The Foundations of Set Theory: Past, Present and Future

Summary:

Cantor: Transfinite counting, Cardinality for infinite sets

Paradoxes $\Rightarrow$ ZFC axioms for set theory

ZFC provides a *foundation* for mathematics

Constructibility, forcing, large cardinals $\Rightarrow$
Many possible interpretations of ZFC
Many possible interpretations of mathematics

Fundamental question: Which are the preferred interpretations?

We provide some answers

Cantor and Zermelo: The basic picture

Georg Ferdinand Ludwig Philipp Cantor

Berlin doctorate 1867 (number theory)
Halle habilitation 1870 (number theory)
Heine $\Rightarrow$ Study of trigonometric series $\Rightarrow$
Set theory

Cantor's results:
Theory of transfinite numbers and cardinality
Algebraic numbers are countable
Real numbers are not countable
1-1 correspondence between n-dimensional space and the real line

Opposition from Kronecker
Support from Dedekind
Mittag-Leffler: "100 years too soon"

Cantor and Zermelo: The basic picture

Transfinite counting
C closed set of reals
C' = limit points of C (Cantor derivative)
$C \supseteq C' \supseteq C'' \supseteq \cdots$
C^∞ = the intersection
$C^\infty \supseteq (C^\infty)'$, maybe strict!
Keep counting: $C^\infty \supseteq C^{\infty+1} \supseteq C^{\infty+2} \supseteq \cdots$!

What is $0, 1, ..., \infty, \infty + 1, ...$?
Wellordering: Linear ordering with no infinite descending sequence

Cantor: Any 2 wellorderings are comparable
Each wellordering is isomorphic to an *ordinal*, a special wellordering ordered by $\in$
$0 = \emptyset, 1 = \{0\}, 2 = \{0, 1\}, ..., \omega = \{0, 1, 2, ...\}$,
$\omega + 1 = \omega \cup \{\omega\}, \ldots$

Cantor and Zermelo: The basic picture

Cantor's assumption: Every set can be wellordered
Therefore every set is bijective with an ordinal (not unique)

Cardinal = Ordinal not bijective with a smaller ordinal
Every set is bijective with a *unique* cardinal, its *cardinality*

Zermelo: Cantor's assumption follows from the Axiom of Choice
So Cantor's theory of cardinality applies to arbitrary sets, assuming the Axiom of Choice

One major gap! What is the cardinality of the continuum?
Continuum Hypothesis (CH):
Every uncountable set of reals has the same cardinality as the set of all reals

Cantor and Zermelo: The basic picture

Paradoxes

Cantor, Burali-Forti, Russell
$x =$ all y such that $y \notin y$
$x \in x \leftrightarrow x \notin x$!

Zermelo's proposal:
Only use established principles of set-formation
Axiomatic theory: Zermelo set theory
ZFC = Zermelo-Fraenkel set theory with the Axiom of Choice

Cantor and Zermelo: The basic picture

The Universe of Sets V

ZFC reduces V to the ordinals and the power set operation:
$V_0 = \emptyset$
$V_{\alpha+1} =$ all subsets of V_α
Limit ordinal λ: $V_\lambda =$ union of the V_α, $\alpha < \lambda$
$V =$ union of the V_α's

Not a "unique" description:
Even fixing the ordinals, there are many interpretations of the power set operation!

An abundance of universes: Constructibility

Gödel, late 1930's

Replace power set operation by a weak power set operation:
$V_{\alpha+1}$ = all subsets of V_α
$L_{\alpha+1}$ = all "simple" subsets of L_α
L = union of the L_α's
L satisfies ZFC
First clearly-described model of ZFC
CH holds in L!

Gödel:
L is *not* the "correct" intepretation of ZFC
It is only a tool for showing consistency with ZFC

There are many other interpretations of ZFC:

An abundance of universes: Forcing

Cohen forcing:
Add new sets to L, preserving ZFC
R is *Cohen over L* iff
R belongs to every open dense set of reals which L can "describe"
Add many Cohen reals to $L \Rightarrow$ Model where CH fails

Solovay forcing:
R in $[0, 1]$ is *random over L* iff
R belongs to every measure 1 subset of $[0, 1]$ which L can "describe"
Using random reals: Model where every "definable" set of reals is Lebesgue measurable

More generally: "Force" using any partial ordering P
(Cohen forcing: Nonempty open sets under inclusion
Solovay forcing: Closed sets of positive measure)
Yields *many* different models of ZFC

An abundance of universes: Large cardinals

Example from measure theory:
Countably additive extension of Lebesgue measure to *all* sets of reals ($\Rightarrow V$ is not L)

Model of ZFC with such a measure $\Leftrightarrow$
Model of ZFC with a *measurable cardinal*

Measurable cardinal: An example of a "large cardinal hypothesis"

Large cardinal hypotheses play a big role in set theory
(they measure "consistency strengths")

An abundance of universes: Large cardinals

More than a measurable cardinal is sometimes needed:

A is *Wadge reducible* to B iff
For some continuous f, $x \in A$ iff $f(x) \in B$

Borel sets = smallest σ-algebra containing all open sets

Σ^1_1 = continuous image of a Borel set
i^1_1 = complement of Σ^1_1 set
Σ^1_{n+1} = continuous image of i^1_n set
i^1_{n+1} = complement of Σ^1_{n+1} set
Projective = Σ^1_n or i^1_n for some n

WP_n: If A, B are Σ^1_n but not i^1_n then
A is Wadge reducible to B and vice-versa

An abundance of universes: Large cardinals

WP_1 has the "strength" of $\#$'s, a large cardinal hypothesis below a measurable cardinal.
WP_2 has the "strength" of a Woodin cardinal, much stronger than a measurable cardinal
WP_n corresponds to $n-1$ Woodin cardinals

An abundance of universes

Summary: Constructibility, forcing and large cardinals yield many different universes, with different mathematics:

- L (Gödel's constructible universe)
  CH true
  Singular cardinal hypothesis true
  A definable, non-measurable set of reals
  Suslin's hypothesis false
  Whitehead conjecture false
  Borel conjecture false
  Borel-isomorphism of non-Borel analytic sets false
  Singular Square principle true

An abundance of universes

- $L[G]$'s (Cohen-style forcing extensions of L)
  CH true, or not!
  Singular cardinal hypothesis still true
  A definable non-measurable set of reals, or not!
  Suslin's hypothesis true, or not!
  Whitehead's conjecture true, or not!
  Borel conjecture true, or not!
  Borel-isomorphism of non-Borel analytic sets still false
  Singular Square principle still true

An abundance of universes

- Big enough K's (Jensen-style core models)
  CH true
  Singular cardinal hypothesis true
  No definable non-measurable set of reals!
  Suslin's hypothesis false
  Whitehead conjecture false
  Borel conjecture false
  Borel-isomorphism of non-Borel analytic sets true!
  Singular Square principle true

An abundance of universes

- $K[G]$'s (Forcing extensions of K)
  Singular cardinal hypothesis true, or not!
  Singular square principle true

- Models with very LARGE cardinals
  Singular square principle false!

- Models where Forcing Axioms hold
  CH false!
  Suslin's hypothesis true!
  Borel's conjecture true!
  Singular cardinal hypothesis true!

What an interesting mess!

Q. Which universes should we prefer?

Preferred universes: Computation theory

ω = the natural numbers

A subset A of ω is *computable* iff there is an algorithm which for any argument n determines whether or not n belongs to A
iff there is a *machine* (Turing machine) which given n as input produces 1 as output if n belongs to A and 0 as output otherwise

ω = the least infinite cardinal number

Now let α be any infinite cardinal number

A subset A of α is α-*computable* iff there is an algorithm which for any argument β determines whether or not *beta* belongs to A
iff there is a *machine* (α-machine) which given β as input produces 1 as output if β belongs to A and 0 as output otherwise

Idea: α-computability should look like (ω-)computability

Preferred universes: Computation theory

Relativised α-computability

A, B subsets of α
A is α-computable *relative to* B iff there is an (α-)machine *with oracle* B which given β as input produces 1 as output if β belongs to A and 0 as output otherwise
Write $A \leq_\alpha B$ for "A is α-computable relative to B"

Fact. For $\alpha = \omega$, $(*)_\alpha$ holds, where:

$(*)_\alpha$: For any A there exist B_0, B_1 such that $A \leq_\alpha B_0$, $A \leq_\alpha B_1$ but $B_0 \not\leq_\alpha B_1$, $B_1 \not\leq_\alpha B_0$.

Question. Does $(*)_\alpha$ hold for all α?

Preferred universes: Computation theory

Theorem

(a) Any universe (model of ZFC) in which $(*)_\alpha$ holds for all α has a subuniverse with "many" measurable cardinals.
(b) Conversely, if there is a universe with "many" measurable cardinals then there is a larger universe in which $(*)_\alpha$ holds for all α.

Therefore (α-)computation theory suggests that we should prefer universes which have a subuniverse with "many" measurable cardinals.

Preferred universes: First-order Model theory

Work in progress (Tapani Hyttinen)
In model theory, one typically studies the class of intepretations or *models* of a set of axioms or *theory* T.
Simplest case: T is "first-order"

Shelah: "Classification theory"
T is "classifiable" iff
T does *not* have the maximum number of uncountable models iff
T is "superstable with NDOP and NOTOP"

Another approach to "classification theory": Consider the *isomorphism relation* $\simeq_T$ for models of T

$M \simeq_T N$ iff M, N are models of T and there is an isomorphism from M onto N

T is "well-behaved" iff $\simeq_T$ is a "simple" equivalence relation

Preferred universes: First-order Model theory

The *countable* models of T form a nice topological space, indeed a separable complete metric space, so we can say:

$\simeq_T$ is *simple on countable models of T* iff $\simeq_T$ is a Borel equivalence relation

For uncountable models, there are different notions of "Borel":

Strictly-Borel $\subseteq \Delta \subseteq$ *Borel**

Theorem

There are universes in which the following are equivalent:
(a) $\simeq_T$ is Δ on (sufficiently large) uncountable models of T.
(b) T is Shelah-classifiable.

The universes of the Theorem are those without "Canary Trees"

Model theory $\Rightarrow$ Prefer universes without Canary Trees!

Preferred universes: Non-first-order Model theory

Current model theory also considers the class of models of a theory T which is *not* first-order: *Abstract elementary classes*

Shelah has made progress with *excellent classes* (classes which obey certain amalgamation conditions):

Fact. Suppose that $\alpha < \beta$ are uncountable cardinal numbers and $\mathcal{C}$ is excellent. If $\mathcal{C}$ has a unique element of size α (up to isomorphism) then the same holds for β.

The question is: When is a class excellent?

Preferred universes: Non-first-order Model theory

Assuming the *Weak Generalised Continuum Hypothesis (WGCH)*:

($*$) If an abstract elementary class has at most $\aleph_n$ models of cardinality $\aleph_n$ for each finite n then it is excellent.

On the other hand, ($*$) fails under a popular but opposing set-theoretic assumption, *Martin's axiom (MA)*. So we have:

Model theory $\Rightarrow$ We should prefer universes which satisfy the Weak GCH over those which satisfy Martin's Axiom!

Preferred universes: Gödel maximality

Two attractive pictures of V:
ause

- Minimal one: $V = L$
- Maximal one: ???

Gödel

Gödel (1964):

"From an axiom in some sense opposite to [V=L], the negation of Cantor's conjecture could perhaps be derived. I am thinking of an axiom which ... would state some maximum property of the system of all sets, whereas [V=L] states a minimum property. Note that only a maximum property would seem to harmonize with the concept of set ..."

The search for maximal universes

How do we find a Maximal Universe?

ause

Problem: V has all sets, so V is trivially maximal

ause

We need to *compare* V to other possible universes

ause

How do we create other possible universes?

ause

Fact. If V were countable, then we could create many other possible universes (by forcing, infinitary logic, ...)

Solution: We *temporarily* treat V as a *countable* universe, embedded into a collection of other possible such universes

The Hyperuniverse

(von Neumann-Zermelo) V is determined by:

- Its Ordinals Ord
- Its Power Set operation $\mathcal{P}$

  $V_0 = \emptyset$
  $V_{\alpha+1} = \mathcal{P}(V_\alpha)$
  $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$

V is countable, so $\text{Ord}(V)$ = some countable ordinal α

Fix α

$\mathcal{H}$ = the Hyperuniverse
$\mathcal{H}$ = *All* countable transitive models of ZFC of ordinal height α

Universe = element of the Hyperuniverse

What is α? We will choose α so that there is a "maximal" Universe

The Search for Maximal Universes

V_0 is an *inner* universe of V_1 iff $V_0 \subseteq V_1$
V_0 is an *outer* universe of V_1 iff $V_1 \subseteq V_0$
V_0, V_1 are *compatible universes* iff
they have a common outer universe

Q. What does it mean for a universe to be "maximal"?

The Search for Maximal Universes: Absoluteness

$\mathcal{L}$ = language of set theory
For a universe W:
$\Phi(W)$ = all sentences of $\mathcal{L}$ which are true in some *inner universe* of W

Obviously: $V \subseteq W \rightarrow \Phi(V) \subseteq \Phi(W)$

Key Definition:
V is *maximal* iff $V \subseteq W \rightarrow \Phi(V) = \Phi(W)$

The *Inner Model Hypothesis* states:
The universe V is maximal

The Inner Model Hypothesis

Is the IMH consistent?

Theorem

(F-Woodin) Assume that there is a Woodin cardinal and a larger inaccessible cardinal. Then there are maximal universes, so the IMH is consistent.

Are large cardinals necessary?

Theorem

(F-Welch) The IMH implies that there are inner models with measurable cardinals of arbitrarily high Mitchell order.

Summary

In summary:

1. Cantor's set theory was highly successful, but suffered from paradoxes.
2. The paradoxes were resolved by the development of axiomatic set theory, ZFC.
3. Constructibility, forcing and large cardinal theory gave rise to an abundance of universes.
4. Ideas from computation theory and model theory, as well as maximality principles in set theory provide criteria for preferring one universe to another.

Will set theory reach a definitive picture of the universe of sets? Only time will tell ...