Stable Axioms of Set Theory

Sy-David Friedman*
Kurt Gödel Research Center for Mathematical Logic
Universität Wien

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Abstract

I discuss criteria for the choice of axioms to be added to ZFC, introducing the criterion of *stability*. Then I examine a number of popular axioms in light of this criterion and propose some new axioms.

1. Criteria for New Axioms

The incompleteness phenomenon is particularly evident in the field of set theory: The standard axiom system ZFC for set theory has a vast range of different types of models. Some people have suggested that this is an essential feature of set theory, because ZFC exhausts our set-theoretic intuition. A more optimistic view is that by increasing our knowledge of set theory, we will arrive at new axioms which are so compelling in their naturalness and in their ability to clarify the structure of the set-theoretic universe that we can assert that our intuition is in fact strong enough to justify adding them to ZFC as standard axioms.

By adopting new axioms, we narrow our view of set theory. Therefore it is important to suggest criteria for doing so. Below are some criteria to consider.

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Naturalness. Axioms should be directly concerned with the structure of the set-theoretic universe V.

Natural axioms typically make assertions about the height or width of the universe, or assert the existence of certain types of elementary embeddings between inner models.

Power. Axioms should explain a lot.

Powerful axioms give us more detail about the structure of V than ZFC alone can provide.

Consistency. Axioms should be consistent (with ZFC).

This criterion presents a problem: How are we to know whether or not a proposed axiom is consistent? By Gödel's second incompleteness theorem there is no way to definitively establish the consistency of an axiom. This leads to:

> *An axiom is deemed to be consistent as long as no proof of its inconsistency is currently known.*

Thus what we accept as consistent can change with time. I see no alternative to this.

The above three criteria are in my view essential to the choice of any new axiom. The next criterion however is not.

Stability. (a) (Syntactic stability) Axioms should be unaffected by small changes. In particular, small changes should not lead to inconsistency. (b) (Semantic stability) A small extension of a model of the axioms should also be a model.

As with naturalness or power, I do not attempt to give here a rigorous definition of stability. In particular, I offer no precise definition of the notion of small extension used to define semantic stability. But surely the addition of one Cohen real must be viewed as a small extension. And the notion of small extension cannot be restricted to just set-generic extensions, as class-forcing provides consistent ways to enlarge the set-theoretic universe in the same way that set-forcing does. Although proper classes do not themselves belong to the universe of sets, via forcing they do produce new sets with important properties that cannot fairly be excluded from consideration.

As we shall see, stability is very restrictive. Indeed, it is violated by almost all of the axioms that have been proposed as candidates for addition to ZFC. Stability is however appealing, as it rules out the choice of axioms which are obtained by slightly weakening inconsistent principles. And as we will see below, there are attractive proposals for stable axioms of considerable naturalness and strength.

2. Examples

I first examine some well-known axioms in terms of the criteria of naturalness, power and stability.

a. V=L

Of course this axiom is natural and very powerful. But by the work of Cohen we know that $V = L$ is easily contradicted by forcing, and therefore the criterion of stability is violated. The same problem exists with any axiom of the form

$$V = L[G]$$

where G is P-generic over L for an L-definable forcing P, as one can similarly violate this easily by further forcing.

b. Large cardinals

Typically these are of the form

There exists $j : V \to M$, where M is "close" to V.

Certainly such axioms are natural and very powerful. They are however unstable. If we require M to equal V, we have a contradiction, by Kunen's theorem [5]. If we only require M to agree with V up to $j(\kappa)$ where κ is the critical point of j, then by stability, we must also allow agreement up to arbitrary iterates of j applied to κ, another contradiction to Kunen's theorem.

c. Determinacy

I am not referring to the full axiom AD, as this contradicts the axiom of choice, but rather to determinacy for sets of reals that are "definable" in some sense. This axiom has proved to be powerful, and in addition appears to be stable.

Unfortunately the existence of strategies for infinite games is not directly concerned with the structure of V, in violation of naturalness. I will however argue below that some definable determinacy is a consequence of natural and stable axioms, even though determinacy itself does not qualify as one.

d. Generic absoluteness

Absoluteness asserts that the truth of certain formulas is not affected by enlarging the universe in certain ways. The classical example of this is Lévy-Shoenfield absoluteness, which says that $\Sigma_1(H(\omega_1))$ formulas (with parameters) are absolute for arbitrary extensions.

Typically, one considers only *set-generic* absoluteness, in which only set-generic extensions of the universe are allowed, and only formulas which are first-order over $H(\omega_2)$. This is to enable the formulation of consistent principles. However such principles fail as soon as more general extensions, such as class-generic extensions, are allowed. Even when restricted to set-generic extensions, instability is present: $\Sigma_1(H(\kappa))$ absoluteness for ccc forcing extensions is inconsistent when κ is greater than $2^{\aleph_0}$. As one typically has $2^{\aleph_0} = \aleph_2$ in the context of set-generic absoluteness principles, inconsistency already occurs when κ is $\aleph_3$.

e. Forcing axioms

The most common such axioms assert that for certain set-forcings P and certain collections X of dense subsets of P, there is a compatible subset G of P which intersects all elements of X. The classical example is Martin's axiom (at ω_1), which asserts this for ccc P and collections X of cardinality ω_1.

As with the set-generic absoluteness principles, these axioms are unstable. For example, one cannot have this forcing axiom for κ-many dense sets with respect to even ccc forcings when κ is at least $2^{\aleph_0}$.

Other types of forcing axioms have also been considered. Foreman, Magidor and Shelah proposed:

Every set-forcing either adds a real or collapses a cardinal.

Little is known about this interesting axiom. A stable version would however require the consideration of more than just set-generic extensions.

An axiom of Chalons (as modified by Larson and then Hamkins) states:

"If a statement with real parameters holds in a set-forcing extension and all further set-forcing extensions, then it holds in V; moreover this property is not only true in V, but also in all set-generic extensions of V."

Woodin proved the consistency of this axiom from large cardinals. Unfortunately, even a weak form of this axiom is inconsistent when "set-forcing" is replaced by "class-forcing". A consistent class-forcing version of this axiom is not known.

f. Strong logics

These are logics whose set of validities is large and remains unchanged by set-forcing. One can obtain such a logic as follows: Say that φ is $**$-provable iff for some set-forcing P, if P belongs to V_α and V_α satisfies ZFC, then V_α^{P*Q} satisfies φ for all Q in V_α^P. Woodin proposes the use of such a strong logic, together with the existence of a proper class of Woodin cardinals. This gives a $**$-complete theory of $H(\omega_1)$ and, assuming that $H(\omega_2)$ is obtained by forcing with Woodin's forcing P_{max} over $L(R)$, gives a $**$-complete theory of $H(\omega_2)$. Therefore under Woodin's assumptions, the theory of $H(\omega_2)$ cannot be changed by set-forcing.

There are several difficulties with this approach.

i. The assumption of the existence of a proper class of Woodin cardinals is left unjustified.

However I will propose below some natural and stable axioms which lead to an *inner model* for this assumption.

ii. Although strong logics are immune to set-forcing, they are not immune to class-forcing.

As mentioned earlier, class-forcing methods provide consistent ways to enlarge the set-theoretic universe in the same way that set-forcing methods do. Therefore adopting as new axioms the validities of a logic with only set-generic absoluteness does not achieve stability. One needs at least a plausible notion of "acceptable class forcing" and a corresponding property of absoluteness for such class forcings.

iii. The axiom asserting that $H(\omega_2)$ is obtained by set-forcing over $L(R)$ is easily contradicted by class-forcing, and therefore as in ii. leads to instability.

3. Some stable axioms of strong absoluteness

As mentioned earlier, the typical absoluteness principles which generalise

Lévy-Shoenfield absoluteness refer exclusively to set-generic extensions, and are unstable. The Lévy-Shoenfield absoluteness principle itself, however, applies to arbitrary extensions. The *strong absoluteness principles* discussed below are in the tradition of Lévy-Shoenfield and impose no genericity requirement on the extensions considered. This leads to the possibility of obtaining stable axioms.

By *extension of* V I shall mean a ZFC model V^* which contains V and has the same ordinals as V. This is best formalised by regarding V as a countable transitive model of ZFC and allowing V^* to range over countable transitive ZFC models which contain V and have the same ordinal height as V.

Any consistent generalisation of Lévy-Shoenfield absoluteness must deal with the following two obstacles:

Counterexample 1. There is a Σ_1 formula with parameters from $H(\omega_2)$ which holds in some set-generic extension V^* of V but not in V.

Counterexample 2. There is a Σ_1 formula with parameters from $H((2^{\aleph_0})^+)$ which holds in some ccc set-generic extension V^* of V but not in V.

Counterexample 1 is witnessed by the formula "ω_1^V is countable". Counterexample 2 is witnessed by the formula "There is a real not in $\mathcal{P}(\omega)^V$".

Let us say that a Σ_1 *absoluteness principle* is a principle asserting the absoluteness of certain Σ_1 formulas with certain parameters with respect to certain extensions of V. Our counterexamples imply that a consistent Σ_1 absoluteness principle must impose some restriction either on the choice of formulas, on the choice of parameters, on the choice of extensions, or a combination of the three.

I offer three proposals. The first allows arbitrary parameters, at the cost of restricting the choice of extensions. The second allows arbitrary extensions, at the cost of restricting the allowable parameters. And the third weakens the parameter restrictions of the second proposal, at the cost of restricting the choice of formulas.

a. Σ_1 absoluteness with arbitrary parameters.

A first attempt to avoid Counterexample 1 is to require that V and V^* have the same ω_1. But Σ_1 absoluteness with parameters from $H(\omega_2)$ even for ω_1-preserving extensions is also inconsistent: Let A be a stationary subset of ω_1. Then the formula which asserts that A contains a CUB subset is Σ_1 and true in a cardinal-preserving (set-generic) extension; therefore Σ_1 absoluteness with parameters from $H(\omega_2)$ for ω_1-preserving extensions implies that A contains a CUB subset. But there are disjoint stationary subsets of ω_1, giving disjoint CUB subsets of ω_1, a contradiction.

Even requiring stationary-preservation at ω_1 (i.e, that stationary subsets of ω_1 in V remain stationary in V^*) results in inconsistency:

Theorem 1. There exists an extension V^* of V which is stationary-preserving at ω_1 such that some Σ_1 sentence with parameters from $H(\omega_2)^V$ true in V^* is false in V.

Proof. By a theorem of Beller-David (see [1]) there is an extension V^* with the same ω_1 as V containing a real R such that $L_\alpha[R]$ fails to satisfy ZFC for each ordinal α. Moreover, V^* is stationary-preserving at ω_1 (see [2]). Now suppose that the Theorem fails. Then there is such a real R in V, as this property of R can be expressed by a Σ_1 sentence with parameters R and ω_1. In particular, ω_1 is not inaccessible to reals. It is easy to see that the failure of the Theorem implies that Σ^1_3-absoluteness holds between V and its stationary-preserving at ω_1 extensions. It then follows from Lemma 7 of [2] that ω_1 is inaccessible to reals after all, contradiction. □

One could continue to make further restrictions on the extension V^*, such as stationary-preservation at ω_1 together with full cardinal-preservation, in the hope of achieving the consistency of $\Sigma_1(H(\omega_2))$ absoluteness (without imposing the requirement that V^* be a set-generic extension of V). But we must also reckon with Counterexample 2.

A possible solution is described by the following. I say that an extension V^* of V *strongly preserves* $H(\kappa)$ iff the $H(\kappa)$ of V^* equals the $H(\kappa)$ of V and all cardinals of V less than or equal to card $(H(\kappa)) = 2^{<\kappa}$ remain cardinals in V^*.

Σ_1 *absoluteness with arbitrary parameters.* Suppose that κ is an infinite cardinal and a Σ_1 formula φ with parameters from $H(\kappa^+)$ holds in an extension V^* of V which strongly preserves $H(\kappa)$. Then φ holds in V.

When κ is ω, this is Lévy-Shoenfield absoluteness. When κ is ω_1, this asserts $\Sigma_1(H(\omega_2))$ absoluteness for extensions which do not add reals and which preserve cardinals up to $2^{\aleph_0}$. Note that in the presence of $\sim$ CH, this axiom does rule out the two standard set-forcings for destroying the stationarity of a subset of ω_1.

b. Σ_1 *absoluteness for arbitrary extensions.*

Counterexample 2 implies that to obtain a consistent version of absoluteness for arbitrary Σ_1 formulas with respect to arbitrary extensions, we must impose some restriction on our choice of parameters. A suitable restriction is perhaps provided by the following definition.

Definition. An *absolute cardinal-formula* is a parameter-free formula of the form

$$\varphi(\kappa) \text{ iff } L(H(\kappa)) \vDash \psi(\kappa),$$

where κ ranges over cardinals. We say that the cardinal κ is *absolute* between V and an extension V^* iff there is an absolute cardinal-formula which has κ as its unique solution in both V and V^*.

Σ_1 *absoluteness for arbitrary extensions.* Suppose that the cardinals $\kappa_1 < \cdots < \kappa_n$ are absolute between V and V^*, where V and V^* have the same cardinals $\leq \kappa_n$. Then any Σ_1 formula with parameters $\kappa_1, \ldots, \kappa_n$ which holds in V^* also holds in V.

Remark. David Asperó and I showed that if one drops the requirement that cardinals up to κ_n are preserved, then the above principle is inconsistent.

c. Cardinality and cofinality absoluteness principles.

Other forms of strong absoluteness result by considering special types of Σ_1 formulas. First I introduce a variant of the notion of absolute parameter.

Definition. Suppose that α is an ordinal, P is a subset of V and V^* is an extension of V. Then α is *weakly absolute relative to parameters in P between*

V *and* V^* iff there is a formula with parameters from P which defines α not only in V, but also in V^*.

For cardinality and cofinality we have the following absoluteness principles.

Cardinal absoluteness. Suppose that α is an ordinal, V^* is an extension of V and α is weakly absolute relative to bounded subsets of α between V and V^*. Then if α is collapsed (i.e., not a cardinal) in V^*, it is also collapsed in V.

Cofinality Absoluteness. Suppose that α is an ordinal, V^* is an extension of V and α is weakly absolute relative to bounded subsets of α between V and V^*. Then if α is singular in V^*, it is also singular in V.

The consistency strength of strong absoluteness principles

I do not know if any of the above principles of strong absoluteness are provably consistent relative to large cardinals. In this subsection I provide some lower bounds on their consistency strength.

Theorem 2. Σ_1 absoluteness with arbitrary parameters implies that the GCH fails at every infinite cardinal, and for regular uncountable κ, there is no κ-Suslin tree.

Proof. Suppose that the GCH held at the infinite cardinal κ. Choose $S \subseteq \kappa^+$ to be a fat-stationary subset of κ^+ which does not contain a CUB subset. (S is *fat-stationary* iff $S \cap C$ contains closed subsets of any ordertype less than κ^+, for each CUB $C \subseteq \kappa^+$.) The existence of such a set is guaranteed by a result of Krueger [4]. Then the forcing P that adds a CUB subset to S using closed subsets of S ordered by end-extension has cardinality κ^+ and, using the fatness of S, is κ^+-distributive. It follows that $H(\kappa^+)$ is strongly preserved by P. But a CUB subset of S witnesses a Σ_1 formula with parameter S not true in the ground model, in contradiction to our hypothesis.

Suppose that there were a κ-Suslin tree T for an uncountable regular cardinal κ. Then forcing with this tree strongly preserves $H(\kappa)$ and adds a witness to a Σ_1 formula with parameter T not witnessed in the ground model, in contradiction to our hypothesis. □

By work of Mitchell [6]:

Corollary 3. Σ_1 absoluteness with arbitrary parameters implies the consistency of a measurable cardinal κ of Mitchell order κ^{++}.

A lower bound for the strength of Σ_1 absoluteness for arbitrary extensions follows from the arguments of [3]:

Theorem 4. Suppose that Σ_1 absoluteness for arbitrary extensions holds. Then there is an inner model with a strong cardinal.

For cardinal absoluteness we have:

Theorem 5. Cardinal absoluteness implies that for each infinite cardinal κ, κ^+ is greater than (κ^+ of HOD).

Proof. If G is generic for the Lévy collapse of κ^+ to ω, then HOD is the same in V and in $V[G]$, by the homogeneity of the forcing. This contradicts our absoluteness hypothesis. $\square$.

By [7] and [8]:

Corollary 6. Cardinal absoluteness implies that there is an inner model with a strong cardinal, and, if there is a proper class of subtle cardinals, there is an inner model with a Woodin cardinal.

It is possible to extend Corollary 6 to obtain inner models with a proper class of Woodin cardinals containing any given set, under the assumption of cardinal absoluteness and a proper class of subtle cardinals. This is more than enough to imply Projective Determinacy.

Corollary 6 also holds for cofinality absoluteness, as the latter implies cardinality absoluteness.

4. Some Final Thoughts

The most important axioms of set theory that have been explored until now have arisen unavoidably out of the need to solve central problems in the field. This is especially true of the large cardinal axioms, which have even provided a measure for the consistency strength of virtually all set-theoretic statements. In my view we should not however impatiently assert that any axiom is "correct" until we can derive it from axioms which meet criteria like the ones discussed above.

I believe that the axioms of finite set theory "capture" the first-order theory of $H(\omega)$, as $H(\omega)$ is the unique well-founded model of finite set theory and no clear examples of ill-founded models are known. PD (projective determinacy) provides attractive axioms for the first-order theory of $H(\omega_1)$. The strong absoluteness axioms of the previous section are stable and natural axioms which lead to inner models with Woodin cardinals, and therefore to PD. Therefore PD follows from natural and stable axioms, and in my view can be judged to be "correct". Although I have not seen a convincing argument that PD "captures" the first-order theory of $H(\omega_1)$, I do believe this to be the case.

Though the axioms of strong absoluteness lead to the existence of *inner models* with Woodin cardinals, they do not produce large cardinals in V. Fortunately, large cardinals in V do not appear to be necessary to obtain "correct" axioms which capture the first-order theory of $H(\omega_2)$, a goal which in my view is still well beyond our reach.

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