Consistency of the Siilver Dichotomy in Generalised Baire Space

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A fundamental result in classical descriptive set theory is *Silver's dichotomy*:

Theorem 1 *(Silver [10]) If a Borel (or even co-analytic) equivalence relation on the reals has uncountably many classes, then it has a perfect set of classes, i.e., there is a perfect closed set of reals, any two distinct elements of which belong to different classes.*

It is convenient to express the conclusion of Silver's theorem in terms of the *continuous reducibility* of equivalence relations. Let id denote the equivalence relation of equality on Cantor space 2^ω. If E, F are equivalence relations on Polish spaces then we say that E is *continuously reducible* to F (written $E \leq_c F$) iff there is a continuous function f such that $E(x, y)$ iff $F(f(x), f(y))$. Then Silver's theorem says that if E is a Borel equivalence relation on the reals with uncountably many classes then id is continuously reducible to E. A more generous notion is *Borel reducibility*, where the "reduction" f is allowed to be Borel (we then write $E \leq_B F$).

In this article we look at Silver's dichotomy in *generalised Baire space.* Let κ be an infinite cardinal such that $\kappa^{<\kappa} = \kappa$. Then the *generalised Baire space* κ^κ associated to κ is the space of functions from κ to κ topologised with basic open sets of the form:

$N_\sigma = \{f : \kappa \to \kappa \mid f \text{ extends } \sigma\}$

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where σ belongs to $\kappa^{<\kappa}$. Our hypothesis on κ implies that this gives a basis for the topology of size κ. Borel sets in this space are obtained by closing the collection of basic open sets under unions and intersections of size κ. We get closure under complements using the fact that the complement of a basic open set is the union of at most κ basic open sets. A function from generalised Baire space to itself is Borel iff the pre-image under this function of any basic open set under the function is Borel. And the *generalised Cantor space* 2^κ associated to κ is the closed subspace of κ^κ consisting of those functions which map κ to 2. As in the classical setting we have the corresponding notions of cotinuous and Borel reducibility (again written as $\leq_c$, $\leq_B$, respectively) of equivalence relations on spaces like 2^κ or κ^κ which are equipped with a notion of Borel set.

As in the classical case, the reducibility of id (the equality relation on 2^κ) to an equivalence relation E on κ^κ can be reformulated as a statement about perfect sets. We say that $X \subseteq \kappa^\kappa$ is *perfect* iff X consists of the κ-branches $[T]$ through a subtree T of $\kappa^{<\kappa}$ which is $< \kappa$-closed and has the property that every node can be extended to a splitting node.

Proposition 2 *Suppose that E is an equivalence relation on κ^κ. Then id is Borel-reducible to E iff id is continuously reducible to E iff there is a perfect set $X \subseteq \kappa^\kappa$ any two distinct elements of which belong to different classes of E.*

Proof. Given $X = [T]$ as above we obtain an order-preserving bijection between $2^{<\kappa}$ and the set of splitting nodes of T; this induces a continuous $\sigma : 2^\kappa \to [T]$ which reduces id to E. Conversely, if f is a Borel function that reduces id to E then f is continuous on a comeager set and this comeager set contains a perfect set; we can thin out this perfect set to a perfect subset whose f-image is the desired perfect set X . □

Now we ask:

Question. Does Silver's dichotomy hold for generalised Baire space κ^κ? I.e., if a Borel equivalence relation E on κ^κ has more than κ classes, is there a continuous reduction of id on 2^κ to E?

The answer is negative in Gödel's L, in a strong sense.

Theorem 3 *(SDF-Hyttinen-Kulikov [2, 3]) Assume $V = L$. Then Silver's dichotomy fails in generalised Baire space for all uncountable regular κ: There are Borel equivalence relations with more than κ classes which lie strictly below id as well as a family of 2^κ Borel equivalence relations including id which are pairwise $\leq_B$-incomparable. If κ is inaccessible then there is a family of 2^κ Borel equivalence relations which are pairwise $\leq_B$-incomparable and $\leq_B$-below id.*

The problem with Silver's dichotomy in L derives from the existence of *weak Kurepa trees* on a regular cardinal κ. These are trees T of height κ with more than κ branches of length κ such that every node of T splits and the α-th level of T has size at most $\text{card}(\alpha)$ for stationary-many ordinals $\alpha < \kappa$. We say that T is *Kurepa* if "stationary-many ordinals" can be replaced by "all infinite ordinals".

Lemma 4 *Suppose $V = L$ and κ is regular and uncountable. Then there exists a weak Kurepa tree on κ. If κ is a successor cardinal then there is a Kurepa tree on κ.*

Proof. Our tree will be a subtree of the binary tree $2^{<\kappa}$. For singular $\alpha < \kappa$ let $\beta(\alpha)$ be the least limit ordinal $\beta > \alpha$ such that α is singular in L_β.

First assume that κ is inaccessible. Then T consists of all $\sigma \in 2^{<\kappa}$ such that:

($*$) For singular cardinals $\alpha \leq |\sigma|$ of cofinality ω, $\sigma|\alpha$ belongs to $L_{\beta(\alpha)}$.

Any node of T can be extended to nodes in T of any greater length (just add 0's). And any node of T of length α splits into two nodes in T of length $\alpha + 1$ so the α-th splitting level consists of nodes of length α. It follows that the α-th splitting level of T has size at most $\text{card}(\alpha)$ for α a singular cardinal of cofinality ω.

Main Claim. T has κ^+ many branches.

Proof of Main Claim. For a limit ordinal β between κ and κ^+ we say that β is *critical* if some subset of κ is definable over L_β but not an element of L_β. The set of critical ordinals is cofinal in κ^+ and for critical β, the Skolem hull of κ in L_β is all of L_β.

Now for each critical β define:

(*) $C_\beta = \{\alpha < \kappa \mid$ The Skolem hull of α in L_β contains no ordinals between α and $\kappa\}$.

Then C_β is a club in κ for each critical β and moreover if $\beta_0 < \beta_1$ are both critical then sufficiently large elements of C_{β_1} are limit points of C_{β_0}; this is because β_0 is an element of the Skolem hull of α in L_{β_1} for a large enough α and therefore so is C_{β_0}.

In particular the C_β's for critical β are distinct. Now we claim that each C_β is a branch through T. For this we need only check that if $\alpha < \kappa$ is a singular cardinal of cofinality ω then $C_\beta \cap \alpha$ belongs to $L_{\beta(\alpha)}$. This is clear if α does not belong to C_β, for then $C_\beta \cap \alpha$ is bounded in α and therfore an element of L_α. Otherwise note that $C_\beta \cap \alpha$ is definable over $L_{\bar\beta+1}$ where $L_{\bar\beta}$ is the transitive collapse of the Skolem hull of α in L_β; as α is regular in $L_{\bar\beta}$ it follows that $\bar\beta$ is less than $\beta(\alpha)$ so $C_\beta \cap \alpha$ is an element of $L_{\beta(\alpha)}$, as desired.

The case of a successor cardinal κ is similar, except one can now obtain a Kurepa tree on κ as all sufficiently large $\alpha < \kappa$ are singular. $\square$ (*Lemma*)

Now note that if T is weak Kurepa then there can be no continuous injection from 2^κ into $[T]$, the set of κ-branches through T: If κ is inaccessible then this would yield a club of $\alpha < \kappa$ such that the α-th level of T has 2^α many nodes and if $\kappa = \gamma^+$ then this would yield an $\alpha < \kappa$ such that T has $2^\gamma = \kappa$-many nodes on level α. In fact there cannot be such an injection which is Borel, as any Borel function is continuous on a comeager set and any comeager set contains a copy of 2^κ.

Finally define xE_Ty iff x, y are not branches through T or $x = y$. Then E_T is a Borel equivalence relation with κ^+ classes yet id cannot Borel reduce to E_T for the reasons given above. And E_T is Borel reducible to id via the reduction that sends each branch of T to itself and the non-branches of T to some fixed non-branch of T. Thus Silver's dichotomy fails at all uncountable regular cardinals in L.

On the other hand, Silver [9] also showed that it is possible to get rid of Kurepa trees on a regular cardinal κ using an inaccessible above κ: If $\lambda > \kappa$ is inaccessible and a Lévy collapse is performed to make λ into κ^+ (using conditions of size less than κ) then in the generic extension there are no Kurepa trees on κ. In fact there not even any weak Kurepa trees on κ in

Silver's model. This suggests that a model like Silver's may obey the Silver dichotomy for κ^κ, provided λ is chosen appropriately. Our main theorem states that this is indeed the case.

To gain further insight into the problem we next consider the following ZFC-provable negative result.

Theorem 5 *Let κ be regular and uncountable. Then there is a Δ^1_1 equivalence relation E with κ^+ classes such that id is not Borel-reducible to E. So the Silver Dichotomy provably fails for Δ^1_1.*

Proof. The relation is $xE^{\rm rank}y$ iff x, y do not code wellorders or x, y code wellorders of the same length. This has exactly κ^+ classes. It is Δ^1_1 because the assumption that κ is uncountable and regular implies that wellfoundedness for linear orders of κ is Δ^1_1 (it is even closed). Suppose T were a perfect tree whose distinct κ-branches were $E^{\rm rank}$-inequivalent. Now let x be a generic branch through T (treating T as a version of κ-Cohen forcing) and let $p \in T$ be a condition forcing that x codes a wellorder of some rank $\alpha < \kappa^+$. Then any sufficiently generic branch through T extending p codes a wellorder of rank α, which contradicts the fact that there are distinct such branches in V. $\square$

So a first step toward obtaining the consistency of Silver's Dichotomy for κ^κ is the following.

Theorem 6 *Assume $\kappa^{<\kappa} = \kappa$. Then the relation $E^{\rm rank}$ of the previous theorem is not Borel.*

Proof. For $\alpha < \kappa^+$ let $\mathcal{L}_\alpha$ denote the forcing to Lévy collapse α to κ (using conditions of size less than κ). If $g : \kappa \to \alpha$ is $\mathcal{L}_\alpha$-generic then g^* denotes the subset of κ defined by $i \in g^*$ iff $g((i)_0) \leq g((i)_1)$ where $i \mapsto ((i)_0, (i)_1)$ is a bijection between κ and $\kappa \times \kappa$.

By induction on Borel rank we show that if B is Borel then there is a club C in κ^+ such that:

$(*)$ For $\alpha \leq \beta$ in C of cofinality κ and (p_0, p_1) a condition in $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$, (p_0, p_1) forces that (g^*_0, g^*_1) belongs to B in the forcing $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$ iff it forces this in the forcing $\mathcal{L}_\alpha \times \mathcal{L}_\beta$.

If $B = U(\sigma_0) \times U(\sigma_1)$ is a basic open set then we may take C to consist of all ordinals greater than κ in κ^+. This is because for any $\alpha \leq \beta$, if (p_0, p_1) belongs to $\mathcal{L}_\alpha \times \mathcal{L}_\beta$ then (p_0, p_1) $\mathcal{L}_\alpha \times \mathcal{L}_\beta$-forces $(g_0^*, g_1^*) \in B$ exactly if (p_0^*, p_1^*) extends (σ_0, σ_1) where p_0^* is the set of i such that $(i)_0, (i)_1$ are in the domain of p_0 and $p_0(((i)_0) \leq p_0((i)_1)$ (similarly for p_1^*); this is independent of the pair α, β.

Inductively, suppose that B is the intersection of Borel sets B_i, $i < \kappa$, of smaller Borel rank. By intersecting clubs obtained by applying $(*)$ to the B_i's we obtain a club C ensuring the desired conclusion for B, as (p_0, p_1) forces $(g_0^*, g_1^*) \in B$ iff for each $i < \kappa$ it forces $(g_0^*, g_1^*) \in B_i$.

Finally if B is the complement of the Borel set B_0 then by induction we have a club C_0 such that for $\alpha \leq \beta$ in C_0 of cofinality κ and $(p_0, p_1) \in \mathcal{L}_\alpha \times \mathcal{L}_\alpha$, (p_0, p_1) $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$-forces $(g_0^*, g_1^*) \in B_0$ iff it $\mathcal{L}_\alpha \times \mathcal{L}_\beta$-forces this. Now thin out the club C_0 to a club C so that for α in C of cofinality κ, if (p_0, p_1) is in $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$ and there is some $\beta \geq \alpha$ in C_0 of cofinality κ and some (q_0, q_1) in $\mathcal{L}_\alpha \times \mathcal{L}_\beta$ below (p_0, p_1) which $\mathcal{L}_\alpha \times \mathcal{L}_\beta$-forces (g_0^*, g_1^*) in B_0 then there is such a (q_0, q_1) in $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$ (which then $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$-forces (g_0^*, g_1^*) in B_0). Then for $\alpha \leq \beta$ of cofinality κ in this thinner club C, (p_0, p_1) $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$-forces (g_0^*, g_1^*) in B iff none of its extensions in $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$ forces (g_0^*, g_1^*) in B_0 in the forcing $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$ iff none of its extensions in $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$ forces (g_0^*, g_1^*) in B_0 in the forcing $\mathcal{L}_\alpha \times \mathcal{L}_\beta$ iff none of its extensions in $\mathcal{L}_\alpha \times \mathcal{L}_\beta$ forces (g_0^*, g_1^*) in B_0 in the forcing $\mathcal{L}_\alpha \times \mathcal{L}_\beta$ iff (p_0, p_1) $\mathcal{L}_\alpha \times \mathcal{L}_\beta$-forces (g_0^*, g_1^*) in B, completing the induction.

It follows that E^{rank} is not Borel, as otherwise we have $g_0^* E^{\text{rank}} g_1^*$ where g_0, g_1 are sufficiently generic for $\mathcal{L}_\alpha \times \mathcal{L}_\beta$ with $\alpha < \beta$. $\square$

Now using an analogous argument we have:

Theorem 7 *Suppose that $0^\#$ exists, κ is regular in L and λ is the κ^+ of V. Then after forcing over L with the Lévy collapse turning λ into κ^+, the Silver Dichotomy holds for κ^κ.*

Proof. Suppose that E is a Borel equivalence relation in the Lévy collapse extension $L[G]$. For simplicity we assume that E has a Borel code in L and therefore has Borel rank less than $(\kappa^+)^L$. Suppose that E has more than κ classes in $L[G]$ and let p be a Lévy collapse condition forcing that the Lévy collapse names $(\sigma_\alpha \mid \alpha < \lambda)$ are pairwise E-inequivalent. We can assume that the σ_α's are of size less than λ and choose $f : \lambda \to \lambda$ in L so that for each $\alpha < \lambda$, σ_α is an $\mathcal{L}_{f(\alpha)}$-name where $\mathcal{L}_\beta$ denotes the part of the Lévy collapse forcing which collapses ordinals less than β to κ. We may assume that for each α, the E-equivalence class of σ_α does not depend on the choice of $\mathcal{L}_{f(\alpha)}$-generic, as otherwise this would fail for a pair of mutual $\mathcal{L}_{f(\alpha)}$-generics and by building a perfect set of mutual $\mathcal{L}_{f(\alpha)}$-generics we obtain a perfect set of distinct E-equivalence classes. It follows that if $\alpha < \beta$ and p belongs to $\mathcal{L}_{f(\alpha)}$ then (p, p) forces in $\mathcal{L}_{f(\alpha)} \times \mathcal{L}_{f(\beta)}$ that σ_α and σ_β are E-inequivalent.

Let I consist of the Silver indiscernibles between κ and λ and for $i < j$ in I let π_{ij} be an elementary embedding from L to L with critical point i, sending i to j. As p, the sequence $(\sigma_\alpha \mid \alpha < \lambda)$ and the function f defined above are constructible, they are L-definable from parameters less than some $i \in I$ together with indiscernibles $\geq \lambda$. Then we have that for $j < k$ in I above i, $\sigma_k = \pi_{jk}(\sigma_j)$ and $f(k) = \pi_{jk}(f(j))$. Let I_0 be the final segment of I consisting of all elements of I greater than i.

In analogy to the previous proof we show that for each Borel B there is a club C contained in I_0 such that:

$(*)$ Suppose that $i_0 < i_1 < \cdots < i_n = j < i_{n+1} = k$ belong to C, $(p_0, p_1) \leq (p, p)$ belongs to $\mathcal{L}_{f(j)} \times \mathcal{L}_{f(j)}$ and is L-definable from the parameters in $i_0 \cup \{i_0, i_1, \ldots, i_n\}$ together with indiscernibles $> j$. Then (p_0, p_1) forces that $(\sigma_j^{g_0}, \sigma_j^{g_1})$ belongs to B in the forcing $\mathcal{L}_{f(j)} \times \mathcal{L}_{f(j)}$ iff $(p_0, \pi_{i_0 i_1}\pi_{i_1 i_2} \cdots \pi_{i_{n-1} i_n}\pi_{i_n i_{n+1}}(p_1))$ forces that $(\sigma_j^{g_0}, \sigma_k^{g_1})$ belongs to B in the forcing $\mathcal{L}_{f(j)} \times \mathcal{L}_{f(k)}$.

Note that the composition $\pi_{i_0 i_1}\pi_{i_1 i_2} \cdots \pi_{i_{n-1} i_n}\pi_{i_n k}$ sends $(i_0, i_1, \ldots, i_n)$ to $(i_1, i_2, \ldots, i_{n+1})$.

We now prove $(*)$ (an appropriate choice of C) by induction on the Borel rank of B. If $B = U(\tau_0) \times U(\tau_1)$ is a basic open set then (p_0, p_1) forces that $(\sigma_j^{g_0}, \sigma_j^{g_1})$ belongs to B iff both p_0 forces that $\sigma_j^{g_0}$ belongs to $U(\tau_0)$ and p_1 forces that $\sigma_j^{g_1}$ belongs to $U(\tau_1)$; as the latter is equivalent to $\pi_{i_0 i_1}\pi_{i_1 i_2} \cdots \pi_{i_{n-1} i_n}\pi_{i_n i_{n+1}}(p_1)$ forces that $\sigma_k^{g_1}$ belongs to $U(\tau_1)$ the conclusion of $(*)$ follows, where we can take C to be the entire club I_0.

Inductively, suppose that B is the intersection of Borel sets B_α, $\alpha < \kappa$, of smaller Borel rank. Then $(*)$ for the B_α's implies ensures $(*)$ for B by intersecting κ-many clubs.

Finally suppose that B is the complement of the Borel set B_0 and the club C_0 witnesses $(*)$ for B_0. Let C consist of all limit points of C_0; we show that C witnesses $(*)$ for B. Suppose that $i_0 < i_1 < \cdots < i_n = j < i_{n+1} = k$ and (p_0, p_1) are as in the hypothesis of $(*)$ where $i_0, \ldots, i_{n+1}$ belong to C. Let π denote the composition $\pi_{i_0 i_1} \pi_{i_1 i_2} \cdots \pi_{i_{n-1} i_n} \pi_{i_n i_{n+1}}$.

If $(p_0, \pi(p_1))$ does not force that $(\sigma_j^{g_0}, \sigma_k^{g_1})$ belongs to B then there is an extension (q_0, q_1) of $(p_0, \pi(p_1))$ in $\mathcal{L}_{f(j)} \times \mathcal{L}_{f(k)}$ which forces that $(\sigma_j^{g_0}, \sigma_k^{g_1})$ belongs to B_0. We may assume that (q_0, q_1) is L-definable from parameters in $i_0 \cup \{i_0, \ldots, i_{n+1}\}$ together with parameters $< i_{-1}$ and indiscernibles $> k$. As i_0 is a limit point of C_0 we can choose $i_{-1} < i_0$ in C_0 (greater than the parameters used in the definition of (p_0, p_1)). Now consider the condition (q_0, q_1^*) in $\mathcal{L}_{f(j)} \times \mathcal{L}_{f(j)}$, where q_1^* is defined in L from $i_{-1} < i_0 < \cdots < i_n$ (together with indiscernibles greater than $i_n = j$) just like q_1 is defined from $i_0 < i_1 < \cdots < i_{n+1}$ (together with the same parameters $< i_{-1}$ and indiscernibles greater than $i_{n+1} = k$). By induction, (q_0, q_1^*) forces that $(\sigma_j^{g_0}, \sigma_j^{g_1})$ belongs to B_0. Moreover (q_0, q_1^*) is an extension of (p_0, p_1) as (q_0, q_1) is an extension of $(p_0, \pi(p_1))$ (this implies that q_1^* is an extension of p_1). So (p_0, p_1) does not force that $(\sigma_j^{g_0}, \sigma_j^{g_1})$ belongs to B.

Conversely, suppose that (p_0, p_1) does not force that $(\sigma_j^{g_0}, \sigma_j^{g_1})$ belongs to B. Then there is an extension (q_0, q_1) of (p_0, p_1) which forces that $(\sigma_j^{g_0}, \sigma_j^{g_1})$ belongs to B_0. We may assume that (q_0, q_1) is definable in L from parameters in $i_0 \cup \{i_0, \ldots, i_n\}$ together with indiscernibles greater than i_n. By induction $(q_0, \pi(q_1))$ forces that $(\sigma_j^{g_0}, \sigma_k^{g_1})$ belongs to B_0 where π is the composition $\pi_{i_0, i_1} \pi_{i_1 i_2} \cdots \pi_{i_{n-1} i_n} \pi_{i_n i_{n+1}}$. This condition extends the condition $(p_0, \pi(p_1))$ and therefore establishes that $(p_0, \pi(p_1))$ does not force that $(\sigma_j^{g_0}, \sigma_j^{g_1})$ belongs to B.

Now apply $(*)$ to the Borel set E, producing a club C. As mentioned before we can assume that (p, p) does $\mathcal{L}_{f(i)} \times \mathcal{L}_{f(i)}$-force $\sigma_i^{\dot{g}_0} E \sigma_i^{\dot{g}_1}$. It follows that for $i < j$ in C, (p, p) also $\mathcal{L}_{f(i)} \times \mathcal{L}_{f(j)}$-forces $\sigma_i E \sigma_j$, as p is not moved by any elementary embedding which is the identity below an element of I_0. But this contradicts our assumption that σ_α, σ_β are forced by (p, p) in $\mathcal{L}_\alpha \times \mathcal{L}_\beta$ to be E-inequivalent when p belongs to $\mathcal{L}_\alpha$ and $\alpha < \beta$. $\square$

I close with two remarks. The first is that if $0^{\#}$ exists and κ is an L-cardinal which is countable in V then the Silver dichotomy holds for κ^κ in some *inner model* with the same cardinals up to κ as L. This is because the Lévy collapse forcing which turns the κ^+ of $V = \omega_1$ of V into the κ^+ of the generic extension has a generic in V (it is built as the limit of countable generics along the indiscernibles less than ω_1 of V). The second remark is that I don't know if the above use of $0^\#$ is necessary. Surely one needs to start with an inaccessible $\lambda > \kappa$ to obtain the Silver dichotomy by forcing over L (preserving cardinals up to κ) but as far as I know it is indeed possible that inaccessibility is sufficient:

Question. Does the consistency of ZFC plus an inaccessible suffice for the consistency of ZFC plus the Silver dichotomy for the generalised Baire space $\omega_1^{\omega_1}$?

References

[1] Friedman, H., and L. Stanley, *A Borel reducibility theory for classes of countable structures*, J. Symb. Logic, 54 (1989), 894–914.

[2] S. Friedman, T. Hyttinen and V. Kulikov, Generalized Descriptive Set Theory and Classification Theory, Memoirs AMS, to appear.

[3] S. Friedman and V. Kulikov, Failures of the Silver Dichotomy in the Generalised Baire Space, Journal of Symbolic Logic, to appear.

[4] S. Gao, Invariant Descriptive Set Theory, Pure and Applied mathematics, CRC Press/Chapman & Hall, 2009.

[5] G. Hjorth and A. Kechris, Recent developments in the theory of Borel reducibility, Fundamenta Mathematicae, volume 170, number 1–2, pages 21–52, 2001.

[6] V. Kanovei, Borel equivalence relations. Structure and classification, Unviersity Lecture Series, 44, American Mathematical Society, 2008.

[7] A. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, Springer-Verlag, 1995.

[8] A. Kechris, *New directions in descriptive set theory*, Bull. Symbolic Logic, 5 (1999), 2, 161–174.

[9] J.H.Silver, The independence of Kurepa's conjecture and two-cardinal conjectures in model theory, in *Axiomatic Set Theory* (D.S.Scott, editor), Proceedings of Symposia on Pure Mathematics XIII, Part I, American Mathematical Society, 1971, pp. 383–390.

[10] J.H.Silver, Counting the number of equivalence classes of Borel and coanalytic equivalence relations, Annals of Mathematical Logic, volume 18, pages 1–18, 1980.