A Simpler Proof of Jensen's Coding Theorem

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Beller-Jensen-Welch [82] provides a proof of Jensen's remarkable Coding Theorem, which demonstrates that the universe can be included in $L[R]$ for some real R, via class forcing. The purpose of this article is to present a simpler proof of Jensen's theorem, obtained by implementing some changes first developed for the theory of strong coding (Friedman [87]).

The basic idea is to first choose $A \subseteq ORD$ so that $V = L[A]$ and then generically add sets $G_\alpha \subseteq \alpha^+, \alpha$ O or an infinite cardinal (O^+ denotes ω) so that G_α codes both G_{α^+} and $A \cap \alpha^+$. Also for limit cardinals α, G_α is coded by $\langle G_{\bar\alpha} | \bar\alpha < \alpha \rangle$. Thus there are two "building blocks" for the forcing, the successor coding and the limit coding. We modify the successor coding so as to eliminate Jensen's use of "generic codes" (this improves an earlier modification of this type, due to Welch and Donder). And we thin out the limit coding so as to eliminate the technical problems causing Jensen's split into cases according to whether or not $O^\#$ exists.

Theorem. *(Jensen) There is a class forcing $\mathcal{P}$ such that if G is $\mathcal{P}$-generic over V then $V[G] \models ZFC + V = L[R], R \subseteq \omega$. If $V \models GCH$ then $\mathcal{P}$ preserves cardinals.*

It is not difficult to class-generically extend V to make GCH true. And any "reshaped" subset of ω_1 can be coded by a real via a CCC forcing. (See Section One below for a definition of "reshaped".) So it suffices to prove that V can be coded by a "reshaped" subset of ω_1, preserving cardinals, assuming the GCH. As a first step, force $A \subseteq ORD$ such that for each infinite cardinal α, $L_\alpha[A] = H_\alpha =$ all sets of hereditary cardinality less than α.

Section One The Successor Coding R^s.

Fix an infinite cardinal α. S_α is defined to be a certain collection of "strings" $s : [\alpha, |s|) \longrightarrow 2, \alpha \leq |s| < \alpha^+$. For s to belong to S_α we require that s is

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"reshaped". This means that for $\eta \leq |s|$, $L[A \cap \alpha, s \restriction \eta] \models \text{card}(\eta) \leq \alpha$. The reshaping of s allows us to code s by a subset of α, in the manner which we now describe.

For $s \in S_\alpha$ define structures $\mathcal{A}_s^0 = L_{\mu_s^0}[A \cap \alpha, s^*]$, $\mathcal{A}_s = L_{\mu_s}[A \cap \alpha, s^*]$ as follows (where $s^* = \{\mu_{s \restriction \eta} | s(\eta) = 1\}$) : If $|s| = \alpha$ then $\mu_s^0 = \alpha$. For $|s| > \alpha$, $\mu_s^0 = \bigcup\{\mu_{s\restriction\eta} | \eta < |s|\}$ and in general μ_s = least p.r. closed ordinal μ greater than μ_s^0 such that $L_\mu[A \cap \alpha, s^*] \models \text{card}(s) \leq \alpha$. These ordinals are well-defined due to the reshaping of s.

For $s \in S_\alpha$ we write $\alpha(s) = \alpha$. Note that if $|s| = \alpha(s)$ then $s = \emptyset$; in this case we think of s as "labelled" with the ordinal $\alpha(s)$, so that there are distinct $s_\alpha \in S_\alpha, \alpha(s_\alpha) = \alpha$.

For later use we also define structures $\widehat{\mathcal{A}}_s$ and $\mathcal{A}'_s$ for $s \in S_\alpha$ as follows: let $\hat{\mu}_s$ = largest p.r. closed μ such that $\mu = \mu_s^0$ or $L_\mu[A \cap \alpha, s^*] \models |s|$ is a cardinal greater than α. Then $\widehat{\mathcal{A}}_s = L_{\hat{\mu}_s}[A \cap \alpha, s^*]$. The ordinal μ'_s and structure $\mathcal{A}'_s$ are defined in the same way, except we replace p.r. closure of μ by the weaker condition $\omega \cdot \mu = \mu$.

For $s \in S_{\alpha^+}$ write $\bar{s} < s$ to mean that $\pi(\bar{s}) = s$ where $\pi : \overline{\mathcal{A}} \longrightarrow \mathcal{A}_s$ is an elementary embedding with some critical point $\alpha(\bar{s}) < \alpha^+$ and where $\pi(\alpha(\bar{s})) = \alpha^+$. Then $\pi = \pi_{\bar{s}s}$ is unique. Let $\bar{s} \leq s$ denote $\bar{s} < s$ or $\bar{s} = s$. We have the following facts:

(a) $\{\alpha(\bar{s}) | \bar{s} < s\}$ is CUB in α^+.

(b) If $\bar{t}$ is a proper initial segment of $\bar{s}$ then $\bar{t} < \pi_{\bar{s}s}(\bar{t}) = t$ and $\pi_{\bar{t}t} = \pi_{\bar{s}s} \restriction \mathcal{A}_{\bar{t}}$.

(c) $\mathcal{A}_s = \bigcup\{\text{Rng}(\pi_{\bar{s}s}) | \bar{s} < s\}$.

Now for $s \in S_{\alpha^+}$ let $b_s = \{\bar{s} | \bar{s} < s\}$. We use the strings $\bar{s} * i$ with $\bar{s} < s \restriction \eta$, $i = 0$ or 1, to code $s(\eta)$. A condition in the successor coding R^s is a pair $(u, \bar{u})$ where:

1) $u \in S_\alpha$

2) $\bar{u} \subseteq \{b_{s\restriction\eta} | s(\eta) = 0\}$, card $(\bar{u}) \leq \alpha$ in $\mathcal{A}_s$.

To define extension of conditions, we need a couple of preliminary definitions. We say that $\bar{u}$ *restrains* $\bar{s} * 1$ if $\bar{s} \in b$ for some $b \in \bar{u}$ and $\bar{s}$ *lies on* u if $u(\alpha(\bar{s})) = 1$ and $u(\langle\alpha(\bar{s}), \eta\rangle) = \bar{s}(\eta)$ for $\eta \in \text{Dom}(\bar{s})$. Also let $\langle Z_\gamma | \gamma < \alpha^+\rangle$ be an L_{α^+}-definable partition of the odd ordinals less than α^+ into α^+ disjoint pieces of size α^+. We use the Z_γ's to code $A \cap \alpha^+$ into G_α. For $u \in S_\alpha$, $u^{\text{even}}(\delta) = u(2\delta)$, $u^{\text{odd}}(\delta) = u(2\delta+1)$.

Extension of conditions for R^s is defined by: $(u_0, \bar{u}_0) \leq (u_1, \bar{u}_1)$ iff u_0 extends

$u_1; \bar{u}_0 \supseteq \bar{u}_1; \bar{u}_1$ restrains $\bar{s}*1$, $\bar{s}*1$ lies on $u_0^{\text{even}} \longrightarrow \bar{s}*1$ lies on u_1^{even}; $\gamma < |u_1|, \gamma \notin A$, $\delta \in Z_\gamma$, $u_0^{\text{odd}}(\delta) = 1 \longrightarrow u_1^{\text{odd}}(\delta) = 1$. Note that $R^s \in \mathcal{A}_s$.

Lemma 1.1. *Suppose G is R^s-generic over $\mathcal{A}_s$ and let $G_\alpha = \bigcup\{u | (u, \bar{u}) \in G$ for some $\bar{u}\}$. Then $G, A \cap \alpha^+, s$ belong to $L_{\mu_s}[G_\alpha]$.*

Proof. We can write $(u, \bar{u}) \in G$ iff $u \subseteq G_\alpha$ and $\bar{s} \in b \in \bar{u}$, $\bar{s}*1$ lies on $G_\alpha^{\text{even}} \longrightarrow \bar{s}*1$ lies on u^{even} and $\gamma < |u|$, $\gamma \notin A, \delta \in Z_\gamma$, $G_\alpha^{\text{odd}}(\delta) = 1 \longrightarrow u^{\text{odd}}(\delta) = 1$. So $G \in L_{\mu_s}[A \cap \alpha^+, G_\alpha, s]$. And $\gamma \in A \cap \alpha^+$ iff $G_\alpha^{\text{odd}}(\delta) = 1$ for unboundedly many $\delta \in Z_\gamma$, so $G, A \cap \alpha^+ \in L_{\mu_s}[G_\alpha, s]$. Finally note that for any $\eta < |s|, \bar{s}$ lies on G_α^{even} for unboundedly many $\bar{s} < s \restriction \eta$ by a density argument using the fact that for $\eta < |s|, (u, \bar{u}) \in R^s$, $b_{s \restriction \eta}$ is almost disjoint from $\{u | u$ extends some $\bar{s} * 1$ restrained by $\bar{u}\}$. So $s(\eta) = 1$ iff $\bar{s} * 1$ lies on G_α^{even} for unboundedly many $\bar{s} < s \restriction \eta$. Thus $s \restriction \eta$ can be recovered by induction on $\eta \leq |s|$, inside $L_{\mu_s}[G_\alpha]$. ⊣

Lemma 1.2. *$R^{<s} = \bigcup\{R^t | t \subseteq s, t \neq s\}$ has the α^{++}-CC in $\widehat{\mathcal{A}}_s$.*

Proof. If $\hat{\mu}_s = \mu_s^0$ then this is vacuous. Otherwise we need only observe that $R^{<s} \in \widehat{\mathcal{A}}_s$ and $(u_0, \bar{u}_0)$, $(u_1, \bar{u}_1)$ incompatible $\longrightarrow u_0 \neq u_1$ and S_α has cardinality α^+ in $\widehat{\mathcal{A}}_s$. ⊣

Lemma 1.3. *R^s is $\leq \alpha$-distributive in $\mathcal{A}_s$.*

Proof. Suppose $(u_0, \bar{u}_0) \in R^s$ and $\langle D_i | i < \alpha \rangle$ are predense on R^s, $\langle D_i | i < \alpha \rangle \in \mathcal{A}_s$. By induction we define conditions $(u_i, \bar{u}_i)$ and elementary submodels M_i of $\mathcal{A}_s$ with $(u_i, \bar{u}_i) \in M_{i+1}$, for $i \leq \alpha$. Choose M_0 to contain α as a subset and to contain $\langle D_i | i < \alpha \rangle, s, A \cap \alpha^+$ as elements. Having defined $(u_i, \bar{u}_i)$ and M_i, choose M_{i+1} to contain M_i as a subset and $(u_i, \bar{u}_i)$ as an element. Choose $(u_{i+1}, \bar{u}_{i+1})$ to extend $(u_i, \bar{u}_i)$, meet D_i, guarantee that if $s(\eta) = 1$, $\eta \in M_i$ then $\bar{s} * 1$ lies on $u_{i+1}^{\text{even}} - u_i^{\text{even}}$ for some $\bar{s} < s \restriction \eta$, guarantee that if $\gamma \in A \cap (M_i \cap \alpha^+)$ then $u_{i+1}^{\text{odd}}(\delta) = 1$ for some $\delta \notin \operatorname{dom} u_i^{\text{odd}}$, $\delta \in Z_\gamma$, and finally choose $\bar{u}_{i+1}$ to contain all $b_{s \restriction \eta}$ with $s(\eta) = 0$, $\eta \in M_i$. The last requirement can be imposed because the facts that $|s|$ has cardinality $\leq \alpha^+$ in $\mathcal{A}_s$, $H_{\alpha^+} \subseteq \mathcal{A}_s$ imply that any subset of $|s|$ of cardinality $\leq \alpha$ belongs to $\mathcal{A}_s$.

For $\lambda \leq \alpha$ limit, $M_\lambda = \bigcup\{M_i | i < \lambda\}$ and $u_\lambda = \bigcup\{u_i | i < \lambda\}$, $\bar{u}_\lambda = \bigcup\{\bar{u}_i | i < \lambda\}$. By construction, u_λ codes $A \cap (M_\lambda \cap \alpha^+)$ as well as $\bar{s} = s \circ \pi^{-1}$ where π is the transitive collapse map for M_λ. Thus the sequence of ordinals $\langle M_i \cap \alpha^+ | i < \lambda \rangle$ is cofinal in $|u_\lambda|$ and belongs to $L[u_\lambda]$, since the entire sequence $\langle \overline{M}_i | i < \lambda \rangle$ can be recovered in $L[u_\lambda]$, $\overline{M}_i$ = transitive collapse (M_i). This shows that u_λ is reshaped, so $(u_\lambda, \bar{u}_\lambda)$ is a condition. Finally note that $(u_\alpha, \bar{u}_\alpha)$ is an extension of $(u_0, \bar{u}_0)$ meeting each of the D_i's. $\dashv$

Corollary 1.4. *$R^{<s}$ is $\leq \alpha$-distributive in $\widehat{\mathcal{A}}_s$.*

Proof. By Lemma 1.2 it suffices to prove $\leq \alpha$-distributivity in $\mathcal{A}_s^0$. This is easily proved by induction on $|s|$, using Lemma 1.3 at successor stages. $\dashv$

Lemma 1.5. *If $D \subseteq R^{<s}$, $D \in \widehat{\mathcal{A}}_s$ is predense and $s \subseteq t \in S_{\alpha^+}$ then D is predense on R^t.*

Proof. It suffices to show that if $D \subseteq R^s$, $D \in \mathcal{A}_s$ is predense, $s \subseteq t \in S_{\alpha^+}$ then D is predense on R^t; for then, as in the proof of Corollary 1.4, we can induct on $|s|$ and use Lemma 1.2.

Suppose D is predense on $R^s, D \in \mathcal{A}_s$ and $(u, \bar{u})$ belongs to R^t. We can extend $(u, \bar{u})$ to guarantee that for some $\bar{t} < t, \bar{u} = \{b_{t \restriction (\eta+1)} | t(\eta) = 0, \eta \in \operatorname{Rng} \pi_{\bar{t}t}\}$, $D, s \in \operatorname{Rng}(\pi_{\bar{t}t})$ and $|u| = \alpha(\bar{t}) + 1$, $u(\alpha(\bar{t})) = 0$, $u \in \mathcal{A}_{\bar{t} \restriction \alpha(\bar{t})}$. Let $(u^*, \bar{u}^*)$ be the least extension of $(u, \bar{u} \cap \mathcal{A}_s) \in R^s$ meeting D. We claim that $(u^*, \bar{u}^* \cup \bar{u})$ is an extension of $(u, \bar{u})$, and this will prove the lemma. Clearly $\gamma < |u|$, $\delta \notin A$, $\delta \in Z_\gamma$, $u^{*\text{odd}}(\delta) = 1 \longrightarrow u^{\text{odd}}(\delta) = 1$, since $(u^*, \bar{u}^*)$ extends $(u, \bar{u} \cap \mathcal{A}_s)$. Suppose $r < t \restriction \eta, t(\eta) = 0$ where $\eta \in \operatorname{Rng} \pi_{\bar{t}t}$ and $r * 1$ lies on $u^{*\text{even}}$. If $\eta < |s|$ then $r * 1$ lies on u^{even}, as desired, since $(u^*, \bar{u}^*)$ extends $(u, \bar{u} \cap \mathcal{A}_s)$. If $\alpha(r) < \alpha(\bar{t})$ then $|r| < \alpha(\bar{t})$ so again $r * 1$ lies on u^{even} since $|u| > \alpha(\bar{t}) > |r * 1|$. If $\alpha(r) = \alpha(\bar{t})$ then $r * 1$ cannot lie on $u^{*\text{even}}$, by choice of u. Finally if $\alpha(r) > \alpha(\bar{t})$ then since $\eta \geq |s|$ we have $\alpha(r) > |u^*|$ by leastness of $(u^*, \bar{u}^*)$. So $r * 1$ cannot lie on $u^{*\text{even}}$. $\dashv$

Section Two Limit Coding.

We begin with a rough indication of the forcing $\mathcal{P}^u$ for coding $u \in S_\alpha, \alpha$ an uncountable limit cardinal, into a subset of α. $\mathcal{P}^u \subseteq \mathcal{A}_u$ consists of $\mathcal{P}^{<u} = \bigcup\{\mathcal{P}^{u \restriction \xi} | \xi < |u|\}$ together with certain $p : \operatorname{Card} \cap \alpha \longrightarrow V$ such that $p(\beta) = (p_\beta, \bar{p}_\beta) \in R^{p_{\beta^+}}$ for $\beta \in \operatorname{dom}(p)$. (We use Card to denote the class of infinite cardinals.) Also for uncountable limit cardinals $\beta < \alpha$ we (inductively) require that $p \restriction \beta \in \mathcal{P}^{p_\beta} - \mathcal{P}^{<p_\beta}$. We also insist that p code u in the following sense: For $\xi < |u|$ and $\beta \in \operatorname{Card} \cap \alpha$

define $M^\xi_\beta = \Sigma_1$ Skolem hull of $\beta \cup \{u \restriction \xi, A \cap \alpha\}$ in $\mathcal{A}_{u\restriction\xi}$ and $b^\xi_\beta = M^\xi_\beta \cap \beta^+$. Then code u by: $u(\xi) = 1$ iff $p^{\text{odd}}_{\beta+}(b^\xi_{\beta+}) = 1$ for sufficiently large $\beta \in \text{Card} \cap \alpha$. Recall that the successor coding $R^{p_{\beta++}}$ makes use of odd ordinals (in the Z_γ's) so the successor and limit codings do not conflict. For $p, q \in \mathcal{P}^u$ we write $p \le q$ iff $p(\beta) \le q(\beta)$ in $R^{p_{\beta+}}$ for each $\beta \in \text{Card} \cap \alpha$.

To facilitate the proofs of extendibility and distributivity for $\mathcal{P}^u$ we thin out the forcing, in a number of ways. For this purpose we need appropriate forms of $\Box$ and $\diamond$, in a relativized form. Jensen observed that his proofs of these principles for L go through when relativized to reshaped strings. Precisely:

Relativized $\Box$ Let $S = \bigcup\{S_\alpha | \alpha$ an infinite cardinal$\}$. There exists $\langle C_s | s \in S\rangle$ such that $C_s \in \mathcal{A}_s$ and:

(a) If $\alpha(s) < |s|$ then C_s is closed, unbounded in μ^0_s, ordertype $(C_s) \le \alpha(s)$. If $|s|$ is a successor ordinal then ordertype $(C_s) = \omega$.

(b) $\nu \in \text{Lim}(C_s) \longrightarrow$ for some $\eta < |s|$, $\nu = \mu^0_{s\restriction\eta}$ and $C_s \cap \nu = C_{s\restriction\eta}$.

(c) Let $\pi : \langle \overline{\mathcal{A}}, \overline{C}\rangle \xrightarrow{\Sigma_1} \langle \mathcal{A}^0_s, C_s\rangle$ and write $\text{crit}(\pi) = \bar\alpha$, $\overline{\mathcal{A}} = L_{\bar\mu}[\overline{A}, \bar s^*]$. If $\pi(\bar\alpha) = \alpha(s)$ then $L[\overline{A}, \bar s^*] \vDash |\bar s|$ is not a cardinal $> \bar\alpha$ and

(c1) $\overline{C} \in L_\mu[\overline{A}, \bar s^*]$ where μ is the least p.r. closed ordinal greater than $\bar\mu$ s.t. $L_\mu[\overline{A}, \bar s^*] \vDash \text{card}(|\bar s|) \le \bar\alpha$.

(c2) π extends to $\pi' : \mathcal{A}' \xrightarrow{\Sigma_1} \mathcal{A}'_s$ where $\mathcal{A}' = L_{\mu'}[\overline{A}, \bar s^*]$, $\mu' =$ largest ordinal either equal to $\bar\mu$ or s.t. $\omega \cdot \mu' = \mu'$ and $L_{\mu'}[\overline{A}, \bar s^*] \vDash |\bar s|$ is a cardinal greater than $\bar\alpha$.

(c3) If $\bar\alpha$ is a cardinal and $\pi(\bar\alpha) = \alpha$ then $\overline{\mathcal{A}} = \mathcal{A}^0_{\bar s}$ and $\overline{C} = C_{\bar s}$.

Relativized $\diamond$ Let $E =$ all $s \in S$ such that $|s|$ limit and ordertype $(C_s) = \omega$. There exists $\langle D_s | s \in E\rangle$ such that $D_s \subseteq \mathcal{A}^0_s$ and:

(a) $D \in \widehat{\mathcal{A}}_s \ne \mathcal{A}^0_s$, $D \subseteq \mathcal{A}^0_s \longrightarrow \{\xi < |s| \,\big|\, s \restriction \xi \in E, D_{s\restriction\xi} = D \cap \mathcal{A}^0_{s\restriction\xi}\}$ is stationary in $\widehat{\mathcal{A}}_s$.

(b) D_s is uniformly Σ_1-definable as an element of $\mathcal{A}'_s$.

(c) If $\mathcal{A}'_s \vDash \alpha^{++}$ exists then $D_s = \emptyset$.

Now we use these combinatorial structures to impose some further restrictions on membership in $\mathcal{P}^u - \mathcal{P}^{<u}$. First some definitions. For $p \in \mathcal{P}^u$ and $\beta \in \text{Card} \cap \alpha$, $(p)_\beta$ denotes $p \restriction \text{Card} \cap [\beta, \alpha)$. $D \subseteq \mathcal{P}^{<u}$ is *predense* if every $p \in \mathcal{P}^{<u}$ is compatible with an element of D and for $\beta \in \text{Card} \cap \alpha, D$ is *β-predense* if every condition

$p \in \mathcal{P}^{<u}$ can be extended to some q such that $p \restriction \beta = q \restriction \beta$ and q *meets* D (i.e., q extends an element of D). And p *reduces* D *below* β if every $q \leq p$ can be further extended to r such that r meets D and $(q)_\beta = (r)_\beta$.

Requirement A. (Predensity Reduction) Suppose $p \in \mathcal{P}^u - \mathcal{P}^{<u}$.

(A1) If $u \in E$ and $D_u \subseteq \mathcal{P}^{<u}$ is β-predense for all $\beta \in \mathrm{Card} \cap \alpha$ then p meets D_u.

(A2) If $|u|$ is a successor ordinal, $D \subseteq \mathcal{P}^{<u}$ is predense and $D \in \mathcal{A}_u^0$ then p reduces D below some $\beta < \alpha$.

Requirement B. (Restriction) For $p \in \mathcal{P}^u$ let $|p|$ denote the least ξ s.t. $p \in \mathcal{P}^{u \restriction \xi}$. If p belongs to $\mathcal{P}^u$ and $\xi < |p|$ then there exists r s.t. $p \leq r$ and $|r| = \xi$.

Requirement C. (Nonstationary Restraint) Suppose $\mathcal{A}_u \vDash \alpha$ inaccessible and $p \in \mathcal{P}^u$. Then there exists a CUB $C \subseteq \alpha$ s.t. $C \in \mathcal{A}_u$ and $\beta \in C \longrightarrow \overline{p}_\beta = \emptyset$.

The remaining Requirement D will be introduced at a later point when we discuss strong extendibility at successor stages.

Extendibility and distributivity for $\mathcal{P}^u$ are stated as follows. Let $q \leq_\beta p$ signify that $q \leq p$ and $q \restriction \beta = p \restriction \beta$. $(\mathcal{P}^{<u})_\beta$ denotes $\{(p)_\beta | p \in \mathcal{P}^{<u}\}$, for $\beta \in \mathrm{Card} \cap \alpha$. <u>$\Delta-$ distributivity</u> for $\mathcal{P}^{<u}$ asserts that if D_β is β^+-predense on $\mathcal{P}^{<u}$ for each $\beta \in \mathrm{Card} \cap \alpha$ then every $p \in \mathcal{P}^{<u}$ can be extended to meet each D_β.

$(*)_u$ $p \in \mathcal{P}^u$, $\beta \in \mathrm{Card} \cap \alpha \longrightarrow \exists q \leq_\beta p \ (q \in \mathcal{P}^u - \mathcal{P}^{<u})$

$(**)_u$ $(\mathcal{P}^{<u})_\beta$ is $\leq \beta$-distributive in $\widehat{\mathcal{A}}_u$ for $\beta \in \mathrm{Card} \cap \alpha$.

And if α is inaccessible in $\mathcal{A}_u^0$ then $\mathcal{P}^{<u}$ is Δ-distributive in $\widehat{\mathcal{A}}_u$.

These are proved by a simultaneous induction on $|u|$. As the base case $|u| = \alpha$ is vacuous we assume from now on that $|u| > \alpha$. The following consequences of predensity reduction are needed in the proof.

Lemma 2.1. *(Chain Condition for $\mathcal{P}^{<u}$) Suppose $(**)_u$ holds. Then $\mathcal{P}^{<u}$ has the α^+-CC in $\widehat{\mathcal{A}}_u$.*

Proof. We may assume that $\widehat{\mathcal{A}}_u \neq \mathcal{A}_u^0$. Suppose $D \subseteq \mathcal{P}^{<u}$ is predense and $D \in \widehat{\mathcal{A}}_u$. Consider $D^* = \{p \in \mathcal{P}^{<u} | p$ reduces D below some $\beta \in \mathrm{Card} \cap \alpha\}$. Then $D^* \in \widehat{\mathcal{A}}_u$.

By $(**)_u$ and Lemma 1.2, D^* is β-predense for all $\beta \in \text{Card} \cap \alpha$. (Use $\leq \beta^+$-distributivity of $(\mathcal{P}^{<u})_{\beta^+}$ and β^{++}-CC of $R^{G_{\beta^+}} \subseteq \beta^{++}$ denoting the $(\mathcal{P}^{<u})_{\beta^+}$-generic, to reduce D below β^+.) Apply relativized $\diamond$ to obtain $\xi < |u|$ such that $u \restriction \xi \in E$, $D_{u\restriction\xi} = D^* \cap \mathcal{A}^0_{u\restriction\xi}$ and $D_{u\restriction\xi}$ is β-predense for all $\beta \in \text{Card} \cap \alpha$. Thus by predensity reduction and restriction, $D^* \cap \mathcal{A}^0_{u\restriction\xi}$ is predense on $\mathcal{P}^{<u}$ and therefore so is $D \cap \mathcal{A}^0_{u\restriction\xi}$, a subset of D of $\widehat{\mathcal{A}}_u$-cardinality $\leq \alpha$. ⊣

Lemma 2.2. *(Persistence for $\mathcal{P}^{<u}$) Suppose $(**)_u$ holds, $D \subseteq \mathcal{P}^{<u}$ is predense, $D \in \widehat{\mathcal{A}}_u$ and $u \subseteq v \in S_\alpha$. Then D is predense on $\mathcal{P}^v$.*

Proof. By restriction, if $p \in \mathcal{P}^v - \mathcal{P}^u$ then p extends some q in $\mathcal{P}^u - \mathcal{P}^{<u}$. By the chain condition for $\mathcal{P}^{<u}$ we can assume that $D \in \mathcal{A}^0_u$ and hence by induction we can assume that $|u|$ is a successor ordinal. But then by predensity reduction, q reduces D below some $\beta \in \text{Card} \cap \alpha$ and hence so does p. In particular p is compatible with an element of D. ⊣

We can now turn to the proofs of $(*)_u$, $(**)_u$.

Lemma 2.3. *Assume $(**)_u$ and $|u|$ a limit ordinal. Then $(*)_u$ holds.*

Proof. We first claim that if $p \in \mathcal{P}^{<u}$ and $\langle D_\beta | \beta_0 \leq \beta < \alpha \rangle \in \mathcal{A}^0_u$, $D_\beta \subseteq \mathcal{P}^{<u}$ β^+-predense for each β then there is $q \leq_{\beta_0} p$ meeting each D_β. We prove this with α replaced by $\beta_1 \in \text{Card} \cap \alpha^+$, by induction on β_1. The base case $\beta_1 = \beta_0^+$ and the case of β_1 a successor cardinal follow easily, using $(**)_u$. If β_1 is singular in $\mathcal{A}^0_u$ then we can choose $\gamma_0 < \gamma_1 < \cdots$ approximating β_1 in length $\lambda < \beta_1$ and consider $\langle E_\delta | \delta < \lambda \rangle$ where $E_\delta =$ all q meeting each D_β, $\lambda \leq \beta < \gamma_\delta$, $|q|$ least so that $\langle D_\beta | \beta_0 \leq \beta < \beta_1 \rangle \in \mathcal{A}^0_{u\restriction|q|}$. Then we are done by induction. If β_1 is inaccessible in $\mathcal{A}^0_u$ then either $\beta_1 = \alpha$, in which case the result follows directly from the second statement of $(**)_u$, or $\beta_1 < \alpha$, in which case we can factor $\mathcal{P}^{<u}$ as $(\mathcal{P}^{<u})_{\beta_1^+} * \mathcal{P}^{G_{\beta_1^+}}$ (where $G_{\beta_1^+}$ denotes $\bigcup\{p_{\beta_1^+} | p \in G\}$, G the generic for $\mathcal{P}^{<u}$). Then choose $(q)_{\beta_1^+} \leq (p)_{\beta_1^+}$ that reduces each D_β, $\beta_0 \leq \beta < \beta_1$ below β_1^+, using $(**)_u$ and the β_1^+-CC of $\mathcal{P}^{G_{\beta_1^+}}$. By induction on α, we can extend q to meet all the D_β's.

Now write $C_u = \{\mu^0_{u\restriction\xi_i} | i < \lambda\}$ and choose a successor cardinal $\beta_0 < \alpha$ to be at least as large as λ and the β given in the statement of $(*)_u$, if $\lambda < \alpha$. Now inductively define a subsequence $\langle \eta_j | j < \lambda_0 \rangle$ of $\langle \xi_i | i < \lambda \rangle$ and conditions $\langle p_j | j < \lambda_0 \rangle$ as follows. First suppose $\lambda < \alpha$. Let p denote the condition given in the statement of $(*)_u$. Set $p_0 = p, \eta_0 =$ least ξ_i s.t. $p \in \mathcal{P}^{<u\restriction\xi_i}$; $p_{j+1} =$ least $q \leq_\beta p_j$ s.t. for all $\gamma, \beta_0 \leq \gamma < \alpha$, q meets all γ^+-predense $D \subseteq \mathcal{P}^{<u\restriction\eta_j}$, $D \in M^{\eta_j}_{\gamma^+} = \Sigma_1$ Skolem hull of $\gamma^+ \cup \{p, \alpha\}$ in $\langle \mathcal{A}^0_{u\restriction\eta_j}, C_{u\restriction\eta_j}\rangle$, $\eta_{j+1} =$ least ξ_i s.t. $p_{j+1} \in \mathcal{P}^{<u\restriction\xi_i}$; $p_\delta =$ g.l.b. $\langle p_j | j < \delta\rangle$, $\eta_\delta = \bigcup\{\eta_j | j < \delta\}$ for limit $\delta \leq \lambda_0$. The ordinal λ_0 is determined by the condition that η_{λ_0} is equal to $|u|$. If $\lambda = \alpha$ then the definition is the same, except in defining p_{j+1} require $p_{j+1} \leq_{\beta\cup\aleph_{i+1}} p_j$ where $\eta_j = \xi_i$ and only require p_{j+1} to meet γ^+-predense D as above for γ between $\beta \cup \aleph_i$ and α.

We must verify that p_δ as defined above is indeed a condition for limit δ. (There is no problem at successor stages, using Lemma 2.2 and the first paragraph of the present proof.) First we show that for $\gamma \in \text{Card} \cap \alpha$, p_{δ_γ} is reshaped. We need only consider $\gamma \geq \beta$ and in case $\lambda = \alpha$ we need only consider $\gamma \geq \beta \cup \aleph_i$ where $\eta_\delta = \xi_i$. By construction if $\gamma \in M^{\eta_\delta}_\gamma = \Sigma_1$ Skolem hull of $\gamma \cup \{p, \alpha\}$ in $\langle \mathcal{A}^0_{u\restriction\eta_\delta}, C_{u\restriction\eta_\delta}\rangle$ then p_{δ_γ} is $\pi[(\mathcal{P}^{<u\restriction\eta_\delta})_\gamma]$ generic over $\text{TC}(M^{\eta_\delta}_\gamma)$ where $\pi : M^{\eta_\delta}_\gamma \longrightarrow TC(M^{\eta_\delta}_\gamma)$ is the transitive collapse. And $|p_{\delta_\gamma}|$ is Σ_1-definably singularized over $\text{TC}(M^{\eta_\delta}_\gamma)$. Write $\text{TC}(M^{\eta_\delta}_\gamma)$ as $\langle \overline{\mathcal{A}}, \overline{C}\rangle$. By genericity and cofinality-preservation for $\pi[(\mathcal{P}^{<u\restriction\eta}\delta)_\gamma]$, p_{δ_γ} codes $\overline{\mathcal{A}}$ and by Relativized $\square$ (c1), $\overline{C}$ is constructible from $\overline{\mathcal{A}}$. So p_{δ_γ} is reshaped. If $M^{\eta_\delta}_\gamma \cap \alpha = \gamma$ then p_{δ_γ} is again reshaped because of Relativized $\square$(c1), (no genericity argument required). Lastly if $\gamma' = \min(M^{\eta_\delta}_\gamma \cap (\text{ORD} - \gamma)) < \alpha$ then use the first argument, but with $\pi[(\mathcal{P}^{<u\restriction\eta_\delta})_{\gamma'}]$ replacing $\pi[(\mathcal{P}^{<u\restriction\eta_\delta})_\gamma]$.

Next we show that $p_\delta \restriction \gamma \in \mathcal{A}_{p_{\delta_\gamma}}$. As $p_\delta \restriction \gamma$ is definable over $\text{TC}(M^{\eta_\delta}_\gamma) \in L[A \cap \gamma, p_{\delta_\gamma}]$ this amounts to showing that $\mu_{p_{\delta_\gamma}}$ is large enough. By $(**)_{u\restriction\eta_\delta}$ and Lemma 2.1 we know that $\mathcal{P}^{<u\restriction\eta_\delta}$ has the α^+-CC in $\widehat{\mathcal{A}}_{u\restriction\eta_\delta}$ and hence (when $M^{\eta_\delta}_\gamma \cap \alpha \neq \gamma$) p_{δ_γ} is in fact ${\pi'}^{-1}[(\mathcal{P}^{<u\restriction\eta_\delta})_{\gamma'}]$-generic over $\mathcal{A}'$, where π' (with domain $\mathcal{A}'$) is the extension of π^{-1} given by Relativized $\square$ (c2) and $\gamma' = \min(M^{\eta_\delta}_\gamma \cap (\text{ORD} - \gamma))$. And thus $\mathcal{A}'[p_{\delta_\gamma}] \vDash |p_{\delta_\gamma}|$ is a cardinal. But by Relativized $\square$ (c1), $\text{TC}(M^{\eta_\delta}_\gamma)$ appears relative to p_{δ_γ} before the next p.r. closed ordinal after the height of $\mathcal{A}'$. So $p_\delta \restriction \gamma \in \mathcal{A}_{p_{\delta_\gamma}}$. If $M^{\eta_\delta}_\gamma \cap \alpha = \gamma$ then no genericity argument is required; we only need Relativized $\square$ (c1).

Requirements B, C are easily checked, the latter using the fact that in case of α inaccessible in $\mathcal{A}^0_u$ we required $p_{j+1} \leq_{\beta\cup\aleph_{i+1}} p_j(\eta_j = \xi_i)$ and therefore can use diagonal intersection of clubs. To check Requirement (A1) note that if $M^{\eta_\delta}_\gamma \cap \alpha \neq \gamma$ then either $p_{\delta_\gamma} \notin E$ or $D_{p_{\delta_\gamma}} = \emptyset$, since $\mathcal{A}'_{p_{\delta_\gamma}} \vDash \gamma^{++}$ exists and we can apply Relativized $\diamond$ (c). If $M^{\eta_\delta}_\gamma \cap \alpha = \gamma$ then $p_{\delta_\gamma} \in E$ iff $u \restriction \eta_\delta \in E$ by Relativized $\square$

(c3) and if these hold then by Relativized $\diamond$ (b), $\pi'[D_{p_{\delta_\gamma}}] = D_{u\restriction\eta_\delta}$, where π' comes from Relativized $\Box$ (c2). So all we need to arrange is that our initial condition p be chosen to meet D_u, in case $u \in E$, and otherwise choose η_0 to be at least ξ_ω, so that $u \restriction \eta_\delta \notin E$ for limit δ. $\dashv$

Lemma 2.4. *Assume $|u|$ limit and $(*)_v, (**)_v$ for $v \subseteq u, v \neq u$. Then $(**)_u$ holds.*

Proof. We may assume that $\widehat{\mathcal{A}}_u \neq \mathcal{A}^0_u$. We need only make a small change in the construction of the proof of Lemma 2.3. Given predense $\langle D_i | i < \beta\rangle$ on $(\mathcal{P}^{<u})_\beta$ in $\widehat{\mathcal{A}}_u$ with $\beta < \alpha$, select $\xi < |u|$ of cofinality $> \beta$ such that $D_i \cap (\mathcal{P}^{<u\restriction\xi})_\beta$ is predense on $(\mathcal{P}^{<u\restriction\xi})_\beta$ for all $i < \beta$ and then choose the continuous sequence $\langle \xi_i | i < \beta\rangle$ from $C_{u\restriction\xi}$ by: $\xi_0 = \omega^{\text{th}}$ element of $C_{u\restriction\xi}$, $\xi_{i+1} =$ least $\xi^* \in C_{u\restriction\xi}$ greater than ξ_i s.t. $q \in (\mathcal{P}^{u\restriction\xi_i})_\beta \longrightarrow \exists r \leq q(r \in (\mathcal{P}^{u\restriction\xi^*})_\beta, r$ meets $D_i)$, $\xi_\lambda = \bigcup\{\xi_i | i < \lambda\}$ for limit $\lambda \leq \beta$. Then $u \restriction \xi_\lambda \notin E$ and $\langle \xi_i | i < \lambda\rangle \in \mathcal{A}_{u\restriction\xi_\lambda}$ for limit λ.

Now repeat the construction of the proof of Lemma 2.3, extending along the ξ_i's instead of along C_u, hitting D_i at stage $i+1$. We can guarantee $\langle D_i \cap (\mathcal{P}^{<u\restriction\xi_i})_\beta | i < \lambda\rangle$ is $\Delta_1\langle \mathcal{A}^0_{u\restriction\xi_\lambda}, C_{u\restriction\xi_\lambda}\rangle$ in our choice of ξ_i's as well, so hitting the D_i's does not interfere with the proof that p_δ is a condition for limit δ. The proof of Δ-distributivity is similar. $\dashv$

Lemma 2.5. *Suppose $(**)_u$ holds and $|u|$ is a successor ordinal. Then $(*)_u$ holds.*

Proof. We may assume that the given p belongs to $\mathcal{A}_v - \mathcal{A}^0_v$ where $v = u \restriction (|u| - 1)$. Write $C_u = \langle \xi_j | j < \omega\rangle$. Now proceed as in the construction of the proof of Lemma 2.3, making successive $\leq_\beta$-extensions below p (where β is given in the statement of $(*)_u$), $p \geq_\beta p_0 \geq_\beta p_1 \geq_\beta \cdots$ so that p_{j+1} meets all γ^+-predense $D \subseteq \mathcal{P}^{<u}$ in $M^{\xi_j}_{\gamma^+}$, where $M^{\xi_j}_{\gamma^+} = \Sigma_1$ Skolem hull of $\gamma^+ \cup \{p, \alpha, \xi_0, \cdots, \xi_{j-1}\}$ in $\mathcal{A}_v \restriction \xi_j$, for all $\gamma \in [\beta, \alpha)$. If we set $\hat{q} =$ g.l.b. $\langle p_i | i \in \omega\rangle$ then $\hat{q}$ meets the requirements for being a condition at all $\gamma \in \text{Card} \cap \alpha^+$ with the exception of γ in $C \cup \{\alpha\}$, $C = \{\gamma | M_\gamma \cap \alpha = \gamma\}$, $M_\gamma = \Sigma_1$ Skolem hull of $\gamma \cup \{p, \alpha\}$ in $\langle \mathcal{A}_v, C_u\rangle$. The reason is that for $\gamma \in \alpha - C$, $T_\gamma = TC(M_\gamma)$ belongs to $\mathcal{A}_{\hat{q}_\gamma}$, since $T_\gamma \vDash |\hat{q}_\gamma|$ is a cardinal and $\hat{q}_\gamma$ is generic over T_γ.

To make $\hat{q}$ into a condition $q \in \mathcal{P}^u$ we must do two things. First extend $\hat{q}_{\gamma^+}$ for $\gamma \geq \beta$ so as to code $u(|v|) = 0$ or 1. This is easily done as there are no conflicts between the successor and limit codings. Second for $\gamma \in C$ we extend $\hat{q}_\gamma$ to $q_\gamma = \hat{q}_\gamma * u(|v|)$. The only remaining question is whether the reatraint $\overline{\hat{q}}_\gamma$ will allow us to do this. But $\gamma \in C \longrightarrow \bar{\hat{q}}_\gamma = \emptyset$ since C is contained in the CUB witnessing Requirement C for $\hat{q}$ at α. $\dashv$

Lemma 2.6. *Suppose* $(*)_u$ *and* $(**)_v, v \subseteq u \neq v$ *hold and* $|u|$ *is a successor. Then* $(**)_u$ *holds.*

Proof. We must show that if $v = u \restriction (|u|-1)$ and $p \in (\mathcal{P}^v)_\beta - (\mathcal{P}^{<v})_\beta$, $\langle D_i | i < \beta \rangle \in \mathcal{A}_v$ are predense on $(\mathcal{P}^v)_\beta$ then there exists $q \leq p$ meeting each D_i. For simplicity we assume $\beta = \omega$.

Definition. Suppose $f(\beta) = M_\beta$ is a function in $\mathcal{A}_v$ from $\text{Card}^+ \cap \alpha$ (Card^+ denotes all successor cardinals) into $\mathcal{A}_v$ such that card $(M_\beta) \leq \beta$ for all $\beta \in \text{Dom}(f)$ and suppose $p \in \mathcal{P}^v$. Then $\Sigma^p_f = \{q \in \mathcal{P}^v | \ \forall\ \beta \in \text{Dom}(f), q(\beta)$ meets all predense $D \subseteq R^{p_{\beta^+}}, D \in M_\beta\}$.

Sublemma 2.7. Σ^p_f *is dense below* p *in* $\mathcal{P}^v$.

Before proving Sublemma 2.7 we establish the Lemma, assuming it. Choose a limit ordinal $\lambda = \omega^\lambda < \mu_v$ such that $\langle D_i | i < \omega \rangle, C_v \in \mathcal{A}_v \restriction \lambda = L_\lambda[A \cap \alpha, v^*]$ and $\Sigma_1 \operatorname{cof}(\mathcal{A}_v \restriction \lambda) = \omega$. Choose a $\Sigma_1(\mathcal{A}_v \restriction \lambda)$ sequence $\lambda_0 < \lambda_1 < \cdots$ cofinal in λ such that $\langle D_i | i < \omega \rangle, C_v, x \in \mathcal{A}_v \restriction \lambda_0$ where x is a parameter defining the λ_i's. Set $M^i_\gamma =$ least $M \prec_{\Sigma_1} A_v \restriction \lambda_i$ such that $\gamma \cup \{x, \langle D_i | i < \omega \rangle, \alpha, C_v\} \subseteq M$, for each $\gamma \in \text{Card}^+ \cap \alpha$. Define $f_i(\gamma) = M^i_\gamma$.

Choose $p = p_0 \geq p_1 \geq \cdots$ successively so that p_{i+1} meets D_i and $\Sigma^{p_i}_{f_i}$. Set $p^* =$ g.l.b. $\langle p_i | i \in \omega \rangle$. We show that p^* is a well-defined condition. If $|v| > \alpha$ then thanks to $(**)_v$ it will suffice to show that if $D \in M^i_\gamma \cap \mathcal{A}^0_v$ is predense on $(\mathcal{P}^{<v})_\gamma$, $\gamma \in \text{Card} \cap \alpha$ then some p_j reduces D below γ. (For then, p^*_γ codes a generic over the transitive collapse of $M^i_\gamma \cap \mathcal{A}^0_v$.) If $|v| = \alpha$ then instead of $\mathcal{P}^{<v} = \emptyset$ use $\mathcal{P}^\alpha = \{p \restriction \beta^+ | \beta \in \text{Card} \cap \alpha,\ p \in \mathcal{P}^v\}$, ordered in the natural way. Note that $\mathcal{P}^\alpha$ is cofinality-preserving, by applying $(**)$ at cardinals $< \alpha$.

Choose $j \geq i$ so that for $k > j, p_k$ reduces D no further than p_j. Let γ' be least so that p_j reduces D below γ'. Then $\gamma' < \alpha$ by Predensity Reduction for p. If $\gamma' \leq \gamma$ then of course we are done. If $\gamma' > \gamma$ is a double successor cardinal then we reach a contradiction since by definition p_{j+1} reduces D further. If $\gamma' = \delta^+$, δ a limit cardinal then by Predensity Reduction at δ, D is reduced below some $\delta' < \delta$, another contradiction. If γ' is a limit cardinal then the same argument applies, replacing γ' by $(\gamma')^+$.

Finally we have:

Proof of Sublemma 2.7. It suffices to show the following.

Strong Extendibility Suppose $g \in \mathcal{A}_v, g(\beta) \in H_{\beta^{++}}$ for all $\beta \in \mathrm{Card} \cap (\beta_0, \alpha)$ and $p \in \mathcal{P}^v$. Then there is $q \leq_{\beta_0} p$ such that $g \upharpoonright \beta \in \mathcal{A}_{q_\beta}$ for all $\beta \in \mathrm{Card} \cap (\beta_0, \alpha]$.

For, Strong Extendibility allows us to extend to a condition q such that for all $\beta \in \mathrm{Card} \cap \alpha$, $g \upharpoonright \beta \in \mathcal{A}_{q_\beta}$, where $g(\beta) = f(\beta) \cap H_{\beta^{++}}$. Then successively extend each $q(\beta)$ to meet predense D in $f(\beta)$.

We now break down Strong Extendibility into the ramified form in which it will be proved. For any μ such that $\mu_v^0 \leq \mu < \mu_v$, $k \in \omega - \{0\}$ and $\beta \in \mathrm{Card} \cap \alpha$ let $M_\beta^{\mu,k} = \Sigma_k$ Skolem hull of $\beta \cup \{\alpha\}$ in $\mathcal{A}_v^* \upharpoonright \mu = \langle L_\mu[A \cap \alpha, v^*, C_v], A \cap \alpha, v^*, C_v \rangle$. (Notice that this structure is Σ_1 projectible to α without parameter.)

$\underline{SE(\mu, k)}$ Suppose $p \in \mathcal{P}^v$ and $\beta_0 \in \mathrm{Card} \cap \alpha$. Then there exists $q \leq_{\beta_0} p$ such that $TC(M_\beta^{\mu,k}) \in \mathcal{A}_{q_\beta}$ for all $\beta \in \mathrm{Card} \cap (\beta_0, \alpha)$.

It suffices to prove $SE(\mu, k)$ for all μ, k as above. We do so by induction on μ and for fixed μ, by induction on k. To verify the base case of this induction we must impose one last requirement on our conditions.

Requirement D Suppose $p \in \mathcal{P}^v - \mathcal{P}^{<v}$ and $g \in \mathcal{A}_v^0$, $g(\beta) \in H_{\beta^{++}}$ for all $\beta \in \mathrm{Card} \cap \alpha$. Then $g \upharpoonright \beta \in \mathcal{A}_{p_\beta}$ for sufficiently large $\beta \in \mathrm{Card} \cap \alpha$.

This requirement is respected by our earlier constructions. Now, if $k = 1$ and μ is a limit ordinal then we can use a $\Sigma_1(\mathcal{A}_v^* \upharpoonright \mu)$ approximation to μ and induction (or Requirement D if $\mu = \mu_v^0$) to obtain $q \leq p$ satisfying the conclusion of $SE(\mu, 1)$, using the Σ_f's for $f \in \mathcal{A}_v^* \upharpoonright \mu$. Similarly if μ is a successor, $k = 1$ then use $\langle \Sigma_k(\mathcal{A}_v^* \upharpoonright \mu - 1) | k \in \omega \rangle$ to approximate $\Sigma_1(\mathcal{A}_v^* \upharpoonright \mu)$, using the Σ_f's, f definable over $\mathcal{A}_v^* \upharpoonright \mu - 1$.

Suppose $k > 1$. By induction we can assume that $\mathrm{TC}(M_\beta^{\mu,k-1}) \in \mathcal{A}_{p_\beta}$ for large enough β. If $C = \{\beta < \alpha | \beta = \alpha \cap M_\beta^{\mu,k}\}$ is unbounded in α then successively extend $p \upharpoonright \beta$ for $\beta \in C$ so that $\mathrm{TC}(M_{\beta'}^{\mu,k}) \in \mathcal{A}_{q_{\beta'}}$ for $\beta' < \beta$. There is no problem at limits since $\mathrm{TC}(M_\beta^{\mu,k}), C \cap \beta \in \mathcal{A}_{p_\beta}$ for $\beta \in C$.

If α is $\Sigma_k(\mathcal{A}_v^* \upharpoonright \mu)$-singular then choose a continuous cofinal $\Sigma_k(\mathcal{A}_v^* \upharpoonright \mu)$ sequence $\beta_0 < \beta_1 < \cdots$ below α of ordertype $\lambda_0 = \mathrm{cof}(\alpha)$. Also choose β_{i+1} large enough so that $M_{\beta_{i+1}}^{\mu,k-1} \models \beta_i$ is defined. This is possible since $\mathcal{A}_v^* \upharpoonright \mu = \bigcup \{M_\beta^{\mu,k-1} | \beta < \alpha\}$. Now define N_β^i for $i < \lambda_0$, $\beta < \beta_i$ to be the Σ_k Skolem hull of β

in $M^{\mu,k-1}_{\beta_i}$. Then $\langle TC(N^i_\beta)|\beta < \beta_i\rangle \in \mathcal{A}_{p_{\beta_i}}$ for $i < \lambda_0$ since it is easily defined from $M^{\mu,k-1}_{\beta_i} \in \mathcal{A}_{p_{\beta_i}}$. Successively λ_0-extend $p \restriction \beta_i$, producing $p = p_0 \geq_{\lambda_0} p_1 \geq_{\lambda_0} \cdots$ where $TC(N^i_\beta) \in \mathcal{A}_{p_{i_\beta}}$ for $\beta \in (\lambda_0, \beta_i)$. This is possible by induction on α, and since $\mathrm{TC}(N^i_\beta)$ is easily defined from $\langle TC(N^i_{\bar\beta})|\bar\beta < \beta\rangle$ for limit $\beta < \beta_i$. (We must also require that p_{i+1} meets $\Sigma^{p_i}_{f_i}$ where $f_i(\beta) = N^i_\beta$.) p_λ is well-defined for limit $\lambda \leq \lambda_0$ and $\mathcal{A}_{p_{\lambda_{0_\beta}}}$ contains $\langle TC(N^i_\beta)|i < \lambda_0\rangle$ and hence $\mathrm{TC}(M^{\mu,k}_\beta)$ for $\beta > \lambda_0$. Then use induction to fill in on $(0, \lambda_0]$ so that $SE(\mu, k)$ is satisfied.

Lastly, there is the intermediate case where α is $\Sigma_k(\mathcal{A}^*_v \restriction \mu)$-regular but $C = \{\beta < \alpha|\beta = \alpha \cap M^{\mu,k}_\beta\}$ is bounded in α. Then $\Sigma_{k+1}(\mathcal{A}^*_v \restriction \mu)$-cof $(\alpha) = \omega$ and we apply induction to produce $p = p_0 \geq p_1 \geq \cdots$ so that $p_{i+1} \restriction [\beta_i, \beta_{i+1}]$ obeys $SE(\mu, k)$ where $\beta_0 < \beta_1 < \cdots$ is a cofinal ω-sequence of successor cardinals below α. Let $q = \text{g.l.b. } \langle p_i|i \in \omega\rangle$.

This completes the proof of Sublemma 2.7 and hence of $(**)_u$. $\dashv$

Section Three Proof of Jensen's Theorem

A condition in $\mathcal{P}$ is a function p from an initial segment of Card into V such that Dom(p) has a maximum $\alpha(p)$, for any $\alpha \in \mathrm{Dom}(p), p(\alpha) = (p_\alpha, \bar{p}_\alpha)$, if $\alpha \in \mathrm{Dom}(p) \cap \alpha(p)$ then $p(\alpha)$ belongs to $R^{p_{\alpha^+}}$, $p(\alpha(p)) = (s(p), \emptyset)$ where $s(p) \in S_{\alpha(p)}$ and for uncountable limit cardinals $\alpha \in \mathrm{Dom}(p)$, $p \restriction \alpha \in \mathcal{P}^{p_\alpha}$. And $q \leq p$ in $\mathcal{P}$ if $\alpha(p) \leq \alpha(q)$, $s(p) \subseteq q_{\alpha(p)}$ and for $\alpha \in \mathrm{Dom}(p) \cap \alpha(p)$, $q(\alpha) \leq p(\alpha)$ in $R^{q_{\alpha^+}}$.

For any $\alpha \in$ Card, $s \in S_\alpha$, $\mathcal{P}^s$ denotes all $p \restriction \alpha$ for $p \in \mathcal{P}$ such that $\alpha(p) = \alpha$ and $s(p) = s$. And $\mathcal{P}^\alpha$ denotes all $p \in \mathcal{P}$ such that $\alpha(p) < \alpha$.

Now suppose α is an uncountable limit cardinal and $s \in S_\alpha$, $|s| = \alpha + 1$. By Lemma 2.2, G $\mathcal{P}$-generic $\longrightarrow G \cap \mathcal{P}^{<s}$ is $\mathcal{P}^{<s}$-generic over $\mathcal{A}^0_s = L_\mu[A \cap \alpha], \mu$ the least p.r. closed ordinal greater than α. As the forcing relation for $\mathcal{P}^{<s}$ restricted to sentences of rank $< \alpha$ belongs to $L_\mu[A \cap \alpha]$, it follows that the forcing relation $p \Vdash \varphi$, $p \in \mathcal{P}$ and φ ranked, is $\langle L[A], A\rangle$-definable: $p \Vdash \varphi$ iff for some α as above, φ has rank $< \alpha, p \in L_\alpha[A]$ and $L_\mu[A \cap \alpha] \models$ "$p \Vdash \varphi$", μ the least p.r. closed ordinal $> \alpha$.

Now note that $\mathcal{P}$ preserves cofinalities, as otherwise $\mathcal{P}^s$ would change cofinalities for some s as above, contradicting Distributivity (Lemmas 2.4, 2.6) and Chain Condition (Lemmas 1.2, 2.1). If G is $\mathcal{P}$-generic then $L[G] = L[X]$ where

$X = G_\omega \subseteq \omega_1$. Finally by Jensen-Solovay [68], X can be coded by a real via a CCC forcing. This completes the proof of Jensen's Coding Theorem, subject to the verification of Relativized $\square$ and $\diamond$.

Section Four Relatived Square and Diamond

For completeness, we prove Relativized $\square$ and $\diamond$. As relativization causes no serious problems, we first establish unrelativized versions, and then afterward indicate what modifications are required. We begin with $\square$.

First we prove $\square$ in the following form:

Global $\square$ Assume $V = L$. Then there exists $\langle C_\mu | \mu$ a singular limit ordinal$\rangle$ such that:

(a) C_μ is CUB in μ

(b) ordertype $(C_\mu) < \mu$

(c) $\bar{\mu} \in \operatorname{Lim} C_\mu \longrightarrow C_{\bar{\mu}} = C \cap \bar{\mu}$.

In the proof we shall take advantage of Jensen's Σ^* theory, as reformulated in Friedman [94]. For the convenience of the reader we describe that theory here.

For simplicity of notation, for limit ordinals μ we let $\widetilde{J}_\mu$ denote J_α where $\omega\alpha = \mu$. So $\text{ORD}(\widetilde{J}_\mu) = \mu$.

Let M denote some $J_\alpha, \alpha > 0$. (More generally, our theory applies to "acceptable J-models".) We make the following definitions, inductively. We order finite sets of ordinals by the maximum difference order: $x < y$ iff $\alpha \in Y$ where α is the largest element of $(y - x) \cup (x - y)$.

1) A Σ_1^* formula is just a Σ_1 formula. A predicate is $\underline{\Sigma_1^*}$ (Σ_1^*, respectively) if it is definable by a Σ_1^* formula with (without, respectively) parameters. $\rho_1^M = \Sigma_1^*$ projectum of M = least ρ s.t. there is a $\underline{\Sigma_1^*}$ subset of $\omega\rho$ not in M and p_1^M = least p s.t. $A \cap \rho_1^M \notin M$ for some A Σ_1^* in parameter p (where p is a finite set of ordinals). $H_1^M = H_{\omega\rho_1^M}^M$ = sets x in M s.t. M-card (transitive closure $(x)) < \omega\rho_1^M$. For any $x \in M$, $M_1(x)$ = First reduct of M relative to $x = \langle H_1^M, A_1(x)\rangle$ where $A_1(x) \subseteq H_1^M$ codes the Σ_1^* theory of M with parameters from $H_1^M \cup \{x\}$ in the natural way: $A_1(x) = \{\langle y, n\rangle |$ the n^{th} Σ_1^* formula is true at $\langle y, x\rangle$, $y \in H_1^M\}$. A good Σ_1^* function is just a Σ_1 function and for any $X \subseteq M$ the Σ_1^* hull (X) is just the Σ_1 hull of X.

(2) For $n \geq 1$, a Σ^*_{n+1} formula is one of the form $\varphi(x) \longleftrightarrow M_n(x) \models \psi$, where ψ is Σ_1. A predicate is $\underline{\Sigma^*_{n+1}}$ (Σ^*_{n+1}, respectively) if it is defined by a Σ^*_{n+1} formula with (without, respectively) parameters. $\rho^M_{n+1} = \Sigma^*_{n+1}$ projectum of $M =$ least ρ such that there is a $\underline{\Sigma^*_{n+1}}$ subset of $\omega\rho$ not in M and $p^M_{n+1} = p^M_n \cup p$ where p is least such that $A \cap \rho^M_{n+1} \notin M$ for some A Σ^*_{n+1} in parameter $p^M_n \cup p$. $H^M_{n+1} = H^M_{\omega\rho^M_{n+1}}$ = sets x in M s.t. M-card (transitive closure (x)) $< \omega\rho^M_{n+1}$. For any $x \in M$, $M_{n+1}(x) = (n+1)$ st reduct of M relative to $x = \langle H^M_{n+1}, A_{n+1}(x)\rangle$ where $A_{n+1}(x) \subseteq H^M_{n+1}$ codes the Σ^*_{n+1} theory of M with parameters from $H^M_{n+1} \cup \{x\}$ in the natural way: $A_{n+1}(x) = \{\langle y, m\rangle |$ the m^{th} Σ^*_{n+1} formula is true at $\langle y, x\rangle, y \in H^M_{n+1}\}$. A good Σ^*_{n+1} function f is a function whose graph is Σ^*_{n+1} with the additional property that for $x \in \text{Dom}(f)$, $f(x) \in \Sigma^*_n$ hull $(H^M_n \cup \{x\})$. The Σ^*_{n+1} hull (X) for $X \subseteq M$ is the closure of X under good Σ^*_{n+1} functions.

Facts. (a) $\varphi, \psi \Sigma^*_n$ formulas $\longrightarrow \varphi \vee \psi, \varphi \wedge \psi$ are Σ^*_n formulas

(b) $\varphi \Sigma^*_n$ or $\prod^*_n$ (= negation of Σ^*_n) $\longrightarrow \varphi$ is Σ^*_{n+1}

(c) $Y \subseteq \Sigma^*_n$ hull $(X) \longrightarrow \Sigma^*_n$ hull $(Y) \subseteq \Sigma^*_n$ hull (X)

(d) f good Σ^*_n function $\longrightarrow f$ good Σ^*_{n+1} function

(e) Σ^*_n hull $(X) \subseteq \Sigma^*_{n+1}$ hull (X)

(f) There is a Σ^*_n relation $W(e, x)$ s.t. if $S(x)$ is Σ^*_n then for some $e \in \omega$, $S(x) \longleftrightarrow W(e, x)$ for all x.

(g) The structure $M_n(x) = \langle H^M_n, A_n(x)\rangle$ is amenable.

(h) $H^M_n = J^{A_n}_{\omega\rho^M_n}$ where $A_n = A_n(0)$.

(i) Suppose $H \subseteq M$ is closed under good Σ^*_n functions and $\pi : \overline{M} \longrightarrow M$, $\overline{M}$ transitive, Range $(\pi) = H$ and $p^M_{n-1} \in H$ (if $n > 1$). Then π preserves Σ^*_n formulas: for $\Sigma^*_n\varphi$ and $x \in \overline{M}$, $\overline{M} \models \varphi(x) \longleftrightarrow M \models \varphi(\pi(x))$. And (for $n > 1$), $\pi(p^{\bar{M}}_{n-1}) = p^M_{n-1}$.

Proof of (i) Note that $H \cap M_{n-1}(\pi(x))$ is Σ_1-elementary in $M_{n-1}(\pi(x))$. And $\pi^{-1}[H \cap M_{n-1}(\pi(x))] = \langle J^A_{\omega\rho}, A(x)\rangle$ for some $\rho, A, A(x)$. But (by induction on n) $A = A^M_{n-1} \cap J^A_{\omega\rho}$, $A(x) = A_{n-1}(x)^{\bar{M}} \cap J^A_{\omega\rho}$. And $\rho = \rho^{\bar{M}}_{n-1}$ using our assumption about the parameter p^M_{n-1}. And $\pi^{-1}(p^M_{n-1}) = \bar{p}$ must be $p^{\bar{M}}_{n-1}$ as $\bar{M} = \Sigma^*_{n-1}$ hull of $H^{\bar{M}}_{n-1} \cup \{p^{\bar{M}}_{n-1}\}$. $\dashv$

Theorem 4.1. *By induction on* $n > 0$:

1) *If* $\varphi(x, y)$ *is* Σ_n^* *then* $\exists y \in \Sigma_{n-1}^*$ *hull* $(H_{n-1}^M \cup \{x\})\varphi(x, y)$ *is also* Σ_n^*.

2) *If* $\varphi(x_1 \cdots x_k)$ *is* $\Sigma_m^*, m \geq n$ *and* $f_1(x), \cdots, f_k(x)$ *are good* Σ_n^* *functions, then* $\varphi(f_1(x) \cdots f_k(x))$ *is* Σ_m^*.

3) *The domain of a good* Σ_n^* *function is* Σ_n^*

4) *Good* Σ_n^* *functions are closed under composition.*

5) *(*Σ_n^* *Uniformization)* *If* $R(x, y)$ *is* Σ_n^* *then there is a good* Σ_n^* *function* $f(x)$ *s.t.* $x \in \text{Dom}(f) \longleftrightarrow \exists y \in \Sigma_{n-1}^*$ *hull* $(H_{n-1}^M \cup \{x\})R(x, y) \longleftrightarrow R(x, f(x))$.

6) *There is a good* Σ_n^* *function* $h_n(e, x)$ *s.t. for each* x, Σ_n^* *hull* $(\{x\}) = \{h_n(e, x) | e \in \omega\}$.

Proof. The base case $n = 1$ is easy (take Σ_0^* hull $(X) = M$ for all X). Now we prove it for $n > 1$, assuming the result for smaller n.

1) Write $\exists y \in \Sigma_{n-1}^*$ hull $(H_{n-1}^M \cup \{x\})\varphi(x, y)$ as $\exists \bar{y} \in H_{n-1}^M \varphi(x, h_{n-1}(e, \langle x, \bar{y} \rangle))$ using 6) for $n - 1$. Since h_{n-1} is good Σ_{n-1}^* we can apply 2) for $n - 1$ to conclude that $\varphi(x, h_{n-1}(e, \langle x, \bar{y} \rangle))$ is Σ_n^*. Since the quantifiers $\exists e \exists \bar{y} \in H_{n-1}^M$ range over H_{n-1}^M they preserve Σ_n^*-ness.

2) $\varphi(f_1(x) \cdots f_k(x)) \longleftrightarrow \exists x_1 \cdots x_k \in \Sigma_{n-1}^*$ hull $(H_{n-1}^M \cup \{x\})$ $[x_i = f_i(x)$ for $1 \leq i \leq k \wedge \varphi(x_1 \cdots x_k)]$. If $m = n$ then this is Σ_n^* by 1). If $m > n$ then reason as follows: the result for $m = n$ implies that $A_n(\langle f_1(x) \cdots f_k(x) \rangle)$ is Δ_1 over $M_{n+1}(x)$. Thus $A_{m-1}(\langle f_1(x) \cdots f_k(x) \rangle)$ is Δ_1 over $M_{m-1}(x)$. So as φ is Σ_m^* we get that $\varphi(f_1(x) \cdots f_k(x))$ is also Σ_1 over $M_{m-1}(x)$, hence Σ_m^*.

3) If $f(x)$ is good Σ_n^* then $\text{dom}(f) = \{x | \exists y \in \Sigma_{n-1}^*$ hull of $H_{n-1}^M \cup \{x\}(y = f(x))\}$ is Σ_n^* by 1).

4) If f, g are good Σ_n^* then the graph of $f \circ g$ is Σ_n^* by 2). And $f \circ g(x) \in \Sigma_{n-1}^*$ hull$(H_{n-1}^M \cup \{x\})$ since the latter hull contains $g(x)$, f is good Σ_n^* and Fact c) holds.

5) Using 6) for $n-1$, let $\overline{R}(x, \bar{y}) \longleftrightarrow R(x, h_{n-1}(\bar{y})) \wedge \bar{y} \in H_{n-1}^M$. Then $\overline{R}$ is Σ_n^* by 2) for $n - 1$ and using Σ_1 uniformization on $(n - 1)$ s.t. reducts we can define a good Σ_n^* function $\bar{f}$ s.t. $\overline{R}(x, \bar{f}(x)) \longleftrightarrow \exists \bar{y} \in H_{n-1}^M \overline{R}(x, \bar{y})$. Let $f(x) = h_{n-1}(\bar{f}(x))$. Then f is good Σ_n^* by 4).

6) Let W be universal Σ_n^* as in Fact f). By 5) there is a good Σ_n^* $g(e, x)$ s.t. $\exists y \in \Sigma_{n-1}^*$ hull$(H_{n-1}^M \cup \{x\})$ $W(e, \langle x, y \rangle) \longleftrightarrow W(e, \langle x, g(e, x) \rangle)$ (and $g(e, x)$ defined $\longrightarrow W(e, \langle x, g(e, x) \rangle))$). Let $h_n(e, x) = g(e, x)$. If $y \in \Sigma_n^*$ hull $(\{x\})$ then for some $e, W(e, \langle x, y' \rangle) \longleftrightarrow y' = y$ so $y = h_n(e, x)$. Clearly $h_n(e, x) \in \Sigma_n^*$ hull $(\{x\})$ since

h_n is good Σ_n^*. ⊣

Now we are ready to prove Global $\square$. Assume $V = L$ and let μ be a singular limit ordinal. Our goal is to define C_μ, a CUB subset of μ. Let $\beta(\mu) \geq \mu$ be the least limit ordinal β such that μ is not regular with respect to $\widetilde{J}_\beta$-definable functions, and let $n(\mu)$ be least so that there is a good $\Sigma^*_{n(\mu)}(\widetilde{J}_{\beta(\mu)})$ partial function from an ordinal less than μ cofinally into μ. Note that $\rho^{\beta(\mu)}_{n(\mu)} \leq \mu$ as otherwise such a partial function would belong to $\widetilde{J}_{\beta(\mu)}$, contradicting the leastness of $\beta(\mu)$. Also $\mu \leq \rho^{\beta(\mu)}_{n(\mu)-1}$, else we have contradicted the leastness of $n(\mu)$.

For $X \subseteq \widetilde{J}_{\beta(\mu)}$ let $H(X) = \Sigma^*_{n(\mu)}$ hull of X in $\widetilde{J}_{\beta(\mu)}$. For some least parameter $q(\mu) \in \widetilde{J}_{\beta(\mu)}$, $H(\mu \cup \{q(\mu)\}) = \widetilde{J}_{\beta(\mu)}$. ("Least" refers to the canonical well-ordering of L.) Also let $\alpha(\mu) = \bigcup\{\alpha < \mu | \alpha = H(\alpha \cup \{q(\mu)\}) \cap \mu\}$. Then (unless $\alpha(\mu) = \bigcup \emptyset = 0$) $\alpha(\mu) = H(\alpha(\mu) \cup \{q(\mu)\}) \cap \mu$ and $\alpha(\mu) < \mu$. To see the latter note that for large enough $\alpha < \mu, H(\alpha \cup \{q(\mu)\})$ contains both the domain and defining parameter for a good $\Sigma^*_{n(\mu)}$ partial function from an ordinal less than μ cofinally into μ.

If $\mu < \beta(\mu)$ let $p(\mu) = \langle q(\mu), \mu, \alpha(\mu)\rangle$ and if $\mu = \beta(\mu)$ let $p(\mu) = \alpha(\mu)$.

We are ready to define C_μ. Let $C^0_\mu = \{\bar{\mu} < \mu |$ For some $\alpha, \bar{\mu} = \bigcup(H(\alpha \cup \{p(\mu)\}) \cap \mu)\}$. Then C^0_μ is a closed subset of μ. If C^0_μ is unbounded in μ then set $C_\mu = C^0_\mu$. If C^0_μ is bounded but nonempty then let $\mu_0 = \bigcup C^0_\mu$ and define $C^1_\mu = \{\bar{\mu} < \mu |$ For some $\alpha, \bar{\mu} = \bigcup(H(\alpha \cup \{p(\mu), \mu_0\}) \cap \mu)\}$. If C^1_μ is unbounded then set $C_\mu = C^1_\mu$. If C^1_μ is bounded but nonempty then let $\mu_1 = \bigcup C^1_\mu$ and define $C^2_\mu = \{\bar{\mu} < \mu |$ For some $\alpha, \bar{\mu} = \bigcup(H(\alpha \cup \{p(\mu), \mu_0, \mu_1\}) \cap \mu)\}$. Continue in this way, defining C^k_μ for $k \in \omega$ until C^k_μ is unbounded or empty. Note that $\alpha_0 > \alpha_1 > \cdots$ where α_k is greatest so that $\bigcup(H(\alpha_k \cup \{p(\mu), \mu_0, \cdots, \mu_{k-1}\}) \cap \mu) = \mu_k$, since $\alpha_k \in H(\alpha_k \cup \{p(\mu), \mu_0, \cdots, \mu_{k-1}, \mu_k\})$. So for some least $k(\mu) \in \omega$, $C^{k(\mu)}_\mu$ is indeed unbounded or empty. If $C^{k(\mu)}_\mu$ is unbounded then set $C_\mu = C^{k(\mu)}_\mu$.

If $C^{k(\mu)}_\mu = \emptyset$ then we choose C_μ to be an ω-sequence cofinal in μ, coding approximations to the structure $\widetilde{J}_{\beta(\mu)}$, as follows. (This is necessary to establish Relativized $\square$ (c).) Note that $H = H(\{p(\mu), \mu_0, \cdots, \mu_{k(\mu)-1}\})$ is cofinal in μ since $C^{k(\mu)}_\mu = \emptyset$. Assume first that $n(\mu) = 1$, when C_μ is more easily described. Then H is also cofinal in $\beta(\mu)$, else $H \in \widetilde{J}_{\beta(\mu)}$ and μ is singular inside $\widetilde{J}_{\beta(\mu)}$. Let $h = h_1(e, x)$ be the canonical good Σ^*_1 Skolem function for $\widetilde{J}_{\beta(\mu)}$, so $H = \{h(e, p) | e \in \omega\}$ where $p = \{p(\mu), \mu_0, \cdots, \mu_{k(\mu)-1}\}$. Let $\bar{\sigma}_n = \max(\{h(e, p) | e < n\} \cap \mu)$ and $\sigma_n =$

$\max(\{h(e,p)|e < n\} \cap \beta(\mu))$. Then $C_\mu = \{\delta_0, \delta_1, \cdots\}$ where δ_n is an ordinal coding TC(Σ_1^* hull $(\bar{\sigma}_n \cup \{p\})$ restricted to σ_n), where TC denotes "transitive collapse". By the Σ_1^* hull of X restricted to σ_n we mean the closure of X under h^{δ_n}, obtained by interpreting the Σ_1^* definition of h in $\widetilde{J}_{\sigma_n}$.

Now suppose $n(\mu) > 1$, $C_\mu^{k(\mu)} = \emptyset$. Then if $\rho(\mu)$ denotes $\rho^\mu_{n(\mu)-1}$, H is cofinal in $\rho(\mu)$, else $H \in \widetilde{J}_{\rho(\mu)}$ and μ is singular in $\widetilde{J}_{\rho(\mu)}$. Let h be the canonical good $\Sigma^*_{n(\mu)}$ Skolem function for $\widetilde{J}_{\rho(\mu)}$ and let $p = \{p(\mu), \mu_0, \cdots, \mu_{k(\mu)-1}\}$. Let $\bar{\sigma}_n = \max(\{h(e,p)|e < n\} \cap \mu)$, $\sigma_n = \max(\{h(e,p)|e < n\} \cap \rho(\mu))$. Then $C_\mu = \{\delta_0, \delta_1, \cdots\}$ where δ_n is an ordinal coding TC($\Sigma^*_{n(\mu)}$ hull $(\bar{\sigma}_n \cup \{p\})$ restricted to σ_n). The $\Sigma^*_{n(\mu)}$ hull of X restricted to σ_n is the closure of X under h^{σ_n}, obtained by replacing the $(n(\mu)-1)$ st reduct $M_{n(\mu)-1}(x)$ by $M_{n(\mu)-1}(x) \restriction \sigma_n$ in the $\Sigma^*_{n(\mu)}$ definition of h. (Recall that $M_n(x) = \langle J^{A_n}_{\omega\rho_n^M}, A_n(x)\rangle$; by $M_n(x) \restriction \sigma$ we mean $\langle \widetilde{J}^{A_n}_\sigma, A_n(x) \cap \widetilde{J}^{A_n}_\sigma\rangle$.)

Clearly C_μ is CUB in μ, and by the same argument used to justify $\alpha(\mu) < \mu$, the ordertype of C_μ is less than μ. (These facts are obvious when $C_\mu^{k(\mu)} = \emptyset$.) So to prove Global $\Box$ we only need to check coherence: $\bar{\mu} \in \operatorname{Lim} C_\mu \longrightarrow C_{\bar{\mu}} = C_\mu \cap \bar{\mu}$.

Lemma 4.2. *$\bar{\mu} \in C^k_\mu \longrightarrow C^k_{\bar{\mu}} = C^k_\mu \cap \bar{\mu}$.*

Proof. First suppose that $k = 0$. Given $\bar{\mu} \in C^0_\mu$ we can choose $\alpha < \bar{\mu}$ such that $\bar{\mu} = \bigcup(H(\alpha \cup \{p(\mu)\}) \cap \mu)$, where H is the operation of taking the $\Sigma^*_{n(\mu)}$ hull. Also let $\rho = \bigcup(H(\alpha \cup \{p(\mu)\}) \cap \rho^\mu_{n(\mu)-1})$. Let $\pi : \widetilde{J}_{\bar{\beta}} \longrightarrow \widetilde{J}_{\beta(\mu)}$ be the inverse to the transitive collapse of $H = \Sigma^*_{n(\mu)}$ hull $(\bar{\mu} \cup \{p(\mu)\})$ restricted to ρ. Note that any $x \in H$ belongs to $H(\mu', \rho') = \Sigma^*_{n(\mu)}$ hull $(\mu' \cup \{p(\mu)\})$ restricted to ρ', for some $\mu' < \bar{\mu}, \rho' < \rho$ and μ', ρ' can be chosen to be in H. It follows that $H \cap \mu = \bar{\mu}$ and therefore when $\mu < \beta(\mu)$, $\pi(\bar{\mu}) = \mu$. Also note that $\Sigma^*_{n(\mu)-1}$ hull $(\rho \cup \{p(\mu)\}) \cap \rho^\mu_{n(\mu)-1} = \rho$, so H is closed under good Σ^*_{n-1} functions. It follows that $\pi : \widetilde{J}_{\bar{\beta}} \longrightarrow \widetilde{J}_{\beta(\mu)}$ is Σ^*_{n-1}-elementary and $\bar{\mu}$ is $\Sigma^*_{n(\mu)-1}(\widetilde{J}_{\bar{\beta}})$-regular, $\Sigma^*_{n(\mu)}(\widetilde{J}_{\bar{\beta}})$-singular. So $\bar{\beta} = \beta(\bar{\mu})$, $n(\mu) = n(\bar{\mu})$. Also $\pi(q(\bar{\mu})) = q(\mu)$. Since $\alpha(\bar{\mu}) < \alpha$ it must be that $\alpha(\bar{\mu}) = \alpha(\mu)$. So $\pi(p(\bar{\eta})) = p(\mu)$. Now it is easy to see that $C^0_{\bar{\mu}} = C^0_\mu \cap \bar{\mu}$.

Now suppose $k = 1$. The above argument shows that $\bar{\mu} \in C^1_\mu \longrightarrow C^0_{\bar{\mu}} = C^0_\mu \cap \bar{\mu}$ and hence, since $\mu_0 < \bar{\mu}$, $\bar{\mu}_0 = \mu_0$. Now again, the above argument shows that $C^1_{\bar{\mu}} = C^1_\mu \cap \bar{\mu}$. The general case $k \geq 0$ now follows similarly. $\dashv$

Coherence now follows easily: if $\bar{\mu} \in \operatorname{Lim} C_\mu$ and $C_\mu = C^k_\mu$ then by Lemma 4.2, $C^k_{\bar{\mu}} = C^k_\mu \cap \bar{\mu}$ is unbounded in $\bar{\mu}$ so $C_{\bar{\mu}} = C^k_{\bar{\mu}}$ and we're done. If $C^k_\mu = \emptyset$ for some k

then $\lim C_\mu = \emptyset$ so coherence is vacuous.

To establish the appropriate relativized form of $\Box$ we need:

Lemma 4.3. *Suppose* $\pi : \langle \widetilde{J}_{\bar\mu}, \bar C\rangle \xrightarrow{\Sigma_1} \langle \widetilde{J}_\mu, C_\mu\rangle$. *Then* $\overline{C} = C_{\bar\mu}$ *and* π *extends uniquely to a* $\Sigma^*_{n(\mu)}$*-elementary* $\tilde\pi : \widetilde{J}_{\beta(\bar\mu)} \longrightarrow \widetilde{J}_{\beta(\mu)}$ *such that* $p(\mu) \in \operatorname{Rng}\tilde\pi$.

Proof. First suppose that $C_\mu = C^k_\mu$ for some k. For $\mu' \in C_\mu$ form $H(\mu')$ as H was formed in the proof of Lemma 2 for $\bar\mu$. Then $\pi(\mu') : \widetilde{J}_{\beta(\mu')} \longrightarrow \widetilde{J}_{\beta(\mu)}$ with range $H(\mu')$ is $\Sigma^*_{n(\mu)-1}$-elementary and $\widetilde{J}_{\beta(\mu)} = \bigcup\{H(\mu')|\mu' \in C_\mu\}$. And $\pi(\mu') \restriction \mu' = id \restriction \mu'$, $\pi(\mu')(p(\mu')) = p(\mu)$. Now let $X = \text{Range}(\pi)$ and form $\widetilde{X} = \Sigma^*_{n(\mu)}$ hull $(X \cup \{p(\mu)\})$ in $\widetilde{J}_{\beta(\mu)}$. If $y \in \widetilde{X}$ then for some $\mu' \in C_\mu$, $y = \pi(\mu')(y')$ where $y' \in \Sigma^*_{n(\mu)}$ hull $((X \cap \widetilde{J}_{\mu'}) \cup \{p(\mu')\})$. In particular if $y \in \widetilde{J}_\mu$ then $y \in \Sigma^*_1$ hull (X) in $\langle \widetilde{J}_\mu, C_\mu\rangle = X$. So the inverse to the transitive collapse of $\widetilde{X} = \tilde\pi$ is a $\Sigma^*_{n(\mu)}$-elementary embedding extending π, with $p(\mu)$ in its range. If $\tilde\pi : \widetilde{J}_{\bar\beta} \longrightarrow \widetilde{J}_{\beta(\mu)}$ then $\bar\mu = \tilde\pi^{-1}(\mu)$ is singular via a $\Sigma^*_{n(\mu)}(\widetilde{J}_{\bar\beta})$ partial function since either $\Sigma^*_{n(\mu)}$ hull of $\mu' \cup \{p(\mu)\}$ in $\widetilde{J}_{\beta(\mu)}$ is unbounded in μ for some $\mu' < \bigcup(\text{Rng}(\pi) \cap \mu)$, in which case we can assume $\mu' \in \operatorname{Rng}\pi$ and by $\Sigma^*_{n(\mu)}$-elementary of $\tilde\pi$ we're done, or if not $\mu^* = \mu \cap \Sigma^*_{n(\mu)}$ hull $(\mu^* \cup \{p(\mu)\})$ in $\widetilde{J}_{\beta(\mu)}$ where $\mu^* = \bigcup(\operatorname{Rng}\pi \cap \mu)$, contradicting the definition of $\alpha(\mu)$. Since $\bar\mu$ is $\Sigma^*_{n(\mu)-1}(\widetilde{J}_{\bar\beta})$-regular, we get $\bar\beta = \beta(\bar\mu)$, $n(\mu) = n(\bar\mu)$. Then the $\Sigma^*_{n(\mu)}$-elementarity of $\tilde\pi$ guarantees that $\overline{C} = C_{\bar\mu}$. The uniqueness of $\tilde\pi$ follows from the fact that $\widetilde{J}_{\beta(\bar\mu)} = \Sigma^*_{n(\mu)}$ hull $(\bar\mu \cup \{p(\bar\mu)\})$ and $\tilde\pi \restriction \bar\mu$ is determined by π.

If $C^k_\mu = \emptyset$ for some k then C_μ was defined as a special ω-sequence cofinal in μ. That definition was made precisely to enable the preceding argument to also apply in this case. $\dashv$

Relativized $\Box$ Let $S = \bigcup\{S_\alpha|\alpha \text{ an infinite cardinal}\}$. There exists $\langle C_s|s \in S\rangle$ such that $C_s \in \mathcal{A}_s$ and:

(a) C_s is closed, unbounded in μ^0_s, ordertype $(C_s) \leq \alpha(s)$.

If $|s|$ is a successor ordinal then ordertype $(C_s) = \omega$.

(b) $\nu \in \text{Lim}(C_s) \longrightarrow$ for some $\eta < |s|$, $\nu = \mu^0_{s\restriction\eta}$ and $C_s \cap \nu = C_{s\restriction\eta}$.

(c) Let $\pi : \langle \overline{\mathcal{A}}, \overline{C}\rangle \xrightarrow{\Sigma_1} \langle \mathcal{A}^0_s, C_s\rangle$ and write $\text{crit}(\pi) = \bar\alpha$, $\overline{\mathcal{A}} = L_{\bar\mu}[\overline{A}, \bar s^*]$. If $\pi(\bar\alpha) = \alpha(s)$ then $L[\overline{A}, \bar s^*] \models |\bar s|$ is not a cardinal $> \bar\alpha$ and

(c1) $\overline{C} \in L_\mu[\overline{A}, \bar s^*]$ where μ is the least p.r. closed ordinal greater than $\bar\mu$ s.t.

$L_\mu[\overline{A}, \bar{s}^*] \vDash \mathrm{card}(|\bar{s}|) \leq \bar{\alpha}$.

(c2) π extends to $\pi' : \mathcal{A}' \xrightarrow{\Sigma_1} \mathcal{A}'_s$ where $\mathcal{A}' = L_{\mu'}[\overline{A}, \bar{s}^*]$, μ' = largest ordinal either equal to $\bar{\mu}$ or s.t. $\omega \cdot \mu' = \mu'$ and $L_{\mu'}[\overline{A}, \bar{s}^*] \vDash |\bar{s}|$ is a cardinal greater than $\bar{\alpha}$.

(c3) If $\bar{\alpha}$ is a cardinal and $\pi(\bar{\alpha}) = \alpha$ then $\overline{\mathcal{A}} = \mathcal{A}^0_{\bar{s}}$ and $\overline{C} = C_{\bar{s}}$.

Relativized $\diamond$ Let E = all $s \in S$ such that ordertype $(C_s) = \omega$. There exists $\langle D_s | s \in E\rangle$ such that $D_s \subseteq \mathcal{A}^0_s$ and:

(a) $D \in \widehat{\mathcal{A}}_s \neq \mathcal{A}^0_s$, $D \subseteq \mathcal{A}^0_s \longrightarrow \{\xi < |s| \,\big|\, s \restriction \xi \in E, D_{s\restriction\xi} = D \cap \mathcal{A}^0_{s\restriction\xi}\}$ is stationary in $\widehat{\mathcal{A}}_s$.

(b) D_s is uniformly Σ_1-definable as an element of $\mathcal{A}'_s$.

(c) If $\mathcal{A}'_s \vDash \alpha^{++}$ exists then $D_s = \emptyset$.

We now make the necessary modifications to obtain Relativized $\Box$. First, if μ is a singular limit ordinal and $\widetilde{J}_\mu \models \alpha$ is the largest cardinal then we thin out C_μ to give it ordertype $\leq \alpha$: By induction on limit $\bar{\mu} \leq \mu$ define $C^*_{\bar{\mu}}$ as follows. For $\bar{\mu} \leq \alpha$, $C^*_{\bar{\mu}} = \bar{\mu}$. Otherwise $C^*_{\bar{\mu}} = \{i^{\text{th}}$ element of $C_{\bar{\mu}} | i \in C^*_{\bar{\mu}_0}$ where $\bar{\mu}_0$ = ordertype $(C_{\bar{\mu}})\}$. This defines C^*_μ. It is easily verified that the C^*_μ enjoy all the properties of the C_μ except they are only defined when μ is a singular limit ordinal such that $\widetilde{J}_\mu \models$ There is a largest cardinal. In addition, ordertype $C^*_\mu \leq \alpha(\mu)$, the largest cardinal of $\widetilde{J}_\mu$.

Now suppose $V = L$, α is a cardinal, $s \in S_\alpha$, $|s| > \alpha$ and s is a 0-*string*, meaning that $s(\mu) = 0$ for all $\eta \in \mathrm{Dom}(s)$. Then we can choose our predicate $A = \emptyset$, define $C_s = C^*_{\mu^0_s}$ and Relativized $\Box$ will hold for such 0-strings. The final comment is that all we have done will relativize to arbitrary strings $s \in S_\alpha$, defined relative to an arbitrary predicate $A \subseteq \mathrm{ORD}$, $H_\alpha = L_\alpha[A]$ for all cardinals α

Now we turn to Relativized $\diamond$. Again we begin with a nonrelativized version. Let α be a cardinal and assume $V = L$.

<u>$\diamond$ on α^+</u> Let E = all $\mu < \alpha^+$ s.t. C_μ has ordertype ω. There exists $\langle D_\mu | \mu \in E\rangle$ s.t. $D_\mu \subseteq \widetilde{J}_\mu$ and:

(a) If $D \subseteq \widetilde{J}_{\alpha^+}$ then $\{\mu \in E | D \cap \widetilde{J}_\mu = D_\mu\}$ is stationary in α^+.

(b) D_μ is uniformly Σ_1 definable as an element of $\widetilde{J}_{\beta'(\mu)}$ where $\beta'(\mu)$ = largest β s.t. either $\beta = \mu$ or $\omega\beta = \beta$ and $\widetilde{J}_\beta \models \mu$ is a cardinal greater than α.

(c) If $\widetilde{J}_{\beta'(\mu)} \models \alpha^{++}$ exists then $D_\mu = \emptyset$.

Proof. For $\mu \in E$ let $D_\mu = \emptyset$ if $\widetilde{J}_{\beta'(\mu)} \models \alpha^{++}$ exists and otherwise let $\langle D_\mu, F_\mu \rangle$ be least in $\widetilde{J}_{\beta'(\mu)}$ such that F_μ is CUB in μ and $\bar{\mu} \in F_\mu \longrightarrow \bar{\mu} \notin E$ or $D_{\bar{\mu}} \neq D_\mu \cap \widetilde{J}_{\bar{\mu}}$. If $\langle D_\mu, F_\mu \rangle$ doesn't exist let $D_\mu = \emptyset$. Properties (b), (c) are clear. To prove (a), suppose it fails and let $\langle D, F \rangle$ be least in $\widetilde{J}_{\alpha^{++}}$ such that $D \subseteq \widetilde{J}_{\alpha^+}$, F is CUB in α^+ and $\mu \in F \longrightarrow \mu \notin E$ or $D_\mu \neq D \cap \widetilde{J}_\mu$. Let σ be least such that $\omega\sigma = \sigma$ and $\langle D, F \rangle \in \widetilde{J}_\sigma$. Then $\widetilde{J}_\sigma \models \alpha^+$ is the largest cardinal. Let $H = \Sigma_1$ Skolem hull of $\{\alpha^+\}$ in $\widetilde{J}_\sigma$ and $\mu = \bigcup(H \cap \alpha^+)$. Then $\widetilde{J}_{\beta'(\mu)}$ is the transitive collapse of the Σ_1 Skolem hull of $\mu \cup \{\alpha^+\}$ in $\widetilde{J}_\sigma$; let $\pi : \widetilde{J}_{\beta'(\mu)} \longrightarrow \widetilde{J}_\sigma$ have range equal to the latter hull. Then we have a contradiction provided $\mu \in E$. But the fact that H is unbounded in μ implies that $C^0_\mu = \emptyset$ so ordertype (C_μ) is ω and $\mu \in E$. $\dashv$

Now as with Relativized $\square$, if $V = L$ and α is a cardinal, $s \in S_\alpha$ is a 0-string in E, $|s| > \alpha$ then we can choose $A = \emptyset$ and define $D_s = D_{\mu^0_s}$ where the latter comes form $\diamond$ on α^+. Finally, relativize everything to arbitrary reshaped strings $s \in S_\alpha$ and an arbitrary predicate $A \subseteq \text{ORD}$, $L_\alpha[A] = H_\alpha$ for all cardinals α.

This completes the proof of Relativized $\square$ and $\diamond$.

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