(\mathscr{P}_\kappa(\kappa^+))^W$, and $C_h^W \subseteq C_{\dot{f}}$.

From h we define a useful function $g : [\kappa^+]^{<\omega} \to [\kappa^+]^{<\kappa}$ in W by induction on $|x|$. Let $g(\{\alpha\}) \in C_h^W$ such that $g(\{\alpha\}) \supseteq h(\{\alpha\}) \cup \{\alpha\}$. If $|x| = n+1$, let $g(x) \in C_h^W$ such that $g(x) \supseteq \bigcup\{g(y) : y \in [x]^n\} \cup h(x)$. In W, let $T \subseteq \kappa^+$ be a tree isomorphic to $2^{<\omega_1}$ satisfying Lemma 3.4 for g. Note: for any branch b through T in $W^{\mathbb{P}}$, $g''[b]^{<\omega}$ is a directed subset (in $W^{\mathbb{P}}$) of C_h^W, hence $\bigcup g''[b]^{<\omega} \in \dot{C}$.

Let $r : \omega_1 \to 2$ be a function in $W^{\mathbb{P}} \setminus W$. We use r to define a new branch through T as follows: Let $\pi : 2^{<\omega_1} \to T$ be a tree isomorphism. Let $\tilde{b} \restriction 0 = \langle\ \rangle$. $\tilde{b} \restriction \alpha = \pi(r \restriction \alpha)$ for $\alpha < \omega_1$. Note that for limit ordinals $\alpha < \omega_1$, $\tilde{b} \restriction \alpha \in W$, since $\mathbb{P}$ adds no new reals. Let $\tilde{b} = \bigcup_{\alpha<\omega_1} \tilde{b} \restriction \alpha$. Let $\tilde{z} = \bigcup g''[\tilde{b}]^{<\omega}$. Then $\tilde{z} \in \dot{C}$. We will show that $\tilde{z} \notin W$.

Claim. Suppose $\alpha = \beta + 1 < \omega_1$. There is a unique α-length branch $\bar{c}$ in T such that $\tilde{z} \cap \bar{c} = \bar{c}$. Moreover, $\tilde{b} \restriction \alpha = \bar{c}$.

Let $\bar{c} = \tilde{b} \restriction \alpha$. $\bar{c} \subseteq \bigcup\{g(\{\zeta\}) : \zeta \in \bar{c}\} \subseteq g''[\bar{c}]^{<\omega} \subseteq \tilde{z}$; so $\bar{c} \cap \tilde{z} = \bar{c}$. Now let $\bar{c}$ be a branch of length α such that $\bar{c} \neq \tilde{b} \restriction \alpha$. Let $\gamma < \alpha$ be least such that $\bar{c}(\gamma) \neq \tilde{b}(\gamma)$. Then $\bar{c} \cap (\tilde{b} \restriction \alpha) = \bar{c} \restriction \gamma$. For all $\alpha \leq \delta < \omega_1$, for any branch b in W extending $\tilde{b} \restriction \delta$, for any branch c in W extending $\bar{c}$, $\bar{c} \cap \bigcup g''[\tilde{b} \restriction \delta]^{<\omega} \subseteq c \cap \bigcup g''[b]^{<\omega} \subseteq c \cap b = \bar{c} \restriction \gamma$. Therefore, $\bar{c} \cap \tilde{z} = \bigcup_{\delta<\omega_1}(\bar{c} \cap \bigcup g''[\tilde{b} \restriction \delta]^{<\omega}) \subseteq \bar{c} \restriction \gamma \subsetneq \bar{c}$. Hence, $\bar{c} \cap \tilde{z} = \bar{c}$ iff $\bar{c} = \tilde{b} \restriction \alpha$. Hence, with $\tilde{z}$ as an oracle we can decode r in W. □

Theorem 3.5 together with Theorem 1.6 of Magidor yield the following equiconsistency.

Theorem 3.6. *The following are equiconsistent:*

1. *There is an ω_1-Erdős cardinal.*
2. *If $\mathbb{P}$ is $\aleph_1$-Cohen forcing, then $(\mathscr{P}_{\aleph_2}(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$ for all $\lambda \geq \aleph_3$.*
3. *If $\mathbb{P}$ adds a new subset of $\aleph_1$ and is $(\aleph_3, \aleph_3, \aleph_1)$-distributive, then $(\mathscr{P}_{\aleph_2}(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$ for all $\lambda \geq \aleph_3$.*

Next we tackle Question 1.4. In preparation, we prove Lemma 3.7.

Lemma 3.7. *Suppose $|2^\omega| < \aleph_\alpha < \kappa$ and κ is ω_1-Erdős. Let $\mathbb{Q} = \mathrm{Col}(\aleph_\alpha, < \kappa)$ and G be $\mathbb{Q}$-generic. Let $\dot{\mathbb{R}}$ be a $\mathbb{Q}$-name for an $\aleph_{\alpha+1}$-closed partial ordering in $V[G]$. Let H be $\dot{\mathbb{R}}$-generic over $V[G]$. Then for each $g : [\aleph_{\alpha+1}]^{<\omega} \to [\aleph_{\alpha+1}]^{<\aleph_\alpha}$ in $V[G][H]$, there is a tree $T \in V$ such that $T \cong 2^{<\omega_1}$ and for all branches b, c in T, $b \cap \bigcup g''[c]^{<\omega} \subseteq b \cap c$; that is, T satisfies Lemma 3.4 (1) for g.*

Proof. Let G be $\mathbb{Q}$-generic over V. κ becomes $\aleph_{\alpha+1}$ in $V[G]$. In $V[G]$, suppose $p \Vdash (\dot{g} : [\aleph_{\alpha+1}]^{<\omega} \to [\aleph_{\alpha+1}]^{<\aleph_\alpha})$, where $p \in \dot{\mathbb{R}}$ and $\dot{g}$ is an $\dot{\mathbb{R}}$-name over $V[G]$. Fix an enumeration $\langle x_\zeta : \zeta < \kappa\rangle$ of $[\kappa]^{<\omega}$ in V such that for each cardinal $\rho < \kappa$ in V, $\langle x_\zeta : \zeta < \rho\rangle$ enumerates $[\rho]^{<\omega}$. In $V[G]$, form a decreasing sequence

$\langle p_\zeta : \zeta < \aleph_{\alpha+1}\rangle$ of elements of $\dot{\mathbb{R}}$ with $p_0 \leq p$ such that for each $\zeta < \aleph_{\alpha+1}$, p_ζ decides $\dot{g}(x_\zeta)$. $\langle p_\zeta : \zeta < \aleph_{\alpha+1}\rangle$ is in $V[G]$, so it evaluates $\dot{g}$ to be some function in $V[G]$, call it h. By Lemma 3.4, there is a tree $T \subseteq \aleph_{\alpha+1}$ with $T \cong 2^{<\omega_1}$ such that for all branches b, c in T, $b \cap \bigcup h''[c]^{<\omega} \subseteq b \cap c$. ($T \in V$ and has the same branches in V, $V[G]$, and $V[G][H]$ since $\mathbb{Q}$ and $\dot{\mathbb{R}}$ are $\aleph_2$-closed.)

Let $\beta = \sup(T)$. Then $\beta < \kappa$ in V. Let $\rho = (|\beta|^+)^V$. Then $\rho < \aleph_{\alpha+1}$ in $V[G]$, and $p_\rho \Vdash (\dot{g} \restriction [\check{\rho}]^{<\omega} = h \restriction [\check{\rho}]^{<\omega})$. Hence, given branches b, c in T, $b \cap \bigcup h''[c]^{<\omega} \subseteq b \cap c$ in $V[G]$, and $p_\rho \Vdash (b \cap \bigcup h''[c]^{<\omega} = b \cap \bigcup \dot{g}''[c]^{<\omega})$, since $c \subseteq \rho$. Thus, for each $p \in \dot{\mathbb{R}}$ there exist a $q \leq p$ and a tree T such that $q \Vdash (T$ satisfies Lemma 3.4 (1) for $\dot{g})$. □

We are now ready to prove an analog of Gitik's Theorem 1.2 for partial orderings which add a new subset of $\aleph_1$.

Theorem 3.8 (Consistency of Global Gitik)**.** *The following are equiconsistent:*

(1) *There is a proper class of ω_1-Erdős cardinals.*
(2) *If $\mathbb{P}$ is $\aleph_1$-Cohen forcing, then $(\mathscr{P}_\kappa(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$ for all regular $\kappa \geq \aleph_2$ and all $\lambda > \kappa$.*
(3) *If $\mathbb{P}$ adds a new subset of $\aleph_1$ and is $\aleph_2$-c.c. (or just satisfies the $(\kappa^+, \kappa^+, <\kappa)$-distributive law for all successor cardinals $\kappa \geq \aleph_2$ and is κ-c.c. for all strongly inaccessible κ), then $(\mathscr{P}_\kappa(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$ for all regular $\kappa \geq \aleph_2$ and all $\lambda > \kappa$.*

Proof. Con(3) $\Longrightarrow$ Con(2): Trivial.

Con(1) $\Longrightarrow$ Con(3): Suppose there is a proper class of ω_1-Erdős cardinals and $|2^\omega| \leq \aleph_2$. We construct an iterated forcing as follows. For indexing reasons, we let $\mathbb{Q}_0$, $\mathbb{Q}_1$, and $\mathbb{Q}_2$ be the trivial partial ordering. Let κ_3 be ω_1-Erdős and $\mathbb{Q}_3 = \mathrm{Col}(\aleph_2, <\kappa_3)$. κ_3 becomes $\aleph_3$ in $V^{\mathbb{Q}_3}$. In $V^{\mathbb{Q}_3}$, let $\kappa_4 > \aleph_3$ be ω_1-Erdős. Let $\dot{\mathbb{P}}_4$ be a $\mathbb{Q}_3$-name for $\mathrm{Col}(\aleph_3, <\kappa_4)$ and $\mathbb{Q}_4 = \mathbb{Q}_3 * \dot{\mathbb{P}}_4$. In general, in $V^{\mathbb{Q}_\alpha}$, let $\kappa_{\alpha+1} > \aleph_\alpha$ be ω_1-Erdős. Let $\dot{\mathbb{P}}_{\alpha+1}$ be a $\mathbb{Q}_\alpha$-name for $\mathrm{Col}(\aleph_\alpha, <\kappa_{\alpha+1})$ in $V^{\mathbb{Q}_\alpha}$ and $\mathbb{Q}_{\alpha+1} = \mathbb{Q}_\alpha * \dot{\mathbb{P}}_{\alpha+1}$. We use reverse Easton support. Let $\mathbb{Q}$ denote the iterated forcing. For each successor ordinal $\alpha > 2$, κ_α becomes $\aleph_\alpha$ in the extension.

If in $V^{\mathbb{Q}}$ $\aleph_\alpha$ is regular, then $\mathbb{Q}_{\alpha+1}$ took care of $\mathscr{P}_{\aleph_\alpha}(\aleph_{\alpha+1})$; that is, Lemma 3.4 (1) holds for $\mathscr{P}_{\aleph_\alpha}(\aleph_{\alpha+1})$ in $V^{\mathbb{Q}_{\alpha+1}}$. Continuing the iteration still preserves this: the remainder forcing is $\aleph_{\alpha+1}$-closed in $V^{\mathbb{Q}_{\alpha+1}}$, so Lemma 3.7 guarantees that Lemma 3.4 (1) still holds for $\mathscr{P}_{\aleph_\alpha}(\aleph_{\alpha+1})$ in $V^{\mathbb{Q}}$. Let $W = V^{\mathbb{Q}}$. The remainder of this direction of the proof follows as that of Theorem 3.5.

Con(2) $\Longrightarrow$ Con(1): The necessity of a proper class of ω_1-Erdős cardinals follows from a natural generalization of Magidor's Theorem 1.6: Let α be an ordinal, and assume there is no ω_1-Erdős cardinal greater than α in K_{DJ}. Then for every ordinal $\beta > \alpha$ one can define in K_{DJ} a countable collection of functions $\mathscr{C}$ on β such that every subset of β containing α as a subset which is closed under $\mathscr{C}$ is a countable union of sets in K_{DJ}. □

4. Equiconsistency for $\mathscr{P}_{\kappa^+}(\aleph_\kappa)$

In the previous section we showed that the existence of an ω_1-Erdős cardinal is equiconsistent with $\mathbb{P}$ forcing $(\mathscr{P}_{\aleph_2}(\lambda))^V$ to be co-stationary in $V^{\mathbb{P}}$ for all (or any) $\lambda \geq \aleph_3$, where $\mathbb{P}$ is $\aleph_1$-Cohen forcing. However, the analog of this does not hold when $\kappa > \aleph_1$. The next theorem implies that when $\kappa > \aleph_1$, κ measurable cardinals are necessary in order to even have a chance at a positive answer to **Quest**(κ, λ) when $\lambda \leq \aleph_\kappa$. This theorem is a strengthening and generalization of Theorem 1.8 of Magidor.

Theorem 4.1. *Assume that there is no inner model with $\aleph_2$ measurable cardinals. Then there is a countable algebra on $\aleph_{\omega_2}$ such that the universe of any subalgebra is the union of $\aleph_1$ sets in K_M, Mitchell's core model for sequences of measures.*

Proof. We use ideas from [7] together with the modern approach to covering (see [9]). Let κ denote $\aleph_{\omega_2}$ and suppose that X is the intersection of $H(\kappa)^{K_M}$ with an elementary submodel of $H(\kappa^+)$. We argue that X is the union of $\aleph_1$ sets in K_M. As in the proof of the Covering Lemma for K_M, let $\bar{K}$ denote the transitive collapse of X and π the isomorphism of $\bar{K}$ onto X.

As in [9], $\bar{K}$ does not move in the comparison of K_M with $\bar{K}$. Let $\bar{N}$ be the result of this comparison on the K_M-side. Then $\bar{N}$ end-extends $\bar{K}$. As in [7], let $\bar{M}$ be the least initial segment of $\bar{N}$ where α decomposes (i.e., for some n, α is included in the n-hull in $\bar{N}$ of some ordinal less than α together with some countable set of parameters). Then π lifts to an (appropriately elementarity) embedding of $\bar{M}$ into M, an element of K_M.

We show by induction on $\alpha \leq \mathrm{Ord}(\bar{K})$ that $\pi[\alpha]$ is the union of $\aleph_1$ sets in K_M. If α has cofinality less than ω_2 then the result is immediate by induction. If $\alpha = \omega_2 = \pi(\omega_2)$ then as ω_2 is not a Jonsson cardinal, π is the identity on ω_2, so the result is trivial. So we may assume that $\pi(\alpha)$ is greater than ω_2 and α has cofinality greater than ω_1.

It will suffice to show that the part of the iteration of K_M to $\bar{N}$ below α is bounded in α. For then, as in [7], $\pi[\alpha]$ is an intial segment of a hull in M of $\pi[\beta]$ for some $\beta < \alpha$ together with countably-many parameters parameters, and therefore by induction is the union of ω_1 sets in K_M.

Suppose that the iteration of K_M below α is unbounded in α. Then some measure is used at least ω_2 times below α, generating a closed set $\bar{C}$ of critical points κ_i, $i < \omega_2$ less than α.

First note that all sufficiently large $\pi(\kappa_i)$ have the same cardinality. Otherwise choose $i < \omega_2$ of cofinality ω_1 such that the cardinality of $\pi(\kappa_j)$ for $j < i$ has no maximum. Then $\pi(\kappa_i)$ must be a cardinal of cofinality ω_1, as it is the least element of X greater than the $\pi(\kappa_j)$, $j < i$. But $\pi(\kappa_i)$ is regular in K_M, and by [9] this yields an inner model with a measurable cardinal of order ω_1, contrary to our hypothesis that no inner model has ω_2 measurables.

Let γ be the cardinality of $\pi(\kappa_i)$ for sufficiently large $i < \omega_2$. We may assume that γ is at least ω_2, as otherwise π is the identity on ω_2 and $\alpha = \pi(\alpha) = \omega_2$. Now apply the following Lemma.

Lemma 4.2. *Suppose that β is greater than ω_2, is not a cardinal, is regular but not measurable in K_M and is not the limit of cardinals which are measurable in K_M. Then the cofinality of β equals the cardinality of β.*

Proof. We have assumed that there is no inner model with ω_2 measurables and therefore the Covering Lemma (see [9]) holds relative to K_M. Suppose that β has cofinality less than its cardinality. The proof of the Covering Lemma shows that β is included in the hull inside an initial segment of K_M of some ordinal less than its cardinality together with a set of indiscernibles associated to measurable cardinals $\leq \beta$. By hypothesis this set of indiscernibles is bounded in β, and therefore β is singular in K_M, contradiction. This proves the Lemma. □

It follows from Lemma 4.2 that all sufficiently large $\pi(\kappa_i)$, $i < \omega_2$, have the same cofinality, and therefore by choice of X, all sufficiently large κ_i have the same cofinality. But this is absurd, as for limit i, the cofinality of κ_i is that of i. This proves the Theorem. □

The previous argument generalizes to show the following.

Theorem 4.3. *Let $\kappa \geq \aleph_2$ be regular and assume that there is no inner model with κ measurable cardinals. Then there is a countable collection $\mathscr{C}$ of functions on $\aleph_\kappa$ such that every subset of $\aleph_\kappa$ closed under $\mathscr{C}$ is the union of $< \kappa$ sets in K_M.*

It follows that if there is no inner model with κ measurables, then in V, any $\mathbb{P}$ which is $(\rho, (\aleph_\kappa)^\kappa)$-distributive for all $\rho < \kappa$ (e.g. adds no new sequences of length less than κ) forces that $(\mathcal{P}_{\kappa^+}(\aleph_\kappa))^V$ is not co-stationary in $V^{\mathbb{P}}$; hence the answer to **Quest**$(\kappa, \aleph_\kappa)$ is negative in a strong sense. Moreover, if $\mathbb{P}$ adds no new $< \kappa$-sequences, then for all $\lambda < \mu \geq \kappa^+$ with μ regular in $V^{\mathbb{P}}$ and $\lambda \geq \aleph_\kappa$, $(\mathscr{P}_\mu(\lambda))^V$ is not co-stationary in $V^{\mathbb{P}}$.

On the other hand, it turns out that *free subsets* for structures with κ many functions solve **Quest**(κ, λ) for $(\lambda, \lambda, \kappa)$-distributive partial orderings. We now review free sets and the equiconsistency results for when they exist.

Definition 4.4. [11] Let $\mathfrak{A}$ be a structure. For any set $X \subseteq |\mathfrak{A}|$, let $\mathfrak{A}[X]$ denote the substructure generated by X in $\mathfrak{A}$. We say that X is *free* in $\mathfrak{A}$ iff for any $y \in X$, $y \notin \mathfrak{A}[X \setminus \{y\}]$. Let ξ, λ, κ be cardinals. $Fr_\xi(\lambda, \kappa)$ *holds* iff for any structure $\mathfrak{A}$ for a language of size $\leq \xi$ with $|\mathfrak{A}| \geq \lambda$, there is a free subset $S \subseteq |\mathfrak{A}|$ with $|S| \geq \kappa$.

Proposition 4.5. *Suppose in V that $Fr_\kappa(\lambda, \kappa)$ holds. Suppose $\mathbb{P}$ adds a new subset of κ and is $(\lambda, \lambda, \kappa)$-distributive. Then $(\mathscr{P}_{\kappa^+}(\lambda))^V$ is co-stationary.*

Proof. Recall that the $(\lambda, \lambda, \kappa)$-d.l. implies preservation of all cardinals $\kappa^+ \leq \theta \leq \lambda$. Let G be $\mathbb{P}$-generic, r a new subset of κ, and $C \subseteq \mathscr{P}_{\kappa^+}(\lambda)$ be club in $V[G]$. Let $f : \kappa \times [\lambda]^{<\omega} \to \lambda$ be such that $C_f \subseteq C$, where $C_f = \{x \in \mathscr{P}_{\kappa^+}(\lambda) : \forall (\alpha, y) \in \kappa \times [x]^{<\omega},\ f(\alpha, y) \in x\}$. By the $(\lambda, \lambda, \kappa)$-distributivity, there is a function $g' : \kappa \times [\lambda]^{<\omega} \to [\lambda]^{\leq \kappa}$ in V such that $\forall (\alpha, x) \in \kappa \times [\lambda]^{\leq \kappa}$, $f(\alpha, x) \in g'(\alpha, x)$. Hence, in V there is a sequence of functions $g_\alpha : [\lambda]^{<\omega} \to \lambda$ $(\alpha < \kappa)$ closed under composition such that $C_{\mathcal{G}} \subseteq C$, where $C_{\mathcal{G}} = \{x \in \mathscr{P}_{\kappa^+}(\lambda) : \forall \alpha < \kappa,\ \forall y \in [x]^{<\omega},\ g_\alpha(y) \in x\}$. Let $\mathfrak{A} = \langle \lambda, g_\alpha \rangle_{\alpha < \kappa}$ and let $I \subseteq \lambda$ be free for $\mathfrak{A}$ of size κ. Enumerate $I = \langle \iota_\alpha : \alpha < \kappa \rangle$. Then we can code r into a subset z of I as follows: Put $\iota_\alpha \in z$ iff $\alpha \in r$. Let $\tilde{z} = \mathfrak{A}[z]$, which is exactly $z \cup \{g_\alpha(x) : \alpha < \kappa,\ x \in [z]^{<\omega}\}$. Then $\tilde{z} \in C_{\mathcal{G}}$, but from $\tilde{z}$ we can decode r, since I is free for $\mathfrak{A}$. □

Shelah showed that starting with κ-many measurable cardinals, one can obtain a model of ZFC in which $\mathrm{Fr}_\kappa(\aleph_\kappa, \kappa)$ holds.

Theorem 4.6 (Shelah [11])**.** *If Con(ZFC + there are κ-many measurable cardinals and $\aleph_\kappa > \kappa$), then Con(ZFC + $Fr_\kappa(\aleph_\kappa, \kappa)$ holds).*

However, Shelah also showed in ZFC that $\aleph_\kappa$ is the least possible λ such that $\mathrm{Fr}_\kappa(\lambda, \kappa)$ can hold [11]. Hence, the free subset property cannot help us in the quest for a model in which $\mathbf{Quest}(\aleph_3, \aleph_4)$ has a positive answer. In the other direction of the equiconsistency of $\mathrm{Fr}_\kappa(\lambda, \kappa)$, Koepke showed that if κ is a cardinal satisfying $\omega_1 \leq \kappa < \aleph_\kappa$ and if also $\mathrm{Fr}_\omega(\aleph_\kappa, \kappa)$ holds, then there is an inner model in which the set of measurable cardinals below $\aleph_\kappa$ has order type $\geq \kappa$ [5].

Theorem 4.1, Proposition 4.5, and Theorem 4.6 yield the following equiconsistency.

Theorem 4.7. *The following are equiconsistent.*

(1) *$\aleph_\kappa > \kappa$ and there are κ measurable cardinals.*
(2) *$\aleph_\kappa > \kappa$, and if $\mathbb{P}$ is the κ-Cohen forcing, then $(\mathscr{P}_{\kappa^+}(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$ for all $\lambda \geq \aleph_\kappa$.*
(3) *$\aleph_\kappa > \kappa$, and if $\mathbb{P}$ adds a new subset of κ and is $(\aleph_\kappa, \aleph_\kappa, \kappa)$-distributive, then $(\mathscr{P}_{\kappa^+}(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$ for all $\lambda \geq \aleph_\kappa$.*

5. Open Problems

We conclude this paper with a list of open problems.

In Theorem 3.8, we obtained the equiconsistency of a generalization of Gitik's Theorem 1.2 for $\aleph_1$-Cohen forcing in particular, and, in general, for all partial orderings which add a new subset of $\aleph_1$ and have certain distributivity properties, e.g. $\aleph_2$-c.c. Was the distributivity necessary?

Open Problem 5.1. Find the equiconsistency of the following statement: Every partial ordering $\mathbb{P}$ which adds a new subset of $\aleph_1$ but no new ω-sequences forces $(\mathscr{P}_\kappa(\lambda))^V$ to be co-stationary for all regular $\kappa \geq \aleph_2$ and all $\lambda \geq \kappa^+$ in $V^{\mathbb{P}}$.

More generally, we would like to know the equiconsistency of Global Gitik for partial orderings which add a new subset of some cardinal μ.

Open Problem 5.2. Find the equiconsistency of the following statement: Every partial ordering $\mathbb{P}$ which adds a new subset of μ but no new $< \mu$-sequences forces $(\mathscr{P}_\kappa(\lambda))^V$ to be co-stationary for all regular $\kappa \geq \mu^+$ and all $\lambda \geq \kappa^+$ in $V^{\mathbb{P}}$.

A variant of Open Problem 5.2 would be to find the equiconsistency for all partial orderings which are $(\kappa^+, \kappa^+, < \kappa)$-distributive for all successor cardinals $\kappa \geq \mu^+$, and κ-c.c. for all strongly inaccessible $\kappa \geq \mu^+$. These conditions would imply that $\mathbb{P}$ preserves all cardinals $> \kappa$.

In Theorems 3.3, 3.6 and 4.7, we found the equiconsistency of forcing the ground model to be co-stationary for partial orderings with a certain amount of distributivity.

Open Problem 5.3. Are the distributivity properties of $\mathbb{P}$ used in Theorems 3.3, 3.6 and 4.7 necessary?

The following are still open for κ-Cohen forcing, or more generally, any forcing which adds a new subset of κ. Theorem 4.3 gives a lower bound of κ measurable cardinals, when $\mathbb{P}$ adds no new $< \kappa$-sequences.

Open Problem 5.4. Find the equiconsistency of $\mathbf{Quest}(\kappa, \lambda)$ for all κ, λ with $\aleph_2 < \kappa^+ < \lambda < \aleph_\kappa$. Of particular interest is when $\lambda = \kappa^{++}$, especially $\mathbf{Quest}(\aleph_2, \aleph_4)$.

A related problem which we have briefly touched on, is when $\mathbb{P}$ does not add a new real but does add a new ω-sequence. Having not solved Question 1.3, nevertheless, we dare to pose an even more general problem.

Open Problem 5.5. Suppose μ is a cardinal and $\nu > 2$ is least such that $\mathbb{P}$ adds a new function $r : \mu \to \nu$. (So $\mathbb{P}$ adds no new subsets of μ; hence, $\nu > 2^\mu$.) Is $(\mathscr{P}_\kappa(\lambda))^V$ necessarily co-stationary in $V^{\mathbb{P}}$ for all cardinals $\kappa < \lambda$ in $V^{\mathbb{P}}$ with κ regular, $\mu^+ \leq \kappa$, and $\lambda \geq \nu$ in $V^{\mathbb{P}}$?

The following is some progress toward an answer to Open Problem 5.5. The proof of Theorem 5.6 is analagous to that of Theorem 3.5, using Lemma 3.4 (2) in place of Lemma 3.4 (1). We do not know if the assumption of large cardinals is necessary if a new ω-sequence is added, as in that case, Magidor's Theorem 1.6 does not apply.

Theorem 5.6. *Assume that in V, $\aleph_1 \leq \nu$, $|\nu^\omega| < \kappa < \lambda$, κ is regular, and λ is ν-Erdős. Let $\mathbb{Q} = Col(\kappa, < \lambda)$, and let G be $\mathbb{Q}$-generic over V and $W = V[G]$. In W, let $\mathbb{P}$ be a partial ordering which adds a new function $r : \omega_1 \to \nu$ and satisfies the $(\kappa^+, \kappa^+, < \kappa)$-distributive law if κ is a successor cardinal, or the κ-c.c. otherwise. Then for all $\mu \geq \kappa^+$, $(\mathscr{P}_\kappa(\mu))^W$ is co-stationary in $W^{\mathbb{P}}$.*

Using the appropriate analog of Theorem 3.7 for Lemma 3.4 (2), and the proof of Theorem 5.6, one can use reverse Easton iteration on Levy collapses to obtain the following global result.

Theorem 5.7. *Suppose $\aleph_1 \leq \nu$ and there is a proper class of ν-Erdős cardinals in V. Then there is a class generic extension W of V in which the following holds: Suppose $\kappa' > \nu$ is regular, and $\mathbb{P}$ adds a new function $r : \omega_1 \to \nu$ and is κ'-c.c. (or just satisfies the $(\kappa^+, \kappa^+, < \kappa)$-d.l. for all successor cardinals $\kappa \geq \kappa'$, and is κ-c.c. for all strongly inaccessible $\kappa \geq \kappa'$). Then $(\mathscr{P}_\kappa(\lambda))^W$ is co-stationary in $W^{\mathbb{P}}$ for each regular $\kappa \geq \kappa'$ and all $\lambda \geq \kappa^+$ in $W^{\mathbb{P}}$.*

Example 5.8 (Namba forcing)**.** Suppose there is a class of ω_2-Erdős cardinals. Let W be a model satisfying CH, $|2^{\aleph_2}| = \aleph_3$, and Theorem 5.7 for $\nu = \aleph_2$. In W, let $\mathbb{N}$ denote Namba forcing. Namba proved that, under CH, $\mathbb{N}$ adds no new subsets of $\aleph_0$ and that $\aleph_2$ is collapsed to $\aleph_1$ [10]. By results of Bukovsky and Coplakova [3], $\mathbb{N}$ collapses $\aleph_3$ to $\aleph_1$. $|2^{\aleph_2}| = \aleph_3$ implies $\mathbb{N}$ is $\aleph_4$-c.c. Let H be $\mathbb{N}$-generic over W. Then $\aleph_1^{W[H]} = \aleph_1^W$ and $\aleph_2^{W[H]} = \aleph_4^W$.

$\mathscr{P}_{\aleph_1}(\lambda) \setminus W$ contains a cone for each cardinal $\lambda > \aleph_1$ in $W[H]$, by Fact 2.3. For each regular $\kappa \geq \aleph_2^{W[H]}$ in $W[H]$, for each cardinal $\lambda > \kappa$ in $W[H]$, $(\mathscr{P}_\kappa(\lambda))^W$ is co-stationary in $W[H]$, by Theorem 5.7.

Remark. Similar results hold for Prikry forcing in a model obtained by collapsing class many κ-Erdős cardinals, where κ is measurable.

References

[1] James E. Baumgartner. On the size of closed unbounded sets. *Annals of Pure and Applied Logic*, 54:195–227, 1991.

[2] A. Beller, R. Jensen, and P. Welch. *Coding the Universe*. Cambridge University Press, 1982.

[3] Lev Bukovsky and Eva Coplakova. Minimal collapsing extensions of models of ZFC. *Annals of Pure and Applied Logic*, 46:265–298, 1990.

[4] Moti Gitik. Nonsplitting subsets of $\mathscr{P}_\kappa(\kappa^+)$. *The Journal of Symbolic Logic*, 50(4):881–894, 1985.

[5] Peter Koepke. Some applications of short core models. *Annals of Pure and Applied Logic*, 37:179–204, 1988.

[6] Sabine Koppelberg. *Handbook of Boolean Algebra*, volume 1. North-Holland, 1989.

[7] Menachem Magidor. Representing sets of ordinals as countable unions of sets in the core model. *Transactions of the American Mathematical Society*, 317(1):91–126, 1990.

[8] Telis K. Menas. On strong compactness and supercompactness. *Annals of Mathematical Logic*, 7:327–359, 1974/75.

[9] William Mitchell. The covering lemma. In *Handbook of Set Theory*.

[10] Kanji Namba. Independence proof of (ω, ω_α)-distributive law in complete Boolean algebras. *Commentarii Mathematici Universitatis Sancti Pauli*, 19:1–12, 1971.

[11] Saharon Shelah. Independence of strong partition relation for small cardinals, and the free-subset problem. *The Journal of Symbolic Logic*, 45(3):505–509, 1980.

[12] Saharon Shelah. *Proper and Improper Forcing*. Springer-Verlag, second edition, 1998.

Kurt Gödel Research Center for Mathematical Logic, Universität Wien, Währingerstrasse 25, 1040 Wien, Austria
E-mail address: dobrinen@logic.univie.ac.at
URL: http://www.logic.univie.ac.at/∼dobrinen/

Kurt Gödel Research Center for Mathematical Logic, Universität Wien, Währingerstrasse 25, 1040 Wien, Austria
E-mail address: sdf@logic.univie.ac.at
URL: http://www.logic.univie.ac.at/∼sdf/