

The Hyperuniverse

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This is joint work with Tatiana Arrigoni.

We discuss the *hyperuniverse approach* to set-theoretic truth, which can be summarised as follows. We aim to clarify what is true in V , the universe of all sets, by creating a context, the *hyperuniverse*, in which we can compare different pictures of the set-theoretic universe. By exploring this “laboratory of possible universes” we develop criteria for preferring certain universes to others. The preferred universes share certain first-order features, which we adopt as being “true in V ”.

This approach contrasts dramatically with a Platonistic conception of V , and also with the views of Shelah, who imposes no preferences at all for one picture of V over another. It also brings the concept of “multiverse”, first explored by Woodin and myself in the contexts of set-forcing and class-forcing, respectively, to its natural and ultimate formulation, embodied by the hyperuniverse.

The Hyperuniverse: The rigorously-defined and richest possible Multiverse

In set theory we have many methods for creating new universes (i.e., well-founded models of ZFC) from old ones: set-forcing, class-forcing, hyperclass-forcing, . . . , model-theoretic methods. This leads to the concept of the *multiverse*, consisting of the different universes that one can obtain (perhaps from

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an initial universe) via these methods. Woodin first isolated this term in the form of the *set-generic multiverse*, in which only the method of set-forcing is permitted. Earlier work of mine ([1]) explored aspects of the *class-generic multiverse*, obtained by closing under class-forcing. These two notions of multiverse are rather different: the former preserves large cardinals notions and does not lead beyond set forcing, whereas the latter can destroy large cardinals and lead to models that are not directly obtainable via class forcing.

The first point to be made in the hyperuniverse programme is the following:

Point 1: The multiverse should be as rich as possible.

I.e., the multiverse should be closed under all conceivable methods for creating new universes. It is not obvious how to do that, because by working with universes that contain all of the ordinals, quantification over outer models which are not set-generic ceases to be first-order.

But note that the collection of countable transitive universes is closed under all known universe-creation methods. Thus if we want to have the broadest picture of the multiverse it is compelling to work not with V but with all countable transitive universes.

Point 2: The multiverse cannot be described in vague terms, but must be given a precise mathematical formulation.

This is a consequence of the basic assumption of the hyperuniverse programme, which is that it makes sense and is indeed desirable to search for preferred universes on the basis of justified criteria for choosing certain universes over others. A skeptic will say that such an approach will not be successful and will not lead to a notion of set-theoretic truth beyond ZFC. But mathematics (and science in general) does not progress through the adoption of pessimistic attitudes; instead it depends on the pursuit of reasonable approaches and an assessment of their degree of success. The burden is on the skeptic to provide a persuasive argument that certain approaches are doomed to failure.

To carry out the search for preferred universes, the concept of multiverse, in which candidates for preferred universes exist, must be given a precise

mathematical formulation. Otherwise one cannot hope to put the multiverse to work for the purpose of arriving at statements with a precise mathematical formulation; only such statements can be added to ZFC as new axioms.

The *hyperuniverse*, which is simply the set of all countable universes (i.e. all countable transitive models of ZFC) fulfills the requirements stated in Points 1 and 2.

What happened to V ? This leads to our third point regarding the hyperuniverse approach.

Point 3: Any first-order property shared by the preferred members of the hyperuniverse is true in V .

By the Löwenheim-Skolem theorem, when we explore the universes within the hyperuniverse we see the full range of possible first-order properties that the full universe V of all sets may satisfy. Naturally, our picture of V is reflected by one of the pictures given by the preferred universes of the hyperuniverse. For this reason, first-order properties shared by all preferred universes will be true in V .

Thus we have a clear and potentially powerful strategy for discovering first-order properties of the universe of all sets: We have a context *closed under arbitrary universe-creation methods* in which we can explore the different possible pictures of V , and then by imposing justifiable preferences for certain universes over others we can discover common first-order properties of these preferred universes which can be regarded as being true in V . This is the hyperuniverse programme.

Preferred universes: Two types of criteria

Which universes should we prefer? I.e., what criteria should we use for choosing certain universes over others? There are two sources of such criteria:

Type 1 criteria: Those that arise directly from set-theoretic practice

These are criteria which prefer universes in which the difficulties in a specific area of set theory are more easily resolved. Here are some examples:

- 1a. The continuum hypothesis (CH). There are areas of analysis in which assuming CH is of great help for answering open questions.
- 1b. $V = L$. This theory yields a powerful infinitary combinatorics which can be used to resolve many problems in set theory.
- 1c. Projective Determinacy (PD). This is the most popular axiom for producing an attractive theory of projective sets of reals.
- 1d. Forcing axioms (such as MA, BPFA, BMM). Like $V = L$, these have great combinatorial strength.

Type 1 criteria are unsatisfying in many respects. They are local, in the sense that they reflect the interests of (and advertise the results of) a specific group of set-theorists. There are as many different such criteria as there are groups of set-theorists. As interests in set theory change, so will these criteria. Thus these criteria do not reflect a broad point of view of set theory and are not stable over time. Thus Type 1 criteria are not justifiable.

Type 2 criteria: Those that arise directly from an unbiased look at the hyperuniverse

These criteria are formulated without reference to set-theoretic practice. In particular, the technical notions of modern set theory, such as forcing, large cardinals, determinacy, combinatorial principles, ... which reflect set-theoretic practice do not appear in Type 2 criteria. Here are some examples and non-examples:

- 2a. Maximality properties of universes (provided they do not mention technical methods like forcing) are the richest source of Type 2 criteria.
- 2b. Reflection principles (which are in fact certain types of maximality principles) can lead to Type 2 criteria.
- 2c. Forcing or determinacy axioms do not qualify. Principles expressed using Woodin's Ω -logic do not qualify, as this logic is explicitly based on the notion of set-forcing.
- 2d. Omniscience and absoluteness principles (clarified below) qualify (provided they do not mention technical methods like forcing).

Point 4: The justifiable criteria for the choice of preferred universes are of Type 2.

However note the risk in pursuing criteria of Type 2: They may lead to the adoption of first-order statements which contradict set-theoretic practice. As an example, consider the criterion of minimality, which says that the preferred universes of the hyperuniverse should be as small as possible. This leads to the preferred choice of just one universe, the minimal model of ZFC, and therefore to the statement that set-models of ZFC do not exist! This is in obvious conflict with set-theoretic practice. The same applies to a weaker notion of minimality, embodied by the axiom $V = L$. Although this does allow for the existence of set-models of ZFC, it does not allow for the existence of inner models of ZFC with measurable cardinals, another conflict with set-theoretic practice (see a further discussion of this point below). Thus:

Point 5: Type 2 criteria which lead to first-order statements in conflict with set-theoretic practice must be rejected.

Examples and a conjectured synthesis

To date, the only justifiable criteria (for the choice of preferred universes) of which I am aware¹ are based either on the principles of *maximality* or *omniscience*. The former is the idea, advocated by Gödel and subsequently by others, that the universe V of all sets should in some sense be as “large as possible”. The latter is instead the idea that the universe should in some sense be able to “see” as much as possible of the full range of alternative universes. These principles can be realised as Type 2 criteria in various ways. I begin with maximality.

A first point to make about maximality is that one cannot have “structural maximality” in the sense that a preferred universe contain all ordinals or all real numbers. This is simply because there is no tallest countable transitive model of ZFC and over any such model we can add new reals to obtain another such model.

The known maximality criteria make use of logic. (This was in fact suggested by Gödel, who talked about the use of fundamental concepts of logic in the search for new axioms.) Let v be a variable that ranges over the

¹Recently other possibilities have occurred to me, which I won’t discuss in these notes. An example is *indiscernibility*, which states that there is a closed unbounded class of good indiscernibles for V (which of course need not be amenable to V).

elements of the hyperuniverse. Maximality criteria express the idea that if a set-theoretic statement with certain parameters holds externally, i.e., in some universe containing v , then it already holds internally, i.e., in some “subuniverse” of v . Different criteria arise depending on what one takes as parameters and what one takes for the concept of “subuniverse”. Below are some examples.

Ordinal (or vertical) maximality

These criteria express the idea that preferred universes are maximal with respect to the ordinals, having fixed the power-set operation. More precisely, let us say that a universe w is a *lengthening* of v if v is a (proper) rank initial segment of w .

Ordinal maximality. v has a lengthening w such that for all first-order formulas φ and subsets A of v belonging to w , if $\varphi(A)$ holds in w then $\varphi(A \cap v_\alpha)$ holds in v_β for some pair of ordinals $\alpha < \beta$ in v .

This is also known as a high-order *reflection principle* and is of the type already considered by Gödel. It leads to the existence of “small” large cardinals, i.e., large cardinal notions consistent with $V = L$ such as inaccessibles, weak compacts, ω -Erdős cardinals,

Remark. Stronger forms of reflection lead to much larger cardinals. These are the “embedding reflection” principles, in which the parameter A is allowed to be a more complex object, such as a hyperclass (class of classes), hyperhyperclass (class of hyperclasses) Carrying this out in the obvious way leads quickly to inconsistency as Koellner has pointed out. Carrying this out in a more sophisticated way, using the concept of embedding, restores consistency and via work of Magidor leads to equivalences with very large cardinals such as supercompact cardinals. However it is not possible to justify embedding reflection principles as unbiased or even natural principles of ordinal maximality, due to the arbitrary nature of the embeddings involved (the relationship between A and its “reflected version” is given by an embedding with no uniqueness properties).

As mentioned, ordinal maximality (reflection) is an old concept and is very popular among set-theorists. It is in perfect accord with set-theoretic practice.

Power set (or horizontal) maximality

In analogy with ordinal maximality, these criteria express the idea that preferred universes are maximal with respect to the power set operation, having fixed the ordinals. More precisely:

Power set maximality. If a parameter-free sentence holds in some outer model of v (i.e., in some universe w containing v with the same ordinals as v) then it holds in some inner model of v (i.e. in some universe v_0 contained in v with the same ordinals as v).

This is equivalent to my *inner model hypothesis (IMH)*, which formally speaking states that by passing to an outer model of v we do not change *internal consistency*, i.e., we do not increase the set of parameter-free sentences which hold in some inner model.

Power-set maximality is relatively new ([2], 2006). Important issues arise when assessing its compatibility with set-theoretic practice. At first sight, it appears to be incompatible with set-theoretic practice, as it refutes the existence of inaccessible cardinals as well as projective determinacy. However this has triggered a re-examination of the roles of large cardinals and determinacy in set-theoretic practice, which may lead to the conclusion that the IMH is in fact compatible with set-theoretic practice after all:

Aside 1: The role of large cardinals in set theory.

Large cardinals arise in set theory in a number of ways: One starts with a model M of ZFC which contains large cardinals and then via forcing produces an outer model $M[G]$ in which some important statement holds. Notice that in the resulting model, large cardinals may fail to exist; they only exist in an inner model. And of course we did not have to assume that M was the full universe V ; it was sufficient for M to be any transitive model with large cardinals. An important part of large cardinal theory consists of Jensen's programme of building nice inner models which realise them. Again, the emphasis here is on inner models for large cardinals, not on their existence in V . Large cardinals are also of great importance as they provide measures of the consistency strengths of statements. A typical consistency lower bound result is obtained by starting with a statement of interest and then

constructing an inner model with a large cardinal. Once again, one sees that set-theoretic practice is concerned with inner models of large cardinals, and not with their existence in the full universe V .

A possible exception could be the use of large cardinals in V , rather than large cardinals in inner models, to prove forms of the axiom of determinacy. However consider the following:

Aside 2: The status of PD (projective determinacy) in set theory.

It is commonly said that since Borel and analytic sets are regular (in the sense that they are measurable and have the Baire and perfect set properties) and PD extends this fact to all projective sets, that PD can be justified as being “true” based on this natural extrapolation. But there is an obvious rebuttal to this argument: Consider Levy-Shoenfield absoluteness, the absoluteness of Σ_2^1 statements with respect to arbitrary outer models. This is provable in ZFC even if one allows arbitrary real parameters. One is then naturally led to conjecture Σ_n^1 absoluteness with arbitrary real parameters. An important, ignored by many set-theorists, is that even Σ_3^1 absoluteness with arbitrary real parameters is provably false! With arbitrary real parameters a consistent principle can only be obtained by making the technical restriction to set-generic outer models; as soon as one relaxes this to class-genericity, the principle becomes inconsistent.

So if one is so easily led to inconsistency when extrapolating from Σ_2^1 to Σ_3^1 absoluteness, how can one feel confident that the extrapolation from Σ_1^1 measurability to Σ_2^1 measurability? More reasonable would be the extrapolation *without parameters*. Indeed, parameter-free Σ_3^1 absoluteness, unlike the version with arbitrary real parameters, is consistent (and indeed follows from the IMH).

Thus a natural conclusion with regard to PD is the following: The regularity of projective sets is a reasonable extrapolation from the regularity of Borel and analytic sets, *provided one does not allow parameters*. Similarly, although PD cannot be justified based on extrapolation, it is plausible that parameter-free PD or even OD (ordinal-definable) determinacy without real parameters is an essential part of set-theoretic practice.

In light of the above two Asides, let us return now to the question of the compatibility of the IMH with set-theoretic practice. If one accepts that the role of large cardinals in set theory is via inner models and that the importance of PD is captured by its parameter-free version then this compatibility is restored: The IMH is consistent both with inner models of large cardinals and with parameter-free PD (indeed with OD-determinacy without real parameters). In particular it is consistent with the regularity of all parameter-free definable projective sets of reals. Allowing arbitrary real parameters makes a big difference and converts a principle compatible with the IMH to one which is not.

Omniscience

I turn now to *omniscience*, the idea that a preferred universe should be able to describe what can be true in alternative universes.

Omniscience principle. Let Φ be the set of sentences with arbitrary parameters which can hold in some outer model of v . Then Φ is first-order definable in v .

I first saw this kind of statement in work of Mack Stanley, where he shows that there are omniscient universes (my terminology, not his), assuming a bit less than the consistency of a measurable cardinal (stationary-many Ramsey cardinals, roughly speaking).

I find omniscience to be very appealing, and not simply a consequence of maximality. Note that the statement of power-set maximality (the IMH) does not allow any parameters, whereas the omniscience principle allows arbitrary set parameters. (A version of the IMH, the strong IMH, has been formulated which does allow certain parameters, but so far this principle has not been shown to be consistent.)

Now where do we stand with regard to possible justified criteria for preferred universes? So far, we have three examples: ordinal-maximality, power-set maximality and omniscience.

A conjectured synthesis

The ideal situation with regard to preferred universes would be that we can combine all of our justifiable criteria into a single consistent criterion, i.e., a criterion that is satisfied by at least one element of the hyperuniverse.

I conjecture that such a synthesis is possible. It is however not as straightforward as simply combining the IMH with ordinal-maximality and omniscience, as it is easy to see that even the first two of these principles, IMH and ordinal-maximality, contradict each other. Instead we propose the following:

Conjecture. Let IMH^* be the IMH restricted to ordinal-maximal universes (i.e., the statement that if a sentence holds in an ordinal-maximal outer model of v then it holds in an inner model of v). Then the conjunction of IMH^* , ordinal-maximality and omniscience is consistent.

A proof of this conjecture is within reach, as it only demands the already-existing method for proving the consistency of the IMH, together with a careful understanding of how Jensen coding can be done over the Dodd-Jensen core model in the presence of large cardinal properties slightly stronger than the existence of Ramsey cardinals.

Some remarks about Woodin's work

Below is a disorganised list of objections to some of the claims of Hugh Woodin in his work on the choice of new axioms for set theory.

Woodin tries to argue that the only basis for the consistency of large cardinal axioms is their truth in V . I explained above why the importance of large cardinals in set theory is derived not from their existence but from their existence in inner models. I think that Woodin is using a false analogy between large infinities and large finite sets: If the existence of large sets is consistent then large sets much exist; this is simply because V_ω has no proper inner models, and therefore existence of large finite sets is the same as their existence in inner models. This is of course not the case with large infinities.

Woodin discusses the consistency of Reinhardt cardinals (without AC) as a serious problem for set theory, because it cannot be justified on the basis of large cardinal axioms with AC. But to date Reinhardt cardinals

have no consequences for set-theoretic practice that cannot be obtained from large cardinal axioms with AC, so this problem is currently of no particular importance.

Ω -logic is a technical construction built on set-forcing. Why set-forcing? Class-forcing and hyperclass-forcing have the same ontological status as set-forcing. By restricting to set-forcing, Woodin is avoiding the real difficulties in the choice of new axioms, that of dealing with justifiable principles that may destroy large cardinal axioms. His discussion of the set-generic multiverse is equally artificial; he should be discussing a much broader multiverse.

Woodin emphasizes the reduction of truth across the set-generic multiverse built from V to truth in V . But he misses the important point that one can also consistently reduce truth across much larger multiverses (such as the entire hyperuniverse) to truth in V . So when he discusses a “multiverse conception of truth” he is using the wrong multiverse. In general, absoluteness issues cannot be justifiably formulated taking only set forcing into account.

There are several problems with Woodin’s discussion of the “ultimate L ”. An obvious point is that this is nothing more than a conjecture, as there is no iterability theory to justify the existence of such a model. Nevertheless, I agree that it is a very desirable goal to try to obtain a model K^∞ with L -like structure that is a good approximation to V (indeed I proposed this idea in the late 1970s in a colloquium talk I gave at Berkeley; Woodin was there). But what justification could one then give for the claim that the axiom $V = K^\infty$ is true? It is unrealistic to expect that truth over K^∞ be invariant under arbitrary model-construction methods.

Woodin refers to $V = L$ as a “limiting axiom”. But this is not the case, as one can consistently add to it axioms asserting the existence of transitive models with very large cardinals. Instead, the claim should be that $V = L$ is too weak, in the sense that it adds no consistency strength to ZFC, and that it conflicts with set-theoretic practice, which demands the existence of inner models for large cardinal axioms.

References

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