The Stable Core

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Abstract

Vopěnka [2] proved long ago that every set of ordinals is set-generic over HOD, Gödel's inner model of hereditarily ordinal-definable sets. Here we show that the entire universe V is class-generic over (HOD, S), and indeed over the even smaller inner model $\mathbb{S} = (L[S], S)$, where S is the *Stability predicate*. We refer to the inner model $\mathbb{S}$ as the *Stable Core of V*. The predicate S has a simple definition which is more absolute than any definition of HOD; in particular, it is possible to add reals which are not set-generic but preserve the Stable Core (this is not possible for HOD by Vopěnka's theorem).

For an infinite cardinal α, $H(\alpha)$ consists of those sets whose transitive closure has size less than α. Let C denote the closed unbounded class of all infinite cardinals β such that $H(\alpha)$ has cardinality less than β whenever α is an infinite cardinal less than β.

Definition 1 *For a finite $n > 0$, we say that α* is n-Stable in β *iff $\alpha < \beta$, α and β are limit points of C and $(H(\alpha), C \cap \alpha)$ is Σ_n elementary in $(H(\beta), C \cap \beta)$.*

The *Stability predicate* S places the Stability notion into a single predicate. S consists of all triples (α, β, n) such that α is n-Stable in β. The Δ_2 definable predicate S describes the "core" of V, in the following sense.

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Theorem 2 *V is generic over $(L[S], S)$ for an $(L[S], S)$-definable forcing. The same is true with $(L[S], S)$ replaced by $(M[S], S)$ for any definable inner model M.*

Note that since S is definable, HOD$[S]$ = HOD. So we get:

Corollary 3 *V is generic over HOD via a forcing which is definable in V.*

In general the inner model $L[S]$ may be strictly smaller than HOD, as illustrated by the next result. For any model N, let S^N denote N's intepretation of the predicate S.

Theorem 4 *(a) Suppose that V is a set-generic extension of M. Then S^M and S^V agree above α for some ordinal α. If V is a P-generic extension of M for a forcing P of size less than the least $\beth$ fixed point of M, then S^M equals S^V.*
(b) Assuming GCH, V has a generic extension of the form $L[R]$, where R is a real not set-generic over V and S^V equals $S^{L[R]}$.

Corollary 5 *It is consistent that $L[S]$ is properly contained in HOD.*

The corollary follows from Theorem 4 by taking V to be L in part (b) of the theorem and observing that in the resulting model $L[R]$, $L[S]$ equals L, R is not set-generic over L but by Vopěnka's theorem, R is set-generic over HOD.

The proof of Theorem 2 comes in two parts. First we show that V can be written as $L[F]$ where F is a function from the ordinals to 2 which "preserves" the Stability predicate S, in the sense that for (α, β, n) in S, α is n-Stable in β relative to F. Then we use this function to prove the genericity of V over $M[S]$ for any definable inner model M. The proof of Theorem 4 is via a refinement of the method of Jensen coding.

Forcing a Stability-preserving predicate

Our aim is to force a function F from the ordinals to 2 which codes V (i.e., $V = L[F]$) and which obeys the following.

(*) Suppose that $0 < n < \omega$ and α is n-Stable in β. Then α is n-Stable in β *relative to F*: $(H(\alpha), C \cap \alpha, F \cap \alpha)$ is Σ_n elementary in $(H(\beta), C \cap \beta, F \cap \beta)$.

To this end we define by induction on $\beta \in C$ a collection $P(\beta)$ of functions from β to 2. For $0 < n < \omega$, we say that β in C is *n-Admissible* iff β is a limit point of C and $(H(\beta), C \cap \beta)$ satisfies Σ_n replacement (with $C \cap \beta$ as an additional unary predicate). If α is n-Stable in some β then α is n-Admissible.

If β is not a limit point of C then $P(\beta)$ consists of all functions $p : \beta \to 2$ such that $p \restriction \alpha$ belongs to $P(\alpha)$ for all $\alpha \in C \cap \beta$. (Such functions exist, assuming that $P(\alpha)$ is nonempty for all $\alpha \in C \cap \beta$, a fact that we will verify.)

Suppose now that β is a limit point of C. Let $P(< \beta)$ denote the union of the $P(\alpha)$, $\alpha \in C \cap \beta$, ordered by extension. Assuming *extendibility for* $P(< \beta)$, i.e. the statement that for $\alpha_0 < \alpha_1 < \beta$ in C, each q_0 in $P(\alpha_0)$ can be extended to some q_1 in $P(\alpha_1)$, this forcing adds a generic function which we denote by $\dot{f} : \beta \to 2$. We say that $p : \beta \to 2$ is *n-generic for $P(< \beta)$* iff $G(p) = \{p \restriction \alpha \mid \alpha \in C \cap \beta\}$ meets every dense subset of $P(< \beta)$ of the form $\{q \in P(< \beta) \mid q \Vdash \varphi \text{ or } q \Vdash \sim \varphi\}$, where φ is a $\Pi_n(H(\beta), C \cap \beta, \dot{f})$ sentence with parameters from $H(\beta)$. We define $P(\beta)$ to consist of all $p : \beta \to 2$ which are n-generic for $P(< \beta)$ for all n such that β is n-Admissible.

Let P be the union of all of the $P(\beta)$'s, ordered by extension.

Lemma 6 *Assume Extendibility for P. Suppose that G is P-generic over V and let F be the union of the functions in G. Then $V = L[F]$ and (*) holds for F. Moreover, V satisfies replacement with F as an additional predicate.*

Proof. Extendibility implies that it is dense to code any set of ordinals into the P-generic function F, from which it follows that V is contained in $L[F]$. As $F \restriction \alpha$ belongs to V for each $\alpha \in C$ it also follows that $L[F]$ is contained in V and therefore $L[F]$ equals V.

Suppose that $0 < n < \omega$ and α is n-Stable in β. The relation $q \Vdash \varphi$ for q in $P(< \beta)$ and $\Pi_1(H(\beta), C \cap \beta, \dot{f})$ sentences φ with parameters from $H(\beta)$ is Π_1 over $(H(\beta), C \cap \beta)$: $q \Vdash \varphi$ iff for all $r \leq q$ and transitive T with $\mathrm{Ord}(T) = \gamma \leq \mathrm{Dom}\ (r)$, $(T, C \cap \gamma, r) \vDash \varphi$. It then follows by induction on $n \geq 1$ that the relation $q \Vdash \varphi$ for q in $P(< \beta)$ and $\Pi_n(H(\beta), C \cap \beta, \dot{f})$ sentences φ with parameters from $H(\beta)$ is Π_n over $(H(\beta), C \cap \beta)$ (and the same for α). As $F \restriction \alpha$ is n-generic for $P(< \alpha)$, it follows that any true $\Pi_n(H(\alpha), C \cap \alpha, F \restriction \alpha)$ sentence φ with parameters from $H(\alpha)$ is forced by some condition $F \restriction \alpha_0$, $\alpha_0 \in C \cap \alpha$. But then as α is n-Stable in β, $F \restriction \alpha_0$ also forces "φ holds in $(H(\beta), C \cap \beta, \dot{f} \restriction \beta)$"; by the n-genericity of $F \restriction \beta$, it follows that φ holds in $(H(\beta), C \cap \beta, \dot{f} \restriction \beta)$ when $\dot{f} \restriction \beta$ is interpreted as the real $F \restriction \beta$. Thus we have proved that α is n-Stable in β relative to F.

To verify replacement relative to F, we need only observe that the above implies that for each n, if α is n-Stable in Ord (i.e., $(H(\alpha), C \cap \alpha)$ is Σ_n elementary in (V, C)) then it remains so relative to F. $\square$

We now turn to extendibility for P.

Lemma 7 *Suppose that $\alpha < \beta$ belong to C and p belongs to $P(\alpha)$. Then p has an extension q in $P(\beta)$.*

Proof. By induction on β. The statement is immediate by induction if β is not a limit point of C.

Suppose that β is a limit point of C but is not 1-Admissible. Then there is a closed unbounded subset D of $C \cap \beta$ of ordertype less than β whose intersection with each of its limit points $\gamma < \beta$ is Δ_1 definable over $(H(\gamma), C \cap \gamma)$. We can assume that both α and the ordertype of D are less than the minimum of D. Now enumerate D as $\beta_0 < \beta_1 < \cdots$ and using the induction hypothesis, successively extend p to $q_0 \subseteq q_1 \subseteq \cdots$ with q_i in $P(\beta_i)$, taking unions at limits. Note that for limit i, q_i is indeed a condition because β_i is not 1-Admissible. The union of the q_i's is the desired extension of p in $P(\beta)$.

Next suppose that β is n-Admissible but not $n+1$-Admissible for some finite $n > 0$:

If β is a limit of n-Stables (i.e., the set of $\alpha < \beta$ which are n-Stable in β is cofinal in β), then proceed as in the previous paragraph: Choose a closed unbounded subset D of $C \cap \beta$ of ordertype less than β consisting of n-Stables in β, whose intersection with each of its limit points $\gamma < \beta$ is Δ_{n+1}

definable over $(H(\gamma), C \cap \gamma)$. Assume that both α and the ordertype of D are less than the minimum of D, enumerate D as $\beta_0 < \beta_1 < \cdots$ and using the induction hypothesis, successively extend p to $q_0 \subseteq q_1 \subseteq \cdots$ with q_i in $P(\beta_i)$, taking unions at limits. For limit i, q_i is indeed a condition because β_i is not $n+1$-Admissible and as it is a limit of n-Stables, q_i is n-generic for $P(<\beta_i)$. The union of the q_i's is the desired extension of p in $P(\beta)$.

If β is not a limit of n-Stables then β must have cofinality ω (else by n-Admissibility, we could find cofinally many n-Stables in β using the fact that β has uncountable cofinality). It suffices to show that any condition q in $P(<\beta)$ can be extended to decide (i.e. force or force the negation of) each of fewer than β-many Π_n sentences with parameters from $H(\beta)$ (given this, we can extend p in ω steps to a condition in $P(\beta)$ which is n-generic). To show this, let $(\varphi_i \mid i < \delta)$ enumerate the given collection of Π_n sentences and if $n > 1$, let D consist of all γ which are limits of $(n-1)$-Stables in β and large enough so that $H(\gamma)$ contains both q and this enumeration. (If $n = 1$ then let D consist of all γ which are limit points of C and large enough so that $H(\gamma)$ contains both q and this enumeration.) Now extend q successively to elements q_i of $P(\gamma_i)$, where $\gamma_{i+1} \geq \gamma_i$ is the least element of D so that either q_i forces φ_i or q_{i+1} forces $\psi_i =$ the negation of φ_i (with corresponding witness to the Σ_n sentence ψ_i), taking unions at limits. For limit i, q_i is a condition as γ_i is not n-Admissible but (in case $n > 1$) is a limit of $(n-1)$-Stables. (The failure of γ_i to be n-Admissible uses the fact that the set of $j < i$ such that q_{j+1} forces the negation of φ_j can be treated as a parameter in $H(\gamma_i)$.) As β is n-Admissible, this construction results in a sequence of q_i's of length δ, whose union it the desired extension of q deciding all of the given Π_n sentences.

Finally, suppose that β is n-Admissible for every finite n. Choose C to be closed unbounded in β so that any $\gamma < \beta$ which is a limit point of C is a limit of n-Stables for every n. (Note that we may choose C to be any cofinal ω-sequence if β has cofinality ω.) Assume that α is less than the least element of C and enumerate C as $\beta_0 < \beta_1 < \cdots$. Then successively extend p to $q_0 \subseteq q_1 \subseteq \cdots$ with q_i in $P(\beta_i)$, taking unions at limits, and note that for limit i, q_i is a condition because its n-genericity follows from the fact that β_i is a limit of n-Stables. The union of the q_i's is the desired q. □

V is generic over the Stability predicate

Now fix a function $F : \mathrm{Ord} \to 2$ as in the last section, i.e. with the following properties:

1. $V = L[F]$, (V, F) satisfies replacement with a predicate for F.
2. If $0 < n < \omega$ and α is n-Stable in β, then α is n-Stable in β relative to F.

We devise a forcing Q definable over $(L[S], S)$ such that for some Q-generic G, $V = L[S, G] = L[G]$ and G is definable over (V, F).

The language $\mathcal{L}$ is defined inductively as follows, where $\dot{F}$ is a unary function symbol.

1. For each ordinal α, "$\dot{F}(\alpha) = 0$" and "$\dot{F}(\alpha) = 1$" are sentences of $\mathcal{L}$.
2. If Φ is a set of sentences of $\mathcal{L}$ and Φ belongs to $L[S]$, then $\bigwedge \Phi$ and $\bigvee \Phi$ are sentences of $\mathcal{L}$.

A sentence φ of $\mathcal{L}$ is *valid* iff it is true when the symbol $\dot{F}$ is replaced by any function that belongs to a set-generic extension of $L[S]$. This notion is $L[S]$-definable and moreover if φ is a sentence of $L[S]$ and M is any outer model of $L[S]$, then φ is valid in $L[S]$ iff it is valid in M[1].

Now let T consist of all sentences of $\mathcal{L}$ of the form

$$\bigwedge(\Phi \cap H(\alpha)) \to \bigwedge(\Phi \cap H(\beta)),$$

where for some $\alpha < \beta$ and $1 < n < \omega$ we have:

(a) Φ is Σ_n definable over $H(\beta) \cap L[S]$ using parameters from $H(\alpha) \cap L[S]$.
(b) α is n-Stable in β (in V).

Note that (a) implies that Φ is Σ_n definable over $(H(\beta), C \cap \beta)$ (using parameters from the $H(\alpha)$ of V). It follows that the sentences in T are true

[1] Indeed, if there is a function witnessing the non-validity of φ in a set-generic extension of M then we may assume that this generic extension is $M[G]$ where G is generic for a Lévy collapse making φ countable; then $L[S][G]$ also has a witness to the non-validity of φ, by Lévy absoluteness. Conversely, if the non-validity of φ is witnessed in a set-generic extension of $L[S]$ then this will happen in $L[S][G]$ where G is Lévy collapse generic over $L[S]$. Choose a condition in the Lévy collapse forcing this and H containing this condition which is Lévy collapse generic over M; then the non-validity of φ is witnessed in $M[H]$, a set-generic extension of M.

when $\dot{F}$ is interpreted as F. Also note that T is $(L[S], S)$ definable, as (b) is expressed by the Stability predicate S.

The desired forcing Q consists of all sentences φ of $\mathcal{L}$ which are consistent with T, in the sense that for no subset T_0 of T is the sentence $\bigwedge T_0 \to \sim \varphi$ valid. The sentences in Q are ordered by: $\varphi \leq \psi$ iff T implies $\varphi \to \psi$.

Lemma 8 *Q has the Ord-chain condition, i.e., any $(L[S], S)$-definable maximal antichain in Q is a set.*

Proof. Suppose that A is an $(L[S], S)$-definable maximal antichain and consider $\Phi = \{\sim \varphi \mid \varphi \in A\}$. Then Φ is also $(L[S], S)$-definable. Choose n so that Φ is Σ_n-definable over $(L[S], S)$ and choose α to be n-Stable in Ord and large enough so that $H(\alpha) \cap L[S]$ contains the parameters in the Σ_n definition of Φ. Then T together with $\Phi \cap H(\alpha)$ implies $\Phi \cap H(\beta)$ for all β greater than α which are n-Stable in Ord and since there are arbitrarily large such β, T together with $\Phi \cap H(\alpha)$ implies all of Φ. It follows that A equals $A \cap H(\alpha)$: Otherwise let φ belong to $A \setminus H(\alpha)$. As $\sim \varphi$ belongs to Φ it is implied by T together with $\Phi \cap H(\alpha)$. But as A is an antichain, T together with φ implies $\Phi \cap H(\alpha)$ and therefore T together with φ implies $\sim \varphi$, contradicting the fact that φ belongs to Q. $\square$

Now it is easy to see that $V = L[F] = L[G]$ where G is Q-generic over $(L[S], S)$: Let G consist of all sentences in Q which are true when $\dot{F}$ is interpreted as F. It is obvious that G intersects all maximal antichains of Q which are sets in $L[S]$, as if the set A is an antichain missed by G then $\bigwedge\{\sim \varphi \mid \varphi \in A\}$ is consistent with T and witnesses the failure of A to be maximal. By Lemma 8 this gives full genericity over $(L[S], S)$.

The above argument was carried out for the ground model $L[S]$. But the same argument can be used for any ground model $M[S]$ provided M is a definable inner model; simply replace n by $n - k - 1$ in (a) above, where M is Σ_k-definable. This completes the proof of Theorem 2.

Preserving S when coding

We sketch the proof of Theorem 4. Part (a) of the theorem is clear, because when applying a set-forcing P, the Stability predicate is not affected above the size of P.

(b) is proved as follows: Again write V as $L[F]$ where F preserves the Stability predicate. Now we describe a version of Jensen coding that produces a real R such that:

i. R is class-generic but not set-generic over (V, F).
ii. V is contained in $L[R]$ and F is definable in $L[R]$ with parameter R.
iii. R preserves the Stability predicate: the S of V equals the S of $L[R]$.

Note that as we have assumed GCH, the class C is just the class of all infinite cardinals.

Let P_0 be the version of Jensen coding defined in [1], Section 4.3, but with the following modification: We require that for limit cardinals α which are n-Admissible, conditions in $P_0^{\emptyset_\alpha} \setminus P_0(<\alpha)$ are n-generic for $P_0(<\alpha)$, i.e., decide all $\Pi_n(H(\alpha), C \cap \alpha, F \restriction \alpha, \dot{G}(<\alpha))$ sentences, where $\dot{G}(<\alpha)$ denotes the $P_0(<\alpha)$-generic. This thinning of the forcing does not affect the proofs of extendibility and distributivity and has the consequence that if G_0 is P_0-generic and α is n-Stable in β relative to F then α is also n-Stable in β relative to F, G_0. As F preserves the Stability predicate, it follows that the P_0-generic real R does as well. □

Some final remarks

Is the Stable Core $\mathbb{S}$ the long sought-after "ultimate core model" of V? To answer this it is necessary to first answer the following questions:

Question 1. Does the existence of large cardinals in V imply their existence in the Stable Core? Is the Stable Core rigid in the sense that there is no nontrivial elementary embedding of it to itself?

As V is generic over the Stable Core there is reason to hope for a positive answer to Question 1.

Question 2. Does the Stable Core satisfy GCH and □ principles?

Unfortunately the Stable Core exhibits no condensation properties which would suggest a positive answer to Question 2. One may however hope to enrich the Stability predicate to obtain condensation and a positive answer to Question 2 for a modified version of the Stable Core.

Regardless of the answers to the above questions, the Stable Core does at least reveal the following: The notion of Stability is fundamental to our understanding of the structure of the set-theoretic universe.

References

[1] Friedman, S., *Fine structure and class forcing*, de Gruyter Series in Logic and its Applications, Volume 3, 2000.

[2] Vopěnka, P. and Hájek, P., *The theory of semisets*, Academia (Publishing house of the Czech Academy of Sciences), Prague, 1972.