The Effective Theory of Borel Equivalence Relations

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Reals = Baire space ${}^{\omega}\omega$, with the natural topology
Basic open sets: $N_s = \{f \mid f \text{ extends } s\}$, $s : n \to \omega$ for some finite n

If E and F are Borel equivalence relations on the reals then
E is *Borel reducible to* F, written $E \leq_B F$, iff
For some Borel function f: $x \; E \; y$ iff $f(x) \; F \; f(y)$

$\leq_B$ is reflexive and transitive
$E \equiv_B F$ iff $E \leq_B F$ and $F \leq_B E$ (equivalence relation)
$[E]_B$ = the equivalence class of E under $\equiv_B$

Object of study: $\mathcal{B}$ = Degrees of Borel equivalence relations under Borel reducibility

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Work of Silver and of Harrington-Kechris-Louveau identifies an interesting initial segment of $\mathcal{B}$:

Theorem

$\mathcal{B}$ has an initial segment

$1 < 2 < \cdots < \omega < R < E_0$

where:

n = Borel equivalence relations with exactly n classes
ω = Borel equivalence relations with exactly $\aleph_0$ classes
R is $({}^{\omega}\omega, =)$ (equality on reals)
E_0 is the equivalence relation xE_0y iff $x(n) = y(n)$ for all but finitely many n

In fact: Any Borel equivalence relation is Borel equivalent to one of the above or lies strictly above E_0 under Borel reducibility.

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Question: What happens if we replace "Borel" by "Lightface Borel"?

Write "Hyp" for "Lightface Borel" ($= \Delta^1_1$). Then we define:

If E and F are Hyp equivalence relations on the reals then
E is *Hyp reducible to* F, written $E \leq_H F$, iff
For some Hyp function f: $x \; E \; y$ iff $f(x) \; F \; f(y)$

$\leq_H$ is reflexive and transitive
$E \equiv_H F$ iff $E \leq_H F$ and $F \leq_H E$ (equivalence relation)
$[E]_H =$ the equivalence class of E under $\equiv_H$

Object of study: $\mathcal{H} =$ Degrees of Hyp equivalence relations under Hyp reducibility

There are some surprises!

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Again we have degrees

$1 < 2 < \cdots < \omega < R < E_0$

defined as follows:

n is represented by $xE^n y$ iff $x(0) = y(0) < n - 1$ or $x(0), y(0) \geq n - 1$
ω is represented by $xE^\omega y$ iff $x(0) = y(0)$
R, E_0 are as before:
xRy iff $x = y$
$xE_0 y$ iff $x(n) = y(n)$ for all but finitely many n

Proposition

There are Hyp equivalence relations strictly between 1 and 2!

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Explanation:

Let E be a Hyp equivalence relation. Recall that the $\mathcal{H}$-degree n is represented by the equivalence relation E^n where:

xE^ny iff $x(0) = y(0) < n - 1$ or $x(0), y(0) \geq n - 1$

Fact 1. E^n is Hyp reducible to E iff at least n distinct E-equivalence classes contain Hyp reals

Proof. Suppose that E^n Hyp reduces to E via the Hyp function f. Each of the n equivalence classes of E^n contains a Hyp real; let $x_0, \ldots, x_{n-1}$ be Hyp, pairwise E^n-inequivalent reals. Then the reals $f(x_i)$, $i < n$, are Hyp, pairwise E-inequivalent reals.
Conversely, if $y_0, \ldots, y_{n-1}$ are Hyp, pairwise E-inequivalent reals then send the E^n-equivalence class of x_i to the real y_i; this is a Hyp reduction of E^n to E. □

Fact 2. E is Hyp reducible to E^2 iff E has at most 2 equivalence classes.

Proof. If E is Hyp reducible to E^2 then E has at most 2 equivalence classes because E^2 has only 2 equivalence classes. Conversely, suppose that the equivalence classes of E are A_0 and A_1. We may assume that A_0 has a Hyp element x. Then A_0 is Hyp as it consists of those reals E-equivalent to x and A_1 is Hyp as it consists of those reals not E-equivalent to x. Now we can reduce E to E^2 by choosing E^2-inequivalent Hyp reals y_0, y_1 and sending the elements of A_0 to y_0 and the elements of A_1 to y_1. $\square$

So to get a Hyp equivalence relation between 1 and 2 we need only find one with two equivalence classes but with all Hyp reals in just one class. This follows from a classical fact from Hyp theory:

Fact 3. There are nonempty Hyp sets of reals which contain no Hyp element.

Proof. Let A be the set of non-Hyp reals. Then A is Σ^1_1 and therefore the projection of a Π^0_1 subset P of Reals $\times$ Reals. P is nonempty. A Hyp real $h = (h_0, h_1)$ in P would give a Hyp real h_0 in A, contradiction. □

In a moment we will ask the harder question: Are there incomparable degrees between 1 and 2?

But first we consider what happens between 1 and 3

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Let E be a Hyp equivalence relation. We have seen:

E^n is Hyp reducible to E iff E has at least n equivalence classes containing Hyp reals

E is Hyp reducible to E^2 iff E has at most 2 equivalence classes

Can we replace 2 by n in this last statement?

Fact 4. E is Hyp reducible to E^n iff E has at most n equivalence classes and each equivalence class is Hyp.

Proof. Each equivalence class of E^n is Hyp. If f is a Hyp function reducing E to E^n then each equivalence class of E is the preimage of a Hyp set under a Hyp function, hence is Hyp. Conversely, if E has at most n classes and each class is Hyp, then we obtain a Hyp reduction of E to E^n by assigning E^n-inequivalent Hyp reals to the different classes of E. $\square$

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Can a Hyp equivalence relation with 3 equivalence classes have a non-Hyp equivalence class? Fortunately, the answer is NO.

Lemma

Suppose that E is a Hyp equivalence relation with countably many classes. Then each equivalence class of E is Hyp.

Proof Sketch. The *Silver dichotomy* states that every Borel (or even boldface Π^1_1) equivalence relation has either countably many classes or a perfect set of equivalence classes (i.e., R is Borel reducible to it). Harrington's proof of this shows: If E is a Hyp equivalence relation with countably many classes, then every real belongs to a Hyp subset of some equivalence class.

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Now let C be the set of codes for Hyp subsets of an equivalence class; then C is Π^1_1. Consider the relation

$R = \{(x, c) \mid c$ belongs to C and x belongs to $H(c)$, the Hyp set coded by c$\}$

Then R is Π^1_1 and can be uniformised by a Π^1_1 function F. As the values of F are numbers, F is Hyp and by Σ^1_1 Separation we can choose a Hyp $D \subseteq C$, D containing Range(F). Now define an equivalence relation E^* on D by:

$d_0 E^* d_1$ iff $H(d_0)$ E $H(d_1)$, i.e., $H(d_0)$ and $H(d_1)$ are subsets of the same E-equivalence class

Then both E^* and its complement are Π^1_1, so E^* is Hyp. And E Hyp reduces to E^* via $x \mapsto F(x)$. But E^* is just a Hyp relation on a Hyp set of numbers, so each of its equivalence classes is Hyp. It follows that also each equivalence class of E is Hyp. $\square$

Corollary

Let E be a Hyp equivalence relation. Then E is Hyp reducible to n iff it has at most n equivalence classes. And E is Hyp reducible to ω iff E has countably many equivalence classes.

There are Hyp equivalence relations between 1 and 3 which are incomparable with 2: Take one with 3 classes, one of which contains all Hyp reals.
There are Hyp equivalence relations strictly between 2 and 3: Take one with 3 classes, only two of which contain Hyp reals.
Similarly, for any $0 < n_0 < n_1$ finite, there are Hyp equivalence relations which are strictly above n_0, strictly below n_1 and incomparable with all n for n between n_0 and n_1.

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Now we discuss the more difficult question: Are there incomparable Hyp equivalence relations between 1 and 2? To answer this we prove:

Theorem

There exists Hyp sets of reals A, B such that for no Hyp function F do we have $F[A] \subseteq B$ or $F[B] \subseteq A$.

Given this Theorem, define E_A to be the equivalence relation with equivalence classes A and $\sim A$ (the complement of A); define E_B similarly. Note that the sets A, B contain no Hyp reals, else there would be a constant Hyp function F mapping one of them into the other. So a Hyp reduction of E_A to E_B would have to send the elements of $\sim A$ (which contains Hyp reals) to elements of $\sim B$, and therefore the elements of A to elements of B, contradicting the Theorem. Similarly with A and B switched.

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Theorem

There exists Hyp sets of reals A, B such that for no Hyp function F do we have $F[A] \subseteq B$ or $F[B] \subseteq A$.

Proof Sketch. First we quote a result of Harrington. For reals a, b and a recursive ordinal α we say that a is α-*below* b iff a is recursive in the α-jump of b.

Fact. For any recursive ordinal α there are Π^0_1 singletons a, b such that a is not α-below b and b is not α-below a.

Now using Barwise Compactness, find a nonstandard ω-model M of ZF^- in which are there are Π^0_1 singletons a, b such that for all recursive α, a is not α-below b and b is not α-below a (i.e., a and b are Hyp incomparable).
Let a, b be the unique solutions in M to the Π^0_1 formulas φ_0, φ_1, respectively.
The desired sets A, B are $\{x \mid \varphi_0(x)\}$ and $\{x \mid \varphi_1(x)\}$.
If F were a Hyp function mapping A into B, then it would send a to an element $F(a)$ of $B \cap M$; but then $F(a)$ must equal b and therefore b is Hyp in a, contradicting the choice of a, b. $\square$

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For the remainder of this talk, fix A, B as in the Theorem: There is no Hyp function F such that $F[A] \subseteq B$ or $F[B] \subseteq A$.

Using A, B we can easily get incomparable Hyp equivalence relations between n and $n+1$ for any finite n, by considering E_A, E_B where the equivalence classes of E_A are A together with a split of $\sim A$ into n classes, each of which contains a Hyp real (similarly for E_B).

We now consider Hyp equivalence relations with infinitely many equivalence classes.

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Recall the Silver and Harrington-Kechris-Louveau dichotomies:

Theorem

(a) (Silver) A Borel equivalence relation is either Borel reducible to ω or Borel reduces R.
(b) (H-K-L) A Borel equivalence relation is either Borel reducible to R or Borel reduces E_0.

How effective are these results? Harrington's proof of (a) and the original proof of (b) show:

Theorem

(a) A Hyp equivalence relation is either Hyp reducible to ω or Borel reduces R.
(b) A Hyp equivalence relation is either Hyp reducible to R or Borel reduces E_0.

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Our sets A, B can be used to show that the Silver and Harrington-Kechris-Louveau dichotomies are *not* fully effective:

Theorem

(a) There are incomparable Hyp equivalence relations between ω and R.
(b) There are incomparable Hyp equivalence relations between R and E_0.

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Proof Sketch. (a) Consider the relations

$E_A(x, y)$ iff ($x \in A$ and $x = y$) or ($x, y \notin A$ and $x(0) = y(0)$)

E_B: The same, with A replaced by B

Now E^ω Hyp reduces to E_A by $n \mapsto (n, 0, 0, ...)$.
Also E_A Hyp reduces to R via the map $G(x) = x$ for $x \in A$, $G(x) = (x(0), 0, 0, ...)$ for $x \notin A$ (same for B)

There is no Hyp reduction of E_A to E_B:
If F were such a reduction then let C be $F^{-1}[\sim B]$. As $\sim B$ is Hyp, C is also Hyp and therefore $A \cap C$ is also Hyp. But $A \cap C$ must be countable as F is a reduction. So if $A \cap C$ were nonempty it would have a Hyp element, contradicting the fact that A has no Hyp element. Therefore F maps A into B, which is impossible by the choice of A, B. By symmetry, there is no Hyp reduction of E_B to E_A.

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(b) Now we define E_A on $R \times R$ by:
$(x, y)E_A(x', y')$ iff $x = x'$ and either $x \notin A$ or $(x \in A$ and $yE_0y')$
E_B: Same, with A replaced by B

We need two Facts:

1. If $h : R \to R$ is Baire measurable and constant on E_0 classes then h is constant on a comeagre set.
2. If $B \subseteq R^2$ is Hyp then so is $\{x \mid \{y \mid (x, y) \in B\}$ is comeagre$\}$.

Now suppose that F were a Hyp reduction of E_A to E_B. Let $\pi(x, y) = x$ for all x and define $h : R \to R$ by: $h(x) = z$ iff $\{y \mid \pi(F(x, y)) = z\}$ is comeagre.

Using 1 and 2, h is a total Hyp function. We claim that $h[A] \subseteq B$, contradicting the choice of A, B: Assume $x \in A$. Then for comeagre-many y, $\pi(F(x, y)) = h(x)$. So if $h(x) \notin B$ then F maps more than one E_A class into a single E_B class, contradiction. By symmetry there is no Hyp reduction of E_B to E_A. □

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Final remarks and questions:

Theorem

(a) For each finite n there are Hyp equivalence relations above n but incomparable with ω.
(b) If a Hyp equivalence relation is above each finite n then it is also above ω.

Questions:
1. If a Hyp equivalence relation is Borel reducible to E_0 must it also be Hyp reducible to E_0? (This is true for finite n, ω, R.)
2. E_1 is the equivalence relation on R^ω defined by $\vec{x} E_1 \vec{y}$ iff $\vec{x}(n) = \vec{y}(n)$ for almost all n. Are there Hyp incomparable Hyp equivalence relations between E_0 and E_1? Kechris-Louveau showed that there are no Borel equivalence relations between E_0 and E_1 in the sense of Borel reducibility.

3. Are there any nodes other than 1? I.e., is there a Hyp equivalence relation with more than one equivalence class which is comparable with all Hyp equivalence relations under Hyp reducibility?
4. Is there a minimal degree? Are there incomparables above each degree?

There is also a jump operation, which requires further study.

THANK YOU!