Research at the Kurt Gödel Research Center (KGRC)

- People at the KGRC: *Set Theory*
  Ajdin Halilović (large cardinals)
  Peter Holy (forcing axioms)
  Radek Honzík (singular cardinal problem)
  Jakob Kellner (iterated forcing)
  Heike Mildenberger (cardinal characteristics of $\mathfrak{c}$)
  Luca Motto Ros (descriptive set theory)
  David Schrittesser (forcing and descriptive set theory)
  Katherine Thompson (universality)
  Asger Törnquist (descriptive set theory)
  Matteo Viale (forcings axioms, singular cardinals)
  Lyubomyr Zdomskyy (combinatorial set theory)

People at the KGRC

- People at the KGRC: *Model Theory*
  Prerna Bihani (stability theory)
  Meeri Kesälä (abstract elementary classes)
  Agatha Walczak-Typke (homogeneous model theory)
- People at the KGRC: *Computation Theory*
  Ekaterina Fokina (computable model theory)
  Sebastiaan Terwijn (theory of randomness)

Three Questions

- Set theory
  Q1. The universe V of all sets has many interpretations. What *should* V look like?

- (Set-theoretic) Model theory
  Q2. How is model theory affected by how we interpret V?

- (Set-theoretic) Computation theory
  Q3. What are the possibilities for infinite computation?

Three lectures

- Lecture 1: The Hyperuniverse and Gödel Maximality.
- Lecture 2: The Internal Consistency and Outer Model programmes.
- Lecture 3: Model Theory and Computation Theory from a set-theoretic perspective.

Lecture 1: The Hyperuniverse and Gödel Maximality

What should the universe V of sets look like?

Many possibilities:

- L (Gödel's constructible universe)
  CH true
  Singular cardinal hypothesis true
  A definable, non-measurable set of reals
  Suslin's hypothesis false
  Whitehead conjecture false
  Borel conjecture false
  Borel-isomorphism of non-Borel analytic sets false
  Singular Square principle true

Interpretations of V

- $L[G]$'s (Cohen-style forcing extensions of L)
  CH true, or not!
  Singular cardinal hypothesis still true
  A definable non-measurable set of reals, or not!
  Suslin's hypothesis true, or not!
  Whitehead's conjecture true, or not!
  Borel conjecture true, or not!
  Borel-isomorphism of non-Borel analytic sets still false
  Singular Square principle still true

Interpretations of V

- Big enough K's (Jensen-style core models)
  CH true
  Singular cardinal hypothesis true
  No definable non-measurable set of reals!
  Suslin's hypothesis false
  Whitehead conjecture false
  Borel conjecture false
  Borel-isomorphism of non-Borel analytic sets true!
  Singular Square principle true

Intepretations of V

- $K[G]$'s (Forcing extensions of K)
  Singular cardinal hypothesis true, or not!
  Singular square principle true
- Models with very LARGE cardinals
  Singular square principle false!
- Models where Forcing Axioms hold
  CH false!
  Suslin's hypothesis true!
  Borel's conjecture true!
  Singular cardinal hypothesis true!

What an interesting mess!

Which universe should we pick?

Minimal and Maximal Universes

Two seductive pictures of V:

- Minimal one: $V = L$
- Maximal one: ???

Gödel and Scott

Gödel (1964):

"From an axiom in some sense opposite to [V=L], the negation of Cantor's conjecture could perhaps be derived. I am thinking of an axiom which ... would state some maximum property of the system of all sets, whereas [V=L] states a minimum property. Note that only a maximum property would seem to harmonize with the concept of set ..."

Gödel and Scott

Scott (1977):

"I see that there are any number of contradictory set theories, all extending the Zermelo-Fraenkel axioms; but the models are all just models of the first order axioms and first-order logic is weak. I still feel that it ought to be possible to have strong axioms which would generate these types of models as submodels of the universe, but where the universe can be thought of as something absolute ... But really pleasant axioms have not been produced by someone else or me, and the suggestion remains speculation. A new idea (or point of view) is needed, and in the meantime all we can do is to study the great variety of models."

The Search for Maximal Universes

How do we find a Maximal Universe?

Problem: V has all sets, so V is trivially maximal!

We need to *compare* V to other possible universes

How do we create other possible universes?

Fact. If V were countable, then we could create many other possible universes (by forcing, infinitary logic, ...)

Solution: We *temporarily* treat V as a *countable* universe, embedded into a collection of other possible such universes

The Hyperuniverse

(von Neumann-Zermelo) V is determined by:

- Its Ordinals Ord
- Its Power Set operation $\mathcal{P}$

  $V_0 = \emptyset$
  $V_{\alpha+1} = \mathcal{P}(V_\alpha)$
  $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$

V is countable, so $\mathrm{Ord}(V) =$ some countable ordinal α

Fix α

$\mathcal{H} =$ the Hyperuniverse
$\mathcal{H} =$ *All* countable transitive models of ZFC of ordinal height α

Universe $=$ element of the Hyperuniverse

What is α? We will choose α so that there is a "maximal" Universe

The Search for Maximal Universes

V_0 is an *inner* universe of V_1 iff $V_0 \subseteq V_1$
V_0 is an *outer* universe of V_1 iff $V_1 \subseteq V_0$
V_0, V_1 are *compatible universes* iff they have
a common outer universe

Q. What does it mean for a universe to be “maximal”?

Maximal = Maximal under inclusion?
NO! Any universe has a larger outer universe

Instead, use truth in inner universes to define maximality:

The Search for Maximal Universes

$\mathcal{L}$ = language of set theory
For a universe W:
$\Phi(W)$ = all sentences of $\mathcal{L}$ which are true in some inner universe of W

Obviously: $V \subseteq W \rightarrow \Phi(V) \subseteq \Phi(W)$

Key Definition:
V is *maximal* iff $V \subseteq W \rightarrow \Phi(V) = \Phi(W)$

The Inner Model Hypothesis

The *Inner Model Hypothesis* states:
The universe V is maximal

Objection! V is not countable!

Three good replies:

- We only treated V as countable *temporarily*. The IMH only says that V should satisfy sentences which are true in countable, maximal universes.
- In the IMH, we could restrict to universes which are inner universes of "forcing extensions" of V; then the IMH is a principle of ordinary "class theory".
- Are you sure that V is not countable? :)
  Maybe we should just figure out which *countable* universes are the good ones.

The Inner Model Hypothesis

Is the IMH consistent?

Theorem

Assume that there is a Woodin cardinal and a larger inaccessible cardinal. Then there are maximal universes, so the IMH is consistent.

In favour of the IMH

Suppose the IMH fails.
Then there is an outer universe W such that $\Phi(V) \subsetneq \Phi(W)$.
I.e. for some statement φ:
φ holds in some inner universe of W but in no inner universe of V

But then V is not big enough; we should replace V by W!

The Inner Model Hypothesis

Against the IMH

1. Socio-Political problem: The IMH is too strong!

The IMH implies:

There are no large cardinals in V
(they exist only in inner universes of V)
$R^\#$ does not exist for some real R

Set-theorists *love* large cardinals and #'s!

What should we do?

What would Barack Obama do?

Barack Obama and The Inner Model Hypothesis

Obama 1: It's time for "change you can believe in"!

I.e., large cardinals can exist in inner models, but not in V
Not so bad!

Obama 2: Negotiate with large-cardinal theorists!

Compromise: The *Relativised IMH*

Let T be ZFC + large cardinals.
IMH relative to T: T holds in V and:
$V \subseteq W$, T holds in $W \rightarrow \Phi(V) = \Phi(W)$

But why assume T?

The Inner Model Hypothesis

2. Mathematical problem: The IMH is not strong enough!

The IMH implies:

Singular cardinal hypothesis true
A definable, non-measurable set of reals
Borel-isomorphism of non-Borel analytic sets false
Singular Square principle true

But:

V satisfies IMH, $V \subseteq W \rightarrow W$ satisfies IMH

So: IMH does not resolve the Continuum Problem

The Strong Inner Model Hypothesis

The Strong IMH
The Strong IMH = The IMH with *absolute* parameters

p is *totally absolute* iff some formula defines p in *all* outer universes
ω is totally absolute

Is $\aleph_1$ totally absolute?
Probably not: $V \subseteq W$ does not imply $\aleph_1^V = \aleph_1^W$

A cardinal κ is *absolute* iff some formula defines κ in all outer universes W with the same cardinals $\leq \kappa$
$\aleph_1$, $\aleph_{99}$, $\aleph_{\omega+1} \cdots$ are absolute

SIMH $\rightarrow$ $c = 2^{\aleph_0}$ is *not* absolute
SIMH $\rightarrow$ $c \neq \aleph_1, \aleph_2, \aleph_3, \cdots$ (strong negation of CH!)
But is the SIMH consistent?

The Strong Inner Model Hypothesis

Theorem

Assuming the existence of a Woodin cardinal and a larger inaccessible cardinal, the SIMH is consistent for the parameter ω_1.

Conjecture: The SIMH is consistent relative to large cardinals.

Gödel revisited

Gödel (1964):
"From an axiom in some sense opposite to [V=L], the negation of Cantor's conjecture could perhaps be derived. I am thinking of an axiom which ... would state some maximum property of the system of all sets, whereas [V=L] states a minimum property. Note that only a maximum property would seem to harmonize with the concept of set ..."

Does the SIMH fulfill the wishes of Gödel and Scott?

Answer: Yes, provided it is consistent!

The Internal Consistency and Outer Model programmes

The IMH: $\Phi(V)$ is maximal

$\Phi(V) =$ All sentences true in some inner universe

φ is *internally consistent* iff φ belongs to $\Phi(V)$, i.e.,
iff φ is true in some inner universe

But what if $V = L$? Then there is only one inner universe!

Assumption: There are inner universes of V with large cardinals

Internal Consistency

A new type of consistency result.

$\text{Con}(\text{ZFC} + \varphi) = \text{ZFC} + \varphi$ is consistent

$\text{Icon}(\text{ZFC} + \varphi) = \text{ZFC} + \varphi$ holds in some inner universe

Consistency result:
$\text{Con}(\text{ZFC} + \text{LC}) \rightarrow \text{Con}(\text{ZFC} + \varphi)$,
where LC is a large cardinal axiom

Internal consistency result:
$\text{Icon}(\text{ZFC} + \text{LC}) \rightarrow \text{Icon}(\text{ZFC} + \varphi)$

Internal consistency is stronger than consistency

Proving Internal Consistency *demands new techniques*

Internal Consistency

Some Internal Consistency Results

Cardinal Exponentiation: F-Ondrejović, F-Honzík

Costationarity of the Ground Model: Dobrinen-F

Global Domination: F-Thompson

Tree Property: Dobrinen-F

Internal Consistency

Cardinal Exponentiation

Easton: Con(ZFC) $\rightarrow$ Con(ZFC $+ 2^\kappa > \kappa^+$ for all regular κ)

Easton uses "Easton product forcing"
This gives *no* internal consistency result.

F-Ondrejović: Instead use "Easton iterated forcing"

Theorem

Icon(ZFC $+ 0^\#$ exists) $\rightarrow$ Icon(ZFC $+ 2^\kappa > \kappa^+$ for all regular κ)

Internal Consistency

Global Domination

κ an infinite regular cardinal.
Suppose $f, g : \kappa \to \kappa$
f *dominates* g iff $f(\alpha) > g(\alpha)$ for sufficiently large $\alpha < \kappa$
$\mathcal{F}$ is a *dominating family* iff every $g : \kappa \to \kappa$ is dominated by some f in $\mathcal{F}$
$d(\kappa) =$ the smallest cardinality of a dominating family
Fact: $\kappa < d(\kappa) \leq 2^\kappa$ for all infinite cardinals κ

Global Domination: $d(\kappa) < 2^\kappa$ for all infinite cardinals κ

Internal Consistency

Cummings-Shelah: Global Domination is consistent
Proof uses Cohen and Hechler forcings
Corollary to their proof:
Icon(ZFC+ Proper class of supercompact cardinals) $\rightarrow$ Icon(ZFC+ Global Domination)

F-Thompson: Use Sacks product forcing instead

Theorem

$Icon(ZFC + 0^{\#}\ exists) \rightarrow Icon(ZFC+\ Global\ Domination)$

Internal Consistency

The Tree Property

κ regular
A κ-*Aronszajn tree* is a tree of height κ with no κ-branch
κ has the *tree property* iff there is no κ-Aronszajn tree

Mitchell: Con(ZFC+ Proper class of weakly compact cardinals) $\rightarrow$ Con(ZFC $+ \alpha^{++}$ has the tree property for all inaccessible α)

Proof uses "Mitchell forcing"
Corollary to proof: Icon(ZFC+ Proper class of supercompact cardinals) $\rightarrow$ Icon(ZFC $+ \alpha^{++}$ has the tree property for all inaccessible α)

Internal Consistency

Dobrinen-F: Use iterated Sacks forcing instead

Theorem

Icon($ZFC + 0^{\#}$ *exists*) $\rightarrow$ *Icon*($ZFC + \alpha^{++}$ *has the tree property for all inaccessible* α)

Further work on Internal Consistency: Singular cardinal problem, cofinality of the symmetric group, embedding complexity

The Outer Model Programme

The IMH gives us a *maximal* universe

$L =$ The *minimal* universe
Attractive consequences of $V = L$:

Generalised Continuum Hypothesis is true
There is a definable wellordering of the universe
Jensen's $\diamondsuit$, $\square$ and Morass are all true

$V = L$ is "mathematically strong"

The Outer Model Programme

What's wrong with $V = L$?

For many interesting statements φ:
$\mathrm{Con}\mathrm{ZFC} \nrightarrow \mathrm{Con}(\mathrm{ZFC} + \varphi)$
But $\mathrm{Con}\mathrm{ZFC} \rightarrow \mathrm{Con}(\mathrm{ZFC} + V = L)$,
so $\mathrm{Con}(\mathrm{ZFC} + V = L) \nrightarrow \mathrm{Con}(\mathrm{ZFC} + \varphi)$

$V = L$ is "consistency weak"

Large cardinals give us consistency strength!

Can we combine $V = L$ with large cardinals, i.e.,
Are there "L-like" universes with large cardinals?

The Outer Model Programme

Two approaches:

Inner model programme: Show that any universe with large cardinals has an L-like inner universe with the same large cardinals

Using fine structure theory and iterated ultrapowers: Produces L-like universes with many Woodin cardinals

Outer model programme: Show that any universe with large cardinals has an L-like outer universe with the same large cardinals

Using iterated forcing: Produces L-like universes for *all* large cardinals!

The Outer Model Programme

Large Cardinals

$j : V \to M$ means: j is an elementary embedding from $(V, \in)$ into the transitive class $(M, \in)$, $j \neq$ identity
Critical point of j = least κ such that $\kappa < j(\kappa)$

j is α-*strong* iff $V_\alpha \subseteq M$
Superstrong means $j(\kappa)$-strong
n-superstrong means $j^n(\kappa)$-strong
ω-*superstrong* means $j^\omega(\kappa)$-strong
$(j^\omega(\kappa) + 1)$-*strong is inconsistent!*

So ω-superstrength is at the edge of inconsistency

κ is ω-*superstrong* iff
κ is the critical point of an ω-supserstrong embedding

The Outer Model Programme

Theorem

Suppose that κ is ω-superstrong. Then there is an outer universe V^* (obtained by forcing) such that:
1. $V^* \models \kappa$ is ω-superstrong.
2. $V^* \models$ There is a definable wellordering of the universe.
3. $V^* \models \diamondsuit$, $\square$ (with restrictions) and Gap-1 Morass.

Universes which are even more L-like:

(Brooke-Taylor)-F: Can have Gap-1 Morasses preserving *all* ω-superstrong cardinals
Asperó-F: Can have a *locally definable* wellordering of the universe
Can have *Strong Condensation*

The Outer Model Programme

Summary:

IMH: V is maximal
IMH_T ($T =$ Large Cardinals): V is maximal relative to LC's
$V = L$: V is minimal
V as above: V is "minimal" relative to Large Cardinals

4 nice alternatives!

What is your choice?

Model Theory and Computation Theory from a set-theoretic perspective

When is Model Theory absolute?

T a countable first-order theory.
$V \subseteq W$ universes of set theory, T in V
Suppose $\mathcal{A}, \mathcal{B}$ are models of T; do we have

$$\mathcal{A} \simeq \mathcal{B} \text{ in } V \text{ iff } \mathcal{A} \simeq \mathcal{B} \text{ in } W?$$

In general, no:
T = Dense Linear Orderings without endpoints
L_0, L_1 non-isomorphic, uncountable models of T in V
Choose $W \supseteq V$ so that L_0, L_1 are countable in W;
then $L_0 \simeq L_1$ in W

But what if the theory T is "nice" model-theoretically and W is a "nice" outer universe of V?

Model Theory

Definition (Shelah): T is *classifiable* iff T is superstable and satisfies both NDOP and NOTOP
Fact: T is classifiable iff models of T of cardinality λ are characterised by their theory in $\mathcal{L}_{\infty,\lambda}$

Definition: An outer universe W of V is *CR-preserving* iff V, W have the same cardinals and the same reals

Model Theory

Theorem

(Baldwin-Laskowski-Shelah) Suppose that T is a countable classifiable first-order theory in V and W is a CR-preserving outer universe of V. Then two models $\mathcal{A}$, $\mathcal{B}$ of T of cardinality $\aleph_2$ are isomorphic in V iff they are isomorphic in W.

Another formulation:
Define:
$I(T,\aleph_2) = \{(\mathcal{A},\mathcal{B}) \mid \mathcal{A}, \mathcal{B}$ are isomorphic models of T of cardinality $\aleph_2\}$
$PI(T,\aleph_2) = \{(\mathcal{A},\mathcal{B}) \mid \mathcal{A}, \mathcal{B}$ are models of T of cardinality $\aleph_2$ which are isomorphic in a CR-preserving outer universe$\}$
Then for countable classifiable first-order T, $I(T,\aleph_2) = PI(T,\aleph_2)$

Model Theory

In particular: If T is classifiable, then $PI(T, \aleph_2)$ is a set in V (even though it refers to arbitrary CR-preserving outer universes of V)

F-Hyttinen-Rautila: The converse also holds for the universe L, assuming that there are enough CR-preserving extensions of L. What does "enough" mean?

Definition: "$0^\#$ exists" iff there is a $j : L \to L$. Equivalently, there is a closed unbounded class I of L-indiscernible ordinals
$0^\# =$ the theory of the structure $(L, \in, I)$, coded as a set of natural numbers

If there is a measurable cardinal then $0^\#$ exists

Model Theory

Theorem

(F-Hyttinen-Rautila) Suppose that $0^{\#}$ *exists. Then the following are equivalent, for countable first-order theories* T *in* L*:*
(1) $PI_L(T, \aleph_2)^L$ *(*L*'s version of* $PI(T, \aleph_2)$*) is a set in* L*.*
(2) T *is classifiable.*
Moreover, if these conditions fail, then the sets $PI_L(T, \aleph_2)$ *are equiconstructible (they belong to the same universes).*

The proof uses stationary sets.
$S \subseteq \aleph_2$ is *stationary* iff $S \cap C$ is nonempty for any closed unbounded $C \subseteq \aleph_2$.

Model Theory

Theorem

Suppose that $0^\#$ exists and let $PNS_L(\aleph_2)$ be the set of stationary $S \subseteq \aleph_2$ in L such that S is not stationary in a CR-preserving outer universe of L. Then $PNS_L(\aleph_2)$ and $0^\#$ are equiconstructible.

Now suppose that $T \in L$ is not classifiable.
Define a function $S \mapsto (\mathcal{A}_S, \mathcal{B}_S)$ in L such that

$$S \in PNS_L(\aleph_2) \text{ iff } (\mathcal{A}_S, \mathcal{B}_S) \in PI_L(T, \aleph_2)$$

Then $0^\#$ is constructible from $PI_L(T, \aleph_2)$.
(The converse can be shown without model theory.)

Current work (F-Hyttinen-(Walczak-Typke)): Extend this work beyond first-order theories, where there is still a good notion of "classifiable". A good context is *Homogeneous Model Theory*.

Computation Theory

ITTM (Infinite Time Turing Machine)

Standard Turing machine, allowed to run transfinitely
Stages of computation are indexed by ordinal numbers

At the start, and at successor steps: Works in the standard way
At a limit stage λ:
(1) Machine is placed into a special "limit state"
(2) Head of the machine is set all the way to the left
(3) For any cell of the tape, a 1 is written iff a 1 appeared there at all sufficiently large stages $\alpha < \lambda$ ("liminf" rule)

Computation ends when the machine reaches the "halting state", if ever; otherwise the computation "diverges"

Use three tapes: Input tape, Work tape and Output tape.
At the start, only 0's are written on the Work and Output tapes.
At each stage, the machine reads the n-th cell of all three tapes, for some n.

ITTM's are powerful.
Consider the Halting Problem $0' = \{e \mid \varphi_e(e) \downarrow\}$, where φ_e is the e-th partial recursive function
There is an ITTM M which gives $0'$ as output:
M computes $\varphi_0(0)$ for 1 step, $\varphi_1(1)$ for 2 steps, etc.
When $\varphi_e(e)$ converges, M writes a 1 in the n-th cell of its output tape
After ω steps, the characteristic function of $0'$ appears on M's output tape

Computation Theory

Similarly: There is an ITTM which computes $0''$ in $\omega + \omega$ steps.
Any set definable in arithmetic can be computed in fewer than ω^ω steps.
In ω^ω steps, the truth set for arithmetic can be computed.

We can go much further:
Let $(X, <)$ be a recursive linear ordering of the natural numbers.
There is an ITTM that successively removes the least element of this linear ordering until only the ill-founded part remains.
Thus there are ITTM's that halt at any recursive ordinal stage and a single ITTM that can compute the set of indices for recursive wellorderings.
So any i_1^1 set of natural numbers is ITTM computable.
The complement of an ITTM computable set is ITTM computable; so any Σ_1^1 set is ITTM computable, and much more.

Computation Theory

However: By absoluteness, any ITTM computable set of natural numbers belongs to Gödel's L, and is in fact Δ^1_2.

Which sets of natural numbers are ITTM computable?
This question will be answered below.

If an ITTM computation does not halt then it must repeat: The configuration (i.e., head position, state and tape contents) is the same as at some earlier stage.
In fact this must happen by a countable stage, because the configuration at stage ω_1 must have occurred already at many countable stages
Once the configuration repeats, it will repeat indefinitely and the machine will never halt.

Computation Theory

Questions about ITTM computations (on the 0 input):

1. What is the least stage by which all computations either halt or repeat?
2. What can appear on the output tape during a halting ITTM computation?
What can appear on the output tape of an arbitrary ITTM computation?
What can appear on the output tape from some point on in some ITTM computation?

Computation Theory

F-Welch: *The Theory Machine*

Recall Gödel's hierararchy of constructible sets:

$L_0 = \emptyset$
$L_{\alpha+1} =$ all subsets of L_α which are definable over $(L_\alpha, \in)$
$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for limit λ.

Using a suitable coding of computations:
The contents of the output tape of an ITTM computation of length α is definable over $(L_\alpha, \in)$
Therefore: If $\alpha < \beta$ are limit ordinals and Theory of $(L_\alpha, \in) =$ Theory of $(L_\beta, \in)$, then the configuration of any ITTM at stage α repeats at stage β
In fact, "Theory" can be replaced by "Σ_2 Theory", as at limit stages, ITTM's perform a Σ_2 operation (the "liminf" operation)

Computation Theory

Conclusion: Every ITTM either halts or repeats by stage Σ*, where* Σ *is least so that for some* $\zeta < \Sigma$*,* $(L_\Sigma, \in)$ *and* $(L_\zeta, \in)$ *have the same* Σ_2 *theory.*

The Theory Machine demonstrates the converse:

Theorem

(F-Welch) There is an ITTM M (the Theory Machine) such that for $\alpha \leq \Sigma$*, the Theory of* $(L_{\omega+\alpha}, \in)$ *(coded as a set of natural numbers) appears on the output tape of M at stage* $\omega^2 \cdot (\alpha + 1)$*. Therefore the configuration of M first repeats at stage* Σ*.*

We can now answer the Questions posed earlier:

Computation Theory

Σ least so that for some $\zeta < \Sigma$, $(L_\zeta, \in)$ and $(L_\Sigma, \in)$ have the same Σ_2 theory
ζ = least such ζ
λ least so that $(L_\lambda, \in)$ and $(L_\Sigma, \in)$ have the same Σ_1 theory.
Then $\lambda < \zeta < \Sigma$

Every ITTM either halts or repeats itself by stage Σ.
There is a machine that first repeats itself at stage Σ.
The supremum of the halting times of ITTM's is λ.
The reals that appear on the output tape of an ITTM are the reals in L_Σ.
The reals that appear on the output tape of a halting ITTM are the reals in L_λ.
The reals that appear on the output tape of an ITTM from some point on are the reals in L_ζ.

Computation Theory

Current work: *Hypermachines*

Use a stronger rule for limit stages of computation

ITTM $= \Sigma_2$-Hypermachine
These reach the first repeat of the Σ_2 Theory of $(L_\alpha, \in)$
n-Hypermachines reach the first repeat of the Σ_n Theory of $(L_\alpha, \in)$

The proofs for $n > 2$ require a deeper analysis of the L_α's
Hypermachines also provide new examples for Descriptive Set Theory: Prewellordering, Uniformisation, Determinacy

α-Hypermachines for transfinite α?
A question for future study