STRICT GENERICITY

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The purpose of this note is to show that unlike for set forcing, an inner model of a class-generic extension need not itself be a class-generic extension. Our counterexample is of the form $L[R]$, where R is a real both generic over L and constructible from $O^{\#}$.

Definition $\langle M, A\rangle$, M transitive is a *ground model* if $A \subseteq M$, $M \models ZFC + A$ Replacement and M is the smallest model with this property of ordinal height $ORD(M)$. G is *literally generic* over $\langle M, A\rangle$ if for some partial -ordering P definable over $\langle M, A\rangle$, G is P-generic over $\langle M, A\rangle$ and $\langle M[H], A, H\rangle \models ZFC + (A, H)$-Replacement for all P-generic H. S is *generic* over M if for some A, S is definable over $\langle M[G], A, G\rangle$ for some G which is literally generic over $\langle M, A\rangle$, and S is *strictly generic* over M if we also require that G is definable over $\langle M[S], A, S\rangle$.

The following is a classic application of Boolean-valued forcing and can be found in Jech [78], page 265.

Proposition 1. If G is P-generic over $\langle M, A\rangle$ where P is an element of M, S definable over $\langle M[G], A\rangle$ then S is strictly generic over M.

Proof Sketch. We can assume that P is a complete Boolean algebra in M and that $S \subseteq \alpha$ for some ordinal $\alpha \in M$. Then $H = G \cap P_0$ is P_0-generic over M, where $P_0 =$ complete subalgebra of P generated by the Boolean values of the sentences "$\hat{\beta} \in \sigma$", where $\beta < \alpha$ and σ is a P-name for S. Then H witnesses the strict genericity of S. $\dashv$

Now we specialize to the ground model $\langle L, \phi\rangle$, under the assumption that $O^{\#}$ exists.

Theorem 2. There is a real $R \in L[O^{\#}]$ which is generic but not strictly generic over L.

Our strategy for proving Theorem 2 comes from the following observation.

Proposition 3. If R is a real strictly generic over L then for some L-amenable A, Sat$(L[R])$ is definable from R, A, where Sat denotes the Satisfaction relation.

Proof. Suppose that A, G witness that R is strictly generic over L. Let G be P-generic over $\langle L, A\rangle$, P definable over $\langle L, A\rangle$, $R \in L[G]$, G definable over $\langle L[R], A\rangle$. Also assume that $\langle L[H], A, H\rangle \models ZFC + (A, H)$-replacement for all P-generic H. The latter implies that the Truth and Definability Lemmas hold for P-forcing, by a result of M. Stanley (See Friedman[95]). Then we have: $L[R] \models \varphi$ iff $\exists p \in G(p \Vdash \varphi$

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holds in $L[\sigma]$) where σ is a P-name for R and therefore $\mathrm{Sat}(L[R])$ is definable from R, $\mathrm{Sat}\langle L, A\rangle$. As A is L-amenable and $O^{\#}$ exists, $\mathrm{Sat}\langle L, A\rangle$ is also L-amenable. $\dashv$

Remarks (a) $\mathrm{Sat}(L[R])$ could be replaced by $\mathrm{Sat}(\langle L[R], A\rangle)$ in Proposition 3, however we have no need here for this stronger conclusion. (b) A real violating the conclusion of Proposition 3 was constructed in Friedman [94], however the real constructed there was not generic over L.

Thus to prove Theorem 2 it will suffice to find a generic $R \in L[O^{\#}]$ such that for each L-amenable A, $\mathrm{Sat}(L[R])$ is not definable (with parameters) over $\langle L[R], A\rangle$. First we do this not with a real R but with a generic class S, and afterwards indicate how to obtain R by coding S.

We produce S using the Reverse Easton iteration $P = \langle P_\alpha | \alpha \leq \infty\rangle$, defined as follows. P_0 =trivial forcing and for limit $\lambda \leq \infty$, Easton support is used to define P_λ (as a direct limit for λ regular, inverse limit otherwise). For singular α, $P_{\alpha+1} = P_\alpha * Q(\alpha)$ where $Q(\alpha)$ is the trivial forcing and finally for regular α, $P_{\alpha+1} = P_\alpha * Q(\alpha)$ where $Q(\alpha)$ is defined as follows: let $\langle b_\gamma | \gamma < \alpha\rangle$ be the L-least partition of the odd ordinals $< \alpha$ into α-many disjoint pieces of size α and we take a condition in $Q(\alpha)$ to be $p = \langle p(0), p(1), \ldots\rangle$ where for some $\alpha(p) < \alpha$, $p(n) : \alpha(p) \longrightarrow 2$ for each n. Extension is defined by: $p \leq q$ iff $\alpha(p) \geq \alpha(q)$, $p(n)$ extends $q(n)$ for each n and $q(n+1)(\gamma) = 1$, $\delta \in b_\gamma \cap [\alpha(q), \alpha(p)) \longrightarrow p(n)(\delta) = 0$. Thus if G is $Q(\alpha)$-generic and $S_n = \bigcup\{p(n) | p \in G\}$ then $S_{n+1}(\gamma) = 1$ iff $S_n(\delta) = 0$ for sufficiently large $\delta \in b_\gamma$.

Now we build a special P-generic $G(\leq \infty)$, definably over $L[O^{\#}]$. The desired generic but not strictly generic class is $S_0 = \bigcup\{p(0) | p \in G(\infty)\}$. We define $G(\leq i_\alpha)$ by induction on $A \in \mathrm{ORD}$, where $\langle i_\alpha | \alpha \in \mathrm{ORD}\rangle$ is the increasing enumeration of $I \cup \{0\}$, I = Silver Indiscernibles. $G(\leq i_0)$ is trivial and for limit $\lambda \leq \infty$, $G(< i_\lambda) = \bigcup\{G(< i_\alpha) | \alpha < \lambda\}$, $G(i_\lambda) = \bigcup\{G(i_{2\alpha}) | \alpha < \lambda\}$ (where $i_\infty = \infty$).

Suppose that $G(\leq i_\lambda)$ is defined, λ limit or 0, and we wish to define $G(\leq i_{\lambda+n})$ for $0 < n < \omega$. If n is even and $G(\leq i_{\lambda+n})$ has been defined then we define $G(\leq i_{\lambda+n+1})$ as follows: $G(< i_{\lambda+n+1})$ is the $L[O^{\#}]$-least generic extending $G(\leq i_{\lambda+n})$. To define $G(i_{\lambda+n+1})$ first form the condition $p \in Q(i_{\lambda+n+1})$ defined by: $\alpha(p) = i_{\lambda+n} + 1$, $p(m) \restriction i_{\lambda+n} = G(i_{\lambda+n})(m) = \bigcup\{q(m) | q \in G(i_{\lambda+n})\}$ for all m, $p(m)(i_{\lambda+n}) = 1$ iff $m > n$. Then $G(i_{\lambda+n+1})$ is the $L[O^{\#}]$-least $Q(i_{\lambda+n+1})$-generic (over $L[G(< i_{\lambda+n+1})]$) containing the condition p. If n is odd and $G(\leq i_{\lambda+n})$ has been defined then we define $G(\leq i_{\lambda+n+1})$ as follows: $G(< i_{\lambda+n+1})$ is the $L[O^{\#}]$-least generic extending $G(\leq i_{\lambda+n})$. To define $G(i_{\lambda+n+1})$, first form the condition $p \in Q(i_{\lambda+n+1})$ by: $\alpha(p) = i_{\lambda+n}$, $p(m)(\gamma) = G(i_{\lambda+n})(m)(\gamma)$ for $\gamma \neq i_{\lambda+n-1}$ and $p(m)(i_{\lambda+n-1}) = 0$ for all m. Then $G(i_{\lambda+n+1})$ is the $L[O^{\#}]$-least $Q(i_{\lambda+n+1})$-generic (over $L[G(< i_{\lambda+n+1})]$) containing the condition p. This completes the definition of $G(\leq \infty)$.

Now for each $i \in I \cup \{\infty\}$ and $n \in \omega$ let $S_n(i) = \bigcup\{p(n) | p \in G(i)\}$ and $S(i) = S_0(i)$, $S = S(\infty)$. We now proceed to show that S is not strictly-generic over L.

Definition. For $X \subseteq \mathrm{ORD}$, $\alpha \in \mathrm{ORD}$ and $n \in \omega$ we say that α is $X - \Sigma_n$ *stable* if $\langle L_\alpha[X], X \cap \alpha\rangle$ is Σ_n-elementary in $\langle L[X], X\rangle$. α is X-*stable* if α is $X - \Sigma_n$ stable for all n.

Lemma 4. For λ limit or 0, n even, $I_{\lambda+n+1}$ is not S-stable.

Proof. Let $i = i_{\lambda+n}$ and $j = i_{\lambda+n+1}$. Note that $S_m(j)$ is defined from $S(j)$ just as $S_m(\infty)$ is defined from $S(\infty) = S$. But $S(j) = S \cap j$ and for $M > n$, $S_m(j) \neq S_m(\infty)$ since $i \in S_m(j)$, $i \notin S_m(\infty)$. So j is not S-stable. $\dashv$

Lemma 5. For L-amenable $A \subseteq$ ORD, $i_{\lambda+n+1}$ is $(S, A) - \Sigma_n$ stable for sufficiently large limit λ, all $n \in \omega$.

Proof. Let $i = i_{\lambda+n+1}$ where λ is large enough to guarantee that i is A-stable. For $p \in P_{i+1} = P_i * Q(i)$ and $m \in \omega$, we let $(p)_m$ be obtained from p by redefining $p(i)(\bar{m}) = \phi$ for $\bar{m} > m$ and otherwise leaving p unchanged.

Claim. Suppose φ is Π_m relative to $S(i)$, B where $B \subseteq i$, $B \in L$. If $p \in P_{i+1}$, $p \Vdash \varphi$ then $(p)_m \Vdash \varphi$.

Proof of Claim. By induction on $m \geq 1$. For $m = 1$, if the conclusion failed then we could choose $q \leq (p)_1$, $q(< i) \Vdash\sim \varphi$ holds of $q(0), B$; then clearly $(q)_0 \Vdash\sim \varphi$, $(q)_0$ is compatible with p, which contradicts the hypothesis that $p \Vdash \varphi$. Given the result for m, if the conclusion failed for $m+1$ then we could choose $q \leq (p)_{m+1}$, $q \Vdash\sim \varphi$. Now write $\sim \varphi$ as $\exists x \psi$, ψ Π_m and we see that by induction we may assume that $(q)_m \Vdash \psi(\hat{x})$ for some x. But $(q)_m, p$ are compatible and $p \Vdash\sim \exists x \psi$, contradiction. $\dashv$ (Claim.)

Now we prove the lemma. Suppose φ is Π_n and true of $(S(i), A \cap i)$. Choose $p \in G(\leq i), p \Vdash \varphi$. Then by the Claim, $(p)_n \Vdash \varphi$. As i is A-stable, $(p)_n \Vdash \varphi$ in $P(\leq \infty)$. By construction $(p)_n$ belongs to $G(\leq \infty)$, in the sense that $(p)_n$ $(< i) \in G(< i) \subseteq G(< \infty)$ and $(p)_n(i) \in G(\infty)$. So φ is true of (S, A). $\dashv$

Theorem 6. S is generic, but not strictly generic, over L.

Proof. By Proposition 3 (which also holds for classes), if S were strictly generic over L then for some L-amenable A we would have that $\mathrm{Sat}\langle L[S], S\rangle$ would be definable over $\langle L[S], S, A\rangle$. But then for some n, all sufficiently large $(S, A) - \Sigma_n$ stables would be S-stable, in contradiction to Lemmas 4,5. $\dashv$

To prove Theorem 2 we must show that an S as in Theorem 6 can be coded by a real R in such a way as to preserve the properties stated in lemmas 4,5. We must first refine the above construction:

Theorem 7. Let $\langle A(i) | i \in I\rangle$ be a sequence such that $A(i)$ is a constructible subset of i for each $i \in I$. Then there exists S obeying Lemmas 4,5 such that in addition, $A(i)$ is definable over $\langle L_i[S], S \cap i\rangle$ for $i \in$ Odd $(I) = \{i_{\lambda+n} | \lambda$ limit or $0, n$ odd$\}$.

Proof. We use a slightly different Reverse Easton iteration: $Q(\alpha)$ specifies $n(\alpha) \leq \omega$ and if $n(\alpha) < \omega$, it also specifies a constructible $A(\alpha) \subseteq \alpha$; then conditions and extension are as before, except we now require that if $n(\alpha) < \omega$ then for p to extend

q, we must have $p(n(\alpha))(2\beta+2)=1$ iff $\beta \in A(\alpha)$, for $2\beta+2 \in [\alpha(q), \alpha(p))$. Then if $n(\alpha) < \omega$, the $Q(\alpha)$-generic will code $A(\alpha)$ definably (though the complexity of the definition increases with $n(\alpha) < \omega$).

Now in the construction of $G(\leq i_\alpha), \alpha \leq \infty$ we proceed as before, with the following additional specifications: $n(i_{\lambda+n}) = n$ for odd n and $n(i_{\lambda+n}) = \omega$ for even n (λ limit or 0). And for odd n we specify $A(i_{\lambda+n})$ to be the $A(i), i = i_{\lambda+n}$ as given in the hypothesis of the Theorem.

Lemma 4 holds as before; we need a new argument for Lemma 5. Note that for $i \in$ Odd(I) it is no longer the case that $P(< i) \Vdash Q(i) = Q(\infty) \cap L_i[G(<i)]$. Let $Q^*(i)$ denote $Q(\infty) \cap L_i[G(<i)]$, i.e., the forcing $Q(i)$ where $n(i)$ has been specified as ω. Define $(p)_m$ as before for $p \in P(\leq i)$.

Claim. Suppose $m \leq n+1$, n is even, $i = i_{\lambda+n+1}$ (λ limit or 0) and φ is Π_m relative to $S(i)$, B with parameters, where $B \subseteq i$, $B \in L$. If $p \in P(\leq i)$ (where $n(i) = n+1$) then $p \Vdash \varphi$ in $P(\leq i)$ iff $(p)_m \Vdash \varphi$ in $P^*(\leq i) = P(<i) * Q^*(i)$ iff $p \Vdash \varphi$ in $P^*(\leq i)$.

Proof. As in the proof of the corresponding Claim in the proof of Lemma 5. If $m = 1$ and $p \Vdash \varphi$ in $P(\leq i)$, then if the conclusion failed, we could choose $q \leq (p)_1$ in $P^*(\leq i)$, $q \Vdash\sim \varphi$; then (we can assume) $(q)_0 \Vdash\sim \varphi$ in $P(\leq i)$, but $(q)_0$ and p are compatible. The other implications are clear, as $P(\leq i) \subseteq P^*(\leq i)$. Given the result for $m \leq n$, φ Π_{m+1} and $p \Vdash \varphi$ in $P(\leq i)$, if the conclusion failed we could choose $q \leq (p)_{m+1}$ in $P^*(\leq i)$, $q \Vdash\sim \varphi$ (indeed, $q \Vdash\sim \psi(x)$ some x, where $\varphi = \forall x \psi$, $\psi \Sigma_m$); then $q \Vdash\sim \varphi$ in $P(\leq i)$, $(q)_m \Vdash\sim \varphi$ in $P^*(\leq i)$, $(q)_m \Vdash\sim \varphi$ in $P(\leq i)$ by induction. But $(q)_m, p$ are compatible in $P(\leq i)$, using the fact that $m \leq n$ and $q \leq (p)_{m+1}$; contradiction. And again the other implications follow, as $P(\leq i) \subseteq P^*(\leq i)$. $\dashv$ (Claim.)

Now the proof of Lemma 5 proceeds as before, using the new version of the Claim. $\dashv$

The choice of $\langle A(i) | i \in I\rangle$ that we have in mind comes from the next Proposition.

Proposition 8. For each n let $A_n = \{\alpha |$ For $i < j_1 < \ldots < j_n$ in $I, \alpha < i$, $(\alpha, j_1 \ldots j_n)$ and $(i, j_1 \ldots j_n)$ satisfy the same formulas in L with parameters $< \alpha\}$. Then any L-amenaable A is Δ_1-definable over $\langle L, A_n\rangle$ for some n.

Proof. For each $i \in I, A \cap L_i$ belongs to L and hence is of the form $t(i)(\vec{j}_0(i), i, \vec{\infty}(n(i)))$ where $t(i)$ is a Δ_0-Skolem term for L, $\vec{j}_0(i)$ is a finite sequence of indiscernibles $< i$ and $\vec{\infty}$ $(n(i))$ is any sequence of indiscernibles $> i$ of length $n(i) \in \omega$. By Fodor's Theorem and indiscernibility we can assume that $t(i) = t$, $\vec{j}_0(i) = \vec{j}_0$ and $n(i) = n$ are independent of i. To see that A is Δ_1-definable over $\langle L, A_{n+1}\rangle$ it suffices to show that for $\vec{i} < \vec{j}$ increasing sequences from A_{n+1} of length $n+1$, $\vec{i}$ and $\vec{j}$ satisfy the same formulas in L with parameters $<$ min $(\vec{i})$. But by definition, for $\alpha < \min(\vec{i})$ and

$\vec{i} = \{i_0, \ldots, i_n\}$, $\vec{j} = \{j_0, \ldots j_n\}$ we get: $L \models \varphi(\alpha, j_0 \ldots j_n) \longleftrightarrow \varphi(\alpha, i_0, j_1 \ldots j_n) \longleftrightarrow \varphi(\alpha, i_0, i_1, j_2 \ldots j_n) \longleftrightarrow \cdots \longleftrightarrow \varphi(\alpha, i_0, \ldots, i_n)$. $\dashv$

Now for $i \in I$ write $i = i_{\lambda+n}, \lambda$ limit or 0, $n \in \omega$ and let $A(i) = A_n \cap i$. Thus by Theorem 7 there is S obeying Lemmas 4,5 such that $A_n \cap i$ is definable over $\langle L_i[S], S \cap i\rangle$ for $i = i_{\lambda+n+1}, n$ even.

Proof of Theorem 2 First observe that as in Friedman [85], we may build $G(\leq \infty)$ to satisfy Theorem 7 for the preceding choice of $\langle A(i) | i \in I\rangle$ and in addition preserve the indiscernibility of Lim I. Then by the technique of Beller-Jensen-Welch [82], Theorem 0.2 we may code $(G(< \infty), S)$ by a real R, where $S = G_0(\infty)$. The resulting R obeys Lemma 4 because S is definable from R; to obtain Lemma 5 for R we must modify the coding of $(G(< \infty), S)$ by R in the following way: for inaccessible κ we require that any coding condition with κ in its domain reduce any dense $D \subseteq P^{<\kappa} = \{q | \alpha(q) < \kappa\}$ strictly below κ, when D is definable over $\langle L_\kappa[G(< \infty), S], G(< \kappa), S \cap \kappa\rangle$. This extra requirement does not interfere with the proofs of extendibility, distributivity for the coding conditions (see Friedman [96]).

Now to obtain Lemma 5 for R argue as follows: Given L-amenable A, choose n and λ large enough so that A is Δ_1-definable from A_n with parameters $< i_\lambda$. Then $i_{\lambda+n+1}$ is $(G(< \infty), S, A) - \Sigma_n$ stable. And also $A \cap i_{\lambda+n+1}$ is definable over $\langle L_i[G(< i), S \cap i], G(< i), S \cap i\rangle$ where $i = i_{\lambda+n+1}$. Thus if φ is Π_n and true of $G(< i), S \cap i, A \cap i$ then φ is forced by some coding condition $p \in P^{<i}$ (p in the generic determined by R) and hence by the $(G(< \infty), S, A) - \Sigma_n$ stability of i, we get that φ is true of $G(< \infty), S, A$. $\dashv$

We built R as in Theorem 2 by perturbing the indiscernibles. However with extra care we can in fact obtain indiscernible preservation.

Theorem 9. There is a real $R \in L[O^\#]$ such that R is generic but not strictly generic over L, L-cofinalities equal $L[R]$-cofinalities and $I^R = I$.

Proof. Instead of using the $i_{\lambda+n}, n \in \omega$ (λ limit or 0) use the $i^n_\alpha, n \in \omega$ where i^n_α = least element of A_n greater than i_α. Thus $\bigcup\{i^n_\alpha | n \in \omega\} = i_{\alpha+1}$ and as above we can construct S to preserve indiscernibles and L-cofinalities and satisfy that no i^n_α, n odd is S-stable, i^{n+1}_α is $(S, A) - \Sigma_n$ stable for large enough α, n (given any L-amenable A) and $A_n \cap i^{n+1}_\alpha$ is definable over $\langle L_i[S], S \cap i\rangle$ for $i = i^{n+1}_\alpha$, n even. Then code $(G(< \infty), S)$ by a real, preserving indiscernibles and cofinalities, requiring as before that for inaccessible κ, any coding condition with κ in its domain reduces dense $D \subseteq p^{<\kappa}$ strictly below κ, when D is definable over $\langle L_\kappa[G(< \kappa), S \cap \kappa], G(< \kappa), S \cap \kappa\rangle$. Then for any L-amenable A, i^{n+1}_α will be $(R, A) - \Sigma_n$ stable for sufficiently large α, n. This implies as before that R is not strictly generic. $\dashv$

Remark 1. A similar argument shows: For any $n \in \omega$ there is a real $R \in L[O^\#]$ which is strictly generic over L, yet G is not $\Sigma_n\langle L[R], R, A\rangle$ whenever $R \in L[G], G$ literally generic over $\langle L, A\rangle$. Thus there is a strict hierarchy within strict genericity, given by the level of definability of the literally generic G from the strictly generic real.

Remark 2. The nongeneric real R constructed in Friedman [94] is strictly generic over some $L[S]$ wher $R \notin L[S]$. The same is true of the real R constructed here to satisfy Theorem 2. This leads to:

Questions (a) Is there a real $R \in L[O^{\#}]$, R not strictly generic over any $L[S]$, $R \notin L[S]$? (b) Suppose R is strictly generic over $L[S]$, S generic over L. Then is R generic over L?

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