Set Theory

1. Basics

We begin with a summary (omitting proofs) of the basics of Zermelo-Fraenkel Set Theory with the Axiom of Choice (ZFC).

Language of ZFC

The only nonlogical symbol is a binary symbol $\in$ for denoting set-theoretic membership. This is combined with the usual logical symbols of a first-order language to form formulas. We also introduce the usual abbreviations for $\exists x\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi \ldots$, as well as:

$$\exists! x\varphi \text{ abbreviates } \exists x\forall y(\varphi^x_y \leftrightarrow x = y)$$

Axioms of ZF (= ZFC without the Axiom of Choice)

1. Extensionality: Two sets are equal iff they have the same elements. Formally, $\forall x\forall y(x = y \leftrightarrow \forall w(w \in x \leftrightarrow w \in y))$.
2. Empty Set: $\emptyset$ exists. Formally, $\exists x\forall y(y \notin x)$.
3. Pairing: $\{x, y\}$ exists. Formally, $\forall x\forall y\exists z\forall w(w \in z \leftrightarrow (w = x \vee w = y))$.
4. Infinity: There is a set which contains $\emptyset$ and is closed under the operation $u \mapsto u \cup \{u\}$. Formally, $\exists x(\emptyset \in x \wedge \forall y(y \in x \rightarrow y \cup \{y\} \in x))$, where $y \cup \{y\} \in x$ abbreviates $\exists z(z \in x \wedge \forall w(w \in z \leftrightarrow (w \in y \vee w = y)))$.
5. Union: For any set x, $\cup x = \{z \mid z \in w \text{ for some } w \in x\}$ exists. Formally, $\forall x\exists y\forall z(z \in y \leftrightarrow \exists w \in x(z \in w))$.
6. Power Set: $\mathcal{P}(x) = \{y \mid y \subseteq x\}$ exists. Formally, $\forall x\exists y\forall z(z \in y \leftrightarrow y \subseteq x)$, where $y \subseteq x$ abbreviates $\forall w(w \in y \rightarrow w \in x)$.
7. Replacement Scheme: If $\varphi(x, y)$ is a formula that defines a function then its range on any set exists. Formally:

$\forall x\exists! y\varphi(x, y) \rightarrow \forall a\exists b\forall y(y \in b \leftrightarrow \exists x \in a\varphi(x, y))$

where φ is any formula whose free variables include x, y but not a, b.

8. Foundation: $\in$ is a well-founded relation. Formally, $\forall x(x \neq \emptyset \rightarrow \exists y \in x\forall z \in x(z \notin y))$.

The following are two important consequences of the ZF axioms.

Comprehension Principle. For any set a and formula $\varphi(x)$ one can form the set $\{x \in a \mid \varphi(x)\}$. Formally:

$$\forall a\exists b\forall x(x \in b \leftrightarrow (x \in a \wedge \varphi(x)))$$

where φ is any formula whose free variables include x but not a, b.

Bounding Principle. If $\varphi(x, y)$ defines a total relation then for any a there is a b such that $\{\langle x, y\rangle \mid \varphi(x, y) \wedge y \in b\}$ is a total relation on a. Formally:

$$\forall x \exists y \varphi(x, y) \rightarrow \forall a \exists b \forall x \in a \exists y \in b \varphi(x, y)$$

where φ is any formula whose free variables include x, y but not a, b.

We discuss some technicalities concerning functions and cartesian products. For any two sets x, y define the ordered pair $\langle x, y\rangle$ to be the set $\{\{x\}, \{x, y\}\}$. A simple exercise is to show that $\langle x, y\rangle = \langle x', y'\rangle$ iff $x = x'$ and $y = y'$. It follows from the Pairing axiom that $\langle x, y\rangle$ exists for any x, y. A *function* is a set f whose elements are ordered pairs with the property: If $\langle x, y\rangle$ and $\langle x, y'\rangle$ are elements of f then $y = y'$. In ZF we can define the notion of function as well as: Dom(f) (domain of f), Ran(f) (range of f), $f \restriction x$ (restriction of f to x). Also, for any two sets a, b define the cartesian product of a, b to be $a \times b = \{\langle x, y\rangle \mid x \in a \wedge y \in b\}$. A simple exercise is to show (using Union, Power Set and Comprehension) that $a \times b$ exists for any two sets a, b.

We can now introduce the final axiom of ZFC:

Axiom of Choice (AC). If every element of x is nonempty then there is a function which selects a unique element from each element of x. Formally, $\forall y \in x(y \neq \emptyset) \rightarrow \exists f(f$ is a function $\wedge$ Dom(f)$= x$ $\forall y \in x(f(y) \in y))$.

ZFC = ZF with the additional axiom AC.

Ordinals

$\langle x, \leq\rangle$ is a *linear ordering (lo)* if it obeys:
$a \leq a$
$a \leq b \wedge b \leq a \rightarrow a = b$
$a \leq b \leq c \rightarrow a \leq c$
For all a, b: $a \leq b \vee b \leq a$
$\langle x, \leq\rangle$ is a *well-ordering (wo)* if it also obeys:
$y \subseteq x, y \neq \emptyset \rightarrow y$ has a $\leq$-least element; i.e., $\exists a \in y \forall b \in y(a \leq b)$.

Cantor classifed the well-orderings. If $\langle x, \leq_x\rangle$ is a wo then an *intial segment of* $\langle x, \leq_x\rangle$ is a wo $\langle x', \leq_{x'}\rangle$ where:

$x' \subseteq x$
$a \in x', b \leq_x a \to b \in x'$
For $a_0, a_1 \in x'$, $a_0 \leq_{x'} a_1$ iff $a_0 \leq_x a_1$.

And $\langle x', \leq'_x \rangle$ is a *proper* intial segment of $\langle x, \leq_x \rangle$ if in addition $x' \neq x$.

Comparability of WO's. If $\langle x, \leq_x \rangle$ and $\langle y, \leq_y \rangle$ are wo's then exactly one of the following is true:

1. $\langle x, \leq_x \rangle$ is isomorphic to a proper initial segment of $\langle y, \leq_y \rangle$.
2. $\langle y, \leq_y \rangle$ is isomorphic to a proper initial segment of $\langle x, \leq_x \rangle$.
3. $\langle x, \leq_x \rangle$ and $\langle y, \leq_y \rangle$ are isomorphic.

Cantor showed that every wo is isomorphic to a unique wo of a special kind. $\langle x, \leq_x \rangle$ is a *quasi-ordinal* iff it is a wo and $<_x = \in$ restricted to $x = \{\langle y, z \rangle \mid y, z \in x \wedge y \in z\}$. A quasi-ordinal $\langle x, \leq_x \rangle$ is an *ordinal* if in addition x is *transitive*: $a \in b \in x \to a \in x$.

Comparability of Ordinals. If α, β are ordinals then $\alpha \in \beta$, $\beta \in \alpha$ or $\alpha = \beta$.

Notation. If α, β are ordinals then we write $\alpha < \beta$ for $\alpha \in \beta$, $\alpha \leq \beta$ for $\alpha < \beta$ or $\alpha = \beta$ and ORD for the class of all ordinals.

Ordinal Facts
(a) An element of an ordinal is an ordinal.
(b) $\emptyset$ is an ordinal.
(c) If α is an ordinal then so is $\alpha \cup \{\alpha\}$, the least ordinal greater than α.
(d) If x is a set of ordinals then $\cup x$ is also an ordinal, the supremum of x in the well-ordering of ordinals.

For natural numbers n define $0 = \emptyset$, $n + 1 = n \cup \{n\} = \{0, 1, \ldots, n\}$. The least infinite ordinal is denoted by ω and is equal to $\{0, 1, \ldots\}$. α is a *successor* ordinal if it is of the form $\beta \cup \{\beta\}$ for some ordinal β; the latter is also written as $\beta + 1$. α is a *limit* ordinal if it is not 0 and is not a successor ordinal. The least infinite ordinal ω is an example.

Classification of wo's. Every wo is isomorphic to an ordinal.

Induction generalises from the natural numbers to the ordinal numbers:

Leastness Principle for ORD.
$\exists \alpha \varphi(\alpha) \to \exists \alpha (\varphi(\alpha) \wedge \forall \beta < \alpha \sim \varphi(\beta))$.

Transfinite Induction.
$(\varphi(0) \wedge \forall\alpha(\forall\beta < \alpha\varphi(\beta) \to \varphi(\alpha))) \to \forall\alpha\varphi(\alpha)$.

Using transfinite induction we can define addition and multiplication on ordinal numbers:

$\alpha + 0 = \alpha$
$\alpha + (\beta + 1) = (\alpha + \beta) + 1$
$\alpha + \lambda = \cup\{\alpha + \beta \mid \beta < \lambda\}$, λ limit

$\alpha \cdot 0 = 0$
$\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha$
$\alpha \cdot \lambda = \cup\{\alpha \cdot \beta \mid \beta < \lambda\}$, λ limit

Note: $1 + \omega = \omega \neq \omega + 1$; $2 \cdot \omega = \omega \neq \omega \cdot 2 = \omega + \omega$.

von Neumann Hierarchy
$V_0 = \emptyset$
$V_{\alpha+1} = \mathcal{P}(V_\alpha)$
$V_\lambda = \cup\{V_\alpha \mid \alpha < \lambda\}$, λ limit

Each V_α is transitive, $\alpha \leq \beta \to V_\alpha \subseteq V_\beta$ and $\alpha \in V_{\alpha+1}$. The function $F(\alpha) = V_\alpha$ is definable.

von Neumann's Theorem. Every set is an element of some V_α.

The *Rank* of a set x is the least ordinal α such that x belongs to $V_{\alpha+1}$.

Cardinals

Using the Axiom of Choice (AC) one can prove:

Theorem 1.1. For every set X there exists $\leq_X$ such that $\langle X, \leq_X\rangle$ is a wo.

Corollary 1.2. For every set X there is an ordinal α and a bijection $f : X \leftrightarrow \alpha$.

Definition. A *cardinal* if an ordinal κ such that $\alpha < \kappa \to$ there is *no* injective function $f : \kappa \to \alpha$.

Remark. If β is not a cardinal then there is $\alpha < \beta$ and a *bijective* function $f : \beta \leftrightarrow \alpha$: If $f : \beta \to \alpha$, $\alpha < \beta$ is injective, then choose $g : \langle \text{ Range}(f), \in\rangle \simeq \langle \bar{\alpha}, \in\rangle$, and replace f by $g \circ f$.

Definition. Card X = cardinality of X is the unique cardinal κ such that there is a bijection $f : X \leftrightarrow \kappa$.

Cantor's Theorem. For any set X, Card $\mathcal{P}(X) >$ Card X.

Proof. Otherwise there is a surjective $f : X \to \mathcal{P}(X)$. But consider the set $Y = \{A \in X \mid a \notin f(a)\}$. Then $f(y) \neq Y$ for all $a \in X$, as otherwise we would have $a \in Y$ iff $a \notin Y$. □

So there is no largest cardinal and every set is in bijective correspondence with a unique cardinal. Define:

$\aleph_0 = \omega$
$\aleph_{\alpha+1}$ = least cardinal greater than $\aleph_\alpha$
$\aleph_\lambda = \cup\{\aleph_\alpha \mid \alpha < \lambda\}$ for limit λ.

These are the infinite cardinals. For which α do we have Card $\mathcal{P}(\omega) = \aleph_\alpha$? We shall discuss this later.

Definition. If κ is a cardinal then κ^+ is the least cardinal greater than κ. A *successor cardinal* is a cardinal of the form κ^+ for some κ; a *limit cardinal* is a nonzero cardinal that is not a successor cardinal. The limit cardinals are $\aleph_0$ together with $\aleph_\lambda$ for limit ordinals λ.

Cardinal Arithmetic

If κ, λ are cardinals then the *cardinal* sum and product $\kappa + \lambda$, $\kappa \cdot \lambda$ are the cardinalities of the *ordinal* sum and product $\kappa + \lambda$, $\kappa \cdot \lambda$.

Theorem 1.3. For nonzero cardinals κ, λ, not both finite:
$\kappa + \lambda = \kappa \cdot \lambda = \max(\kappa, \lambda)$.

So addition and multiplication of cardinals is not very interesting. However cardinal *exponentiation* is very interesting, as we will now see.

Definition. For cardinals κ, λ, κ^λ is the cardinality of the set $\{f \mid f : \lambda \to \kappa\}$.

For example, $2^{\aleph_0} = 2^\omega$ is the cardinality of the set of functions f from the natural numbers N into $\{0, 1\}$. Of course this is the same as the cardinality of $\mathcal{P}(N)$. It is also the same as the cardinality of the set of real numbers:

Proposition 1.4. The set of real numbers has cardinality $2^{\aleph_0}$.

What is $2^{\aleph_0}$? It turns out that this question cannot be answered in ZFC.

Gödel: If ZFC is consistent then so is $\mathrm{ZFC} + 2^{\aleph_0} = \aleph_1$.
Cohen: If ZFC is consistent then so is $\mathrm{ZFC} + 2^{\aleph_0} = \aleph_2$.

The *Continuum Hypothesis (CH)* is the statement that $2^{\aleph_0} = \aleph_1$. Thus it follows that both CH and $\sim$ CH are consistent with ZFC (assuming of course that ZFC is consistent). There is a similar situation at other infinite cardinals. The *Generalised Continuum Hypothesis (GCH)* is the statement that $2^\kappa = \kappa^+$ for every infinite cardinal κ. Gödel's work also showed that the GCH is conistent with ZFC. But the general behaviour of the function $\kappa \mapsto 2^\kappa$ is very difficult to determine. For example, we have:

Silver: If $2^\alpha = \alpha^+$ for every $\alpha < \kappa = \aleph_{\aleph_1}$ then $2^\kappa = \kappa^+$.

And there are results stating that Silver's result does not hold with $\aleph_{\aleph_1}$ replaced by $\aleph_\omega$. However there is some restriction in the latter case:

Shelah: If $2^{\aleph_n} < \aleph_\omega$ for every finite n then $2^{\aleph_\omega} < \aleph_{\aleph_4}$.

The Lévy Hierarchy

The Δ_0 *formulas* form the least set of formulas containing the atomic formulas $x \in y, x = y$ and closed under $\sim, \wedge$ and bounded quantification $\forall x \in y$. Now define:

$\Sigma_0 = \Pi_0 = \Delta_0$
A Σ_{n+1} formula is one of the form $\exists x_1 \cdots \exists x_m \varphi$, where φ is Π_n
A Π_{n+1} formula is one of the form $\forall x_1 \cdots \forall x_m \varphi$, where φ is Σ_n.

Definability of Σ_n Satisfaction. For each n there is a formula $\mathrm{Sat}_n(i, s)$ such that if $i = \#\varphi$, φ is Σ_n and s is a function with domain $i + 1$ then:

$$\mathrm{ZF}^- \vdash \mathrm{Sat}_n(i, s) \leftrightarrow \varphi(s(0), \ldots, s(i))$$

(where if φ has free variables $x_0, \ldots, x_j$ then $\varphi(s(0), \ldots, s(i))$ is obtained from φ by replacing x_k by $s(k)$).

Tarski observed that ther is *no* formula $\text{Sat}(i)$ such that if $i = \#\varphi$, φ an arbitrary sentence then $\text{ZF}^- \vdash \text{Sat}(i) \leftrightarrow \varphi$. The same applies to any recursive theory containing ZF^-.

Using the previous result we can formulate the Reflection Principles. The expression

$$M \prec_n V$$

means that for Σ_n formulas $\varphi(x_1, \ldots, x_m)$ and $a_1, \ldots, a_m \in M$:
$M \vDash \varphi(a_1, \ldots, a_m)$ iff $\varphi(a_1, \ldots, a_m)$ is true.

Theorem 1.5. For each n, ZF proves the n-th Reflection Principle RP_n:

$$\forall\alpha\exists\beta > \alpha \; V_\beta \prec_n V.$$

The Universe of Constructible Sets

Gödel's universe of constructible sets is defined via the following hierarchy:

$L_0 = \emptyset$
$L_{\alpha+1} = \text{Def } L_\alpha$
$L_\lambda = \cup\{L_\alpha \mid \alpha < \lambda\}$ for limit ordinals λ.

We say that x is *constructible* if for some ordinal α, $x \in L_\alpha$. This is often abbreviated as "$x \in L$", where $L = \cup\{L_\alpha \mid \alpha \in \text{ORD}\}$, but it is important to keep in mind that L is not a set, but what can be referred to as a "proper class" of sets.

We need to know that this hierarchy is definable in an "absolute" way, in the following sense. Let ZF^- be the finite subtheory of ZF obtained by restricting the Replacement scheme to formulas with only 100 quantifiers.

Fact. If M, N are transitive sets and both $\langle M, \in\rangle$ and $\langle N, \in\rangle$ are models of ZF^- then for every ordinal $\alpha \in M \cap N$, $L_\alpha^M = L_\alpha^N$, where L_α^M, L_α^N are the interpretations of L_α in M, N, respectively.

We now show that in a certain sense, L is a "model" of ZF. For each formula φ, define a formula φ_L as follows:

$(x \in y)_L = (x \in y)$
$(x = y)_L = (x = y)$
$(\varphi \wedge \psi)_L = (\varphi_L \wedge \psi_L)$
$(\sim \varphi)_L = \sim (\varphi_L)$
$(\forall x \varphi)_L = \forall x(x \in L \rightarrow \varphi_L)$.

Then φ expresses the property "φ is true in L". We have:

Theorem 1.6. ZF $\vdash \varphi_L$ for each axiom φ of ZF.

Corollary 1.7. Let $V = L$ be the sentence $\forall x(x \in L)$. Then assuming that ZF is consistent, the theory ZF $+ V = L$ is also consistent.

Proof of Corollary from Theorem. Suppose that ZF $+ V = L$ were inconsistent. Then ZF $\vdash \sim (V = L)$. By the Theorem, ZF proves φ_L whenever φ is an axiom of ZF, and therefore ZF proves φ_L whenever φ is a theorem of ZF. So ZF proves $(\sim (V = L))_L$. But

$$(\sim (V = L))_L = \sim ((V = L)_L) = \sim (\forall x(x \in L))_L = \sim \forall x(x \in L \rightarrow x \in L)$$

which is the negation of a valid sentence. It follows that ZF is inconsistent, against our hypothesis. □

Proof of Theorem. φ_L is easy to check when φ is an axiom of ZF, except for the Power Set Axiom and Replacement. For example if φ is the Union Set Axiom, we must only show that if $x \in L$ then $\cup x$ is in L, for then by absoluteness $(\cup x)^L = \cup x$. But if x belongs to L_α then $\cup x = \{y \mid y$ is an element of an element of $x\}$ is definable over L_α, and therefore belongs to $L_{\alpha+1}$.

For Power Set: Suppose that x belongs to L_α and define $\mathcal{P}_L(x)$ to be $\{y \in L \mid y \subseteq x\}$. We must show that $\mathcal{P}_L(x)$ belongs to some L_β. Define a function $f : \mathcal{P}_L(x) \rightarrow$ ORD by $f(y) =$ the least ordinal γ such that $y \in L_\gamma$. By the Replacement axiom there is an ordinal β such that $\mathcal{P}_L(x) \subseteq L_\beta$, and therefore $\mathcal{P}_L(x) \in$ Def $L_\beta = L_{\beta+1}$.

For Replacement: Suppose that x belongs to L and $f : x \rightarrow L$ is an L-definable function. We want to show that there is an ordinal α such that Range f belongs to L_α. If f is Σ_n-definable in L then it is enough to find an ordinal α such that $L_\alpha \prec_n L$, Range $f \subseteq L_\alpha$ and the parameters in the formula that defines f belong to L_α. Using the Reflection Principle we can choose such an α, with $L_\alpha \prec_n L$ replaced by $V_\alpha \prec_m V$ for any m. By choosing m large enough we get $L^{V_\alpha} \prec_n L$ and therefore by absoluteness $L_\alpha \prec_n L$. $\square$

One of Gödel's famous results is that if ZF is consistent then so is ZFC = ZF + AC. By the previous corollary, any statement provable in the theory ZF + $V = L$ is consistent with ZF, so this follows from:

Theorem 1.8. ZF + $V = L \vdash$ AC.

2. Hyperfine Structure Theory

Names and Locations

For any $\alpha \in \mathrm{ORD}$, $\varphi(u, \vec{v})$ a first–order formula with $n+1$ free variables, and $\vec{x}$ a sequence from L_α of length n, let $I(\alpha, \varphi, \vec{x})$ denote $\{y \in L_\alpha \mid L_\alpha \models \varphi(y, \vec{x})\}$. Thus we can think of the above triples $(\alpha, \varphi, \vec{x})$ as *names* for elements of L. A central idea in our theory is to also view $(\alpha, \varphi, \vec{x})$ as a *location* for the structure $L_{(\alpha,\varphi,\vec{x})}$ in the fine hierarchy with an associated *hull operation* $L_{(\alpha,\varphi,\vec{x})}\{\cdot\}$ which approximates the usual Skolem hull operation on subsets of L_α. Before we define these notions we first discuss the ordering of names (=locations) and prove a condensation result for "constructibly–closed" subsets of L_α.

Well-order names and constructible sets in the standard way as follows: Consider $\in$–formulae built using $\neg$, $\wedge$, $\vee$ and the existential quantifier $\exists$. We agree that every formula φ has a distinguished variable used for the I–operation and for existential quantifications. When we write $\varphi(u, \vec{x})$, we intend that u is distinguished in φ; then $\exists u\varphi$ with any choice of distinguished variable is a new permitted formula. Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be an ω–ordering of permitted formulas, subformulas appearing earlier, which we assume to be fixed throughout this article.

We take $<_0$ to be the vacuous ordering on $L_0 = \emptyset$. If $<_\alpha$ is defined as a wellordering of L_α then order sequences from L_α by $\vec{x} <_\alpha^{\mathrm{lex}} \vec{y}$ iff $\vec{x}$ is lexicographically less then $\vec{y}$, using $<_\alpha$ on the components of $\vec{x}$ and $\vec{y}$. Names $(\beta, \varphi, \vec{x})$ where $\beta \leq \alpha$ are ordered by:

$(\beta, \varphi_m, \vec{x}) \mathrel{\tilde{<}} (\gamma, \varphi_n, \vec{y})$ iff
$(\beta < \gamma) \vee$
$(\beta = \gamma \wedge m < n) \vee$
$(\beta = \gamma \wedge m = n \wedge \vec{x} <_\beta^{\mathrm{lex}} \vec{y})$.

And for $y \in L_{\alpha+1}$ let $N(y)$ denote the $\tilde{<}$–least $(\beta, \varphi, \vec{x})$ such that $I(\beta, \varphi, \vec{x}) = y$. Then define $y <_{\alpha+1} z$ iff $N(y) \mathrel{\tilde{<}} N(z)$. Finally for limit λ set $<_\lambda = \bigcup_{\alpha<\lambda} <_\alpha$. Thus we obtain a wellordering $<_L = \bigcup_{\alpha \in \mathrm{ORD}} <_\alpha$ of L and a wellordering $\tilde{<}$ of names $(\alpha, \varphi, \vec{x})$ used to denote elements of L.

By an α–location we understand a location s of the form $s = (\alpha, \varphi, \vec{x})$. The $\tilde{<}$–smallest α–location is $(\alpha, \varphi_0, \vec{0})$ with $\vec{0}$ a vector of 0's of appropriate length. The $\tilde{<}$–successor of s is denoted by s^+.

Constructible Operations and Basic Constructible Closures.

The basic constructible operations are I and N as defined above and a Skolem function:

(Interpretation)
For a name $(\alpha, \varphi, \vec{x})$, set $I(\alpha, \varphi, \vec{x}) = \{y \in L_\alpha \mid L_\alpha \models \varphi(y, \vec{x})\}$.
(Naming)
For $y \in L$, let $N(y)$ be the $\widetilde{<}$–least name $(\alpha, \varphi, \vec{x})$ such that $I(\alpha, \varphi, \vec{x}) = y$.
(Skolem Function)
For a name $(\alpha, \varphi, \vec{x})$, let $S(\alpha, \varphi, \vec{x})$ be the $<_L$– least $y \in L_\alpha$ such that $L_\alpha \models \varphi(y, \vec{x})$, and set $S(\alpha, \varphi, \vec{x}) = 0$ if such a y does not exist.

As we do not assume that α is a limit ordinal and therefore do not have pairing, we make the following nonstandard definition.

Definition. For $X \subseteq L$ and $\vec{x}$ a finite sequence we write $\vec{x} \in X$ if each component of $\vec{x}$ belongs to X. If $(\alpha, \varphi, \vec{x})$ is a name we write $(\alpha, \varphi, \vec{x}) \in X$ to mean that $\alpha \in X$ and $\vec{x} \in X$.

A set or class $X \subseteq L$ is *constructibly closed*, written $X \triangleleft L$, iff X is closed under I, N and S, i.e.,

$$\begin{aligned} (\alpha, \varphi, \vec{x}) \in X \quad &\longrightarrow \quad I(\alpha, \varphi, \vec{x}) \in X \text{ and } S(\alpha, \varphi, \vec{x}) \in X, \\ y \in X \quad &\longrightarrow \quad N(y) \in X. \end{aligned}$$

For $X \subseteq L$ let $L\{X\}$ denote the $\subseteq$–smallest $Y \supseteq X$ such that $Y \triangleleft L$.

Clearly each L_α is constructibly closed.

Proposition 2.1. Let X be constructibly closed and let $\pi : X \cong M$ be the Mostowski collapse of X onto the transitive set M. Then there is an ordinal α such that $M = L_\alpha$, and π preserves I, N, S and $<_L$:

$$\pi : (X, \in, <_L, I, N, S) \cong (L_\alpha, \in, <_L, I, N, S).$$

Proof. We prove this for $X \subseteq L_\gamma$, by induction on γ. The cases $\gamma = 0$ and γ limit are easy. Let $\gamma = \beta + 1$ and $X \subseteq L_{\beta+1}$ but $X \nsubseteq L_\beta$. Closure under N and I implies that $X = \{I(\beta, \varphi, \vec{x}) \mid \vec{x} \text{ from } X \cap L_\beta\}$. Inductively let $\pi : X \cap L_\beta \cong L_\alpha$. Closure under S and the fact that β belongs to X imply that $X \cap L_\beta$ is elementary in L_β. It follows that π extends to $\tilde{\pi}: X \cong L_{\alpha+1}$. Preservation of I, N, S and $<_L$ follows also from the elementarity of $X \cap L_\beta$ in L_β. $\square$

The Hyperfine Hierarchy.

Definition. Let s be a location, $s = (\alpha, \varphi_m, \vec{x})$. Set

$$L_s = (L_\alpha, \in, <_L, I, N, S, S^{L_\alpha}_{\varphi_0}, S^{L_\alpha}_{\varphi_1}, \ldots, S^{L_\alpha}_{\varphi_m} \restriction \vec{x}, \emptyset, \emptyset, \ldots)$$

where $S^{L_\alpha}_\varphi(\vec{y}) = S(\alpha, \varphi, \vec{y})$, $S^{L_\alpha}_{\varphi_m} \restriction \vec{x}$ is the restricted Skolem function $S^{L_\alpha}_{\varphi_m} \restriction \{\vec{y} \mid \vec{y} <^{\text{lex}}_\alpha \vec{x}\}$ and $\emptyset, \emptyset, \ldots$ are empty functions.

$(L_s \mid s$ is a location) is the *hyperfine constructible hierarchy.*

Each structure of the hyperfine hierarchy possesses an associated hull operator.

Definition. Let $s = (\alpha, \varphi_m, \vec{x})$ be a location. A set $Y \subseteq L_\alpha$ is *closed* in L_s, written $Y \triangleleft L_s$, if Y is an algebraic substructure of L_s, i.e., if Y is closed under I, N, S, $S^{L_\alpha}_{\varphi_0}$, $S^{L_\alpha}_{\varphi_1}$,..., $S^{L_\alpha}_{\varphi_m} \restriction \vec{x}$.
For a set $X \subseteq L_\alpha$ let $L_s\{X\}$ be the $\subseteq$–smallest set Y such that $Y \triangleleft L_s$ and $Y \supseteq X$. We call $L_s\{X\}$ the L_s*–hull* of X.

The hyperfine hierarchy is a very slow growing hierarchy which nonetheless satisfies full condensation. This is the basis for its applications to fine structure theory.

Condensation. Let $s = (\alpha, \varphi_m, \vec{x})$ be a location and suppose X is a set such that $X \triangleleft L_s$. Then there is a unique isomorphism

$\pi: (X, \in, <_L, I, N, S, S^{L_\alpha}_{\varphi_0}, S^{L_\alpha}_{\varphi_1}, \ldots, S^{L_\alpha}_{\varphi_m} \restriction \vec{x}, \emptyset, \ldots) \cong$
$L_{\overline{s}} = (L_{\overline{\alpha}}, \in, <_L, I, N, S, S^{L_{\overline{\alpha}}}_{\varphi_0}, S^{L_{\overline{\alpha}}}_{\varphi_1}, \ldots, S^{L_{\overline{\alpha}}}_{\varphi_{\overline{m}}} \restriction \vec{\overline{x}}, \emptyset, \ldots)$.

Proof. Let $\pi: X \cong L_{\overline{\alpha}}$ be given by Proposition 1. Note that X is φ_i–elementary in L_α for $i \leq m$, since X is closed under the Skolem functions for every proper subformula of φ_i. Hence $\pi^{-1}: L_{\overline{\alpha}} \to L_\alpha$ is φ_i–elementary for $i \leq m$. Let $r = (\overline{\alpha}, \varphi_i, \vec{w})$ be a location such that $\pi^{-1}(r) := (\alpha, \varphi_i, \pi^{-1}(\vec{w})) \tilde{<} (\alpha, \varphi_m, \vec{x})$. Then $z := S^{L_\alpha}_{\varphi_i}(\pi^{-1}(\vec{w}))$ belongs to X and $L_\alpha \models \varphi_i(z, \pi^{-1}(\vec{w}))$ iff $L_{\overline{\alpha}} \models \varphi_i(\pi(z), \vec{w})$. Moreover, if there is $\overline{z} \in L_{\overline{\alpha}}$ such that $L_{\overline{\alpha}} \models \varphi_i(\overline{z}, \vec{w})$, then

$\pi(z)$ is the $<_L$–minimal such element, because $\overline{z} <_L \pi(z)$ and $L_{\overline{\alpha}} \models \varphi_i(\overline{z}, \vec{w})$ imply $L_\alpha \models \varphi_i(\pi^{-1}(\overline{z}), \pi^{-1}(\vec{w}))$ and $\pi^{-1}(\overline{z}) <_L z$, contradicting the definition of S_{φ_i}. Hence

$$\pi(z) = \pi(S^{L_\alpha}_{\varphi_i}(\pi^{-1}(\vec{w}))) = S^{L_{\overline{\alpha}}}_{\varphi_i}(\vec{w})$$

as required. The location $\overline{s}$ of the condensed structure is defined as the $\widetilde{<}$–smallest strict upper bound of all r such that $\pi^{-1}(r) \widetilde{<} s$ and $\overline{s} = \widetilde{<}\text{–}\sup\{r \mid \pi^{-1}(r) \widetilde{<} s\}$. □

Usually, we shall have $\overline{m} = m$ in the proposition, except when for every $\vec{w} \in L_{\overline{\alpha}}$ of the right length

$$\pi^{-1}(\vec{w}) <^{\text{lex}} \vec{x}.$$

In that case we have $\overline{m} = m + 1$ and $\vec{\overline{x}} = \vec{0}$, i.e., $\overline{s} = (\overline{\alpha}, \varphi_{m+1}, \vec{0})$ and

$$L_{\overline{s}} = (L_{\overline{\alpha}}, \in, <_L, I, N, S, S^{L_{\overline{\alpha}}}_{\varphi_0}, S^{L_{\overline{\alpha}}}_{\varphi_1}, \ldots, S^{L_{\overline{\alpha}}}_{\varphi_m}, \emptyset, \ldots)$$

observing that $S^{L_{\overline{\alpha}}}_{\varphi_{m+1}} \restriction \vec{0} = \emptyset$.

The condensation situation in proposition 2 is often written as $\pi\colon X \cong L_{\overline{s}}$.

The slow growth of the $L_{\overline{s}}$–hierarchy is expressed by a finiteness property which says that at successor locations essentially only one more point enters the hulling process, and by continuity properties saying that at limit locations we just collect results of previous processes.

Finiteness Property. Let $s = (\alpha, \varphi, \vec{x})$ be an α–location. Then there exists $z \in L_\alpha$ such that for any $X \subseteq L_\alpha$:

$$L_{s^+}\{X\} \subseteq L_s\{X \cup \{z\}\}.$$

Proof. The expansion from L_s to L_{s^+} provides us with at most one new Skolem value in forming hulls, namely $S^{L_\alpha}_\varphi(\vec{x})$. Take this $S^{L_\alpha}_\varphi(\vec{x})$ to be z. □

Monotonicity. (i) Suppose that s_0 and s_1 are α–locations such that $s_0 \widetilde{\leq} s_1$. Then $L_{s_0}\{X\} \subseteq L_{s_1}\{X\}$ for all $X \subseteq L_\alpha$.
(ii) Suppose that α_0 and α_1 are ordinals such that $\alpha_0 < \alpha_1$. If s_0, s_1 are α_0– and α_1–locations, respectively, and $X \subseteq L_{\alpha_0}$ then $L_{s_0}\{X\} \subseteq L_{s_1}\{X \cup \{\alpha_0\}\}$.

Proof. Clear from the definitions. □

For the continuity property we have to distinguish among three kinds of limit locations:

Continuity.
(a) If α is a limit ordinal, $s = (\alpha, \varphi_0, \vec{0})$, and $X \subseteq L_\alpha$ then

$$L_s\{X\} = L\{X\} = \bigcup_{\beta<\alpha} L_{(\beta,\varphi_0,\vec{0})}\{X \cap L_\beta\}.$$

(b) If $s = (\alpha + 1, \varphi_0, \vec{0})$ and $X \subseteq L_\alpha$ then

$$\begin{aligned} L_s\{X \cup \{\alpha\}\} \cap L_\alpha &= L\{X \cup \{\alpha\}\} \cap L_\alpha \\ &= \bigcup\{L_r\{X\} \mid r \text{ is an } \alpha\text{–location}\}. \end{aligned}$$

(c) If $s = (\alpha, \varphi, \vec{x})$ is a $\tilde{<}$–limit, $s \neq (\alpha, \varphi_0, \vec{0})$, and $X \subseteq L_\alpha$ then

$$L_s\{X\} = \bigcup\{L_r\{X\} \mid r \text{ is an } \alpha\text{–location, } r \,\tilde{<}\, s\}.$$

Proof. (a) is clear from the definitions since the hull operators considered only use the functions I, N, S.
(b) The first equality is clear. The other is proved by two inclusions.
($\supseteq$) If z is an element of the right hand side, z is obtained from elements of X by successive applications of I, N, S and $S^{L_\alpha}_{\varphi_n}$ for $n < \omega$. Since $S^{L_\alpha}_{\varphi_n}(\vec{y}) = S(\alpha, \varphi_n, \vec{y})$, z is also obtainable from elements of $X \cup \{\alpha\}$ using only the I, N and S operations. Hence z belongs to the left hand side.
($\subseteq$) Conversely, consider $z \in L\{X \cup \{\alpha\}\} \cap L_\alpha$. There is a finite sequence computing z in $L\{X \cup \{\alpha\}\}$:

$$y_0,\ y_1,\ \ldots\ ,\ y_k = z$$

such that each y_j is an element of $X \cup \{\alpha\}$ or y_j is obtained from $\{y_i \,|\, i < j\}$ by using I, N, S:

$$y_j = I(\beta, \varphi_n, \vec{y}) \quad \text{or} \quad y_j = S(\beta, \varphi_n, \vec{y}) \quad \text{or} \quad y_j \text{ is a component of } N(y)$$

for some $\beta, \vec{y}, y \in \{y_i \,|\, i < j\}$.
We show by induction on $j \leq k$:

$$\text{if } y_j \in L_\alpha \text{ then } y_j \in U = \bigcup\{L_r\{X\} \mid r \text{ is an } \alpha\text{–location}\}.$$

Case 1: $y_j \in X \cup \{\alpha\}$. Then our claim is obvious.
Case 2: $y_j = I(\beta, \varphi_n, \vec{y})$ (as in the first of the three ways of obtaining y_j from $\vec{y} \in \{y_i \mid i < j\}$, displayed above). If $\beta < \alpha$, then $\beta, \vec{y} \in U$ by the induction hypothesis and hence $y_j \in U$. If $\beta = \alpha$, then $\vec{y} \in U$ by the induction hypothesis. Setting

$$\psi(v, \vec{w}) = \forall u \, (u \in v \longleftrightarrow \varphi_n(u, \vec{w}))$$

with distinguished variable v we obtain $y_j = S_{\psi}^{L_\alpha}(\vec{y}) \in U$.
Case 3: $y_j = S(\beta, \varphi_n, \vec{y})$ (the second way of obtaining y_j). If $\beta < \alpha$, then $\beta, \vec{y} \in U$ and $y_j \in U$. If $\beta = \alpha$, then $\vec{y} \in U$ and $y_j = S_{\varphi_n}^{L_\alpha}(\vec{y}) \in U$.
Case 4: y_j is a component of $N(y_i)$ for some $i < j$ (the third way of obtaining y_j).
Case 4.1: $y_i \in L_\alpha$. Then $y_i \in U$ by the induction hypothesis. As U is closed under N, we get $N(y_i) \in U$, i.e., each component of $N(y_i)$ belongs to U.
Case 4.2: $y_i \in L_{\alpha+1} \setminus L_\alpha$. Then $y_i = \alpha$, or $y_i = I(\alpha, \psi, \vec{y})$ for some $\vec{y} \in \{y_h \mid h < i\}$. Since $\alpha = I(\alpha, \text{“}u \text{ is an ordinal”}, \emptyset)$, we may assume the latter. $N(y_i)$ will be of the form $(\alpha, \chi, (c_0, \ldots, c_{m-1}))$. We obtain c_0 in U as follows: If

$$\chi_0(v_0, \vec{w}) = \exists v_1 \ldots \exists v_{m-1} \forall u \, (\chi(u, v_0, v_1, \ldots, v_{m-1}) \longleftrightarrow \psi(u, \vec{w}))$$

with distinguished variable v_0 then $c_0 = S_{\chi_0}^{L_\alpha}(\vec{y}) \in U$, since, inductively, $\vec{y} \in U$. We obtain c_1 in U as follows: If

$$\chi_1(v_1, \vec{w}) = \exists v_2 \ldots \exists v_{m-1} \forall u \, (\chi(u, v_0, v_1, \ldots, v_{m-1}) \longleftrightarrow \psi(u, \vec{w}))$$

with distinguished variable v_1 then $c_1 = S_{\chi_1}^{L_\alpha}(c_0 {}^\frown \vec{y}) \in U$. Proceeding like this we see that $y_j \in U$.
(c) is again obvious from the definitions. □

This completes our list of basic properties of the hull operations associated with the hyperfine hierarchy. They are sufficient to establish Jensen's Square Principle in L, which we consider next.

A Proof of Square

Theorem 2.2. (Jensen) Assume $V = L$. There exists a sequence $\langle C_\beta \mid \beta \text{ singular} \rangle$ such that
(i) C_β is closed unbounded in β

(ii) C_β has ordertype less than β
(ii) if $\overline{\beta}$ is a limit point of C_β then $\overline{\beta}$ is singular and $C_{\overline{\beta}} = C_\beta \cap \overline{\beta}$.

Proof. Let β be singular. The following claim gives a reformulation of the singularity of β:

Claim 1. There is a location $s = (\gamma, \varphi, \vec{x})$, $\gamma \geq \beta$, and a finite set $p \subseteq L_\gamma$ such that

$$\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_s\{\overline{\beta} \cup p\}\}$$

is bounded in β.

Proof. Choose α less than β and a function $f{:}\,\alpha \to \beta$ cofinally. Choose $\gamma \in \mathrm{ORD}$ such that $f \in L_\gamma$. Set $p = \{f\}$ and $s = (\gamma, \varphi_{n+1}, \vec{0})$ where n is a natural number choosen such that $\varphi_n \equiv v_0 = v_1(v_2)$ with distinguished variable v_0. If $\alpha \leq \overline{\beta} < \beta$ then

$$\beta \cap L_s\{\overline{\beta} \cup p\} \supseteq \beta \cap L_s\{\alpha \cup p\} \supseteq f''\alpha.$$

Hence $\beta \cap L_s\{\overline{\beta} \cup p\}$ is cofinal in β, and so $\beta \cap L_s\{\overline{\beta} \cup p\} \neq \overline{\beta}$.

Let $s = s(\beta)$ be $\widetilde{<}$–minimal satisfying Claim 1, together with the finite set $p \subseteq L_\gamma$. We show that s is a $\widetilde{<}$–limit which can be nicely approximated from below.

Claim 2. s is a limit location.

Proof. Assume that $s = r^+$. By the Finiteness Property there exists a $z \in L_\gamma$ such that if $\overline{\beta}$ is less than β then

$$L_s\{\overline{\beta} \cup p\} \subseteq L_r\{\overline{\beta} \cup p \cup \{z\}\}.$$

So

$$\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_r\{\overline{\beta} \cup p \cup \{z\}\}\} \subseteq \{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_s\{\overline{\beta} \cup p\}\}.$$

Hence $\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_r\{\overline{\beta} \cup p \cup \{z\}\}\}$ is bounded in β, contradicting the minimality of s.

Claim 3. $s \neq (\beta, \varphi_0, \vec{0})$.

Proof. Assume that $s = (\beta, \varphi_0, \vec{0})$. Choose β_0 less than β such that $p \subseteq L_{\beta_0}$. If $\beta_0 \leq \overline{\beta} < \beta$ then

$$\overline{\beta} \subseteq \beta \cap L_s\{\overline{\beta} \cup p\} \subseteq \beta \cap L\{\overline{\beta} \cup p\} \subseteq \beta \cap L_{\overline{\beta}} = \overline{\beta},$$

contradicting the fact that s and p satisfy the requirements in Claim 1.

Claim 4. $s \neq (\gamma, \varphi_0, \vec{0})$ for limit γ.

Proof. Assume that there is a limit ordinal γ such that $s = (\gamma, \varphi_0, \vec{0})$. Choose γ_0 less than γ such that $p \subseteq L_{\gamma_0}$ and $\gamma_0 \geq \gamma$, and set $s_0 = (\gamma_0, \varphi_0, \vec{0})$. Then

$$\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_{s_0}\{\overline{\beta} \cup p\}\} \subseteq \{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_s\{\overline{\beta} \cup p\}\}.$$

Hence $\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_{s_0}\{\overline{\beta} \cup p\}\}$ is bounded below β, contradicting the minimality of s.

In defining C_β we shall consider three special cases and a generic case. In the special cases, β will have cofinality ω and we can pick any ω–sequence cofinal in β as C_β.

Special Case 1. $s = (\alpha + 1, \varphi_0, \vec{0})$ for some α.

Every element of $L_{\alpha+1}$ can be "named" by α and finitely many elements of L_α. So we may assume that p is of the form $p = q \cup \{\alpha\}$ with $q \subseteq L_\alpha$.
Define a strictly increasing sequence $(\beta_n \mid n < \omega)$ of ordinals less than β recursively: Let

$$\beta_0 = \max\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_s\{\overline{\beta} \cup p\}\} < \beta.$$

Given β_n choose β_{n+1} greater than β_n least such that

$$\beta_{n+1} = \beta \cap L_{(\alpha, \varphi_n, \vec{0})}\{\beta_{n+1} \cup q\}.$$

Since $s = (\alpha, \varphi_n, \vec{0}) \tilde{<} (\alpha+1, \varphi_0, \vec{0})$, the definition of s implies that β_{n+1} exists below β. Let $\beta_\omega = \bigcup_{n<\omega} \beta_n$. Then

$$\begin{aligned} \beta \cap L_s\{\beta_\omega \cup p\} &= \beta \cap L_s\{\beta_\omega \cup q \cup \{\alpha\}\} \\ &= \beta \cap \bigcup\{L_r\{\beta_\omega \cup q\} \mid r \text{ is an } \alpha\text{–location}\} \\ &= \bigcup_{n<\omega} \beta \cap L_{(\alpha, \varphi_n, \vec{0})}\{\beta_\omega \cup q\} \end{aligned}$$

$$= \bigcup_{n<\omega} \beta \cap L_{(\alpha,\varphi_n,\vec{0})}\{\beta_{n+1} \cup q\}$$
$$= \bigcup_{n<\omega} \beta_{n+1} = \beta_\omega;$$

the second equality uses Continuity (b), the third and fourth use the monotonicity property of our hulls. Now by the definition of β_0 we must have $\beta_\omega = \beta$. Hence setting

$$C_\beta = \{\beta_n \mid n < \omega\}$$

we get a cofinal subset of β. This finishes Special Case 1.

Now assume that $s = (\gamma, \varphi, \vec{x}) \neq (\gamma, \varphi_0, \vec{0})$.

Claim 5. There is a finite $\overline{p} \subseteq L_\gamma$ such that $L_s\{\beta \cup \overline{p}\} = L_\gamma$.

Proof. By Condensation, there are a unique function π and a unique location $\overline{s}$ such that $\pi\colon L_s\{\beta \cup p\} \cong L_{\overline{s}}$. Then we have $L_{\overline{s}} = L_{\overline{s}}\{\beta \cup \overline{p}\}$ where $\overline{p} = \pi''p$. As $\pi \upharpoonright \beta = \mathrm{id}$, we can conclude that $\beta \cap L_s\{\overline{\beta} \cup p\} = \beta \cap L_{\overline{s}}\{\overline{\beta} \cup \overline{p}\}$ holds for all $\overline{\beta}$ less than β. Hence

$$\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_{\overline{s}}\{\overline{\beta} \cup \overline{p}\}\} = \{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_s\{\overline{\beta} \cup p\}\}$$

is bounded below β. Then $\overline{s} = s$ by the $\tilde{<}$–minimality of s, and so $L_s = L_s\{\beta \cup \overline{p}\} = L_\gamma$.

Let $<^*$ be the canonical wellorder of finite subsets of L derived from $<_L$: $p_0 <^* p_1 \longleftrightarrow p_0 \neq p_1$ and the $<_L$-maximal element of $p_0 \triangle p_1$ belongs to p_1. Choose a $<^*$-minimal $p(\beta) \subseteq L_\gamma$ such that $p(\beta)$ satisfies Claim 5. Since in particular the old parameter p is generated by $\beta \cup p(\beta)$ we have

Claim 6. $\{\overline{\beta} < \beta \mid \overline{\beta} = \beta \cap L_s\{\overline{\beta} \cup p(\beta)\}\}$ is bounded below β. Let $\beta_0 < \beta$ be the maximum of this set.

By Claim 6, $p(\beta)$ satisfies the requirements in Claim 1 and we may denote $p(\beta)$ by p without danger of confusion.

We have to examine which locations below s are computed in $L_s\{X\}$: for $Y \subseteq L_\gamma$ we write $r = (\gamma, \psi, \vec{y}) \epsilon Y$ if $\vec{y} \in Y$. We say that a subset Y of L_γ is *bounded below* s, if there is $s_0 \tilde{<} s$ such that if $r \tilde{<} s$ and $r \epsilon Y$, then $r \tilde{<} s_0$. The $\tilde{<}$-least such s_0 is called the *$\tilde{<}$-least upper bound of Y below s*. Note that if in addition $Y = L_s\{Z\}$ then we get $L_s\{Z\} = L_{s_0}\{Z\}$.

Special Case 2. $L_s\{\alpha \cup p\}$ is bounded below s for every $\alpha < \beta$.

Define a strictly increasing sequence $(\beta_n \mid n < \omega)$ of ordinals less than β recursively: Let β_0 be defined as in Claim 6. Given β_n, set

$$\beta_{n+1} = \bigcup(\beta \cap L_s\{(\beta_n + 1) \cup p\}).$$

By Special Case 2, there is $r \,\widetilde{<}\, s$ such that

$$L_s\{(\beta_n + 1) \cup p\} = L_r\{(\beta_n + 1) \cup p\}.$$

The minimality of s implies that $\beta \cap L_r\{(\beta_n + 1) \cup p\}$ cannot be cofinal in β, and so β_{n+1} is less than β. Let $\beta_\omega = \bigcup_{n<\omega} \beta_n$. Then

$$\beta_\omega \subseteq \beta \cap L_s\{\beta_\omega \cup p\} \subseteq \bigcup_{n<\omega} \beta \cap L_s\{(\beta_n + 1) \cup p\} \subseteq \bigcup_{n<\omega} \beta_{n+1} = \beta_\omega,$$

and since β_ω is greater than β_0 we have $\beta_\omega = \beta$. Hence setting

$$C_\beta = \{\beta_n \mid n < \omega\}$$

we get a cofinal subset of β. This finishes Special Case 2.

Now assume that $L_s\{\alpha_0 \cup p\}$ is unbounded below s for some α_0 less than β. Choose $\alpha_0 = \alpha_0(\beta)$ least with this property. We would like to use α_0 to steer the singularisation of β and obtain ordertype$(C_\beta) \leq \max\{\alpha_0, \omega\} < \beta$. If α_0 is neither a limit ordinal nor zero we have to look for another steering ordinal. In this case we write $\alpha_0 = \alpha'_0 + 1$, and we choose a least $\alpha_1 = \alpha_1(\beta)$ less than α_0 such that

$$L_s\{\alpha_1 \cup p \cup \{\alpha'_0\}\}$$

is unbounded below s. If $\alpha_1 = \alpha'_1 + 1$, then we choose a least $\alpha_2 = \alpha_2(\beta)$ less than α_1 such that

$$L_s\{\alpha_2 \cup p \cup \{\alpha'_0, \alpha'_1\}\}$$

is unbounded below s. Continuing this way we find a natural number $k = k(\beta)$ such that $\alpha = \alpha(\beta) = \alpha_k(\beta)$ is a limit ordinal or zero.

Special Case 3. $\alpha = 0$.

One easily sees that $L_s\{p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$ is a countable set. Since $\alpha = 0$, it is unbounded below s. So s has "cofinality ω" in the ordering of locations and we can find a strictly increasing sequence $(s_n \mid n < \omega)$ of γ–locations converging towards s. Define a strictly increasing sequence $(\beta_n \mid n < \omega)$ of ordinals less than β recursively: Let β_0 be defined as in Claim 6. Given β_n, choose β_{n+1} greater than β_n minimal such that

$$\beta_{n+1} = \beta \cap L_{s_{n+1}}\{\beta_{n+1} \cup p\}.$$

β_{n+1} exists, since $s_{n+1} \mathrel{\widetilde{<}} s$. Let $\beta_\omega = \bigcup_{n<\omega} \beta_n$. Then

$$\beta_\omega = \bigcup_{n<\omega} \beta_{n+1} = \bigcup_{n<\omega} \beta \cap L_{s_{n+1}}\{\beta_{n+1} \cup p\} = \beta \cap L_s\{\beta_\omega \cup p\},$$

hence the definition of β_0 implies $\beta_\omega = \beta$. Setting

$$C_\beta = \{\beta_n \mid n < \omega\}$$

we get a cofinal subset of β. This finishes Special Case 3.

So, finally, we arrive at the general case:

General Case. $s = (\gamma, \varphi, \vec{x}) \neq (\gamma, \varphi_0, \vec{0})$, and $L_s\{\alpha \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$ is unbounded below s where α is a limit ordinal less than β.

Define sequences $(\beta_i(\beta) \mid i \leq \alpha)$ and $(s_i \mid 0 < i \leq \alpha)$ recursively: Let $\beta_0 < \beta$ be defined as in Claim 6. For each $0 < i \leq \alpha$ let s_i be the $\widetilde{<}$–least upper bound of $L_s\{i \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$ below s, and let $\beta_i = \beta_i(\beta)$ be the least ordinal greater than β_0 such that

$$\beta_i = \beta \cap L_{s_i}\{\beta_i \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}.$$

If $i < \alpha$ then $\beta_i < \beta$ because $s_i \mathrel{\widetilde{<}} s$; also $s_\alpha = s$, $\beta_\alpha = \beta$ and

Claim 7. If $0 < i < j < \alpha$ then $s_i \mathrel{\widetilde{\leq}} s_j$ and $\beta_i \leq \beta_j$.

Claim 8. $\{\beta_i \mid i < \alpha\}$ is closed unbounded in β.

Proof. Let $\overline{\alpha} \leq \alpha$ be a limit ordinal. We only have to show that $\beta_{\overline{\alpha}} = \bigcup_{i<\overline{\alpha}} \beta_i$ and since $\beta_{\overline{\alpha}} \geq \beta_i$ for $i < \overline{\alpha}$ it suffices to see that

$$\begin{aligned}\bigcup_{i<\overline{\alpha}} \beta_i &= \bigcup_{i<\overline{\alpha}} \beta \cap L_{s_i}\{\beta_i \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\} \\ &= \beta \cap L_{s_{\overline{\alpha}}}\{\textstyle\bigcup_{i<\overline{\alpha}} \beta_i \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}\end{aligned}$$

so that $\bigcup_{i<\overline{\alpha}} \beta_i$ satisfies the defining property of $\beta_{\overline{\alpha}}$.
C_β will now be defined as an endsegment of such β_i's for which important elements of the preceding construction are visible below β_i or s_i. Let $I(\beta)$ be the set of those ordinals i that satisfy the following properties (1) — (5):

(1) $0 < i < \alpha$, and if $l \leq k$ then $\beta_i \geq \alpha'_l$.
(2) s_i is a γ–location.
(3) $j < \beta_j$ for $i \leq j < \alpha$.
(4) If $l < k$ and t is the $\widetilde{<}$–least upper bound of $L_s\{\alpha'_l \cup p \cup \{\alpha'_0, \dots, \alpha'_{l-1}\}\}$ below s then $s_i \widetilde{>} t$.
(5) If $\beta < \gamma$ then $\beta \in L_{s_i}\{\beta_i \cup p\}$.

Using the following facts (i) — (iv) the reader can easily show that there is i_0 less than α such that an ordinal i less than α satisfies the conditions (1) — (5) if and only if $i > i_0$, i.e., $I(\beta)$ is a final segment of α.

(i) $L_s\{\alpha \cup p \cup \{\alpha'_0, \dots, \alpha'_{k-1}\}\}$ is unbounded below s.
(ii) $\alpha < \beta$ and $\beta = \bigcup\{\beta_i | i < \alpha\}$ where $(\beta_i \mid i < \alpha)$ is (weakly) increasing.
(iii) $L_s\{\alpha'_l \cup p \cup \{\alpha'_0, \dots, \alpha'_{l-1}\}\}$ is bounded below s for all $l \leq k$.
(iv) If $\beta < \gamma$ then $\beta \in L_s\{\beta \cup p\} = L_\gamma$.

So set

$$C_\beta = \{\beta_i \mid i \in I(\beta)\}.$$

Claim 9. C_β is closed unbounded in β and ordertype$(C_\beta) \leq \alpha < \beta$.

This completes the definition of the system $\langle C_\beta \mid \beta \text{ singular}\rangle$, and we are left with proving the coherence property. Fix $\overline{\beta}$ less than β such that $\overline{\beta}$ is a limit point of C_β. We have to show that $\overline{\beta}$ is singular and $C_{\overline{\beta}} = C_\beta \cap \overline{\beta}$. β falls under the General Case, as ordertype$(C_\beta) > \omega$. Let $\overline{\alpha}$ be the least ordinal η such that $\overline{\beta} = \beta_\eta$. Then $\overline{\alpha}$ is a limit ordinal and $\overline{\beta}$ is singular since $\mathrm{cf}(\beta_{\overline{\alpha}}) \leq \overline{\alpha} < \beta_{\overline{\alpha}}$. By condensation there is an isomorphism

$$\pi\colon L_{s_{\overline{\alpha}}}\{\overline{\beta} \cup p\} \cong L_{\overline{s}}.$$

Let $q = \pi'' p$ and $\overline{\gamma} = \alpha(\bar{s})$.

Claim 10. $\pi \restriction \overline{\beta} = \mathrm{id}$. If s is a β–location then $\overline{s}$ is a $\overline{\beta}$–location while if s is a γ–location and $\gamma > \beta$ then $\pi(\beta) = \overline{\beta}$.

Proof. If $\gamma > \beta$ then $\beta \in L_{s_{\overline{\alpha}}}\{\overline{\beta} \cup p\}$ and $\overline{\beta} = \beta \cap L_{s_{\overline{\alpha}}}\{\overline{\beta} \cup p\}$.

Claim 11. $\overline{s} = s(\overline{\beta})$.

Proof. If $\beta_0 < \delta < \overline{\beta}$ then $\delta \neq \beta \cap L_{s_{\overline{\alpha}}}\{\delta \cup p \cup \{\alpha'_0 \ldots \alpha'_{k-1}\}\}$ and therefore $\delta \neq \overline{\beta} \cap L_{\overline{s}}\{\delta \cup q \cup \{\alpha'_0 \ldots \alpha'_{k-1}\}\}$. It follows that $s(\overline{\beta}) \mathrel{\widetilde{\leq}} \overline{s}$.
Conversely if $r \mathrel{\widetilde{<}} \overline{s}$ and $\overline{q}$ is a finite subset of $L_{\alpha(r)}$ then $\pi^{-1}(r) \mathrel{\widetilde{<}} s_i$ and $\pi^{-1}{''}\overline{q} \subseteq L_{s_i}\{\beta_i \cup p\}$ for sufficiently large i less than $\overline{\alpha}$, since the s_i's are unbounded below $s_{\overline{\alpha}}$, the β_i's are unbounded in $\overline{\beta}$ and $L_{\overline{s}}\{\overline{\beta} \cup q\} = L_{\alpha(\overline{s})}$. As $\beta_i = \beta \cap L_{s_i}\{\beta_i \cup p\}$ we get $\beta_i = \overline{\beta} \cap L_r\{\beta_i \cup \overline{q}\}$ for β_i's cofinal in $\overline{\beta}$ and so $r \mathrel{\widetilde{<}} s(\overline{\beta})$. Therefore $\overline{s} \mathrel{\widetilde{\leq}} s(\overline{\beta})$.

Claim 12. $\overline{\beta}$ does not fall under Special Case 1.

Claim 13. $q = p(\overline{\beta})$.

Proof. As $L_{\overline{s}}\{\overline{\beta} \cup q\} = L_{\overline{\gamma}}$, we get $q \geq^* p(\overline{\beta})$. Assume $q >^* p(\overline{\beta})$. As $p(\overline{\beta})$ satisfies the requirements in Claim 5 at $\overline{\beta}$, we get $q \subseteq L_{\overline{s}}\{\overline{\beta} \cup p(\overline{\beta})\}$, hence $p = \pi^{-1}{''}q \subseteq L_s\{\overline{\beta} \cup \pi^{-1}{''}p(\overline{\beta})\}$. So $\pi^{-1}{''}p(\overline{\beta}) <^* p = \pi^{-1}{''}q$ and $\pi^{-1}{''}p(\overline{\beta})$ satisfies the requirements in Claim 5, contrary to the minimal choice of $p = p(\beta)$.
Now $L_{s_{\overline{\alpha}}}\{\overline{\alpha} \cup p\} = L_s\{\overline{\alpha} \cup p\}$ is unbounded below $s_{\overline{\alpha}}$. Hence $L_{\overline{s}}\{\overline{\alpha} \cup q\}$ is unbounded below $\overline{s}$, and $\overline{\alpha} < \overline{\beta}$. Hence

Claim 14. $\overline{\beta}$ does not fall under Special Case 2.

Claim 15. If $j < k$ then $\alpha_j(\beta) = \alpha_j(\overline{\beta})$.

Proof. By induction on $j < k$.
By definition, $\alpha_j(\beta)$ is the smallest ν s.t. $L_s\{\nu \cup p \cup \{\alpha'_i \mid i < j\}\}$ is unbounded below s. Now $L_s\{\overline{\alpha} \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$ is unbounded below $s_{\overline{\alpha}}$, so $L_{\overline{s}}\{\overline{\alpha} \cup q \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$ is unbounded below $\overline{s}$. Hence $L_{\overline{s}}\{\alpha_j(\beta) \cup q \cup \{\alpha'_0, \ldots, \alpha'_{j-1}\}\}$ is unbounded below $\overline{s}$, as $\overline{\alpha} \cup \{\alpha'_j \ldots \alpha'_{k-1}\} \subseteq \alpha_j(\beta)$. Conversely, the definition of $I(\beta)$ implies that $L_s\{\alpha'_j \cup p \cup \{\alpha'_0, \ldots, \alpha'_{j-1}\}\}$ is bounded below s by some $s' \mathrel{\widetilde{<}} s_{\overline{\alpha}}$, hence by some location in $L_{s_{\overline{\alpha}}}\{\overline{\beta} \cup p\}$. So $L_{\overline{s}}\{\alpha'_j \cup q \cup \{\alpha'_0, \ldots, \alpha'_{j-1}\}\}$ is bounded below $\overline{s}$ by some location less than $\overline{s}$. So $\alpha_j(\beta) = \alpha_j(\overline{\beta})$.

Claim 16. $\alpha_k(\overline{\beta}) = \overline{\alpha}$.

Proof. The set $L_{\overline{s}}\{\overline{\alpha} \cup q \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$ is unbounded below $\overline{s}$. If we take α' less than $\overline{\alpha}$, then $L_{s_{\overline{\alpha}}}\{\alpha' \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$ is bounded below $s_{\overline{\alpha}}$, by the minimality of $\overline{\alpha}$. So we have $\alpha_k(\overline{\beta}) = \overline{\alpha}$.

Claim 17. $\overline{\beta}$ does not fall under Special Case 3,

since $\overline{\alpha} \neq 0$. So we are again in the General Case.

Claim 18. If $i < \overline{\alpha}$ then $\beta_i(\beta) = \beta_i(\overline{\beta})$.

Proof. By definition, $\beta_0 = \beta_0(\beta)$ is the largest δ less than β such that $\delta = \beta \cap L_s\{\delta \cup p\}$. From the definition of $\overline{\beta} = \beta_{\overline{\alpha}}$ we infer that β_0 is the largest δ less than $\overline{\beta}$ such that $\delta = \overline{\beta} \cap L_{s_{\overline{\alpha}}}\{\delta \cup p\}$. As $L_{s_{\overline{\alpha}}}\{\overline{\beta} \cup p\} \cong L_{\overline{s}}\{\overline{\beta} \cup q\}$ by a map which is the identity on $\overline{\beta}$, we see that β_0 is the largest δ less than $\overline{\beta}$ such that $\delta = \overline{\beta} \cap L_{\overline{s}}\{\delta \cup q\}$, which is the definition of $\beta_0(\overline{\beta})$.
Now consider $0 < i < \overline{\alpha}$. Then

$s_i(\beta)$ is the $\tilde{<}$–least upper bound of $L_s\{i \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$ below s.

By the definition of $s_{\overline{\alpha}}$ we get that

$s_i(\beta)$ is the $\tilde{<}$–least upper bound of $L_{s_{\overline{\alpha}}}\{i \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$ below $s_{\overline{\alpha}}$.

Moreover,

$s_i(\overline{\beta})$ is the $\tilde{<}$–least upper bound of $L_{\overline{s}}\{i \cup q \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$ below $\overline{s}$.

Now $\beta_i(\beta)$ is the minimal ordinal greater than β_0 such that

$$\beta_i(\beta) = \beta \cap L_{s'}\{\beta_i(\beta) \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$$

for all $s' \tilde{<} s_{\overline{\alpha}}(\beta)$ with $s' \in L_{s_{\overline{\alpha}}}\{i \cup p \cup \{\alpha'_0 \ldots \alpha'_{k-1}\}\}$, and $\beta_i(\overline{\beta})$ is the minimal ordinal greater than β_0 such that

$$\beta_i(\overline{\beta}) = \overline{\beta} \cap L_{\overline{s}'}\{\beta_i(\overline{\beta}) \cup q \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$$

for all $\overline{s}' \tilde{<} \overline{s}$ with $\overline{s}' \in L_{\overline{s}}\{i \cup q \cup \{\alpha'_0 \ldots \alpha'_{k-1}\}\}$. By the above and the fact that $\pi \upharpoonright \overline{\beta} = \mathrm{id}$ we have $\beta_i(\beta) = \beta_i(\overline{\beta})$ as required.
Now one easily checks that each ordinal i less than $\overline{\alpha}$ satisfies the defining properties of $I(\beta)$ (cf. (1) — (5) above) if and only if it satisfies the corresponding defining properties of $I(\overline{\beta})$. So we get $I(\overline{\beta}) = I(\beta) \cap \overline{\alpha}$, and this immediately implies the coherence property. □

3. Set-Forcing

The method of forcing provides a way to construct extensions of Gödel's model L. Cohen invented this method to demonstrate the unprovability of the continuum hypothesis (CH) in ZFC and of the axiom of choice (AC) in ZF; as AC, CH hold in L we obtain in this way two striking examples of undecidable propositions. Cohen's method was extended by Solovay to provide a very general and powerful technique for enlarging any transitive ZFC model M, given the choice of a pre-ordering (i.e., reflexive and transitive binary relation) $P \in M$.

Let M be a transitive model of ZF, either a set or a class. The case that interests us most is when M is L, but the forcing method does not require such a restriction. Let $P \in M$ be a pre-ordering; our plan is to do the following:

1. We define what it means for $G \subseteq P$ to be P-generic over M.
2. We describe, for each $G \subseteq P$, a transitive $M[G] \supseteq M \cup \{G\}$.
3. We prove that if G is P-generic over M then $M[G]$ is a model of ZF and, assuming AC holds in M, that AC holds in $M[G]$.

P-Generic Sets

We assume that $P = (P, \leq)$ has a greatest element, which we call 1^P. We think of $p \leq q$ as meaning "p is at least as strong as q."

Definition. p, q are *compatible* if for some r, $r \leq p$ and $r \leq q$. $D \subseteq P$ is *dense* if $\forall p \exists q (q \leq p$ and $q \in D)$. $G \subseteq P$ is *P-generic over M* if:

1. $p, q \in G \longrightarrow p, q$ are compatible.
2. $p \geq q \in G \longrightarrow p \in G$.
3. $D \subseteq P, D$ dense, $D \in M \longrightarrow G \cap D \neq \emptyset$.

Assumption. We assume that for each $p \in P$ there exists $G \subseteq P, p \in G$, G P-generic over M.

Our Assumption is vacuous if M is countable as we can list the dense $D \in M$ as $D_0, D_1, \ldots$, define $p_0 = p$, $p_n \geq p_{n+1} \in D_n$ and take $G = \{p | p_n \leq p$ for some $n\}$.

The Extension $M[G]$

We define $M[G]$ to consist of sets which have "names" in M, interpreted using G.

A *name* is a set $\sigma \in M$ consisting of pairs $\langle \tau, p \rangle$ where τ is a name and $p \in P$. Equivalently, a name is an element of $\cup\{\text{Name}_\alpha | \alpha \in \text{ORD}(M)\}$ where $\text{Name}_0 = \emptyset$, $\text{Name}_{\alpha+1} =$ All subsets of $\text{Name}_\alpha \times P$ in M, $\text{Name}_\lambda = \cup\{\text{Name}_\alpha | \alpha < \lambda\}$ for limit λ.

The interpretation of the name σ is $\sigma^G = \{\tau^G | \langle \tau, p \rangle \in \sigma, p \in G\}$. Then $M[G] = \{\sigma^G | \sigma \text{ a name}\}$.

Lemma 3.1. Suppose $1^P \in G \subseteq P$.

1. $M \subseteq M[G]$, $G \in M[G]$,
2. $M[G]$ is transitive, $\text{ORD}(M[G]) = \text{ORD}(M)$.
3. If $M \cup \{G\} \subseteq N$, N a model of ZF then $M[G] \subseteq N$.

Proof.

1. For $a \in M$ define $\hat{a} = \{\langle \hat{b}, 1^P \rangle | b \in a\}$ and then $\hat{a}^G = a$. Also $G = \gamma^G$ where $\gamma = \{\langle \hat{p}, p \rangle | p \in P\}$.
2. If $a \in \sigma^G \in M[G]$ then by definition $a = \tau^G \in M[G]$ for some τ; so $M[G]$ is transitive. By induction on Rank $\sigma =$ least α such that $\sigma \in \text{Name}_{\alpha+1}$, it follows that the von Neumann rank of σ^G is at most Rank $\sigma \in \text{ORD}(M)$. So $\text{ORD}(M[G]) \subseteq \text{ORD}(M)$.
3. For each $\alpha \in \text{ORD}(M)$, the inductive definition of σ^G for Rank $\sigma < \alpha$ can be carried out in N. □

Definition. Suppose p belongs to P, $\varphi(v_1 \dots v_n)$ is a formula and $\sigma_1 \dots \sigma_n$ are names. We write $p \Vdash \varphi(\sigma_1 \dots \sigma_n), p$ *forces* $\varphi(\sigma_1 \dots \sigma_n)$, iff whenever $G \subseteq P$ is P-generic over M and $p \in P$, we have $M[G] \vDash \varphi(\sigma_1^G \dots \sigma_n^G)$. And we write $P \Vdash \varphi(\sigma_1 \dots \sigma_n)$ for $1^P \Vdash \varphi(\sigma_1 \dots \sigma_n)$.

The key to forcing is to establish the Definability and Truth lemmas. The Definability lemma, much like Gödel's Completeness Theorem equating nonconstructive semantical validity with semiconstructive syntactical provability, says that the forcing relation is M-definable for each φ (as a property of $p, \sigma_1 \dots \sigma_n$). The Truth lemma says that P-generic G are in fact "generic" in the intuitive sense: If $\varphi(\sigma_1^G \dots \sigma_n^G)$ is true in $M[G]$ then for some $p \in G$, it is true in every $M[H], H$ P-generic and containing p.

Definability Lemma. For any φ, the relation "$p \Vdash \varphi(\sigma_1 \dots \sigma_n)$" is definable in M.

Truth Lemma. If G is P-generic over M then $M[G] \vDash \varphi(\sigma_1^G \dots \sigma_n^G) \longleftrightarrow \exists p \in G\ (p \Vdash \varphi(\sigma_1 \dots \sigma_n))$.

Our proof strategy for these lemmas is indirect: We define a relation $\Vdash^*$ for which the Definability Lemma is clear, prove the Truth Lemma for $\Vdash^*$ and finally show $\Vdash = \Vdash^*$.

Definition of $\Vdash^*$. We say that $D \subseteq P$ is *dense* $\leq p$ if $\forall q \leq p \exists r (r \leq q, r \in D)$.

1. $p \Vdash^* \sigma \in \tau$ iff $\{q | \exists \langle \pi, r \rangle \in \tau$ such that $q \leq r, q \Vdash^* \sigma = \pi\}$ is dense $\leq p$.

2. $p \Vdash^* \sigma = \tau$ iff for all $\langle \pi, r \rangle \in \sigma \cup \tau$, $p \Vdash^* (\pi \in \sigma \longleftrightarrow \pi \in \tau)$.

3. $p \Vdash^* \varphi \wedge \psi$ iff $p \Vdash^* \varphi$ and $p \Vdash^* \psi$.

4. $p \Vdash^* \sim \varphi$ iff $\forall q \leq p (\sim q \Vdash^* \varphi)$.

5. $p \Vdash^* \forall x \varphi$ iff for all names $\sigma, p \Vdash^* \varphi(\sigma)$.

Note that circularity is avoided in (a), (b) as max(Rank σ, Rank τ) goes down (in at most three steps) when these definitions are applied. Also all quantifiers in (a), (b) are bounded, as P is a set, so the above definition can be carried out in M and the Definability Lemma does hold for $\Vdash^*$.

Technical Lemma.

1. $p \Vdash^* \varphi, q \leq p \longrightarrow q \Vdash^* \varphi$.
2. If $\{q | q \Vdash^* \varphi\}$ is dense $\leq p$ then $p \Vdash^* \varphi$.
3. If $\sim p \Vdash^* \varphi$ then $\exists q \leq p$ $(q \Vdash^* \sim \varphi)$.

Proof.

1. Clear, by induction on φ, as dense $\leq p \longrightarrow$ dense $\leq q$.
2. Again by induction on φ. The proof uses the following facts: If $\{q | D$ is dense $\leq q\}$ is dense $\leq p$ then D is dense $\leq p$; if $\{q | q \Vdash^* \sim \varphi\}$ is dense $\leq p$ then $\forall q \leq p (\sim q \Vdash^* \varphi)$, using (a).
3. Immediate by (b). □

We are ready to prove the Truth Lemma for $\Vdash^*$.

Lemma 3.2. For G P-generic over M:

$$M[G] \models \varphi(\sigma_1^G \ldots \sigma_n^G) \longleftrightarrow \exists p \in G (p \Vdash^* \varphi(\sigma_1 \ldots \sigma_n)).$$

Proof. By induction on φ.

$\sigma \in \tau :(\longrightarrow)$ If $\sigma^G \in \tau^G$ then choose $\langle \pi, r \rangle \in \tau$ such that $\sigma^G = \pi^G$ and $r \in G$. By induction we can choose $p \in G$, $p \leq r$, $p \Vdash^* \sigma = \pi$. Then $p \Vdash^* \sigma \in \tau$. $(\longleftarrow)$ If $p \in G, \{q | \exists \langle \pi, r \rangle \in \tau$ such that $q \leq r$, $q \Vdash^* \sigma = \pi\} = D$ is dense $\leq p$ then by genericity we can choose $q \in G$, $\langle \pi, r \rangle \in \tau$ such that $q \leq r$, $q \Vdash^* \sigma = \pi$; then by induction $\sigma^G = \pi^G$ and as $r \geq q \in G$ we get $r \in G$ and hence by definition of $\tau^G, \pi^G \in \tau^G$. So $\sigma^G \in \tau^G$.

$\sigma = \tau :(\longrightarrow)$ Suppose $\sigma^G = \tau^G$. Consider $D = \{p |$ Either $p \Vdash^* \sigma = \tau$ or for some $\langle \pi, r \rangle \in \sigma \cup \tau$, $p \Vdash^* \sim (\pi \in \sigma \longleftrightarrow \pi \in \tau)\}$. Then D is dense, using the definition of $p \Vdash^* \sigma = \tau$. By genericity there is $p \in G \cap D$ and by induction it must be that $p \Vdash^* \sigma = \tau$. $(\longleftarrow)$ Suppose $p \in G$, $p \Vdash^* \sigma = \tau$. Then by induction, $\pi^G \in \sigma^G \longleftrightarrow \pi^G \in \tau^G$ for all $\langle \pi, r \rangle \in \sigma \cup \tau$. So $\sigma^G = \tau^G$.

$\varphi \wedge \psi$: Clear by induction, using the fact that $p, q \in G \longrightarrow \exists r \in G(r \leq p$ and $r \leq q)$.

$\sim \varphi$: Clear by induction, using the density of $\{p | p \Vdash^* \varphi$ or $p \Vdash^* \sim \varphi\}$.

$\forall x \varphi$:($\longrightarrow$) Suppose $M[G] \vDash \forall x \varphi$. As in the proof of ($\longrightarrow$) for $\sigma = \tau$, there is $p \in G$ such that either $p \Vdash^* \forall x \varphi$ or for some $\sigma, p \Vdash^* \sim \varphi(\sigma)$. By induction the latter is impossible so $p \Vdash^* \forall x \varphi$. ($\longleftarrow$) Clear by induction. □

Lemma 3.3. $\Vdash^* = \Vdash$.

Proof. $p \Vdash^* \varphi(\sigma_1 \ldots \sigma_n) \longrightarrow p \Vdash \varphi(\sigma_1 \ldots \sigma_n)$. And $\sim p \Vdash^* \varphi(\sigma_1 \ldots \sigma_n) \longrightarrow q \Vdash^* \sim \varphi(\sigma_1 \ldots \sigma_n)$ for some $q \leq p \longrightarrow \sim p \Vdash \varphi(\sigma_1 \ldots \sigma_n)$ using our Assumption about the existence of generics. □

ZFC and Cofinalities in $M[G]$

Theorem 3.4. If G is P-generic over M then $M[G]$ is a model of ZF. If M satisfies AC then so does $M[G]$.

Proof. As $M[G]$ is transitive and contains ω, it is a model of all ZF axioms with the possible exception of pairing, union, power and replacement.

For pairing, given σ_1^G, σ_2^G consider $\sigma = \{\langle \sigma_1, 1^P \rangle, \langle \sigma_2, 1^P \rangle\}$. Then $\sigma^G = \{\sigma_1^G, \sigma_2^G\}$.

For union, given σ^G consider $\pi = \{\langle \tau, p \rangle | p \Vdash \tau \in \cup\sigma, \text{Rank } \tau < \text{Rank } \sigma\}$. By the Truth Lemma, $\pi^G = (\cup\sigma^G) \cap \{\tau^G | \text{Rank } \tau < \text{Rank } \sigma\}$. As any element of $\cup\sigma^G$ is of the form $\tau^G, \text{Rank } \tau < \text{Rank } \sigma$ we get $\pi^G = \cup\sigma^G$.

For power, given σ^G consider $\pi = \{\langle \tau, p \rangle | p \Vdash \tau \subseteq \sigma, \text{Rank } \tau \leq \text{Rank } \sigma\}$. Then $\pi^G = \mathcal{P}(\sigma^G) \cap \{\tau^G | \text{Rank } \tau \leq \text{Rank } \sigma\}$. Now suppose that $\tau^G \subseteq \sigma^G$, with no restriction on Rank τ. Form the name τ^* by replacing each $\langle \tau_0, p \rangle \in \tau$ by all of the $\langle \tau_0^*, q \rangle$ such that $\text{Rank } \tau_0^* < \text{Rank } \sigma, q \leq p, q \Vdash \tau_0^* = \tau_0$. Then $\text{Rank } \tau^* \leq \text{Rank } \sigma$ and $\tau^{*G} = \tau^G$ since if $\langle \tau_0, p \rangle \in \tau$, $p \in G$ then $\tau_0^G \in \sigma^G$ and hence there is $q \leq p$, $q \in G$, $q \Vdash \tau_0 = \tau_0^*$ where $\text{Rank } \tau_0^* < \text{Rank } \sigma$; conversely, if $q \leq p, q \Vdash \tau_0^* = \tau_0$ and $q \in G$ then $p \in G$ and $\tau_0^{*G} = \tau_0^G$. So we conclude that $\pi^G = \mathcal{P}(\sigma^G) \cap M[G]$.

For replacement, given $f : \sigma^G \longrightarrow M[G], f$ definable (with parameters) in $M[G]$ consider $\pi_\alpha = \{\langle \tau, p \rangle | \text{Rank } \tau < \alpha$ and for some σ_0, $\text{Rank } \sigma_0 < \text{Rank } \sigma, p \Vdash \sigma_0 \in \sigma \wedge f(\sigma_0) = \tau\}$. Then $\pi_\alpha^G = \text{Range } (f) \cap \{\tau^G | \text{Rank } \tau < \alpha\}$.

Now choose $\alpha \in \text{ORD}(M)$ so large that if $p \in P$, Rank $\sigma_0 <$ Rank σ and $p \Vdash f(\sigma_0) = \tau$ for some τ, then there is such a τ of Rank $< \alpha$. This is possible by replacement in M. Then $\pi_\alpha^G =$ Range (f).

Finally if M satisfies AC, we can well-order σ^G in $M[G]$ by first choosing a well-ordering of names of Rank $<$ Rank σ in M, and then comparing $x, y \in \sigma^G$ by comparing the least names σ_x, σ_y such that $\sigma_x^G = x, \sigma_y^G = y$. $\square$

It does not follow that $M, M[G]$ have the same cardinals. We now turn to conditions on P which guarantee that cardinals (indeed, cofinalities) are preserved. Assume that AC holds in M and hence also in $M[G]$.

Definition. An *antichain* is a set $A \subseteq P$ such that $p \neq q$ in $A \longrightarrow p, q$ are incompatible. For regular, uncountable κ, P is κ-cc (κ*-chain condition*) if every antichain has cardinality $< \kappa$.

Lemma 3.5. If P is κ-cc in M and cof $(\alpha) \geq \kappa$ in M then cof $(\alpha) \geq \kappa$ in $M[G]$.

Proof. It suffices to show that if $f : \beta \longrightarrow \gamma$ belongs to $M[G]$ then there is $g : \beta \longrightarrow P(\gamma)$ in M such that for each $\beta_0 < \beta$, $f(\beta_0) \in g(\beta_0)$, Card $(g(\beta_0)) < \kappa$ in M. Let $\sigma^G = f$ and define g by $g(\beta_0) = \{\gamma_0 < \gamma | p \Vdash \sigma$ is a function and $\sigma(\hat{\beta}_0) = \hat{\gamma}_0$, for some $p\}$. $\square$

Definition. If $D \subseteq P$ and $p \in P$ then we say that p *meets* D if $p \leq q \in D$ for some q. For regular, uncountable κ, P is κ*-distributive* if whenever $p \in P$ and $\langle D_i | i < \beta \rangle$ are dense subsets of $P, \beta < \kappa$ then $\exists q \leq p$ (q meets each D_i).

Lemma 3.6. If P is κ-distributive in M and cof $(\alpha) \geq \kappa$ in M then cof $(\alpha) \geq \kappa$ in $M[G]$.

Proof. It suffices to show that if $f : \beta \longrightarrow \gamma$, $\beta < \kappa$ belongs to $M[G]$ then it belongs to M. Let $\sigma^G = f$ and note that for each $\beta_0 < \beta$, $D_{\beta_o} = \{q|$ For some $\gamma_0 < \gamma$, $q \Vdash \sigma$ a total function $\longrightarrow \sigma(\hat{\beta}_0) = \hat{\gamma}_0\}$ is dense. If $p \in G$, $p \Vdash \sigma$ total and p meets each D_{β_0} then $f(\beta_0) =$ unique $\gamma_0, p \Vdash \sigma(\hat{\beta}_0) = \hat{\gamma}_0$; so $f \in M$. $\square$

There is one more condition for cofinality preservation to consider, which is best motivated by an example. Suppose that κ is regular and that the ground model M is L. Let P consist of all functions p on $I = \{0\} \cup$ All infinite cardinals $< \kappa$ such that for all $\alpha \in I, p(\alpha)$ is a bounded subset of α^+ (we take $0^+ = \omega$). Order P by $p \leq q \longleftrightarrow$ For each $\alpha \in I, q(\alpha)$ is an initial segment of $p(\alpha)$. For inaccessible κ, P is neither κ^+-cc nor κ^+-distributive, yet "cofinality $> \kappa$" is preserved when forcing with P. This is because P is Δ-distributive at κ, a concept that we now define.

Definition. Let κ be regular. We say that $d \subseteq P$ is *predense* $\leq p$ if $q \leq p \longrightarrow q$ is compatible with an element of d. If $D \subseteq P$ is dense then p α^+*-reduces* D if there exists $d \subseteq D$, Card $(d) \leq \alpha^+, d$ predense $\leq p$. P is Δ*-distributive at* κ if whenever $\langle D_i | i < \kappa \rangle$ are dense subsets of P and $p \in P$, there is $q \leq p, q$ i^+-reduces D_i for each i. (We take $i^+ = \omega$ for finite i.)

Lemma 3.7. If P is Δ-distributive at κ in M and cof $(\alpha) \geq \kappa^+$ in M then cof $(\alpha) \geq \kappa^+$ in $M[G]$.

Proof. It suffices to show that if $f : \kappa \longrightarrow \gamma$ belongs to $M[G]$ then there is $g : \kappa \longrightarrow \mathcal{P}(\gamma)$ in M such that Card $(g(i)) \leq \kappa$, $f(i) \in g(i)$ for each $i < \kappa$. Let $\sigma^G = f$ and note that $D_i = \{p |$ For some $\bar{\gamma} < \gamma$, $p \Vdash \sigma$ total $\longrightarrow \sigma(\hat{i}) = \hat{\bar{\gamma}}\}$ is dense for each i. Let $p \in G, p \Vdash \sigma$ total, p i^+-reduces D_i for each i. Then the desired g is $g(i) = \{\bar{\gamma} < \gamma | q \Vdash \sigma(\hat{i}) = \hat{\bar{\gamma}}$ for some $q \leq p\}$. □

Corollary 3.8. If for some κ, P is either both κ-distributive and κ^+-cc, or both Δ-distributive at κ and κ^{++}-cc then P preserves cofinalities.

The above Lemmas are the basic tools for proving cofinality preservation.

GCH Preservation

Given that cofinalities are preserved, we can ask what further conditions we need on P to guarantee that GCH, if true in M, will remain true in $M[G]$ for P-generic G. The basic fact is the following.

Lemma 3.9. If $M \vDash 2^\kappa = \kappa^+$, $P \in M$ and either P is κ^+-distributive or P is κ^+-preserving, Card $(P) \leq \kappa^+$ then G P-generic over $M \longrightarrow M[G] \vDash 2^\kappa = \kappa^+$.

Proof. This is clear if P is κ^+-distributive as then $\mathcal{P}(\kappa)$ in $M[G] = \mathcal{P}(\kappa)$ in M. Now if P is a κ^+-preserving forcing of cardinality $\leq \kappa^+$ choose $f : P \xrightarrow{1-1} \kappa^+$ and let $P_\alpha = f^{-1}[\alpha]$ for $\alpha < \kappa^+$. If $\sigma^G \subseteq \kappa$ then there is $\alpha < \kappa^+$ such that for all $i < \kappa$, $i \in \sigma^G \longleftrightarrow \exists p \in P_\alpha \cap G(p \Vdash \hat{i} \in \sigma)$. Thus σ^G is uniquely determined by $\alpha, \langle S_i | i < \kappa\rangle$ where $\alpha < \kappa^+$, $S_i = \{p \in P_\alpha | p \Vdash \hat{i} \in \sigma\}$ and hence in $M[G]$ there are at most κ^+-many such σ^G. □

Cohen's Results

Theorem 3.10. If ZF is consistent then so is ZFC+ $\sim$ CH.

Proof. First suppose that ZF has a countable transitive model N; then so does ZFC for we can take $M = (L)^N$. Now take $P \in M$ to consist of all $p : F_p \longrightarrow 2$, F_p a finite subset of $\omega \times \aleph_2^M$, ordered by $p \leq q \longleftrightarrow p$ extends q as a function. If G is P-generic over M (such G exist by the assumption that M is countable) then $\cup G : \omega \times \aleph_2^M \longrightarrow 2$, since for each $(n, \alpha) \in \omega \times \aleph_2^M$ the set $D = \{p | (n, \alpha) \in F_p\}$ is dense. Also $\alpha < \beta < \aleph_2^M \longrightarrow G_\alpha \neq G_\beta$ where $G_\alpha(n) = (\cup G)(n, \alpha)$. So $M[G] \vDash \text{ZFC} + 2^{\aleph_0} \geq \aleph_2^M$. Thus to get $\sim$ CH in $M[G]$ we only need $\aleph_2^M = \aleph_2^{M[G]}$, which will follow if we can show that P is $\aleph_1$-cc in M.

Claim. P is $\aleph_1$-cc in M.

Suppose A were an uncountable antichain and choose F maximal so that $F \subseteq F_p$ for uncountably many $p \in A$. We may assume that $p \restriction F$ is constant for $p \in A$. But then for any $p \in A$ choose $p \neq q \in A$ such that $F_q \cap F_p = F$ and we see that p, q are compatible, contradiction.

Now to prove the Theorem notice the following: The above shows that if ZF_{n+17}(= ZF with only Σ_{n+17} Replacement) has a countable transitive model then so does ZF_n + AC+ $\sim$ CH. But in ZF we can prove that ZF_{n+17} has a countable transitive model, so if ZF + AC+ $\sim$ CH were inconsistent we would get an inconsistency in ZF. □

Theorem 3.11. If ZF is consistent then so is ZF+ $\sim$ AC.

Proof. As in the previous Theorem, it will suffice to show that if ZF$+V = L$ has a countable transitive model M then so does ZF+ $\sim$ AC. Let $P \in M$ be the pre-ordering of all $p : F_p \longrightarrow 2$ where F_p is a finite subset of $\omega \times \omega$, ordered by $p \leq q \longleftrightarrow p$ extends q. If G is P-generic over M then $\cup G : \omega \times \omega \longrightarrow 2$ and $n \neq m \longrightarrow G_n \neq G_m$ where $G_n(i) = (\cup G)(i, n)$.

For any $m, n \in \omega$ define $\pi_{mn} : P \longrightarrow P$ as follows: if $p \in P$ then $\pi_{mn}(p)$ agrees with p except it sends (i, m) to $p(i, n)$ and (i, n) to $p(i, m)$. Then

$G_{mn} = \{\pi_{mn}(p) | p \in G\}$ is P-generic over M and $M[G] = M[G_{mn}]$. It follows that if $f : \omega \longrightarrow S = \{G_n | n \in \omega\}$ is definable in $M[G]$ with parameters from $M \cup \{S, G_0, G_1, \ldots\}$ then Range (f) is finite: If the formula φ defining f does not have $f(k) = G_m$ as a parameter, choose $p \in G, p \Vdash f$ is a function, $f(k) = G_m$; then for large enough $n \geq m$, p and $\pi_{mn}(p)$ are compatible and together force $f(k)$ to equal both G_m and G_n, contradiction.

Let $N = \cup\{t \in M[G] | t$ transitive and $x \in t \longrightarrow x$ is definable in $M[G]$ with parameters from $M \cup \{S, G_0, G_1, \ldots\}\}$. We have shown that $f : \omega \longrightarrow S, f \in N \longrightarrow$ Range (f) finite and clearly $S \in N$. So we need only show that N is a model of ZF. Note that N is a transitive, definable (with parameter G) subclass of $M[G]$, since by the Reflection Principle, $N = \cup\{t \in M[G] | t$ transitive and $x \in t \longrightarrow$ for some $\alpha \in \text{ORD}(M), x$ is definable in $V_\alpha^{M[G]}$ with parameters from $M \cup \{S, G_0, G_1, \ldots\}\}$. The axioms of extensionality, foundation, empty and infinity obviously hold in N. Pairing and union hold as these are definable operations and the transitive closure (TC) of $\{x, y\}$ is $TC\{x\} \cup TC\{y\}$, $TC(\cup x) \subseteq TC(x)$. For power, use the definability of N to get $x \in N \longrightarrow P(x) \cap N \in N$. Finally, for replacement use replacement in $M[G]$ and the definability of N. □

Iterated Set-Forcing

First we consider two-step iteration.

Let P be a notion of forcing and $\dot{(Q)}$ a P-name such that $1^P \Vdash \dot{Q}$ is a pre-ordering. There is a notion of forcing $P * \dot{Q}$ with the property that forcing with $P * \dot{Q}$ is the same as first forcing with P and then in the extension by P forcing with $\dot{Q}$. We define:

$P * \dot{Q} = \{(p, q) \mid p \in P, \text{Rank } q < \text{Rank } \dot{Q} \text{ and } p \Vdash q \in \dot{Q}\}$
$(p_0, q_0) \leq (p_1, q_1)$ iff $p_0 \leq p_1$ and $p_0 \Vdash q_0 \leq q_1$.

Then $P * \dot{Q}$ is a pre-ordering, called the *two-step iteration of P and $\dot{Q}$*.

Lemma 3.12. Let G be P-generic over V and $Q = \dot{Q}^G$, a notion of forcing in $V[G]$. If H is Q-generic over $V[G]$ then

$$G * H = \{(p, q) \in P * \dot{Q} \mid p \in G \text{ and } p^G \in H\}$$

is $P * \dot{Q}$-generic over V and $V[G * H) = V[G][H]$.

Proof. If $D \in V[G]$ is dense on $P * \dot{Q}$ then define $D_1 = \{q^G \mid (p, q) \in D$ for some $p \in G\}$.

Claim. D_1 is dense in $Q = \dot{Q}^G$.

To prove the Claim, suppose that q^G belongs to Q, Rank $q <$ Rank $\dot{Q}$. Consider the set $\{p \in P \mid$ For some q_1, $p \Vdash q_1 \leq q_0$ and $(p, q_1) \in D\}$. Since D is dense in $P * \dot{Q}$, it follows that the latter set is dense in P. By the genericity of G there is $p \in G$ belonging to this set and therefore q^G has an extension in D_1.

Now since D_1 is dense and belongs to $V[G]$ it follows from the genericity of H that there is $q \in H$ belonging to D_1. But then there is $p \in G$ such that (p, q) belongs to D and to $P * \dot{Q}$, as desired. □

Lemma 3.13. Let K be $P * \dot{Q}$-generic over V. Then the set $G = \{p \in P \mid (p, q) \in K$ for some $q\}$ is P-generic over V and the set $H = \{q^G \mid (p, q \in K$ for some $p\}$ is $Q = \dot{Q}^G$-generic over $V[G]$. Moreover $K = G * H$.

Proof. If $D \in V$ is dense on P then $D_1 = \{(p, q) \mid p \in D\}$ is dense on $P * \dot{Q}$ and it follows that $D \cap G$ is nonempty. And, if $D \in V[G]$ is dense on Q we may choose a name $\dot{D}$ such that $\dot{D}^G = D$ and $1^P \Vdash \dot{D}$ is dense in $\dot{Q}$. Then $\{(p, q) \in P * \dot{Q} \mid p \Vdash q \in \dot{D}\}$ is dense in $P * \dot{Q}$ and it follows that $H \cap D$ is nonempty. The equality $K = G * H$ is clear, using the compatibility of K. □

It follows from the Lemmas that $V[G * H] = V[G][H]$.

Lemma 3.14. Let κ be regular. If P has the κ-cc and $1^P \Vdash \dot{Q}$ has the κ-cc then $P * \dot{Q}$ has the κ-cc.

Proof. Assume that (p_α, q_α), $\alpha < \kappa$ are mutually incompatible. Let G be P-generic over V and $Z = \{\alpha \mid p_\alpha \in G\}$. Whenever α and β belongs to Z, we have that q_α^G and q_β^G are incompatible in $Q = \dot{Q}^G$. As Q has the κ-cc in $V[G]$ it follows that Z has cardinality less than κ in $V[G]$. But as P has the κ-cc in V it follows that for some $\gamma < \kappa$, $1^P \Vdash Z$ is a subset of γ; but this contradicts the fact that $p_\gamma \Vdash \gamma \in Z$. □

Now we turn to transfinite iterations. We shall introduce sequences $\langle P_\beta \mid \beta < \alpha\rangle$ of forcing notions so that $P_{\beta+1} = P_\beta * \dot{Q}_\beta$ for $\beta < \alpha$. At limits we will take "direct limits".

Definition. Let α be a nonzero ordinal. P_α is an *iteration of length α with finite support* iff it is a set of α-sequences with the following properties:

(i) If $\alpha = 1$ then for some forcing notion Q_0, P_1 is the set of all sequences $\langle p(0)\rangle$ of length 1, where $p(0) \in Q_0$. And $\langle p(0)\rangle \leq \langle q(0)\rangle$ iff $p(0) \leq q(0)$.

(ii) If $\alpha = \beta + 1$ then $P_\beta = \{p \restriction \beta \mid p \in P_\alpha\}$ is an iteration of length β and there is some name $\dot{Q}_\beta$ such that $1^{P_\beta} \Vdash \dot{Q}_\beta$ is a forcing notion and:

$p \in P_\alpha$ iff $p \restriction \beta \in P_\beta$ and $1^{P_\beta} \Vdash p(\beta) \in \dot{Q}_\beta$

$p \leq q$ in P_α iff $p \restriction \beta \leq q \restriction \beta$ in P_β and $p \restriction \beta \Vdash p(\beta) \leq q(\beta)$.

(iii) If α is a limit ordinal then for all $\beta < \alpha$, $P_\beta = \{p \restriction \beta \mid p \in P_\alpha\}$ is an iteration of length β and:

$p \in P_\alpha$ iff $p \restriction \beta \in P_\beta$ for all $\beta < \alpha$ and

$1^{P_\beta} \Vdash p(\beta) = 1^{\dot{Q}_\beta}$ for all but finitely many $\beta < \alpha$

Also: $p \leq q$ in P_α iff $p \restriction \beta \leq q \restriction \beta$ in P_β for all $\beta < \alpha$.

Notation. $\leq_\beta$ denotes the ordering of P_β, $\Vdash_\beta$ denotes the forcing relation of P_β and $\Vdash_\beta \varphi$ denotes $1^{P_\beta} \Vdash_\beta \varphi$. An easy exercise is the following.

Fact. If G is P_α-generic over V then for $\beta < \alpha$, $G \restriction \beta = \{p \restriction \beta \mid p \in G\}$ is P_β-generic over V.

Theorem 3.15. Let P_α result from the iteration of finite support of $\langle \dot{Q}_\beta \mid \beta < \alpha\rangle$. If $\Vdash_\beta \dot{Q}_\beta$ has the $\aleph_1$-cc for each $\beta < \alpha$ then P_α has the $\aleph_1$-cc.

Proof. By induction on α. If $\alpha = \beta + 1$ then $P_\alpha = P_\beta * \dot{Q}_\beta$ and the result follows from our earlier Lemma. Now suppose that α is a limit ordinal and for each $p \in P_\alpha$ let supp (p) denote the support of p, i.e. the set of $\beta < \alpha$ such that $p(\beta) \neq 1^P_\beta$.

Case 1. cof $\alpha \neq \aleph_1$. Let $W \subseteq P_\alpha$ be a set of size $\aleph_1$. Since cof $\alpha \neq \aleph_1$ there is a $\beta < \alpha$ and $Z \subseteq W$ of size $\aleph_1$ such that supp $(p) \subseteq \beta$ for all $p \in Z$. Then $\{p \restriction \beta \mid p \in Z\}$ is a set of size $\aleph_1$ in P_β and since by induction P_β has the $\aleph_1$-cc there are p and q in Z such that $p \restriction \beta$ and $q \restriction \beta$ are compatible in $P\beta$. But then p and q are compatible. So W is not an antichain.

Case 2. cof $\alpha = \aleph_1$. Let $\langle \alpha_\xi \mid \xi < \aleph_1\rangle$ be a continuous increasing sequence with limit α and $W = \{p_\xi \mid \xi < \aleph_1\}$ a subset of P_α of size $\aleph_1$. For each limit $\xi < \aleph_1$ there is $\gamma(\xi) < \xi$ such that supp $(p) \cap \alpha_x i \subseteq \alpha_{\gamma(\xi)}$. By Fodor's Theorem there is a stationary $S \subseteq \aleph_1$ and some $\gamma < \aleph_1$ such that supp $(p_\xi) \cap \alpha_\xi \subseteq \alpha_\gamma$

for all $\xi \in S$. Also we can construct an uncountable set $Z \subseteq S$ so that for any $\xi < \eta$ in Z, supp $(p_\xi) \subseteq \alpha_\eta$.

Now consider the set $\{p_\xi \restriction \alpha_\gamma \mid \xi \in Z\}$. This is an uncountable subset of P_{α_γ} and so there are $\xi < \eta$ in Z such that $p_\xi \restriction \alpha_\gamma$ and $p_\eta \restriction \alpha_\gamma$ are compatible. Let $q \in P_{\alpha_\gamma}$ be stronger than both of these conditions. Now define $r \in P_\alpha$ as follows.

$r(i) = q(i)$ if $i < \alpha$
$r(i) = p_\xi(i)$ if $\alpha_\gamma \leq i < \alpha_\eta$
$r(i) = p_\eta(i)$ if $\alpha_\eta \leq i < \alpha$.

Then r is stronger than both p_ξ and p_η and therefore p_ξ and p_η are compatible. So W is not an antichain. □

Suslin's Problem

Suslin asked whether there is a complete, dense linear ordering without endpoints, without an uncountable set of pairwise disjoint intervals and not isomorphic to the real line. It turned out the answer is Yes in L, but the answer is No in an extension of L obtainable through iteration with finite support.

An equivalent version of Suslin's question is the following: Is there a Suslin Tree? The latter is an uncountable partially-ordered set $(T, <_T)$ such that the predecessors of each element of T are well-ordered by $<_T$ and $(T, <_T)$ has no uncountable chain or antichain.

Notice that a Suslin tree is a partial-ordering and therefore can be used as a forcing notion. If T is a Suslin tree with the property that each $t \in T$ has uncountably many extensions in T, then forcing with T adds an $\aleph_1$-branch through T and therefore T will not be Suslin in the generic extension.

Theorem 3.16. In L, there is an iteration with finite support P of length $\aleph_2$ such that if G is P-generic over L then in $L[G]$ there are no Suslin trees.

Proof. We construct P as the iteration of $\langle \dot{Q}_\alpha \mid \alpha < \aleph_2 \rangle$ where at each stage $\Vdash_\alpha \dot{Q}_\alpha$ is $\aleph_1$-cc. Thus P is also $\aleph_1$-cc and all cofinalities are preserved.

We define $\dot{Q}_\alpha$ by induction on $\alpha < \aleph_2$. Fix a function π mapping $\aleph_2$ onto $\aleph_2 \times \aleph_2$ so that if $\pi(\alpha) = (\beta, \gamma)$ then β, $\gamma \leq \alpha$. Assuming for the moment that P_α is an $\aleph_1$-cc forcing of size $\leq \aleph_1$, it follows $\Vdash_\alpha 2^{\aleph_1} = \aleph_2$ and therefore there at most $\aleph_2$ nonisomorphic Suslin trees in a P_α-generic extension. Since P_α is $\aleph_1$-cc there are at most $\aleph_2$ P_α-names for Suslin trees. Let $\pi(\alpha)$ be (β, γ). Then $\dot{Q}_\alpha$ is defined to be the γ-th P_β-name for a Suslin tree.

We assumed that P_α is an $\aleph_1$-cc forcing of size $\leq \aleph_1$ for each $\alpha < \aleph_2$. We now prove this inductively. Clearly it holds for limit stages since we are taking direct limits. At the successor stage $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$ we have $\Vdash \dot{Q}_\alpha$ has cardinality $\aleph_1$, as $\dot{Q}_\alpha$ is a name for a Suslin tree. Every name for an element of $\dot{Q}_\alpha$ can be represented as a function from an antichain of P_α into $\aleph_1$, and since P_α is $\aleph_1$-cc there are at most $\aleph_1^{\aleph_0} = \aleph_1$ such names. It follows that $P_{\alpha+1}$ has size at most $\aleph_1$.

Now we claim that there are no Suslin trees in a P-generic extension $L[G]$. Let G_α denote $G \restriction P_\alpha$ for each $\alpha < \aleph_2$.

Claim. If X is a subset of $\aleph_1$ in $L[G]$ then $X \in L[G_\alpha]$ for some $\alpha < \aleph_2$.

Proof of Claim. A name for X is determined by an $\aleph_1$-sequence of maximal antichains, and therefore by the $\aleph_1$-cc, by a name of size $\aleph_1$.

Now suppose there were a Suslin tree in $L[G]$. Then there would be a Suslin tree T with the property that each $t \in T$ has uncountably many extensions in T. By the Claim we can assume that T belongs to $L[G_\alpha]$ for some $\alpha < \aleph_2$ and therefore by construction at some stage β of the iteration, we force with T. But then T is not Suslin in $L[G]$, contradiction. □

Countable Support Iteration

Iterations with countable support are defined just like iterations with finite support, but with the condition at limit stages α given as follows:

$p \in P_\alpha$ iff $p \restriction \beta \in P_\beta$ for all $\beta < \alpha$ and
$1^{P_\beta} \Vdash p(\beta) = 1^{\dot{Q}_\beta}$ for all but *countably* many $\beta < \alpha$.

This type of iteration is needed when one wishes to use forcings which are not $\aleph_1$-cc. Typically one performs an iteration of length $\aleph_2$, using forcings of size $\aleph_1$. To show that cardinals above $\aleph_1$ are preserved one uses:

Proposition 3.17. Let P be a countable support iteration of length $\aleph_2$ such that for $\beta < \aleph_2$, $P \upharpoonright \beta$ has a dense subset of size at most $\aleph_1$. Then P has the $\aleph_2$-cc.

Proof. If $\langle p_\xi \mid \xi < \aleph_2 \rangle$ are conditions in P then there is a stationary set $S \subseteq \aleph_2$ consisting of ordinals of uncountable cofinality such that for $\xi \in S$, supp $(p_\xi) \cap \xi$ is bounded by a fixed ordinal $\gamma < \aleph_2$. But then we can choose two conditions p_ξ and p_η whose restrictions to γ are compatible and whose supports above γ are disjoint. It follows that these conditions are compatible and therefore the original sequence cannot enumerate the elements of an antichain. $\square$

How does one show that $\aleph_1$ is preserved in a countable support iteration? Shelah isolated a condition on the forcings used in the iteration, called *properness*, which guarantees preservation of $\aleph_1$ and is preserved through countable support iteration.

Definition. P is *proper* iff player II has a winning strategy in the following game: Player I begins by selecting a condition p and choosing a name $\dot{A}_0$ for a countable set of ordinals. Player II chooses an ordinal β_0. At the n-th move, I plays a name $\dot{A}_n$ for a countable set of ordinals and II plays an ordinal β_n. Now II wins the game iff for some $q \leq p$:

$$(*)\ q \Vdash \text{For all } n \text{ and } \alpha \text{ in } \dot{A}_n,\ \alpha = \beta_k \text{ for some } k.$$

Notice that if II has a winning strategy in the above game, then every countable set of ordinals in a P-generic extension of V is a subset of a set of ordinals which is countable in V. Thus properness implies that $\aleph_1$ is preserved.

Theorem 3.18. Let P_γ be a countable support iteration of length γ of $\dot{Q}_\beta$, $\beta < \gamma$ such that for every $\beta < \gamma$, $\Vdash_\beta \dot{Q}_\beta$ is proper. Then P_γ is proper.

Proof. We actually prove something stronger than stated, to facilitate an inductive argument. A winning strategy σ for II in the properness game is *good* iff for every sequence of moves p, $\dot{A}_0$, …, $\dot{A}_n$, … of player I, σ produces a sequence $\langle \beta_n \mid n \in \omega \rangle$ such that for some $q \leq p$ obeying $(*)$ above: supp $(q) \subseteq \{\beta_n \mid n \in \omega\}$.

Claim. (a) For all $\eta < \gamma$, $\Vdash_\eta$ II has a good winning strategy in the proper game for $P_{\eta\gamma} = \{p \restriction [\eta, \gamma) \mid p \in P_\gamma\}$.
(b) Suppose that γ has cofinality ω and $\langle \gamma_n \mid n \in \omega \rangle$ is an increasing sequence cofinal in γ. Let $R_0 = P_{\gamma_0}$ and $\dot{R}_{n+1} = P_{\gamma_n \gamma_{n+1}}$ for each $n \in \omega$. Then P is equivalent to the ω-iteration of the $\dot{R}_n$'s.

To treat the case of the Claim (a) when γ is a successor ordinal we need:

Lemma 3.19. Suppose that P is proper and $\Vdash_P \dot{Q}$ proper. Then $P * \dot{Q}$ is proper.

Proof of Lemma. It is not difficult to show that in the definition of properness, we can equally well use the game where I plays names for single ordinals, rather than countable sets of ordinals. We shall prove the lemma using this modified version of the game.

Let σ be a winning strategy for II in the game on P and let $\dot{\tau}$ be such that $\Vdash_P \dot{\tau}$ is a winning strategy for II on $\dot{Q}$. We describe a winning strategy for II on $P*\dot{Q}$: Player I starts by selecting a condition $(p, \dot{q}) \in P*\dot{Q}$ and a name $\dot{\alpha}_0$ for an ordinal. We describe II's response, an ordinal γ_0. The $P * \dot{Q}$-name $\dot{\alpha}_0$ can be identified with a P-name for a $\dot{Q}$ name. Apply II's strategy $\dot{\tau}$ in the $\dot{Q}$-game where I begins with $\dot{q}$ and $\dot{\alpha}_0$. Let $\dot{\beta}_0$ be II's response. Now consider the game on P and use σ to respond when I plays p and $\dot{\beta}_0$. The result is γ_0.

At the nth move, I plays a $P * \dot{Q}$-name $\dot{\alpha}_n$. Identify $\dot{\alpha}_n$ with a P-name for a $\dot{Q}$-name and apply $\dot{\tau}$ to get $\dot{\beta}_n$. Now in the game on P we use σ to produce an ordinal γ_n, when I plays $\dot{\beta}_n$ as his nth move.

Since $\dot{\tau}$ is a winning strategy we have

$$p \Vdash_P \exists q' \leq \dot{q},\ q' \Vdash_Q \forall n \exists m\ \dot{\alpha}_n = \dot{\beta}_m.$$

Therefore there is a $\dot{q}'$ such that $p \Vdash \dot{q}' \leq \dot{q}$ and

$$(p, \dot{q}) \Vdash \forall n \exists m\ \dot{\alpha}_n = \dot{\beta}_m.$$

Since σ is a winning strategy, there is $p' \leq p$ such that

$$p' \Vdash \forall m \exists k\ \dot{\beta}_m = \gamma_k.$$

Putting this together we get

$$(p', \dot{q}') \Vdash \forall n \exists k \ \dot{\alpha}_n = \gamma_k,$$

and therefore the strategy described above is a winning strategy for II in the game on $P * \dot{Q}$. This proves the Lemma.

Notice that the proof of the Lemma shows that if II has a good winning strategy in the game on $P_{\eta\gamma}$ and $\Vdash_{\eta\gamma} \dot{Q}_\gamma$ is proper, then II has a good winning strategy in the game on $P_{\eta\gamma+1}$, the successor step in the proof of the Claim, part (a).

Now we prove the Claim, part (a) for limit γ. It suffices to prove it when $\eta = 0$: When $\eta < \gamma$ is arbitrary, we will be able to carry out the same proof in a generic extension via P_η, because by the Claim, part (b), $P_{\eta\gamma}$ is a countable support iteration of proper forcings in this generic extension.

If γ has cofinality ω, fix an increasing ω-sequence $\langle \gamma_i \mid i < \omega \rangle$ cofinal in γ.

Player I starts the game on P_γ by selecting $p \in P_\gamma$ and a P_γ-name $\dot{\alpha}_0$. If γ has uncountable cofinality we choose some $\gamma_0 < \gamma$ and consider $R_0 = P_{\gamma_0}$. (Otherwise γ_0 has already been chosen.) Let $p_0 = p \restriction \gamma_0$. There are R_0-names $\dot{\alpha}_0^0$ and $\dot{s}_0$ such that $p_0 \Vdash_{R_0} \dot{s}_0 \leq p \restriction [\gamma_0, \gamma)$ and $(p_0, \dot{s}_0) \Vdash \dot{\alpha}_0^0 = \dot{\alpha}_0$. We start the game on R_0 by letting I play p_0 and an R_0-name for the countable set $\{\dot{\alpha}_0\} \cup \mathrm{supp}\ (\dot{s}_0)$. Player II uses a good winning strategy σ_0 to return an ordinal β_0. This is II's first move in the game on P_γ.

At the nth move I chooses a P_γ-name $\dot{\alpha}_n$. If γ has uncountable cofinality we choose some $\gamma_n \in (\gamma_{n-1}, \gamma)$ greater than β_{n-1}. Let $\dot{R}_n = P_{\gamma_{n-1}\gamma_n}$. Let $\dot{p}_n$ be a name for a condition in $\dot{R}_n$ so that $\langle p_0, \dots, \dot{p}_{n-1} \rangle \Vdash \dot{p}_n = \dot{s}_{n-1} \restriction [\gamma_{n-1}, \gamma_n)$. There are $R_0 * \cdots * \dot{R}_n$-names $\dot{\alpha}_n^n$ and $\dot{s}_n$ such that $\langle p_0, \dots, \dot{p}_n \rangle \Vdash \dot{s}_n \in P_{\gamma_n\gamma}$, $\dot{s}_n \leq \dot{s}_{n-1} \restriction [\gamma_n, \gamma)$ and $(\langle p_0, \dots, \dot{p}_n \rangle, \dot{s}_n) \Vdash \dot{\alpha}_n = \dot{\alpha}_n^n$. We start the game on $\dot{R}_n$ by letting I play $\dot{p}_n$ and $\dot{A}_n^n$, where $\dot{A}_n^n = \{\dot{\alpha}_n^n\} \cup \mathrm{supp}\ (\dot{s}_n)$.

II uses a good winning strategy $\dot{\sigma}_n$ to play $\dot{\alpha}_n^{n-1}$. Then we continue the $\dot{R}_{n-1}$-game by letting I play $\dot{\alpha}_n^{n-1}$, to which II responds $\dot{\alpha}_n^{n-2}$. And so on, until II plays (by σ_0 in the R_0-game) an ordinal β_n.

It remains to show that the strategy described above is a good winning strategy for II in the P_γ-game. Let $\gamma_\infty = \lim_n \gamma_n$ and $S = \{\beta_n \mid n \in \omega\}$. We can obtain a sequence $p = \langle \dot{q}_n \mid n \in \omega \rangle$ in the ω-iteration of the $\dot{R}_n$'s such that $q \leq \langle \dot{p}_n \mid n \in \omega \rangle$ and

$$q \Vdash \forall n \exists k \ \dot{\alpha}_n^n = \beta_k.$$

Since the $\dot{\sigma}_n$ are good winning strategies, it follows that R and P_{γ_∞} are equivalent forcings, and that q is a condition in P_{γ_∞} with supp $q \subseteq S$. Let us identify q with $q * \langle 111 \cdots \rangle \in P_\gamma$ (if $\gamma_\infty < \gamma$). Since $S \subseteq \gamma_\infty$ and for every n, $q \restriction n \Vdash$ supp $\dot{s}_n \subseteq S$, we have

$$q \leq (\langle p_0, \ldots, \dot{p}_n \rangle, \dot{s}_n).$$

It follows that $q \leq p$, $q \Vdash \forall n \exists k \ \dot{\alpha}_n = \beta_k$ and supp $q \subseteq S$. Hence the strategy given is a good winning strategy, as desired. This proves the Claim, part (a).

We now prove the Claim, part (b). Let $\gamma = \lim_n \gamma_n$ and let P_γ be the proper iteration of length γ of $\langle \dot{Q}_\xi \mid \xi < \gamma \rangle$. For each n let $\dot{R}_n = P_{\gamma_{n-1}\gamma_n}$ (and $R_0 = P_{\gamma_0}$). Let R be the ω-iteration of the $\dot{R}_n$. We want to show that R and P_γ are equivalent forcings. For any $p \in P_\gamma$ let $r = \langle r_n \mid n \in \omega \rangle$ where $r_n = p \restriction [\gamma_{n-1}, \gamma_n)$. Thus P_γ embeds into R; it suffices to show that P_γ embeds into R densely.

Thus let $r = \langle \dot{r}_n \mid n \in \omega \rangle$ be a condition in R. We wish to find $p \in P$ such that $p \leq r$. By the induction hypothesis, each $\dot{R}_n$ has a good winning strategy $\dot{\sigma}_n$. We use these good strategies to produce p.

Play the proper games on the $\dot{R}_n$ simultaneously for all $n \in \omega$. The game on $\dot{R}_n$ begins with the condition $\dot{r}_n$. The moves of I are names for countable sets of ordinals; the moves of II are according to the strategy $\dot{\sigma}_n$.

At step 0, start the game for R_0. I plays r_0 and a name $\dot{A}_0^0$ for the support of $\dot{r}_1$. II responds with β_0. At step 1 we start the game on $\dot{R}_1$ in an R_0-generic extension. I plays $\dot{r}_1$ and a name $\dot{A}_1^1$ for the support of $\dot{r}_2$. II responds with $\dot{\alpha}_1^0$. Continue the game on R_0: I plays $\dot{\alpha}_1^0$ and II responds with β_1. At step n, we start the game on $\dot{R}_n$ in a $R_0 * \cdots * R_{n-1}$-generic extension. I plays $\dot{r}_n$ and a name $\dot{A}_n^n$ for the support of $\dot{r}_{n+1}$. II responds with $\dot{\alpha}_n^{n-1}$. Then, playing the game on $\dot{R}_{n-1}$, I plays $\dot{\alpha}_n^{n-1}$ and II responds with $\dot{\alpha}_n^{n-2}$. And so on, until II plays β_n in the game on R_0.

Since the $\dot{\sigma}_n$ are good winning strategies there exists a condition $q = \langle \dot{q}_n \mid n \in \omega \rangle \in R$, stronger than $\langle \dot{r}_n \mid n \in \omega \rangle$, such that for each n, $q \restriction n$ forces:

$\dot{q}_n \Vdash \forall \alpha$ played by I $\exists \beta$ played by II such that $\alpha = \beta$, and the support of $\dot{q}_n$ is included in the set of all ordinals played by II.

Let $S_{\{}\beta_n \mid n \in \omega\}$. It follows jthat for every n, $q \restriction n \Vdash$ supp $(\dot{q}_n) \subseteq S$. We conclude the proof by constructing a condition $p \in P_\gamma$ so that $p = q$ (under the embedding of P into R). This we do by induction on $\xi < \gamma$: If $\xi \notin S$ we let $p(\xi) = 1$ and if $\xi \in S$ then we let $p(\xi)$ be the condition $\dot{t} \in \dot{Q}_\xi$ so that $p \restriction \xi \Vdash \dot{t} = \dot{q}_n(\xi)$, where n is the unique n for which $\gamma_{n-1} \leq \xi < \gamma_n$. For each n we have $p \restriction \gamma_{n-1} \Vdash p \restriction [\gamma_{n-1}, \gamma_n) = \dot{q}_n$ and so $p = q$. $\square$

The Borel Conjecture

Properness can be used to establish the consistency of Borel's Conjecture concerning sets of strong measure 0. Let X be a subset of $[0,1]$. X has *strong measure* 0 if for every sequence $\langle \epsilon_n \mid n \in \omega \rangle$ of positive reals there exists a sequence $\langle I_n \mid n \in \omega$ of intervals with length $I_n \leq \epsilon_n$ such that $X \subseteq \cup_n I_n$. Borel conjectured that strong measure 0 sets are in fact countable. This contradicts CH, but Laver proved the consistency of Borel's Conjecture using a countable support iteration of *Laver forcing.*

Laver forcing is defined as follows. A set $p \subseteq \omega^{<\omega}$ is a *tree* iff it is closed under initial segments. A tree p is a *Laver tree* iff for some $s \in p$ (called the *stem* of p):

1. For all $t \in p$ either $t \subseteq s$ or $s \subseteq t$.
2. For all $t \in p$ extending s the set $S(t) = \{a \mid t * a \in p\}$ (the set of successors of t in p) is infinite.

Laver forcing consists of all Laver trees, partially ordered by inclusion. If G is generic then $f = \bigcup\{s \mid s \text{ is the stem of some } p \in G\}$ is a function from ω into ω, a *Laver real.* It is easy to show that $V[G] = V[f]$.

By an earlier Proposition, if we iterate Laver forcing for $\aleph_2$ steps over L, we will have the $\aleph_2$-cc and therefore preserve all cofinalities greater than $\aleph_1$. To show that this iteration preserves cofinality ω_1 it suffices to show that Laver forcing is proper.

Lemma 3.20. Laver forcing is proper.

Proof. Define the relations $\leq_n$ as follows. Consider a canonical enumeration of $\omega^{<\omega}$ in which s appears before t when $s \subseteq t$ and $s * a$ appears before $s * (a+1)$ for $a \in \omega$. If p is a Laver tree then part of p above the stem is isomorphic to $\omega^{<\omega}$ and so we have a canonical enumeration of it $\langle s^p_i \mid i \in \omega \rangle$, where s^p_0 is the stem of p. Note that if $q \leq p$ and $s^q_n = s^p_m$ then $n \leq m$. We define:

$q \leq_n p$ iff p and q have the same stem and $s^p_i = s^q_i$ for all $i \leq n$.

It is easy to show that if $p_0 \geq_0 p_1 \geq_1 p_2 \geq_2 \ldots$ then $p = \bigcap_n p_n$ is a Laver tree, called the *fusion of the fusion sequence* $\langle p_n \mid n \in \omega \rangle$.

Fact. If $p \Vdash \dot\alpha \in \mathrm{ORD}$ then there are $q \leq_n p$ and a countable $A \subseteq \mathrm{ORD}$ such that $q \Vdash \dot\alpha \in A$.

Proof of Fact. We assume that $n = 0$, as the proof for general n is almost the same. If p is a Laver tree, $n \in \omega$, $q \leq p$ and the stem of t is maximal among $\{s^p_0, \ldots, s^p_n\}$ then

$$r = q \cup \{u \in p \mid u \not\subseteq t \text{ and } t \not\subseteq u\}$$

is a Laver tree $\leq_n p$, called *the n-amalgamation of q into p*. This has the obvious generalisation to *the amalgamation of* $\{q_1, \ldots, q_k\}$ *into* p when the q_i extend p and their stems are all the maximal nodes among $\{s^p_0, \ldots, s^p_n\}$ (for a uniquely determined n).

We construct a fusion sequence $\langle p_n \mid n \in \omega \rangle$ with $p_0 = p$ and finite sets A_n so that the fusion of this sequence forces $\dot\alpha \in \bigcup_n A_n$. At stage n we already have p_n. Let $t_1, \ldots, t_k$ be all the maximal nodes among $s^{p_n}_0, \ldots, s^{p_n}_n$. For each $i \in \{1, \ldots, k\}$ if there exists $q_i \leq p_n$ with stem t_i and an ordinal α^i_n so that $q_i \Vdash \dot\alpha = \alpha^i_n$ then we choose such q_i and α^i_n. Let A_n be the collection of all the α^i_n chosen and let p_{n+1} be the amalgamation of $\{q_1, \ldots, q_k\}$ into p_n. (If q_i did not exist, then we take it to be the collection of nodes in p_n compatible with t_i.) We have $p_{n+1} \leq_n p_n$.

Let p_∞ be the fusion of the p_n's and $A = \bigcup_n A_n$. To prove that $p_\infty \Vdash \dot\alpha \in A$, let $q \leq p_\infty$. There are a condition $\bar q \leq q$ and $\alpha \in \mathrm{ORD}$ such that $\bar q \Vdash \dot\alpha = \alpha$. Let n be large enough so that the stem of $\bar q$ is among $K = \{s^{p_n}_0, \ldots, s^{p_n}_n\}$. There is $t \in \bar q$ that is a maximal node in K and therefore one of the nodes considered at stage n, say $t = t_i$. Let r consist of those nodes of $\bar q$ which are compatible with t. As r and α satisfy the requirements for choosing q_i in the definition of p_{n+1} we indeed have chosen q_i and α^i_n. Because $r \leq q_i$ it must be the case that $\alpha = \alpha^i_n$ and so $r \Vdash \dot\alpha \in A$. So by a density argument, $p_\infty \Vdash \dot\alpha \in A$. This proves the Fact.

Now we can show that II wins the proper game for Laver forcing (in the version where I plays a condition p and names for single ordinals, II plays countable sets of ordinals and II wins iff there is $q \leq p$ which forces all the names to be in the union of the sets played). At the start of the game let I select p_0 and the ordinal name $\dot{\alpha}_0$. By the Fact there is $p_1 \leq_0 p_0$ and a countable B_0 such that $p_1 \Vdash \dot{\alpha} \in B_0$. At the nth move, when I plays $\dot{\alpha}_n$ there are $p_{n+1} \leq_n p_n$ and a countable set B_n with $p_{n+1} \Vdash \dot{\alpha}_n \in B_n$. Then the fusion of the p_n's verifies that II wins the game. □

The Main Lemma needed to verify that Borel's Conjecture holds in the Laver model (obtained via an $\aleph_2$-iteration of Laver forcing over L) is the following.

Main Lemma. If GCH holds in V and X is an uncountable set of reals in V then X does not have strong measure 0 in $V[G]$ where G is generic over V for the $\aleph_2$-iteration of Laver forcing.

We content ourselves with a proof of the following simpler version.

Theorem 3.21. If GCH holds in V and X is an uncountable set of reals in V then X does not have strong measure 0 in $V[G]$ where G is generic over V for (a single application of)Laver forcing.

Proof. We show that if f is the Laver real and $\epsilon_n = I/f(n)$ then for some n_0, X cannot be covered by intervals of lengths $\epsilon_{n_0}, \epsilon_{n_0+1}, \ldots$.

Lemma 3.22. Let $p \Vdash \varphi_1 \vee \cdots \vee \varphi_k$. Then there is $q \leq p$ with the same stem as p such that $p \Vdash \varphi_i$ for some i.

Proof of Lemma 3.22. Recall that for $t \in p$, $S(t)$ denotes the set of $a \in \omega$ such that $t * a$ belongs to p. Let s be the stem of p and assume that the Lemma fails. Then there are only finitely many $a \in S(s)$ such that some $q \leq_0 p \restriction s*a$ has the desired property. By removing these finitely many nodes and their extensions, we get $p_1 \leq_0 p$. For each $s * a \in p_1$ there are only finitely many $b \in S(s * a)$ such that some $q \leq_0 p_1 \restriction s * a * b$ has the desired property. By removing all such b and their extensions we get $p_2 \leq_1 p_1$. Continue in this way to form a fusion sequence with limit r. Then if $t \in r$ there is no $q \leq_0 r \restriction t$ with the desired property. But then no extension of r forces any φ_i, a contradiction.

Lemma 3.23. Let p be a condition with stem s and $\dot{x}$ a name for a real. Then there is $q \leq_0 p$ and a real u such that for every $\epsilon > 0$, for all but finitely many $a \in S(s)$,

$$q \restriction s * a \Vdash |\dot{x} - u| < \epsilon.$$

Proof of Lemma 3.23. Let $\{t_n \mid n \in \omega\}$ be an enumeration of $\{s * a \mid a \in S(s)\}$. For each n we can choose $q_n \leq_0 p \restriction t_n$ and an interval $J_n = [m/n, (m+1)/n]$ so that $q_n \Vdash \dot{x} \in J_n$. There is a sequence $\langle k_n \mid n \in \omega\rangle$ so that the J_{k_n}'s form a decreasing sequence converging to a unique real u. Let $q = \cup_n q_{k_n}$. Then q is as desired.

Lemma 3.24. Let p be a condition with stem s and let $\langle \dot{x}_n \mid n \in \omega\rangle$ be a sequence of names for reals. Then thre is $q \leq_0 p$ and a set of reals $\{u_t \mid t \in q, t \supseteq s\}$ such that for every $\epsilon > 0$ and every $t \in q$ extending s, for all but finitely many $a \in S(t)$:

$$q \restriction t + a \Vdash |\dot{x}_k - u_t| < \epsilon.$$

where $k = \text{length } t - \text{length } s$.

Proof of Lemma 3.24. By repeated application of Lemma 3.23. First we get $p_1 \leq_0 p$ and u_s. Then for every immediate successor t of s in p_1 we get $q_t \leq_0 p_1 \restriction t$ and u_t; let $p_2 = \cup_t q_t$. Continue to get a fusion sequence $p \geq_0 p_1 \geq_1 p_2 \geq_2 \ldots$ and let $q = \cap_n p_n$.

We are now ready to prove the Theorem. Let $X \in V$ be a subset of $[0,1]$ and $p \Vdash X$ has strong measure 0. We show that X is countable. Let s be the stem of p, of length n. Let f be the Laver real. Consider the sequence $\epsilon_k = 1/f(k)$, $k \geq n$. There exists a sequence of intervals $\dot{I}_k$, $k \geq n$ of length ϵ_k so that $X \subseteq \bigcup_{k \geq n} \dot{I}_k$. For each $k \geq n$ let $\dot{x}_k$ be the center of I_k.

Let $q \leq_0 p$ be a condition obtained by Lemma 3 applied to p and $\langle \dot{x}_k \mid k \geq n\rangle$ and let $\{u_t \mid t \in q, q \supseteq s\}$ be the resulting reals. We shall show that $X \subseteq \{u_t \mid t \in q, q \supseteq s\}$.

Let $v \notin \{u_t \mid t \in q, q \supseteq s\}$. Since $p \Vdash X \subseteq \bigcup_{k \geq n} \dot{I}_k$ it suffices to find some $r \leq q$ such that $r \Vdash v \notin \dot{I}_k$ for all $k \geq n$. We construct r by induction on the levels of q; at stage $k \geq n$ we guarantee that $r \Vdash v \notin \dot{I}_k$.

The first step is as follows: Let $\epsilon = (1/2) \cdot |v - u_s|$. For all but finitely many $a \in S(s)$, $q \restriction s * a \Vdash |\dot{x}_n - u_s| < \epsilon$. Also, for each a, $q \restriction s * a \Vdash \dot{f}(n) = a$ and so $q \restriction s + a \Vdash \dot{\epsilon}_n = 1/a$; thus, for all but finitely many a, $q \restriction s * a \Vdash \acute{} \dot{x}_n - v| > \dot{\epsilon}_n$, i.e., $q \restriction s{*}a \Vdash v \notin \dot{I}_n$. Thus, by removing finitely many immediate successors of s we ensure that $r \Vdash v \notin \dot{I}_n$. We continue in this way to get $r \leq q$ such that $r \Vdash v \notin \bigcup_{k \geq n} \dot{I}_k$. $\square$

4. Class Forcing

Under the assumption of "large cardinal axioms" it can be shown that there are reals that are not generic over L for set-forcing. The standard example is the real called $0^\#$, which causes dramatic effects when added to L: In $L[0^\#]$ all successor L-cardinals are collapsed and indeed the cardinals of $L[0^\#]$ are indiscernible in L.

Solovay asked if $0^\#$ provides the only counterexample to the universality of set-forcing over L. For our present purposes we can pose his question as follows:

Is it consistent that for some real R, L and $L[R]$ have the same cardinals but R belongs to no set-generic extension of L?

The positive answer to this question was provided by Jensen, who developed a powerful new type of forcing, in which a generic real is created by forcing over L with a *class* partial-ordering. We shall next develop the general theory of class-forcing and establish this result of Jensen.

Let M be a transitive set or class satisfyingg ZF, and $A \subseteq M$. We say that $\langle M, A\rangle$ is a *model of ZF* if M is a model of ZF and the scheme of replacement holds in M for formulas which mention A as a predicate. In addition we require $\langle M, A\rangle$ to be a *ground model*, which means that $\langle M, A\rangle$ satisfies: $V = L(A) = \cup\{L(A \cap V_\alpha)|\alpha \in \text{ORD}\}$. Any ZF model $\langle M, A\rangle$ is easily modified to a ground model $\langle M, A^*\rangle$ (with the same definable predicates) by taking A^* to be $\{\langle 0, x\rangle | x \in A\} \cup \{\langle 1, V_\alpha^M\rangle | \alpha \in \text{ORD}(M)\}$. This "minimality" property of M relative to A is needed to guarantee that M is definable as a predicate in all of its extensions $\langle M[G], A, G\rangle$.

A partial ordering P is a *class forcing* for M (or an *M-forcing*) if for some ground model $\langle M, A\rangle$, P (with its ordering) is definable with parameters over $\langle M, A\rangle$. Assume that this is the case and that P has a greatest element 1^P.

Definition. $G \subseteq P$ is P-generic over $\langle M, A\rangle$ iff:

$p, q \in G \longrightarrow p, q$ are compatible.
$p \geq q \in G \longrightarrow p \in G$.
If $D \subseteq P$ is dense and $\langle M, A\rangle$-definable (with parameters) then $G \cap D \neq \emptyset$.

We make the same Assumption as before, that for each $p \in P$ there exists G such that $p \in G$ and G is P-generic over $\langle M, A\rangle$. (This is provable when M is countable.) We will discuss (and dispense with) this Assumption later.

Define names and $M[G]$ as before. We have the following:

Lemma 4.1. (a) $M \subseteq M[G]$ and $M[G]$ is transitive, $\mathrm{ORD}(M[G]) = \mathrm{ORD}(M)$.
(b) $G \cap V_\alpha \in M[G]$ for each $\alpha \in \mathrm{ORD}(M)$ and if $M \subseteq N$, $\langle N, G\rangle$ is amenable and N is a model of ZF then $M[G] \subseteq N$ and M is definable over $\langle N, A\rangle$.

Proof. (a) Exactly as before.
(b) For each $\alpha \in \mathrm{ORD}(M), G \cap V_\alpha = \gamma_\alpha^G$ where $\gamma_\alpha = \{\langle \hat{p}, p\rangle | p \in P \cap V_\alpha\}$, so $G \cap V_\alpha \in M[G]$. Under the assumptions on N we can define σ^G as an element of N, for each name σ; M is definable over $\langle N, A\rangle$ as it equals $L(A)^N$. $\square$

Define $\Vdash$ and $\Vdash^*$ as before. We would like to carry out the earlier argument to show that the Truth and Definability lemmas hold for $\Vdash$. But we immediately run into trouble: We do not know that the Definability lemma holds for $\Vdash^*$. The problem is in (a), (b) of the definition of $\Vdash^*$:

(a) $p \Vdash^* \sigma \in \tau$ iff $\{q | \exists \langle \pi, r\rangle \in \tau$ such that $q \leq r$, $q \Vdash^* \sigma = \pi\}$ is dense $\leq p$.

(b) $p \Vdash^* \sigma = \tau$ iff for all $\langle \pi, r\rangle \in \sigma \cup \tau$, $p \Vdash^* (\pi \in \sigma \longleftrightarrow \pi \in \tau)$ iff for all $\langle \pi, r\rangle \in \sigma \cup \tau$, $\{q | q \Vdash^* (\pi \in \sigma \wedge \pi \in \tau)$ or $q \Vdash^* (\pi \notin \sigma \wedge \pi \notin \tau)\}$ is dense $\leq p$.

As P may now be a proper class these clauses involve unbounded quantifiers, and therefore lead to definitions of $p \Vdash^* \sigma \in \tau, p \Vdash^* \sigma = \tau$ whose quantifier complexity may increase with the ranks of σ, τ.

By introducing a further condition on P we can control the quantifier complexity of the relations $p \Vdash^* \sigma \in \tau$, $p \Vdash^* \sigma = \tau$ and therefore obtain the Definability lemma for $\Vdash^*$. In the discussion below, "definable" always means "definable with parameters" unless we say otherwise.

Pretameness Condition. P is *pretame* iff whenever $\langle D_i | i \in a\rangle$ is an $\langle M, A\rangle$-definable sequence of dense classes, $a \in M$ and $p \in P$ then there is $q \leq p$ and $\langle d_i | i \in a\rangle \in M$ such that $d_i \subseteq D_i$ and d_i is predense $\leq q$ for each i.

Proposition. Suppose that for each $p \in P$ there is $G \subseteq P$ such that $p \in G$, G is P-generic over $\langle M, A\rangle$ and $\langle M[G], A, G\rangle$ is a model of ZF $-$ Power . Then P is pretame.

Proof. Given $\langle D_i | i \in a\rangle$ and p as in the statement of pretameness choose G such that $p \in G$, G P-generic over $\langle M, A\rangle$ and consider $f(i) =$ least rank of an element of $G \cap D_i$. If pretameness failed for $p, \langle D_i | i \in a\rangle$ then for every $q \leq p$ and $\alpha \in \mathrm{ORD}(M)$ there would be $r \leq q$ and $i \in a$ with r incompatible with each element of $D_i \cap V_\alpha$. But then by genericity, no ordinal of M can bound the range of f, so replacement fails in $\langle M[G], A, G, M\rangle$. As $\langle M, A\rangle$ is a ground model, replacement fails in $\langle M[G], A, G\rangle$. $\square$

Thus pretameness is necessary for a reasonable notion of class forcing. We now prove the Definability lemma for $\Vdash^*$ assuming pretameness. By *formula* we now mean a formula in the language of set theory with the addition of the unary predicate symbols $\underline{A}, \underline{G}$. Of course $\langle M[G], A, G\rangle \vDash \underline{A}(\sigma^G)$ iff $\sigma^G \in A$, $\langle M[G], A, G\rangle \vDash \underline{G}(\sigma^G)$ iff $\sigma^G \in G$. And extend the definition of $\Vdash^*$ by adding:

(f) $p \Vdash^* \underline{A}(\sigma)$ iff $p \Vdash^* \sigma \in \hat{a}_\alpha$, where $a_\alpha = A \cap V_\alpha$, $\alpha = \mathrm{Rank}\ \sigma + 1$.
(g) $p \Vdash^* \underline{G}(\sigma)$ iff $p \Vdash^* \sigma \in \gamma_\alpha$, where $\gamma_\alpha = \{\langle \hat{p}, p\rangle | p \in P \cap V_\alpha\}$, $\alpha = \mathrm{Rank}\ \sigma + 1$.

Theorem 4.2. If P is pretame then for any formula φ, the relation "$p \Vdash^* \varphi(\sigma_1 \ldots \sigma_n)$" of $p, \sigma_1 \ldots \sigma_n$ is $\langle M, A\rangle$-definable.

Proof. It suffices to show that the relations $p \Vdash^* \sigma \in \tau$ and $p \Vdash^* \sigma = \tau$ are $\langle M, A\rangle$-definable, for then we may induct on teh structure of φ. Note that by modifying A if necessary, we may assume that the relations "$x = V^M_\alpha$," "p, q are compatible," "d is predense below p," as well as $(P, \leq)$, are Δ_1-definable over $\langle M, A\rangle$.

Using pretameness we shall define a function F from pairs $(p, \sigma \in \tau)$, $(p, \sigma = \tau)$ into M such that:

(a) $F(p, \sigma \in \tau) = (i, d)$ where $d \in M$ is a nonempty subset of $P(\leq p) = \{q \in P \mid q \leq p\}$ and either ($i = 1$ and $q \Vdash^* \sigma \in \tau$ for $q \in d$) or ($i = 0$ and $q \Vdash^* \sigma \notin \tau$ for $q \in d$).
(b) The same holds for $\sigma = \tau$, $\sigma \neq \tau$ instead of $\sigma \in \tau, \sigma \notin \tau$.
(c) F is Σ_1-definable over $\langle M, A\rangle$.

Given this we can define $p \Vdash^* \sigma \in \tau$ by: $p \Vdash^* \sigma \in \tau$ iff for all $q \leq p$, $F(q, \sigma \in \tau) = (1, d)$ for some d. This holds because $p \Vdash^* \sigma \in \tau$ iff $\{q | q \Vdash^* \sigma \in \tau\}$ is dense $\leq p$. Similarly we can define $p \Vdash^* \sigma = \tau$.

Now define F by induction on $\sigma \in \tau$, $\sigma = \tau$. We consider the two cases separately.

$\sigma \in \tau$

Given p, search for $\langle \pi, r\rangle \in \tau$ and $q \le p$, $q \le r$ such that $F(q, \sigma = \pi) = (1, d)$ for some d. If such exist, let $F(p, \sigma \in \tau) = (1, e)$ where e is the union of all such d which appear by the least possible stage α (i.e., this Σ_1 property is true in $\langle V_\alpha^M, A \cap V_\alpha^M\rangle$, α least). If not then for each $\langle \pi, r\rangle \in \tau$, $D(\pi, r) = \cup\{d|$ For some $q \le r, F(q, \sigma = \pi) = (0, d)\} \cup \{q | q$ incompatible with $r\}$ is dense below p. So also search for $\langle d(\pi, r) | \langle \pi, r\rangle \in \tau\rangle \in M$ and $q \le p$ such that $d(\pi, r) \subseteq D(\pi, r)$ for each $\langle \pi, r\rangle$ and each $d(\pi, r)$ is predense $\le q$; if this latter search terminates then set $F(p, \sigma \in \tau) = (0, e)$, where e consists of all such q witnessed by the least possible stage α. One of these searches must terminate (by pretameness) and hence $F(p, \sigma \in \tau)$ is defined and either of the form $(1, e)$ where $q \in e \longrightarrow q \le p$, $q \Vdash^* \sigma \in \tau$, or of the form $(0, e)$ where $q \in e \longrightarrow q \le p, q \Vdash^* \sim (\sigma \in \tau)$.

$\sigma = \tau$

Given p, search for $\langle \pi, r\rangle \in \sigma \cup \tau$ and $q \le p, q \le r$ such that for some i, d, q' and e, $F(q, \pi \in \sigma) = (i, d), q' \in d, F(q', \pi \in \tau) = (1 - i, e)$. If this search terminates then set $F(p, \sigma = \tau) = (0, f)$ where f is the union of all such e which appear by the least possible stage α. If this search fails then for each $\langle \pi, r\rangle \in \sigma \cup \tau$, $D(\pi, r) = \cup\ \{e|$ For some $q \le r$, some i, d, q', $F(q, \pi \in \sigma) = (i, d), q' \in d$, $F(q', \pi \in \tau) = (i, e)\} \cup \{q|q$ is incompatible with $r\}$ is dense $\le p$. So also search for $\langle d(\pi, r) | \langle \pi, r\rangle \in \sigma \cup \tau\rangle \in M$ and $q \le p$ such that for each $\langle \pi, r\rangle$, $d(\pi, r) \subseteq D(\pi, r)$ and $d(\pi, r)$ is predense $\le q$. If this latter search terminates then $q \Vdash^* \sigma = \tau$ for all such q and let $F(p, \sigma = \tau) = (1, f)$, where f consists of all such $q \le p$ witnessed to obey the above by the least stage α. □

The previous Theorem was proved independently by M. Stanley. The author does not know if the assumption of pretameness is necessary for this result.

Now that we have the Definability lemma for $\Vdash^*$ we can prove the Truth lemma for $\Vdash^*$ as we did before; the two new clauses (f), (g) cause no difficulty. Then we infer that $\Vdash = \Vdash^*$ as before.

Pretameness is sufficient to verify that ZF $-$ Power is preserved:

Lemma 4.3. If P is pretame and G is P-generic over $\langle M, A\rangle$ then $\langle M[G], A, G\rangle$ is a model of ZF $-$ Power . If M is a model of AC then so is $M[G]$.

Proof. This is exactly as before, except for the verifications of replacement, union. For replacement, suppose $f : \sigma^G \longrightarrow M[G]$, f definable (with parameters) in $\langle M[G], A, G\rangle$ and choose $p \in G$, $p \Vdash f$ is a total function on σ. Then for each σ_0 of Rank $<$ Rank σ, $D(\sigma_0) = \{q|$ For some τ, $q \Vdash \sigma_0 \in \sigma \longrightarrow f(\sigma_0) = \tau\}$ is dense $\leq p$. Thus by pretameness we get that for each $q \leq p$ there is $r \leq q$ and $\alpha \in \mathrm{ORD}(M)$ such that $D_\alpha(\sigma_0) = \{s|s \in V_\alpha$ and for some τ of Rank $< \alpha$, $s \Vdash \sigma_0 \in \sigma \longrightarrow f(\sigma_0) = \tau\}$ is predense $\leq r$ for each σ_0 of Rank $<$ Rank σ. By genericity there is $q \in G$ and $\alpha \in \mathrm{ORD}(M)$ such that $q \leq p$, $D_\alpha(\sigma_0)$ is predense $\leq q$ for each σ_0 of Rank $<$ Rank σ. Thus Range $(f) = \pi^G$ where $\pi = \{\langle\tau, r\rangle|\text{Rank } \tau < \alpha, r \in V_\alpha, r \Vdash \tau \in \text{Range } (f)\}$. So Range $(f) \in M[G]$.

For union, given σ^G consider $\pi = \{\langle\tau, p\rangle \mid p \Vdash \tau \in \cup\sigma\}$. This is not a set, but for each α we may consider $\pi_\alpha = \pi \cap V_\alpha^M$. By replacement in $\langle M[G], A, G\rangle$, π_α^G is constant for sufficiently large $\alpha \in \mathrm{ORD}(M)$. For such α we have $\pi_\alpha^G = \cup\sigma^G$. □

Thus pretameness is equivalent to ZF$-$ Power preservation. P is *tame* iff P is pretame and in addition 1^P forces the Power Set Axiom. Thus tameness is equivalent to ZF preservation.

Examples

We describe the four basic examples of tame class forcing: Easton, Long Easton, Reverse Easton and Amenable forcing.

We now fix our ground model $\langle M, A\rangle$ to just be $\langle L, \emptyset\rangle$, and maintain the Assumption that for each forcing P considered, P-generic classes exist containing any given condition in P (where P-generic means P-generic over $\langle L, \emptyset\rangle$). We shall later consider the question of generic class existence and will show how to eliminate this Assumption, when establishing first-order properties of P-generic class.

Easton Forcing

Easton extended Cohen's independence proof for CH to all regular cardinals, showing that the function $f(\kappa) = 2^\kappa$ can exhibit any reasonable behavior for regular κ. To do so he developed a class forcing for adding generic subsets to all regular κ simultaneously. We describe here a version of his technique, where we explicitly add only one generic subset to each regular κ, thereby preserving GCH.

A condition in P is a function $p : \alpha(p) \longrightarrow L$ where $\alpha(p) \in$ ORD and for $\alpha < \alpha(p)$, $p(\alpha) = \emptyset$ unless α is infinite and regular, in which case $p(\alpha) \in 2^{<\alpha} = \{f : \beta \longrightarrow 2 | \beta < \alpha\}$. In addition we require that p has *Easton support* which means that for inaccessible κ, $\{\alpha < \kappa | p(\alpha) \neq \emptyset\}$ is bounded in κ. Extension is defined by: $p \leq q$ iff $\alpha(p) \geq \alpha(q)$, $\alpha < \alpha(q) \longrightarrow p(\alpha)$ extends $q(\alpha)$. The key to analyzing P is to observe that for each infinite regular κ, P is isomorphic to $P(\leq \kappa) \times P(> \kappa)$, where $P(\leq \kappa) = \{p \restriction [0, \kappa] | p \in P\}$, $P(> \kappa) = \{p \restriction (\kappa, \infty) | p \in P\}$, ordered in the natural way. Note that $P(\leq \kappa)$ is κ^+-cc (indeed has cardinality κ) and $P(> \kappa)$ is κ^+-distributive (indeed is κ^+-*closed*: decreasing κ-sequences of conditions have lower bounds).

Long Easton Forcing

This is like Easton forcing, except we drop the Easton support requirement. There are two types of Long Easton forcing, depending upon whether or not the forcing is trivial at inaccessibles. We begin with the simpler case, called Long Easton forcing at Successors. We treat ω as a successor cardinal in this discussion: $0^+ = \omega$.

A condition in P is a function $p : \alpha(p) \longrightarrow L$, $\alpha(p) \in$ ORD where $p(\alpha) = \emptyset$ unless α is a successor cardinal, in which case $p(\alpha) \in 2^{<\alpha}$. Extension is defined by $p \leq q$ iff $\alpha(p) \geq \alpha(q)$ and for each $\alpha < \alpha(q)$, $p(\alpha)$ extends $q(\alpha)$. For any infinite regular κ we can factor P as $P(\leq \kappa) \times P(> \kappa)$ and $P(> \kappa)$ is κ^+-distributive. However if κ is inaccessible, $P(\leq \kappa) = P(< \kappa)$ is not κ^+-cc.

Now we consider (unrestricted) Long Easton forcing, where we redefine P so as to allow $p(\alpha) \in 2^{<\alpha}$ for any infinite regular α, not just successor cardinals. Then for any infinite *successor* cardinal κ we can factor P as $P(\leq \kappa) \times P(> \kappa)$ and the analysis of Easton forcing shows that P is tame and preserves "cofinality $> \kappa$" for *successor* cardinals κ. However P is *not* cofinality-preserving in general. A cardinal κ is *Mahlo* if κ is inaccessible and $\{\alpha < \kappa | \alpha \text{ inaccessible}\}$ is stationary in κ.

Theorem 4.4. Suppose G is P-generic over L and κ is L-regular. Then $(\kappa^+)^{L[G]} = (\kappa^+)^L$ iff κ is not Mahlo in L.

Proof. Let $G = \langle G_\alpha | \alpha \text{ infinite, regular}\rangle$ be P-generic. For each $\alpha < \kappa$ consider $A_\alpha \subseteq \kappa$ defined by: $\beta \in A_\alpha \longleftrightarrow \alpha \in G_\beta$.

Claim. Suppose κ is Mahlo. Then $\{A_\alpha | \alpha < \kappa\} \subseteq L$ but for no $\gamma < (\kappa^+)^L$ do we have $\{A_\alpha | \alpha < \kappa\} \subseteq L_\gamma$.

Proof of Claim. For any $\alpha < \kappa$ and condition p, we can extend p to q so that $\alpha < \bar{\kappa} < \kappa, \bar{\kappa}$ regular $\longrightarrow p(\bar{\kappa})$ has length greater than α. Thus A_α is forced to belong to L.

Given $\gamma < (\kappa^+)^L$ and a condition p, define $f(\bar{\kappa}) = \text{length}(p(\bar{\kappa}))$ for regular $\bar{\kappa} < \kappa$. As κ is Mahlo, f has stationary domain and hence by Fodor's Theorem we may choose $\alpha < \kappa$ such that $\text{length}(p(\bar{\kappa}))$ is less than α for stationary many regular $\bar{\kappa} < \kappa$. Then p can be extended so that A_α is guaranteed to be distinct from the κ-many subsets of κ in L_γ.

Thus κ^+ is collapsed if κ is Mahlo. Conversely, if κ is not Mahlo, then choose a CUB $C \subseteq \kappa$ consisting of cardinals which are not inaccessible (we may assume that κ is a limit cardinal). Suppose that $\langle D_\alpha | \alpha \in C \rangle$ is a definable sequence of dense classes. Given p we can successively extend $p(\geq \alpha^+), \alpha \in C$ so that $\{q \leq p | q, p \text{ agree} \geq \alpha^+, q \in D_\alpha\}$ is predense $\leq p$. There is no difficulty in obtaining a condition at a limit stage less than κ precisely because conditions are trivial at limit points of C. Thus we have shown that $P(< \kappa) \times P(> \kappa)$ preserves κ^+ as κ-many dense classes can be simultaneously reduced to predense subsets of size $< \kappa$. Finally $P \simeq P(< \kappa) \times P(> \kappa) \times P(\kappa)$ and $P(\kappa)$ preserves κ^+ as it has size κ. $\square$

Remark. Full cofinality-preservation does hold for *Thin* Easton forcing, defined like Long Easton forcing but with the requirement that for inaccessible $\kappa, \{\alpha < \kappa | p(\alpha) \neq \emptyset\}$ is *nonstationary* in κ.

Reverse Easton Forcing

Our third class forcing example is a type of iteration of set forcings first considered by Silver. Define the iteration $\langle P(< i) | i \leq \infty \rangle$ in L by: $P(< 0) = \{\emptyset\}$, the trivial forcing; $P(\leq i) \simeq P(< i) * P(i)$ where $P(i)$ is (a formula for) the trivial forcing unless $i \geq \omega$ is regular, in which case $P(i)$ is (a formula for) the forcing $2^{<i} = \{p : \alpha \longrightarrow 2 | \alpha < i\}$, ordered by extension; for i limit we take $P(< i) = \text{Inverse Limit } \langle P(< j) | j < i \rangle$ if i is singular and Direct Limit $\langle P(< j) | j < i \rangle$ if i is regular (or if $i = \infty$).

Fact 1. For each $i < \infty$, $P(\leq i)$ has a dense subordering which is a set of cardinality $\leq i^+$ (by convention, $0^+ = \omega$).

Fact 2. For κ regular and infinite, $P(\leq \kappa)$ is κ^+-cc.

We now state the Factoring Property. For $\alpha \le \beta \le \infty$ we let $P[\alpha, \beta)$ be (a formula for) the iteration of length $\beta - \alpha$ stages defined just like P, except beginning at index α and ending after $\beta - \alpha$ stages. Then $P(< \alpha) * P[\alpha, \beta)$ consists of pairs (p, q) where $p \in P(< \alpha)$ and q is a $P(< \alpha)$-name for a condition in the iteration $P[\alpha, \beta)$.

Fact 3. (Factoring Property) $P(< \beta)$ is isomorphic to $P(< \alpha) * P[\alpha, \beta)$.

Fact 4. For κ regular and infinite, $P(\le \kappa) \Vdash P[\kappa + 1, \infty)$ is κ^+-closed (descending sequences of length $\le \kappa$ have lower bounds).

These are all the facts needed to establish tameness and cofinality-preservation for $P =$ Direct Limit $\langle P(< i) | i < \infty \rangle$.

Amenable Class Forcing

Our fourth and final basic example of class forcing is where one has κ-distributivity for every κ. Tameness and preservation of cofinalities follow easily. Note that in this case one adds a generic class but no new sets, so GCH preservation is trivial.

A simple example is $P =$ all functions $p : \alpha \longrightarrow 2$, $\alpha \in \mathrm{ORD}$, ordered by extension. Another is $P =$ all closed sets of ordinals, ordered by end extension.

We pose the question: For which class forcings P defined in L can we construct P-generic classes? We will make sense of this question using Silver's theory of indiscernibles for L, which will lead us to some unexpected answers.

Construction of Generic Classes

Recall that we imposed the Assumption that P-generic classes exist for any class forcing defined over a ground model $\langle M, A \rangle$. This is true when M is countable, but not in general. We now drop this Assumption and study in detail, for the case of forcings defined over L, the problem of generic class existence. We will see that there is a natural condition, *L-rigidity,* shared by all tame class-generic extensions of L and if this property fails in V then there is a least inner model in which it fails, $L[0^{\#}]$. We then use $L[0^{\#}]$ to provide a criterion for deciding which class forcings P defined over L have generic classes, by defining such a P to be *relevant* if it has a generic definable in $L[0^\#]$. Finally, we determine which of the basic class-forcing examples are relevant, using properties of the L-indiscernibles provided by $0^\#$.

First we must verify that if a P-generic G exists then the model $\langle M[G], A, G\rangle$ does behave as earlier described under the various hypotheses on P discussed there. This is not immediate as we in fact do need the Assumption to prove some of the basic facts about $\Vdash$ such as the fact that $\Vdash$ and $\Vdash^*$ coincide, as well as the Definability and Truth lemmas for $\Vdash$. However note the following:

Proposition 4.5. Suppose φ is a first-order property true in $\langle M[G], A, G\rangle$ whenever M is countable and G is P-generic over $\langle M, A\rangle$ for a forcing P definable over $\langle M, A\rangle$. Then φ is true for all such $\langle M[G], A, G\rangle$, without the assumption that M is countable.

Proof. Given an arbitrary $\langle M[G], A, G\rangle$ let $\langle \bar{M}[\bar{G}], \bar{A}, \bar{G}\rangle$ be the transitive collapse of a sufficiently elementary countable submodel and apply the hypothesis about φ and elementarity to conclude that φ holds in $\langle M[G], A, G\rangle$. □

Thus when establishing first-order properties of $\langle M[G], A, G\rangle$ for P-generic G, we may in fact use our earlier Assumption. Consequently:

Theorem 4.6. If P is one of the basic examples of class forcing over L (Easton, Long Easton at Successors, Reverse Easton, Amenable) then P is tame and preserves both cofinalities and the GCH.

Rigidity

Which forcings P defined in L have generic classes? Of course if $V = L$ then for no nontrivial P does there exist a P-generic class, however we declare this hypothesis to be too restrictive. A necessary condition for every $p \in P$ to belong to a P-generic, as we have seen, is that P be tame, and for any such P it is consistent that a P-generic class exists. However, the possibility that a P-generic class exist for every tame P which is L-definable without parameters is ruled out by the following result.

Proposition 4.7. There exist tame forcings P_0, P_1 which are L-definable without parameters such that if G_0, G_1 are P_0, P_1-generic over L, respectively, then $\langle L[G_0, G_1], G_0, G_1\rangle$ is not a model of ZF.

Proof. For any ordinal α, let $n(\alpha)$ be the least n such that L_α is not a model of Σ_n-replacement, if such an n exists. Let $S_0 = \{\alpha | n(\alpha)$ exists and is even$\}$. P_0 consists of all closed p such that $p \subseteq S_0$, ordered by $p \le q$ iff q is an initial segment of p.

Note that S_0 is unbounded in ORD: Given α, let β be least such that $\beta > \alpha$ and $L_\beta \vDash \Sigma_1$-Replacement. Then $n(\beta) = 2$ so $\beta \in S_0$. If $G_0 \subseteq P_0$ is P_0-generic over L then $\cup G_0$ is therefore a closed unbounded subclass of ORD contained in S_0. To show that P_0 is tame, it suffices to show that it is κ-distributive for every L-regular κ : If $\langle D_i | i < \kappa\rangle$ is an L-definable sequence of classes dense on P_0 and $p \in P_0$ then choose n odd so that $\langle D_i | i < \kappa\rangle$ is Σ_n definable and choose $\langle \alpha_i | i < \kappa\rangle$ to be first κ-many α such that L_α is Σ_n-elementary in L and $\kappa, p, x \in L_\alpha$ where x is the defining parameter for $\langle D_i | i < \kappa\rangle$. We can define $p \ge p_0 \ge p_1 \ge \dots$ so that p_{i+1} meets D_i and $\max(p_i) = \alpha_i$, using the Σ_n-elementarity of L_{α_i} in L. As $n(\alpha_i) = n+1$ and $n+1$ is even, we may define p_λ to be $\cup\{p_i | i < \lambda\} \cup \{\alpha_\lambda\}$ for limit $\lambda \le \kappa$ and we see that $q = p_\kappa \le p$ meets each D_i.

Now define P_1 in the same way, but using $S_1 = \{\alpha | n(\alpha)$ is defined and odd$\}$. Then P_1 is also tame yet if G_0, G_1 are P_0, P_1-generic over L (respectively) then $\cup G_0, \cup G_1$ are disjoint CUB subclasses of ORD. □

So we need a criterion for choosing L-definable forcings for which we can have a generic. Our approach is to isolate a "property of transcendence" (#) such that:

(a) In tame class-generic extensions of L, (#) fails.
(b) If (#) is true in V then there is a least inner model $L(\#)$ satisfying (#).

Then our criterion for generic class existence is: P has a generic iff it has one definable over $L(\#)$.

Definition. An amenable $\langle L, A\rangle$ is *rigid* if there is no nontrivial elementary embedding $\langle L, A\rangle \longrightarrow \langle L, A\rangle$. L is *rigid* if $\langle L, \emptyset\rangle$ is rigid.

We shall take (#) to be: L is not rigid. First we demonstrate property (b) above, i.e., that there is a least model in which L is not rigid (if there is one at all).

Theorem 4.8 If L is not rigid then there exists a CUB class C of ordinals which are L-indiscernible: If φ is an n-ary formula, $\alpha_1 \dots \alpha_n$ and $\beta_1 \dots \beta_n$ are increasing n-tuples from C then $L \vDash \varphi(\alpha_1 \dots \alpha_n) \longleftrightarrow \varphi(\beta_1 \dots \beta_n)$.

Proof. We need a lemma.

Lemma 4.9. Suppose there exists $j : L \longrightarrow L$. Then there exists such a j which is definable (with parameters) and such that every cardinal λ of L-cofinality greater than κ satisfying $\bar{\lambda} < \lambda \longrightarrow \text{Card }(\bar{\lambda}^\kappa)^L < \lambda$ is a fixed point of j, where $\kappa = \text{crit }(j) = \text{least } \alpha$ such that $j(\alpha) \neq \alpha$. ("crit " stands for "critical point".)

Proof of Lemma. We use the ultrapower construction. Define an ultrafilter U on $\mathcal{P}(\kappa) \cap L$ by: $X \in U$ iff $\kappa \in j(X)$. Then there is an elementary embedding $k : L \longrightarrow \text{Ult }(L, U)$ where $\text{Ult }(L, U)$ is the ultrapower L^κ/U defined using functions $f : \kappa \longrightarrow L$ which belong to L. Thus an element of $\text{Ult }(L, U)$ is $[f] = \{g : \kappa \longrightarrow L | g \in L \text{ and for some } X \in U,\ \alpha \in X \longrightarrow g(\alpha) = f(\alpha)\}$, with $E = \in$-relation of $\text{Ult }(L, U)$ defined in the natural way: $[f]E[g]$ iff $\{\alpha | f(\alpha) \in g(\alpha)\} \in U$.

The map $[f] \longmapsto j(f)(\kappa)$ gives an elementary embedding from $\text{Ult }(L, U)$ into L and hence $\text{Ult }(L, U)$ is well-founded and isomorphic to L. If $h : \text{Ult }(L, U) \simeq L$ then $j^* = h \circ k : L \longrightarrow L$ is definable with parameters κ, U. If λ has L-cofinality greater than κ then k (and hence j^*) is continuous at λ since any constructible $f : \kappa \longrightarrow \lambda$ is bounded by the constant function $c_{\bar{\lambda}}$ with value $\bar{\lambda}$ for some $\bar{\lambda} < \lambda$ (hence $[f]E[c_\lambda] \longrightarrow [f]E[c_{\bar{\lambda}}]$ for some $\bar{\lambda} < \lambda$). But if $[f]E[c_{\bar{\lambda}}]$ then $\{[g] | [g]E[f]\}$ has size at most $\text{Card }(\bar{\lambda}^\kappa)^L$, and if this is smaller than λ then $j^*[\lambda] \subseteq \lambda$ and hence by continuity $j^*(\lambda) = \lambda$.

If L is not rigid then there is $j : L \longrightarrow L$ with critical point κ such that every limit cardinal of cofinality $> \kappa$ is a fixed point of j. It follows that if $F = \{\alpha | \alpha \text{ a limit cardinal of cofinality} > \kappa\}$ then $\kappa \notin \text{Hull}(\kappa \cup F)$ where Hull denotes the Skolem hull in L.

For any class of ordinals G let G^* denote $\{\alpha \in G | \alpha = \text{ordertype }(\alpha \cap G)\}$. Then define inductively: $F_0 = F$, $F_{\alpha+1} = (F_\alpha)^*$, $F_\lambda = (\bigcap\{F_\alpha | \alpha < \lambda\})^*$ for limit λ. For any α, H_α denotes $\text{Hull}(\kappa \cup F_\alpha)$. And $\langle \kappa_\alpha | \alpha \in \text{ORD}\rangle$ is defined by: $\kappa_0 = \kappa, \kappa_{\alpha+1} = \min(H_\alpha - \kappa), \kappa_\lambda = \cup\{\kappa_\alpha | \alpha < \lambda\}$ for limit λ.

Claim 1. For every $\alpha, \kappa_\alpha < \kappa_{\alpha+1}$.

Proof. We may assume that α is not 0. As $\kappa_{\alpha+1}$ belongs to $\text{Hull}(\kappa \cup F_\alpha)$ it is a fixed point of the isomorphism $L \simeq H_{<\alpha} = \text{Hull}(\kappa \cup \bigcap\{F_\beta | \beta < \alpha\})$. But $H_{<\alpha} \cap [\kappa, \kappa_\alpha) = \emptyset$, so κ_α is *not* a fixed point of this isomorphism, using $\kappa < \kappa_\alpha$.

Claim 2. Let $\pi_{\alpha\beta}: L \simeq \text{Hull}(\kappa_\alpha \cup F_\beta)$. Then $\pi_{\alpha\beta}$ fixes κ_γ when $\gamma < \alpha$ or when γ is a successor ordinal $> \beta + 1$. Also $\pi_{\alpha\beta}(\kappa_\alpha) = \kappa_{\beta+1}$.

Proof. $\gamma < \alpha \longrightarrow \kappa_\gamma < \kappa_\alpha$, so clearly $\pi_{\alpha\beta}$ fixes κ_γ. If $\beta + 1 < \gamma$, γ successor then $\kappa_\gamma \in \text{Hull}(\kappa \cup F_{\gamma-1})$, so κ_γ is a fixed point of $\pi_{\alpha\beta}$.

As $\kappa_{\beta+1} \in \text{Hull}(\kappa_\alpha \cup F_\beta) = H$, we have $\kappa_\alpha \leq \pi_{\alpha\beta}(\kappa_\alpha) \leq \kappa_{\beta+1}$. Conversely, suppose that $\kappa_\alpha \leq \delta < \kappa_{\beta+1}, \delta \in H$; we derive a contradiction. Write $\delta = t(\vec{\xi}, \vec{\eta})$ where the components of $\vec{\xi}$ are less than κ_α and the components of $\vec{\eta}$ belong to F_β. Choose $\bar{\alpha}+1 \leq \alpha$ least so that the components of $\vec{\xi}$ are less than $\kappa_{\bar{\alpha}+1}$. Then $L \vDash \exists \vec{\xi}$ with components $< \kappa_{\bar{\alpha}+1}(\kappa_{\bar{\alpha}+1} \leq t(\vec{\xi}, \vec{\eta}) < \kappa_{\beta+1})$. Let $\pi: L \simeq \text{Hull}(\kappa \cup F_{\bar{\alpha}})$. Then $\pi(\kappa) = \kappa_{\bar{\alpha}+1}$, $\pi(\vec{\eta}) = \eta$, $\pi(\kappa_{\beta+1}) = \kappa_{\beta+1}$. So $L \vDash \exists \vec{\xi}$ with components $< \kappa(\kappa \leq t(\vec{\xi}, \vec{\eta}) < \kappa_{\beta+1})$, contradicting the defintion of $\kappa_{\beta+1}$.

Now for any two increasing n-tuples $\alpha_1 \ldots \alpha_n$ and $\beta_1 \ldots \beta_n$ with $\alpha_n < \beta_1$ we can obtain $\pi: L \longrightarrow L$ such that $\pi(\kappa_{\alpha_i}) = \kappa_{\beta_i+1}$ for all i, by taking $\pi_{\alpha_1\beta_1} \circ \ldots \circ \pi_{\alpha_n\beta_n}$. This implies that $C = \{\kappa_\alpha | \alpha \in \text{ORD}\}$ is a class of L-indiscernibles. $\square$

Now we introduce $0^\#$. As before, Hull denotes Skolem hull in L.

Theorem 4.10. Suppose L is not rigid. Then there is a unique CUB class sense that $L = \text{Hull}(I)$. Moreover I is unbounded in every uncountable cardinal and if $0^\# =$ First-Order theory of $\langle L, \in, i_1, i_2, \ldots \rangle$ (where the first ω elements $i_1, i_2, \ldots$ of I are introduced as constants) then we have the following:

(a) $0^\# \in L[I]$, I is $\Delta_1(L[0^\#])$ in the parameter $0^\#$ and I is unbounded in α whenever $L_\alpha[0^\#] \vDash \Sigma_1$ replacement.
(b) $0^\#$, viewed as a real, is the unique solution to a Π^1_2 formula (i.e., a formula of the form $\forall x \exists y \psi$, where x, y vary over reals and ψ is arithmetical).
(c) If $f : I \longrightarrow I$ is increasing, $f \neq$ identity then there is a unique $j : L \longrightarrow L$ extending f with critical point in I, and every $j : L \longrightarrow L$ is of this form.
(d) If $\langle L, A \rangle$ is amenable then A is $\Delta_1(L[0^\#])$, $\langle L, A \rangle$ is not rigid and a final segment of I is a class of $\langle L, A \rangle$-indiscernibles.

Remarks. (i) As I is closed and is unbounded in every uncountable cardinal it follows that every uncountable cardinal belongs to I and $0^\# =$ First-Order theory of $\langle L, \in, \aleph_1, \aleph_2, \ldots \rangle$. (ii) The Σ^1_2-absoluteness of L implies that the unique solution to a Σ^1_2 formula is constructible; so in a sense (b) is best possible. (iii) I is a class of *strong* indiscernibles: If $\vec{i}, \vec{j}$ are increasing tuples from I of the same length and $x < \min(\vec{i})$, $\min(\vec{j})$ then for any φ, $L \vDash \varphi(x, \vec{i}) \longleftrightarrow \varphi(x, \vec{j})$. In fact the proof below shows that any unbounded class I of L-indiscernibles such that $I \cap \operatorname{Lim} I \neq \emptyset$ is necessarily a class of strong indiscernibles.

Proof. There exists a CUB class C of L-indiscernibles. Let $\pi : \mathrm{Hull}(C) \simeq L$ and we see that $I = \pi[C]$ is a CUB class of generating L-indiscernibles. Note that $\alpha \in I \longrightarrow L_\alpha \prec L$ and therefore $L = \Sigma_1\text{-Hull}(I)$. For any Σ_1 $\varphi(x, y_1 \ldots y_n)$ let t_φ be the term $\mu x \varphi(x, y_1 \ldots y_n)$, intended to name the L-least x such that $L \vDash \varphi(x, y_1 \ldots y_n)$, if it exists, and 0 otherwise. Then L is described as the Ehrenfeucht-Mostowski model consisting of all terms $t_\varphi(j_1 \ldots j_n)$ (with $j_1 \ldots j_n \in I$ substituting for the variables $y_1 \ldots y_n$), with terms identified as dictated by $\mathrm{Thy}\langle L, \in, i_1, i_2, \ldots\rangle$ = First-Order theory of $\langle L, \in, i_1, i_2, \ldots\rangle$. Thus I is uniquely determined by Thy $\langle L, \in, i_1, i_2, \ldots\rangle$. But if I^* is another CUB class of generating L-indiscernibles we get $I \cap I^*$ infinite (and in fact CUB), hence $\mathrm{Thy}\langle L, \in, i_1, i_2, \ldots\rangle = \mathrm{Thy}\langle L, \in, i_1^*, i_2^*, \ldots\rangle$. So I is unique. Also note that I is a class of *strong* L-indiscernibles in the sense that $x < \min(\vec{i}), \min(\vec{j})$, $\vec{i}$ and $\vec{j}$ of the same length from I implies that $L \vDash \varphi(x, \vec{i}) \longleftrightarrow \varphi(x, \vec{j})$ for any formula φ; if not then we get $\vec{i} < \min(\vec{j})$ with $\{x < \min(\vec{i}) | L \vDash \varphi(x, \vec{i})\} \neq \{x < \min(\vec{i}) | L \vDash \varphi(x, \vec{j})\}$ and $\min(\vec{i})$ a limit point of I. But then we can get $\vec{i} < \vec{j}_0 < \vec{j}_1 < \ldots$ of length ORD with $\alpha < \beta \longrightarrow \{x < i_0 | L \vDash \varphi(x, \vec{j}_\alpha)\} \neq \{x < i_0 | L \vDash \varphi(x, \vec{j}_\beta)\}$; this is absurd because there are only set-many choices for subsets of i_0.

It follows from the strong indiscernibility of I that $t(\vec{i}, \vec{j}) < \min(\vec{j})$ implies $t(\vec{i}, \vec{j}) < I$-successor to $\max(\vec{i})$. Hence for all $i \in I \cup \{0\}$, $\mathrm{Hull}(i \cup \{i, j_1, j_2 \ldots\}) \supseteq L_{i^*}$ where $i < i^* \leq j_1 < j_2 < \ldots$ are ω-many elements of I, $i^* = I$-successor to i. So Card (L_{i^*}) = Card (i) and it follows that uncountable cardinals belong to $\operatorname{Lim} I$. Moreover if $i \in \operatorname{Lim} I$ then $L_i = \mathrm{Hull}(I \cap i)$ and L_i is isomorphic to the natural Ehrenfeucht-Mostowski model built from $I \cap i$, using $0^\# =$ Thy $\langle L, \in, i_1, i_2, \ldots\rangle$ to determine when to identify two terms $t_{\varphi_0}(\vec{i}_0)$, $t_{\varphi_1}(\vec{i}_1)$. We now verify (a)–(d).

(a) Clearly $0^\# \in L[I]$ as $0^\# =$ Thy $\langle L_{i_\omega}, \in, i_1, i_2, \ldots\rangle$ where $i_n = n$th indiscernible. If α is $0^\#$-admissible (i.e., $L_\alpha[0^\#] \vDash \Sigma_1$ replacement) then for any limit $\lambda < \alpha$, $L_{i_\lambda} \simeq$ Ehrenfeucht-Mostowski model $M(0^\#, \lambda)$ built from λ indiscernibles and therefore belongs to $L_\alpha[0^\#]$, as Σ_1-replacement gives us the Mostowski collapse. So $\alpha = i_\alpha = \alpha$th indiscernible and $\lambda \longmapsto \langle L_{i_\lambda}, \{i_\beta | \beta < \lambda\}\rangle$ is $\Delta_1(L_\alpha[0^\#])$. Hence I is $\Delta_1(L[0^\#])$ (with parameter $0^\#$).

(b) $0^{\#} = \text{Thy}\ \langle L, \in, i_1, i_2, \ldots\rangle$ has the property that for every countable limit $\lambda, M(0^{\#}, \lambda)$ is well-founded and if $\pi : M(0^{\#}, \lambda) \simeq L_{i_\lambda}$, $\pi(\beta^{\text{th}}$ indiscernible in $M(0^{\#}, \lambda)) = i_\beta$ then $\{i_\beta | \beta < \lambda\}$ is CUB in i_λ. This is a Π^1_2 property as it says $\forall$ relation R on ω (R a well-ordering $\longrightarrow M(0^{\#}, <_R)$ is well-founded and is a model of φ) where φ is first-order. But if 0^* obeys this property then $M(0^*, \text{ORD}) \simeq L$ and $0^* = \text{Thy}\ \langle L, \in, i^*_1, i^*_2 \ldots\rangle$ where $I^* = \{i^*_\beta | \beta \in \text{ORD}\}$ is a CUB class of generating L-indiscernibles. We have seen that $I = I^*$ and so $0^* = 0^{\#}$.

(c) If $f : I \longrightarrow I$ is increasing, $f \neq$ identity then define $j : L \longrightarrow L$ by $j\ (t_\varphi(j_1 \ldots j_n)) = t_\varphi(f(j_1) \ldots f(j_n))$. This is well-defined since I is a class of L-indiscernibles. j must be the identity on i = the critical point of f = the least $i, f(i) > i$, as $t_\varphi(j_1 \ldots j_n, k_1 \ldots k_m) = t_\varphi(j_1 \ldots j_n, f(k_1) \ldots f(k_m))$ when $t_\varphi(j_1 \ldots j_n, k_1, \ldots k_m) < k_1$. So the critical point of j = the critical point of f belongs to I. Clearly j is unique, given f. If $j : L \longrightarrow L$ is arbitrary then α = the critical point of j belongs to I, as α = critical point of j^* where $j^*(i) = i$ for unboundedly many $i \in I$ and thus if $\alpha \notin I$ we get $\alpha = t_\varphi(x, \vec{i})$, $x < \alpha < \vec{i}$, $j^*(\vec{i}) = \vec{i}$ and thus $j^*(\alpha) = \alpha$, contradicting α = critical point of j^*. Now note that if $i \in I$ then $j(i)$ is the critical point of some $j^* : L \longrightarrow L$ as $i \notin \text{Hull}(i \cup (I - (i+1)))$ implies $j(i) \notin \text{Hull}(j(i) \cup J)$ where $J = j[I - (i+1)]$ so $k : L \simeq \text{Hull}(j(i) \cup J)$ has critical point $j(i)$. So $j(i) \in I$.

(d) If $\langle L, A\rangle$ is amenable then for each $i \in I$ we may write $A \cap i = t_{\varphi_i}(\vec{j}_i, i, \vec{k}_i)$ where $\vec{j}_i < i < \vec{k}_i$ are all from I. By Fodor's Theorem $(\varphi_i, \vec{j}_i)$ is constant on an unbounded subclass of I and hence by indiscernibility we may assume that $A \cap i = t_\varphi(\vec{j}, i, \vec{k}_i)$ for all $i \in I, i > \max(\vec{j})$ where the choice of $\vec{k}_i \in I - (i+1)$ does not matter. Thus $I - (\max(\vec{j}) + 1)$ is a class of $\langle L, A\rangle$-indiscernibles and A is $\Delta_1(L[0^{\#}])$ in parameters $\vec{j}, 0^{\#}$. We get $j : \langle L, A\rangle \longrightarrow \langle L, A\rangle$ by shifting I above $\vec{j}$. $\square$

In case the conclusion of this Theorem holds (i.e. in case L is not rigid) we say that "$0^{\#}$ exists" and refer to I as the *Silver Indiscernibles*. Note that if L is not rigid then $L[0^{\#}]$ is the smallest inner model in which L is not rigid.

The next theorem shows that L is rigid in its tame class-generic extensions.

Theorem 4.11. Suppose that G is P-generic over $\langle L, A\rangle$ and P is tame. Then $L[G] \models 0^{\#}$ does not exist.

Proof. Suppose $p_0 \in P, p_0 \Vdash I =$ Silver indiscernibles is unbounded and $i < j$ in $I \longrightarrow L_i \prec L_j$. Suppose that $p \leq p_0, p \Vdash \hat{\alpha} \in I$. Then $L_\alpha \prec L$ as this is true in any P-generic extension $\langle L[G], A, G\rangle, p \in G$. (By Löwenheim-Skolem we can assume that such a G exists for the sake of this argument.) Thus an L-Satisfaction predicate is definable over $\langle L, A\rangle$ as $L \models \varphi(x)$ iff for some $p \in P$ below p_0, some α with $x \in L_\alpha, p \Vdash \varphi(\hat{x})$ is true in L_α. This is a contradiction if $A = \emptyset$, for then L-satisfaction would be L-definable. But note that for any A such that $\langle L, A\rangle$ is amenable we can apply the same argument, using the fact that $\langle L_\alpha, A \cap L_\alpha\rangle \prec \langle L, A\rangle$ for α in a final segment of I.

The previous result was proved independently by A. Beller.

The most important sufficient condition for the existence of $0^\#$ is expressed by Jensen's Covering Theorem, to which we turn next. A set X is *covered in L* if there is a constructible Y such that $X \subseteq Y$, Card $Y =$ Card X.

Covering Theorem. Suppose there exists an uncountable set of ordinals which is not covered in L. Then $0^\#$ exists.

Using this result we can show:

Theorem 4.12. Each of the following is equivalent to the existence of $0^\#$:
(a) L is not rigid.
(b) Some uncountable set of ordinals is not a subset of a constructible set of the same cardinality.
(c) Some singular cardinal is regular in L.
(d) $\kappa^+ \neq (\kappa^+)^L$ for some singular cardinal κ.
(e) Every constructible subset of ω_1 either contains or is disjoint from a closed, unbounded subset of ω_1.
(f) There exists $j : L_\alpha \longrightarrow L_\beta$, crit $(j) = \kappa$, $\kappa^+ \leq \alpha$.
(g) There exists $j : L_\alpha \longrightarrow L_\beta$, crit $(j) = \kappa, (\kappa^+)^L \leq \alpha, \alpha \geq \omega_2$.

Proof. It is straightforward to show that these all follow from the existence of $0^\#$. Also (a) implies the existence of $0^\#$ by an earlier result. Conditions (c), (d) each easily imply (b), and we get $0^\#$ from (b) by the Covering Theorem. Condition (e) implies (a), since $L \longrightarrow L \simeq$ Ult (L, U), where U consists of all constructible subsets of ω_1 containing a closed unbounded subset. To see that (f) implies the existence of $0^\#$, define an ultrafilter U on constructible subsets of κ by: $X \in U$ iff $\kappa \in j(X)$. Then Ult (L, U) is well-founded, for if not then by Löwenheim-Skolem there would be an infinite descending chain in Ult (L_{κ^+}, U) which contradicts $\kappa^+ \leq \alpha$.

Finally we show that (g) implies the existence of $0^\#$. Define U as before by: $X \in U$ iff $\kappa \in j(X)$. First suppose that κ is at least ω_2. We shall argue that U is *countably complete*, i.e. that if $\langle X_n | n \in \omega\rangle$ belong to U then $\cap\{X_n | n \in \omega\}$ is nonempty. (This gives $0^\#$ as it implies that Ult (L, U) is well-founded.) By the Covering Theorem, there is $F \in L$ of cardinality ω_1 such that $X_n \in F$ for each n. Then as we have assumed that $\kappa \geq \omega_2$, F has L-cardinality less than κ. We may assume that F is a subset of $\mathcal{P}(\kappa) \cap L$, and hence as α is an L-cardinal, F belongs to L_α and there is a bijection $h : F \longleftrightarrow \gamma$ for some $\gamma < \kappa, h \in L_\alpha$. But then $F^* = \{X \in F | \kappa \in j(X)\}$ belongs to L_α as $X \in F^* \longleftrightarrow \kappa \in j(h^{-1})(h(X))$ and F^* has nonempty intersection as $j(F^*) =$ Range $(j \restriction F^*)$ and $\kappa \in \cap j(F^*)$. Thus $\{X_n | n \in \omega\}$ has nonempty intersection since it is a subset of F^*. If κ is less than ω_2 then we have $\alpha \geq \omega_2 \geq \kappa^+$ so we have a special case of (f). $\square$

The author does not know if "ω_2" can be replaced by "ω_1" in (g) of the previous theorem.

Relevant Forcing

We showed that L is rigid in its tame class generic extensions and that if L is not rigid then there is a least inner model $L[0^\#]$ in which L is not rigid. We now use these facts to provide a criterion for generic class existence for class forcings over L.

Definition. A forcing P defined over a ground model $\langle L, A\rangle$ is *relevant* if there is a G P-generic over $\langle L, A\rangle$ which is definable (with parameters) over $L[0^\#]$. P is *totally relevant* if for each $p \in P$ the same is true for $P(\leq p) = P$ restricted to conditions extending p.

Assume that $0^\#$ exists. Then any $L[0^\#]$-*countable* $P \in L$ is totally relevant, as there are only countably many constructible subsets of P (using the fact that ω_1 is inaccessible in L). Note that this includes the case of any forcing $P \in L$ definable in L without parameters.

The situation is far less clear for uncountable $P \in L$. The next result treats the case of κ-Cohen forcing.

Proposition 4.13. Suppose κ is L-regular and let $P(\kappa)$ denote κ-Cohen forcing in L: Conditions are constructible $p : \alpha \longrightarrow 2$, $\alpha < \kappa$ and $p \leq q$ iff p extends q.

(a) If κ has cofinality ω in $L[0^\#]$ then $P(\kappa)$ is totally relevant.
(b) If κ has uncountable cofinality in $L[0^\#]$ then $P(\kappa)$ is not relevant.

Proof. Let j_n denote the first n Silver indiscernibles $\geq \kappa$.

(a) We use the fact that $P(\kappa)$ is κ-distributive in L. Let $\kappa_0 < \kappa_1 < \dots$ be an ω-sequence in $L[0^\#]$ cofinal in κ. Then any $D \subseteq P(\kappa)$ in L belongs to $\mathrm{Hull}(\kappa_n \cup j_n)$ for some n, where Hull denotes Skolem hull in L. As $\mathrm{Hull}(\kappa_n \cup j_n)$ is constructible of L-cardinality $< \kappa$ we can use the κ-distributivity of $P(\kappa)$ to choose $p_0 \geq p_1 \geq \dots$ successively below any $p \in P(\kappa)$ to meet all dense $D \subseteq P(\kappa)$ in L.

(b) Note that in this case $\kappa \in \mathrm{Lim}\, I$, as otherwise $\kappa = \cup\{\kappa_n | n \in \omega\}$ where $\kappa_n = \cup(\kappa \cap \mathrm{Hull}(\bar{\kappa} + 1 \cup j_n)) < \kappa, \bar{\kappa} = \max(I \cap \kappa)$, and hence κ has $L[0^\#]$-cofinality ω. Suppose $G \subseteq P(\kappa)$ were $P(\kappa)$-generic over L. For any $p \in P(\kappa)$ let $\alpha(p)$ denote the domain of p. Define $p_0 \geq p_1 \geq \dots$ in G so that $\alpha(p_{n+1}) \in I$ and p_{n+1} meets all dense $D \subseteq P(\kappa)$ in $\mathrm{Hull}(\alpha(p_n) \cup j_n)$. Then $p = \cup\{p_n | n \in \omega\}$ meets all dense $D \subseteq P(\kappa)$ in $\mathrm{Hull}(\alpha \cup j)$ where $\alpha = \cup\{\alpha(p_n) | n \in \omega\} \in I$, $j = \cup\{j_n | n \in \omega\}$. But then p is $P(\alpha)$-generic over L, as every constructible dense $\bar{D} \subseteq P(\alpha)$ is of the form $D \cap P(\alpha)$ for some D as above. So p is not constructible, contradicting $p \in G$. □

As a consequence we see that the basic class forcing examples of Easton and Long Easton forcing are not relevant. However, we can rescue these forcings by restricting to successor cardinals, thereby not adding κ-Cohen sets for κ of uncountable $L[0^\#]$-cofinality. *Easton forcing at Successors* is defined as follows: Conditions are constructible $p : \alpha(p) \longrightarrow L$ where for $\alpha < \alpha(p)$, $p(\alpha) = \emptyset$ unless α is a successor cardinal of L, in which case $p(\alpha) \in \alpha$-Cohen forcing; we also require that if α is L-inaccessible then $\{\beta < \alpha | p(\beta) \neq \emptyset\}$ is bounded in α. Extension is defined in the natural way: $p \leq q$ iff $p(\alpha)$ extends $q(\alpha)$ for each $\alpha < \alpha(q)$. *Long Easton forcing at Successors* is obtained from Easton forcing at Successors by dropping the support condition.

Theorem 4.14. Let P be Easton forcing at Successors, the basic example of Reverse Easton forcing or Long Easton forcing at Successors. Then P is totally relevant.

Indiscernible Preservation

Though we have shown Easton at Successors and Reverse Easton to be totally relevant, we can further ask for a generic class that preserves indiscernibles. This is important in the context of Jensen coding, as we can only code a class by a real (in $L[0^{\#}]$) if the class preserves (a periodic subclass of the) indiscernibles.

It is too much to ask that every condition p be included in a generic class that preserves indiscernibles, as p itself may not (only $2^{\aleph_0}$ subclasses of L can).

Definition. A class $A \subseteq L$ *preserves indiscernibles* if I is a class of indiscernibles for the structure $\langle L[A], A\rangle$.

Theorem 4.15. For each of Easton at Successors, Reverse Easton, Thin Easton at Successors, Coherent Easton at Successors and Long Easton at Successors there is a generic class G that preserves indiscernibles.

The Coding Theorem

The class forcings discussed in the previous two chapters provide examples of set-theoretic universes which neither contain $0^{\#}$ nor are obtainable by forcing over L by the traditional method of forcing, with sets of conditions. Notice however that these universes are "locally set-generic" over L: Each of their sets belongs to an intermediate set-generic extension of L.

Solovay posed three questions the solutions to which require use of a new kind of forcing, where *sets* are produced using a *class* of forcing conditions. Jensen developed this technique to prove his Coding Theorem, which says that any universe can be class-generically extended to one of the form $L[R]$, R a real. We now introduce the Solovay questions and prove a special case of the Coding Theorem, in which we assume that $0^{\#}$ is not present in the universe to be coded.

Three Questions of Solovay

Solovay's three problems each demand the existence of a real that neither constructs $0^{\#}$, nor is in a set-generic extension of L.

Definition. If x, y are sets of ordinals then we write $x \leq_L y$ for $x \in L[y]$ and $x <_L y$ for $x \leq_L y, y \not\leq_L x$.

The Genericity Problem. Does there exist a real $R <_L 0^\#$ such that R does not belong to a set-generic extension of L?

It was to affirmatively answer this question that Jensen proved his Coding Theorem. Roughly speaking he showed that if G is generic for Easton forcing at Successors and G preserves indiscernibles then there is a real $R <_L 0^\#$, obtained by class forcing over $\langle L[G], G\rangle$, such that $L[G] \subseteq L[R]$ and G is definable over $L[R]$. Then R does not belong to a set-generic extension of L as $L[G]$ is not included in any set-generic extension of L.

Solovay's second problem concerns definability of reals.

Definition. R is an *Absolute Singleton* if for some formula φ, R is the unique solution to φ in every inner model containing R.

Shoenfield's Absoluteness Theorem states that if φ is Π^1_2 (i.e., of the form $\forall R \exists S \psi$, ψ arithmetical) then $\varphi(R) \longleftrightarrow M \vDash \varphi(R)$ where M is any inner model containing R. Thus any Π^1_2-Singleton (i.e., the unique solution to a Π^1_2 formula) is an Absolute Singleton; $0^\#$ is an example. Also 0 is trivially an example. Solovay asked if there are any examples lying strictly between these two.

The Π^1_2-Singleton Problem. Does there exists a real R, $0 <_L R <_L 0^\#$ such that R is a Π^1_2-Singleton?

Note that it follows from the Covering Theorem (relative to R) that if $R <_L 0^\#$ then $R^\# \in L[0^\#]$ where $R^\#$ is defined relative to $L[R]$ the way we defined $0^\#$ relative to L. In particular $L_{\aleph_1}[R]$ is elementary in $L[R]$ and therefore if R is in a P-generic extension of L, $P \in L$ then there is such a P in $L_{\aleph_1}$. As $\aleph_1$ is inaccessible in L, there are only countably-many subsets of P in L and therefore we can build a P-generic containing any condition in P. So we conclude that if R is a nonconstructible real in a P-generic extension of L then R cannot be a Π^1_2-Singleton, as there must be other P-generic extensions with reals $R' \neq R$ satisfying any Π^1_2 formula satisfied by R. This is why the Π^1_2-Singleton Problem requires Jensen's method: An affirmative answer to the Π^1_2-Singleton Problem implies an affirmative answer to the Genericity Problem.

Solovay's third problem concerns Admissibility Spectra. Let T be a subtheory of ZF and R a real. The *T-spectrum* of R, $\Lambda_T(R)$, is the class of all ordinals α such that $L_\alpha[R] \vDash T$. A general problem is to characterize the possible T-spectra of reals for various theories T. An important special case is when $T = T_0 = (ZF$ without the Power Set Axiom and with Replacement restricted to Σ_1 formulas). We may refer to T_0 as "admissibility theory," as an ordinal α is R-admissible if and only if it is ω or belongs to the T_0-spectrum of R. We refer to the T_0- spectrum of R as the *admissibility spectrum* of R and denote it by $\Lambda(R)$.

There are some basic facts which limit the possibilities for $\Lambda(R)$: First, if R belongs to a set-generic extension of L then $\Lambda(R)$ contains $\Lambda - \beta$ for some ordinal β, where $\Lambda = \Lambda(0)$. This is because if $\alpha \in \Lambda$, $P \in L_\alpha$ then $L_\alpha[G] \vDash T_0$ for P-generic G. Second, if $0^\# \leq_L R$ then $\Lambda(R) - \beta \subseteq L$-inaccessibles for some β. This is because if $0^\# \in L_\beta[R]$ then every α in $\Lambda(R) - \beta$ is in $\Lambda(0^\#)$ and hence is a Silver indiscernible.

Thus to get a nontrivial admissibility spectrum for R without $0^\#$ we need Jensen's methods. An ordinal is *recursively inaccessible* if it is admissible and also the limit of admissibles.

The Admissibility Spectrum Problem. Does there exist a real $R \leq_L 0^\#$ such that $\Lambda(R) =$ the recursively inaccessible ordinals?

Of course we must in fact have $R <_L 0^\#$ as otherwise $\Lambda(R)$ is too small.

The Coding Theorem without $0^\#$

We prove the following result of Jensen.

Theorem 4.16. Suppose that $A \subseteq$ ORD and $\langle L[A], A\rangle$ is a model of ZFC + GCH + $0^\#$ does not exist. Then there is an $\langle L[A], A\rangle$-definable class forcing P such that if $G \subseteq P$ is P-generic over $\langle L[A], A\rangle$:

(a) $\langle L[A, G], A, G\rangle$ is a model of ZFC + GCH.
(b) $L[A, G] = L[R]$ for some real R and A, G are definable over $L[R]$ from the parameter R.
(c) $L[A]$ and $L[R]$ have the same cofinalities.

The proof makes use of the following consequence of the Covering Theorem.

Fact. Assume that $0^\#$ does not exist. If $j : L_\alpha \longrightarrow L_\beta$ is Σ_1-elementary, $\alpha \geq \omega_2$ and $\kappa =$ critical point of j then $\alpha < (\kappa^+)^L$.

We make the following assumption about the predicate A: If H_α, α an infinite $L[A]$-cardinal, denotes $\{x \in L[A] \mid$ transitive closure (x) has $L[A]$-cardinality $< \alpha\}$ then we assume that $H_\alpha = L_\alpha[A]$. This is easily arranged using the fact that the GCH holds in $L[A]$.

The basic idea of the proof is simple. Let Card denote all infinite $L[A]$-cardinals. Also Card $^+ = \{\alpha^+ \mid \alpha \in$ Card $\}$ and Card $' =$ all uncountable limit cardinals. If $a \subseteq \alpha^{++}$, $\alpha \in$ Card we can attempt to "code" a by $b \subseteq \alpha^+$ as follows. We associate a subset b_ξ of α^+ to each $\xi < \alpha^{++}$ and design b so that $\xi \in a$ iff b, b_ξ are almost disjoint, i.e. have intersection bounded in α^+. There is a natural forcing R^a for doing this, invented by Solovay. A condition in R^a is a pair $(p, \bar{p})$ where p is a bounded subset of α^+ and $\bar{p}$ consists of at most α-many b_ξ's with $\xi \in a$. When extending $(p, \bar{p})$ to $(q, \bar{q})$, q must end-extend p, $\bar{q}$ must contain $\bar{p}$ and $q - p$ must be disjoint from all b_ξ in $\bar{p}$.

Of course the forcing R^a does not really code a by a subset of α^+ without some assumptions about the b_ξ's. For example each b_ξ should be almost disjoint from the union of α-many other b_ξ's; this is easy to arrange. More seriously, we need to know how to find b_ξ in $L[a \cap \xi]$ in a uniform way, so that a can be inductively recovered from our generic $b \subset \alpha^+$. The latter is possible only if $\xi < \alpha^{++} \longrightarrow L[a \cap \xi] \models$ Card $(\xi) \leq \alpha^+$. If this fails then we must first "reshape" a to make it true, by forcing with bounded subsets of α^{++} which do have this property up to their supremum.

It is not clear that the forcing for the purpose of reshaping a is cardinal-preserving unless we can apply it in $L[c]$, where c is an already-reshaped subset of α^{+++}. Jensen's solution to this problem is to both reshape $A \cap \alpha^+$ and code $A \cap \alpha^+$ into a subset of α, for all α simultaneously. Then in effect, the forcing to reshape $A \cap \alpha^+$ takes place in $L[c]$ where c is a reshaped subset of α^{++} that codes $L[A]$.

As suggested in the previous paragraph there is a forcing analogous to R^a for coding a reshaped $a \subseteq \alpha^+$ into a subset of α, for α a limit cardinal. Thus if we combine all of these forcings we obtain a single forcing P for coding A by a real. A condition is of the form $p = \langle (p_\alpha, p^*_\alpha) \mid \alpha \in$ Card $, \alpha \leq \alpha(p)\rangle$ where p_α is a (reshaped) bounded subset of α^+, p^*_α is the "restraint" imposed on p_α to ensure that p_{α^+} is coded, and where for $\alpha \in$ Card $'$ we require that $\langle p_{\bar{\alpha}} \mid \bar{\alpha} \in$ Card $\cap\, \alpha\rangle$ code p_α.

Proof of Theorem 4.16. Let α belong to Card.

Definition. (Strings) S_α consists of all $s : [\alpha, |s|) \longrightarrow 2$, $\alpha \leq |s| < \alpha^+$ such that $|s|$ is a multiple of α and for all $\eta \leq |s|$, $L_\delta[A \cap \alpha, s \restriction \eta] \models \text{Card}\ (\eta) \leq \alpha$ for some $\delta < (\eta^+)^L \cup \omega_2$.

Thus for α equal to ω or ω_1, elements of S_α are "reshaped" in the natural sense, but for $\alpha \geq \omega_2$ we insist that $s \in S_\alpha$ be "quickly reshaped" in that $\eta \leq |s|$ be collapsed relative to $A \cap \alpha$, $s \restriction \eta$ before the next L-cardinal. This will be important when we use $\sim 0^\#$ to establish cardinal-preservation, via the above-mentioned *Fact*. The requirement that $|s|$ be a multiple of α is a technical convenience. Elements of S_α are called "strings". Note that we allow the empty string $\emptyset_\alpha \in S_\alpha$, where $|\emptyset_\alpha| = \alpha$. For s, t in S_α we write $s < t$ for $s \leq t$, $s \neq t$.

Definition. (Coding Structures) For $s \in S_\alpha$ define $\mu^{<s}$, μ^s inductively by: $\mu^{<\emptyset_\alpha} = \alpha$, $\mu^{<s} = \cup\{\mu^t \mid t < s\}$ for $s \neq \emptyset_\alpha$ and $\mu^s =$ least limit of limit ordinals $\mu > \mu^{<s}$ such that $L_\mu[A \cap \alpha, s] \models s \in S_\alpha$. And $\mathcal{A}^s = L_{\mu^s}[A \cap \alpha, s]$.

Thus by definition, when $\alpha \geq \omega_2$ there is $\delta < \mu^s$ such that $L_\delta[A \cap \alpha, s] \models \text{Card}\ (|s|) \leq \alpha$ and $L_{\mu^s} \models \text{Card}\ (\delta) \leq |s|$. The requirement "limit of limit ordinals" on μ^s is a technical convenience.

Definition. (Coding Apparatus) For $\alpha > \omega$, $s \in S_\alpha$, $i < \alpha$ let $H^s(i) = \Sigma_1$ Skolem hull of $i \cup \{A \cap \alpha, s\}$ in $\mathcal{A}^s$ and $f^s(i) = \text{ordertype}\ (H^s(i) \cap \text{ORD})$. For $\alpha \in \text{Card}^+$, $b^s = \text{Range}\ (f^s \restriction B^s)$ where $B^s = \{i < \alpha \mid i = H^s(i) \cap \alpha\}$.

Note that if $s < t$ belong to S_α then Range f^s, Range f^t are almost disjoint in the sense that their intersection is bounded in α. The choice of $f^s \restriction B^s$ rather than f^s is a technical convenience.

Using the above we will construct a tame, cofinality-preserving forcing P for coding $\langle L[A], A\rangle$ by a subset G_ω of ω_1 which is reshaped in the sense that proper initial segments of (the characteristic function of) G_ω belong to S_ω. Then as G_ω can be coded into a real by a ccc forcing of size ω_1 by the Solovay technique mentioned earlier, the theorem follows.

Definition. (A Partition of the Ordinals) Let B, C, D, E denote the classes of ordinals congruent to $0, 1, 2, 3$ mod 4, respectively. For any ordinal α, α^B denotes the α^{th} element of B, when B is listed in increasing order and for any set of ordinals X, X^B denotes $\{\alpha^B \mid \alpha \in X\}$. Similarly for C, D, E.

Definition. (The Successor Coding) Suppose $\alpha \in$ Card and $s \in S_{\alpha^+}$. A *condition in* R^s is a pair (t, t^*) where $t \in S_\alpha, t^* \subseteq \{b^{s\restriction\eta} \mid \eta \in [\alpha^+, |s|)\} \cup |t|$, Card $(t^*) \leq \alpha$. Extension of conditions is defined by: $(t_0, t_0^*) \leq (t_1, t_1^*)$ iff $t_0 \subseteq t_1$, $t_1^* \subseteq t_0^*$ and:

1. $|t_1| \leq \gamma^B < |t_0|, \gamma \in b^{s\restriction\eta} \in t_1^* \longrightarrow t_0(\gamma^B) = 0$ or $s(\eta)$.
2. $|t_1| \leq \gamma^C < |t_0|, \gamma = \langle\gamma_0, \gamma_1\rangle$, $\gamma_0 \in A \cap t_1^* \longrightarrow t_0(\gamma^C) = 0$.

In (b) above, $\langle\cdot,\cdot\rangle$ is an L-definable pairing function on ORD so that Card $(\langle\gamma_0, \gamma_1\rangle) =$ Card $\gamma_0 +$ Card γ_1 in L for infinite γ_0, γ_1. An R^s-generic over $\mathcal{A}^s$ is determined by a function $T : \alpha^+ \longrightarrow 2$ such that $s(\eta) = 0$ iff $T(\gamma^B) = 0$ for sufficiently large $\gamma \in b^{s\restriction\eta}$ and such that for $\gamma_0 < \alpha^+ : \gamma_0 \in A$ iff $T(\langle\gamma_0, \gamma_1\rangle^C) = 0$ for sufficiently large $\gamma_1 < \alpha^+$.

Now we come to the definition of the limit coding, which incorporates the idea of "coding delays." Suppose $s \in S_\alpha$, $\alpha \in$ Card $'$ and $p = \langle(p_\beta, p_\beta^*) \mid \beta \in$ Card $\cap \alpha\rangle$ where $p_\beta \in S_\beta$ for each $\beta \in$ Card $\cap \alpha$. A natural definition of "p codes s" would be: For $\eta < |s|$, $p_\beta(f^{s\restriction\eta}(\beta)) = s(\eta)$ for sufficiently large $\beta \in$ Card $\cap \alpha$. There are a number of problems with this definition however. First, to avoid conflict with the Successor Coding we should use $f^{s\restriction\eta}(\beta)^D$ instead of $f^{s\restriction\eta}(\beta)$. Second, to lessen conflict with codings at $\beta \in$ Card $' \cap \alpha$ we only require the above for $\beta \in$ Card $^+ \cap \alpha$. However there are still difficulties in making sure that the coding of s is consistent with the coding of p_β by $p \restriction \beta$ for $\beta \in$ Card $' \cap \alpha$.

We introduce coding delays to facilitate extendibility of conditions. The rough idea is to code not using $f^{s\restriction\eta}(\beta)^D$, but instead just after the least ordinal $\geq f^{s\restriction\eta}(\beta)^D$ where p_β takes the value 1. In addition, we "precode" s by a subset of α, which is then coded with delays by $\langle p_\beta \mid \beta \in$ Card $\cap \alpha\rangle$; this "indirect" coding further facilitates extendibility of conditions.

Definition. Suppose $\alpha \in$ Card , $X \subseteq \alpha$, $s \in S_\alpha$. Let $\tilde{\mu}^s$ be defined just as we defined μ^s but with the requirement "limit of limit ordinals" replaced by the weaker condition "limit ordinal". Then note that $\tilde{\mathcal{A}}^s = L_{\tilde{\mu}^s}[A \cap \alpha, s]$ belongs to $\mathcal{A}^s$, contains s and $\Sigma_1\mathrm{Hull}(\alpha \cup \{A \cap \alpha, s\})$ in $\tilde{\mathcal{A}}^s = \tilde{\mathcal{A}}^s$. Now X *precodes* s if X is the Σ_1 theory of $\tilde{\mathcal{A}}^s$ with parameters from $\alpha \cup \{A \cap \alpha, s\}$, viewed as a subset of α.

Definition. (Limit Coding) Suppose $s \in S_\alpha$, $\alpha \in$ Card $'$ and $p = \langle(p_\beta, p_\beta^*) \mid \beta \in$ Card $\cap \alpha\rangle$ where $p_\beta \in S_\beta$ for each $\beta \in$ Card $\cap \alpha$. We wish to define

"p codes s". First we define a sequence $\langle s_\gamma \mid \gamma \leq \gamma_0\rangle$ of elements of S_α: Let $s_0 = \emptyset_\alpha$. For limit $\gamma \leq \gamma_0$, $s_\gamma = \cup\{s_\delta \mid \delta < \gamma\}$. Now suppose s_γ is defined and let $f_p^{s_\gamma}(\beta) =$ least $\delta \geq f^{s_\gamma}(\beta)$ such that $p_\beta(\delta^D) = 1$, if such a δ exists. If for cofinally many $\beta \in \text{Card}^+ \cap \alpha$, $f_p^{s_\gamma}(\beta)$ is undefined, then set $\gamma_0 = \gamma$. Otherwise define $X \subseteq \alpha$ by: $\delta \in X$ iff $p_\beta((f_p^{s_\gamma}(\beta)+1+\delta)^D) = 1$ for sufficiently large $\beta \in \text{Card}^+ \cap \alpha$. If Even (X) precodes an element t of S_α extending s_γ such that $\mathcal{A}^t$ contains X and $f_p^{s_\gamma}$, then set $s_{\gamma+1} = t$. Otherwise let $s_{\gamma+1}$ be $s_\gamma * X^E$, if $f_p^{s_\gamma}$ belongs to $\mathcal{A}^{s_\gamma * X^E}$; if not, then again $\gamma_0 = \gamma$. Now p *exactly codes* s if $s = s_\gamma$ for some $\gamma \leq \gamma_0$ and p *codes* s if $s \leq s_\gamma$ for some $\gamma \leq \gamma_0$.

Note that the Successor Coding only restrains p_β from taking certain nonzero values, so there is no conflict between the Successor Coding and these delays. The advantage of delays is that they give us more control over *where* the limit coding takes place, thereby enabling us to avoid conflict between the limit codings at different cardinals.

Definition. (The Conditions) A *condition in* P is a sequence $p = \langle (p_\alpha, p^*_\alpha) \mid \alpha \in \text{Card}, \alpha \leq \alpha(p)\rangle$ where $\alpha(p) \in \text{Card}$ and:

1. $p_{\alpha(p)}$ belongs to $S_{\alpha(p)}$ and $p^*_{\alpha(p)} = \emptyset$.

2. For $\alpha \in \text{Card} \cap \alpha(p)$, (p_α, p^*_α) belongs to $R^{p_{\alpha^+}}$.

3. For $\alpha \in \text{Card}'$, $\alpha \leq \alpha(p)$, $p \restriction \alpha$ belongs to $\mathcal{A}^{p_\alpha}$ and exactly codes p_α.

4. For $\alpha \in \text{Card}'$, $\alpha \leq \alpha(p)$ if α is inaccessible in $\mathcal{A}^{p_\alpha}$ then there exists a CUB$C \subseteq \alpha$, $C \in \mathcal{A}^{p_\alpha}$ such that $p^*_\beta = \emptyset$ for $\beta \in C$.

For $\alpha \in \text{Card}$, $P^{<\alpha}$ denotes the set of all conditions p such that $\alpha(p) < \alpha$. Conditions are ordered by: $p \leq q$ iff $\alpha(p) \geq \alpha(q)$, $p(\alpha) \leq q(\alpha)$ in $R^{p_{\alpha^+}}$ for $\alpha \in \text{Card} \cap \alpha(p) \cap (\alpha(q)+1)$ and $p_{\alpha(p)}$ extends $q_{\alpha(p)}$ if $\alpha(q) = \alpha(p)$. Also for $s \in S_\alpha$, $\omega < \alpha \in \text{Card}$, P^s denotes $P^{<\alpha}$ together with all $p \restriction \alpha$ for conditions p such that $\alpha(p) = \alpha$, $p_{\alpha(p)} \leq s$. To order conditions in P^s, first define p^+ for p in P^s as follows: $p^+ = p$ for $p \in P^{<\alpha}$; for $p \in P^s - P^{<\alpha}$, $p^+ \restriction \alpha = p$ and $p^+(\alpha) = (s \restriction \eta, \emptyset)$ where η is least such that $p \in P^{s\restriction\eta}$. Now $p \leq q$ in P^s iff $p^+ \leq q^+$ in P. Finally, $P^{<s} = \cup\{P^{s\restriction\eta} \mid \eta < |s|\} \cup P^{<\alpha}$.

It is worth noting that (3) above implies that f^{p_α} dominates the coding of p_α by $p \restriction \alpha$, in the sense that f^{p_α} strictly dominates each $f^{p_\alpha\restriction\eta}_{p\restriction\alpha}$, $\eta < |p_\alpha|$

on a tail of Card $^+ \cap \alpha$. The purpose of (d) is to guarantee that extendibility of conditions at (local) inaccessibles is not hindered by the Successor Coding (see the proof of Extendibility below).

We now embark on a series of lemmas which together show that P preserves cofinalities and that if G is P-generic over $\langle L[A], A\rangle$ then for some reshaped $X \subseteq \omega_1$, $L[A,G] = L[X]$ and A is $L[X]$-definable from the parameter X.

Distributivity for R^s Suppose $\alpha \in$ Card , $s \in S_{\alpha^+}$. Then R^s is α^+-distributive in $\mathcal{A}^s$: If $\langle D_i \mid i < \alpha\rangle \in \mathcal{A}^s$ is a sequence of dense subsets of R^s and $p \in R^s$ then there is $q \leq p$ such that q meets each D_i.

Proof. Choose $\mu < \mu^s$ to be a large enough limit ordinal such that p, $\langle D_i \mid i < \alpha\rangle$ and $\mathcal{A}^{<s}$ belong to $\mathcal{A} = L_\mu[A \cap \alpha^+, s]$. Let $\langle \alpha_i \mid i < \alpha\rangle$ enumerate the first α elements of $\{\beta < \alpha^+ \mid \beta = \alpha^+ \cap \Sigma_1\text{Hull of}(\beta \cup \{p, \langle D_i \mid i < \alpha\rangle, \mathcal{A}^{<s}\})$ in $\mathcal{A}\}$.

Now write p as (t_0, t_0^*) and successively extend p to (t_i, t_i^*) for $i \leq \alpha$ as follows: (t_{i+1}, t_{i+1}^*) is the least extension of (t_i, t_i^*) meeting D_i such that: (a) t_{i+1}^* contains $\{b^{s\restriction\eta} \mid \eta \in H_i \cap [\alpha^+, |s|)\}$ where $H_i = \Sigma_1\text{Hull of}\alpha_i \cup \{p, \langle D_i \mid i < \alpha\rangle, \mathcal{A}^{<s}\}$ in $\mathcal{A}$. (b) If $b^{s\restriction\eta} \in t_i^*, s(\eta) = 1$ then $t_{i+1}(\gamma^\beta) = 1$ for some $\gamma \in b^{s\restriction\eta}, \gamma > |t_i|$. (c) If $\gamma_0 \notin A, \gamma_0 < |t_i|$ then $t_{i+1}(\langle\gamma_0, \gamma_1\rangle^C) = 1$ for some $\gamma_1 > |t_i|$.

The lemma reduces to:

Claim. (t_λ, t_λ^*) = greatest lower bound to $\langle(t_i, t_i^*) \mid i < \lambda\rangle$ exists for limit $\lambda \leq \alpha$.

Proof of Claim. We must show that $t_\lambda = \cup\{t_i \mid i < \lambda\}$ belongs to S_α. Note that $\langle t_i \mid i < \lambda\rangle$ is definable over $\overline{H}_\lambda$ = transitive collapse of H_λ and by construction, t_λ codes $\overline{H}_\lambda$ definably over $L_{\bar{\mu}_\lambda}[t_\lambda]$, where $\bar{\mu}_\lambda$ = height of $\overline{H}_\lambda$. So t_λ is reshaped, as $|t_\lambda|$ is definably singular over $L_{\bar{\mu}_\lambda}[t_\lambda]$. By the *Fact*, $\bar{\mu}_\lambda < (|t_\lambda|^+)^L$ if $\alpha \geq \omega_2$. So t_λ belongs to S_α. □

The next lemma illustrates the use of coding delays.

Extendibility for P^s. Suppose that α is a limit cardinal, s belongs to S_α, and $p \in P^s$. Suppose also that $X \subseteq \alpha$ belongs to $\mathcal{A}^s$. Then there exists $q \leq p$ in P^s such that $X \cap \beta \in \mathcal{A}^{q_\beta}$ for each $\beta \in$ Card $\cap \alpha$.

Proof. By induction on α. Let $Y \subseteq \alpha$ be chosen so that Even(Y) precodes s and Odd(Y) is the Σ_1 theory of $\mathcal{A}$ with parameters from $\alpha \cup \{A \cap \alpha, s\}$, where $\mathcal{A}$ is an initial segment of $\mathcal{A}^s$ of limit height large enough to extend $\widetilde{\mathcal{A}}^s$ and contain X, p. For $\beta \in$ Card $\cap\, \alpha$ let $\overline{\mathcal{A}}_\beta$ be the transitive collapse of $H_\beta = \Sigma_1\text{Hull}(\beta \cup \{A \cap \alpha, s\})$ in $\mathcal{A}$ and suppose that β is large enough so that H_β contains p. If $H_\beta \cap \alpha = \beta$ then Even $(Y \cap \beta)$ precodes $s_\beta \in S_\beta$ where s_β is the pre-image of s under the natural embedding $\overline{\mathcal{A}}_\beta \longrightarrow \mathcal{A}$. If $H_\beta \cap \alpha \neq \beta$ then $|p_\beta| < (\beta^+)^{\overline{\mathcal{A}}_\beta}$, in which case f^{p_β} is dominated by the function $g(\gamma) = (\gamma^+)^{\overline{\mathcal{A}}_\gamma}$ on a final segment of Card $^+ \cap\, \beta$.

Now define q as follows: If Even($Y \cap \beta$) precodes $s_\beta \in S_\beta$, then $q_\beta = s_\beta$. For other $\beta \in$ Card $' \cap\, \alpha$, $q_\beta = p_\beta * (Y \cap \beta)^E$. For $\beta \in$ Card $^+ \cap\, \alpha$, $q_\beta = p_\beta * \vec{0} * 1 * (Y \cap \beta)^D$ where $\vec{0}$ has length $g(\beta)$.

As $g \restriction \beta$ and $Y \cap \beta$ are definable over $\overline{\mathcal{A}}_\beta$ for $\beta \in$ Card $' \cap\, \alpha$ we get $g \restriction \beta, Y \cap \beta \in \mathcal{A}^{s_\beta}$ when Even $(Y \cap \beta)$ precodes $s_\beta \in S_\beta$. Also $g \restriction \beta, Y \cap \beta \in \mathcal{A}^{q_\beta}$ for other $\beta \in$ Card $' \cap\, \alpha$ as Odd $(Y \cap \beta)$ codes $\overline{\mathcal{A}}_\beta$. And note that for sufficiently large $\beta \in$ Card $' \cap\, \alpha, g \restriction \beta$ dominates f^{p_β} on a final segment of Card $^+ \cap\, \beta$ (and hence $q \restriction \beta$ exactly codes q_β), unless Even $(Y \cap \beta)$ precodes s_β and $s_\beta = p_\beta$, in which case $q \restriction \beta$ exactly codes $q_\beta = s_\beta$ because $p \restriction \beta$ does.

So we conclude that for sufficiently large $\beta \in$ Card $' \cap\, \alpha, q \restriction \beta$ exactly codes q_β and $X \cap \beta \in \mathcal{A}^{q_\beta}$. Apply induction on α to obtain this for all $\beta \in$ Card $' \cap\, \alpha$. Finally, note that the only problem in verifying $q \leq p$ is that the restraint p^*_β may prevent us from making the extension q_β of p_β when $q_\beta = s_\beta$ and Even $(Y \cap \beta)$ precodes s_β. But property (4) in the definition of condition guarantees that $p^*_\beta = \emptyset$ for β in a CUB$C \subseteq \alpha$, $C \in \mathcal{A}^s$. We may assume that $C \in \mathcal{A}$ and hence for sufficiently large β as above we get $\beta \in C$ and hence $p^*_\beta = \emptyset$. So $q \leq p$ on a final segment of Card $\cap\, \alpha$, and we may again apply induction to get $q \leq p$ everywhere. □

We come now to the verification of distributivity for P^s. Before we can state and prove this property we need some preliminary definitions.

Definition. Suppose $i < \beta \in$ Card and $D \subseteq P^s$, $s \in S_{\beta^+}$. D is i^+*-predense* on P^s if $\forall p \in P^s \exists q \in P^s (q \leq p, q$ meets D and $q \restriction i^+ = p \restriction i^+)$. $X \subseteq$ Card $\cap\, \beta^+$ is *thin* if for each inaccessible $\gamma \leq \beta$, $X \cap \gamma$ is not stationary in γ. A function $f :$ Card $\cap\, \beta^+ \longrightarrow V$ is *small* if for each $\gamma \in$ Card $\cap\, \beta^+$, Card $(f(\gamma)) \leq \gamma$ and Support $(f) = \{\gamma \in$ Card $\cap\, \beta^+ \mid f(\gamma) \neq \emptyset\}$ is thin. If $D \subseteq P^s$ is predense and $p \in P^s$, $\gamma \in$ Card $\cap\, \beta^+$ we say that p *reduces* D *below* γ if for some $\delta \leq \gamma$ in Card $^+$, every $q \leq p$ can be extended to $r \leq q$ such that r

meets D and $r \restriction [\delta, \beta] = q \restriction [\delta, \beta]$. Finally, for $p \in P^s$, f small, f in $\mathcal{A}^s$ we define Σ^p_f to consist of all $q \leq p$ in P^s such that whenever $\gamma \in$ Card $\cap\, \beta^+$, $D \in f(\gamma)$, and D is predense on $P^{p_{\gamma^+}}$, we have that q reduces D below γ.

Distributivity for P^s. Suppose $s \in S_{\beta^+}$, $\beta \in$ Card .

1. If $\langle D_i \mid i < \beta \rangle \in \mathcal{A}^s$, D_i i^+-dense on P^s for each $i < \beta$ and $p \in P^s$ then there is $q \leq p$ such that q meets each D_i.

2. If $p \in P^s$, f small, f in $\mathcal{A}^s$ then there exists $q \leq p$, $q \in \Sigma^p_f$.

Proof. We demonstrate 1 and 2 by a simultaneous induction on β. If $\beta = \omega$ or belongs to Card $^+$ then by induction, 1 and 2 reduce to the following: If S is a collection of β-many predense subsets of P^s, $S \in \mathcal{A}^s$ then $\{q \in P^s \mid q$ reduces each $D \in S$ below $\beta\}$ is dense on P^s. The latter follows, since P^s factors as $R^s * Q$ where $R^s \Vdash Q$ is β^+-cc, and hence any $p \in P^s$ can be extended to $q \in P^s$ such that $D^q = \{r \mid r \cup q(\beta)$ meets $D\}$ is predense $\leq q \restriction \beta$ for each $D \in S$.

Now suppose that β is inaccessible. We first show that 2 holds for f, provided $f(\beta) = \emptyset$. First select a CUB $C \subseteq \beta$ in $\mathcal{A}^s$ such that $\gamma \in C \to f(\gamma) = \emptyset$ and extend p so that $f \restriction \gamma$, $C \cap \gamma$ belong to $\mathcal{A}^{p_\gamma}$ for each $\gamma \in$ Card $\cap\, \beta^+$. Then we can successively extend p on $[\beta_i^+, \beta_{i+1}]$ in the $L[A]$-least way so as to meet Σ^p_f on $[\beta_i^+, \beta_{i+1}]$, where $\langle \beta_i \mid i < \beta \rangle$ is the increasing enumeration of C. At limit stages λ, we still have a condition, as the sequence of first λ extensions belongs to $\mathcal{A}^{p_{\beta_\lambda}}$. The final condition, after β steps, is an extension of p in Σ^p_f.

Now we prove 1 in this case. Suppose $p \in P^s$ and $\langle D_i \mid i < \beta \rangle \in \mathcal{A}^s$ and D_i is i^+-dense on P^s for each $i < \beta$. Let $\mu_0 < \mu^s$ be a large enough limit ordinal so that $\langle D_i \mid i < \beta \rangle$, p and $\tilde{\mu}^s$ belong to $L_{\mu_0}[A \cap \beta^+, s]$. For $i < \beta$, μ_i denotes $\mu_0 + \omega \cdot i < \mu^s$. For any γ we let $H_i(\gamma)$ denote $\Sigma_1\text{Hull}(\gamma \cup \{\langle D_i \mid i < \beta \rangle, p, \tilde{\mu}^s, s, A \cap \beta^+\})$ in $L_{\mu_i}[A \cap \beta^+, s]$.

Let $f_i :$ Card $\cap\, \beta \to V$ be defined by: $f_i(\gamma) = H_i(\gamma)$ if $i < \gamma \in H_i(\gamma)$ and $f_i(\gamma) = \emptyset$ otherwise. Then each f_i is small in $\mathcal{A}^s$ and we inductively define $p = p^0 \geq p^1 \geq \cdots$ in P^s as follows: $p^{i+1} = L[A]$-least $q \leq p^i$ such that:

(a) $q(\beta)$ meets all predense $D \subseteq R^s$, $D \in H_i(\beta)$.
(b) q meets $\Sigma^{p^i}_{f_i}$ and D_i.
(c) $q \restriction i^+ = p^i \restriction i^+$.

For limit $\lambda \leq \beta$ we take p^λ to be the greatest lower bound to $\langle p^i \mid i < \lambda \rangle$, whose existence is guaranteed by the following Claim.

Claim. p^λ is a condition in P^s, where $p^\lambda(\gamma) = (\cup\{p^i_\gamma \mid i < \lambda\}, \cup\{p^{i\,*}_\gamma \mid i < \lambda\})$ for each $\gamma \in$ Card $\cap \beta^+$.

Suppose that γ belongs to $H_\lambda(\gamma) \cap \beta$. First we verify that $p^\lambda_\gamma = \cup\{p^i_\gamma \mid i < \lambda\}$ belongs to S_γ. Let $\bar{H}_\lambda(\gamma)$ be the transitive collapse of $H_\lambda(\gamma)$ and write $\bar{H}_\lambda(\gamma)$ as $L_{\bar{\mu}}[\bar{A}, \bar{s}], \bar{P} =$ image of $P^s \cap H_\lambda(\gamma)$ under transitive collapse, $\bar{\beta} =$ image of β under collapse. Also write $\bar{P}$ as $\bar{R}^{\bar{s}} * P^{\bar{G}_{\bar{\beta}}}$ where $\bar{G}$ denotes an $\bar{R}^{\bar{s}}$-generic (just as P^s factors as $R^s * P^{G_\beta}, G_\beta$ denoting an R^s-generic).

Now the construction of the p^i's (see conditions (a), (b)) was designed to guarantee: (i) $\bar{G}_{\bar{\beta}} = \{\bar{p} \in R^{\bar{s}} \mid \bar{p}$ is extended by some $\bar{p}^i(\bar{\beta}), i < \lambda\}$ is $R^{\bar{s}}$-generic over $\bar{H}_\lambda(\gamma)$, where $\bar{p}^i =$ image of p^i under collapse, and (ii) For each $\bar{\delta}$ in (Card $^+$ of $\bar{H}_\lambda(\gamma)$), $\gamma < \bar{\delta} < \bar{\beta}$, $\{\bar{p} \mid \bar{p}$ is extended by some $\bar{p}^i \restriction [\gamma, \bar{\delta})$ in $\bar{P}^{\bar{p}^i_{\bar{\delta}}}_\gamma\}$ is $\bar{P}^{\bar{G}_{\bar{\delta}}}_\gamma$-generic over $\mathcal{A}^{\bar{G}_{\bar{\delta}}} = \cup\{\mathcal{A}^{\bar{p}^i_{\bar{\delta}}} \mid i < \lambda\}$, where $\bar{P}^{\bar{G}_{\bar{\delta}}}_\gamma = \cup\{\bar{P}^{\bar{p}^i_{\bar{\delta}}}_\gamma \mid i < \lambda\}$ and $\bar{P}^{\bar{p}^i_{\bar{\delta}}}_\gamma$ denotes the image under collapse of $P^{p^i_\delta}_\gamma = \{q \restriction [\gamma, \delta) \mid q \in P^{p^i_\delta}\}$, $\bar{\delta} =$ image of δ under collapse.

Note. We do *not* necessarily have property (ii) above for $\bar{\delta} = \bar{\beta}$, and this is the source of our need for $\sim 0^\#$ in this proof.

By induction, we have the distributivity of P^t for $t \in S_\delta$, $\delta \in$ Card $^+ \cap \beta$, and hence that of $\bar{P}^{\bar{t}}$ for $\bar{t} \in \bar{S}_{\bar{\delta}}$, $\bar{\delta} \in$ (Card $^+$ of $\bar{H}_\lambda(\gamma)$), $\bar{\delta} < \bar{\beta}$. So the "weak" genericity of the preceding paragraph implies that:

(d) $L_{\bar{\mu}}[A \cap \gamma, p^\lambda_\gamma] \vDash |p^\lambda_\gamma|$ is Σ_1-singular.

Also:

(e) $L_{\bar{\beta}}[A \cap \gamma, p^\lambda_\gamma] \vDash |p^\lambda_\gamma|$ is a cardinal.

Thus $p^\lambda_\gamma \in S_\gamma$ (by (d)) provided we can show that when $\gamma \geq \omega_2$, $\bar{\mu} < (|p^\lambda_\gamma|^+)^L$. But $\bar{H}_\lambda(\gamma) \xrightarrow{\sim} H_\lambda(\gamma)$ gives a Σ_1-elementary embedding with critical point $|p^\lambda_\gamma|$, so by the *Fact*, this is true.

The key point is that we also get $p^\lambda \restriction \gamma \in \mathcal{A}^{p^\lambda_\gamma}$, since $p^\lambda \restriction \gamma$ is definable over $\bar{H}_\lambda(\gamma)$ and we defined $\mathcal{A}^{p^\lambda_\gamma}$ to be large enough to contain $\bar{H}_\lambda(\gamma)$, since $L_{\bar{\beta}} \vDash |p^\lambda_\gamma|$ is a cardinal by (e) and $\bar{\beta}$ is a cardinal of $L_{\bar{\mu}}$.

The previous argument applies also if $\gamma = \beta$, using the distributivity of R^s, or if $\gamma = \beta \cap H_\lambda(\gamma)$, using the fact that p^λ_β collapses to p^λ_γ. If $\gamma < \gamma^* = \min(H_\lambda(\gamma) \cap [\gamma, \beta))$ then we can apply the first argument to get the result for γ^*, and then the second argument to get the result for γ.

Finally, to prove the Claim we must verify the restraint condition 4 in the definition of P. Suppose γ is inaccessible and for $i < \lambda$ let C^i be the least CUB subset of γ in $\mathcal{A}^{p^i_\gamma}$ disjoint from $\{\bar{\gamma} < \gamma \mid {p^i_{\bar{\gamma}}}^* \neq \emptyset\}$. If $\lambda < \gamma$ then $\cap\{C^i \mid i < \lambda\}$ witnesses the restraint condition for p^λ at γ. If $\gamma < \lambda$ then the restraint condition for p^λ at γ follows by induction on λ. And if $\gamma = \lambda$ then $\Delta\{C^i \mid i < \lambda\}$ witnesses the restraint condition for p^λ at γ, where Δ denotes diagonal intersection.

Thus the Claim and therefore 1 is proved in case β is inaccessible. To verify 2 in this case, note that as we have already proved 2 when $f(\beta) = \emptyset$, it suffices to show: If $\langle D_i \mid i < \beta\rangle \in \mathcal{A}^s$ is a sequence of dense subsets of P^s then every $p \in P^s$ cna be extended to $q \in P^s$ that reduces each D_i below β. But using 1 we see that $D^*_i = \{q \mid q \text{ reduces } D_i \text{ below } i^+\}$ is i^+-dense for each $i < \beta$, so again by 1 there is $q \leq p$ reducing D_i below i^+ for each i.

We are now left with the case where β is singular. The proof of 1 can be handled using the ideas from the inaccessible case as follows. Choose $\langle \beta_i \mid i < \lambda_0\rangle$ to be a continuous and cofinal sequence of cardinals $< \beta$, $\lambda_0 < \beta_0$. As before, we first we argue that $p \in P^s$ can be extended to meet Σ^p_f for any small f in $\mathcal{A}^s$ provided $f(\beta) = \emptyset$: Extend p if necessary so that for each $\gamma \in \text{Card} \cap \beta^+$, $f \restriction \gamma$ and $\{\beta_i \mid \beta_i < \gamma\}$ belong to $\mathcal{A}^{p_\gamma}$. Now perform a construction like the one used in the inaccessible case, successively extending p this time on $[\beta_0, {\beta_i}^+]$ so as to meet Σ^p_f on $[\beta_0, {\beta_i}^+]$ as well as $\Sigma^{p^i}_{f_i}$'s defined on $[\beta_0, {\beta_i}^+]$, to guarantee that p^λ is a condition for limit $\lambda \leq \lambda_0$. Note that each extension is made on a bounded initial segment of $[\beta_0, \beta)$ and therefore by induction $\Sigma^p_f, \Sigma^{p^i}_{f_i}$ can be met on these intervals. The result is that p can be extended to meet Σ^p_f on a final segment of Card $\cap\, \beta$ and therefore by induction can be extended to meet Σ^p_f. Second, use the density of Σ^p_f when $f(\beta) = \emptyset$ to carry out the proof of 1 as we did in the inaccessible case. And again, the general case of 2 follows from 1. This completes the proof. □

The argument of the previous lemma also shows:

Lemma 4.17. *P is Δ-distributive at κ for all regular κ.*

Thus P is tame and preserves cofinalities. As $L[A, G] = L[X]$ where

$X \subseteq \omega_1$, we also have GCH-preservation. This completes the proof of the Coding Theorem in the $\sim 0^\#$ case.

Theorem 4.18. Let P be the forcing used above, when $A = \emptyset$. Then there is a class G which is P-generic over L, which is definable in $L[0^\#]$ and which preserves indiscernibles.

Proof. For any indiscernible i let j_n be the first n indiscernibles $\geq i$. Then define $s_n \in S^{i^+}$ and $p^n \in P^{s_n}$ inductively, meeting the following conditions: $s_0 = \emptyset$ and p^n is the trivial condition. $s_{n+1} = \pi_i(p^n)_{i^+}$ where $\pi_i : L \to L$ is an elementary embedding with critical point i, and p^{n+1} is the least $q \leq p^n$ in P^{s_n} meeting $\Sigma^{p^n}_{f_n}$ where for $\beta \in \text{Card} \cap i^+$, $f_n(\beta) = \text{Hull}(\beta \cup j_n)$ if $\beta \in \text{Hull}(\beta \cup j_n)$ and $f_n(\beta) = \emptyset$ otherwise. (When $\beta = i$ we take $p^n_{\beta^+}$ to be s_n.) Let $G^i_0 = \{p \mid p$ is extended by some $p^n\}$.

G^i_0 need *not* be P^{s_n}-generic over $\mathcal{A}^{s_n}$ as all conditions in G^i_0 have empty restraint at indiscernibles $< i$. But notice that for $i_0 < i_1 < \cdots < i_n \leq i$ in I, $G^{i_0}_0 \cup \cdots \cup G^{i_n}_0$ is a compatible set of conditions. We take G^i to be $\{p \mid p$ is extended by $q_0 \wedge \cdots \wedge q_n$ for some choice of $q_l \in G^{i_l}_0$, $i_0 < \cdots < i_n \leq i$ in $I\}$. Now we claim that G^i is P^{s_n}-generic over $\mathcal{A}^{s_n}$ for each n. Indeed, if D is predense on P^{s_n} and belongs to $\mathcal{A}^{s_n}$, $D \in \text{Hull}(\{k_0, \cdots, k_m\} \cup j_n)$ with $k_0 < \cdots < k_m < i$ in I then p^{n+1} reduces D below k^+_m, p^{n+2} reduces D below $k^+_{m-1}, \cdots$ and eventually we get p^{n+m+2} in G^i meeting D.

It follows that $G^i(< i) = G^i \cap P^i$ is generic over L_i (for L_i-definable dense sets) and hence G is P-generic over L where $G = \cup\{G^i(< i) \mid i \in I\}$. Clearly G preserves indiscernibles. □

Corollary 4.19 (to proof). If $A \subseteq \text{ORD}$ preserves indiscernibles and $L[A]$ satisfies GCH then there is a real $R \in L[A, 0^\#]$ such that R preserves indiscernibles and A is definable in $L[R]$. If $L[A] \vDash \text{GCH}$ then $L[A], L[R]$ have the same cofinalities.

In fact, it is possible to characterize those $A \subseteq \text{ORD}$ which are coded by reals R such that $0^\# \not\leq_L R$:

Definition. For $\alpha, \beta < \omega_1$, $\beta \neq 0$ let $I_{\alpha,\beta} = \{i_{\alpha+\beta\cdot\gamma} \mid \gamma \in \text{ORD}\}$ where $\langle i_\alpha \mid \alpha \in \text{ORD}\rangle$ is the increasing enumeration of I.

Corollary 4.20. If $A \subseteq \text{ORD}$ and for some $\alpha, \beta < \omega_1$ the class $I_{\alpha,\beta}$ forms a generating class of indiscernibles for $\langle L[A], A\rangle$ then A is definable in $L[R]$ for some real R such that $0^\# \notin L[R]$.

One can use the preceding Corollary to show that $A \subseteq \mathrm{ORD}$ is definable in $L[R]$ for some real R, $0^{\#} \notin L[R]$ iff $I_{\alpha,\beta}$ forms a class of indiscernibles for $\langle L[A], A\rangle$ for some $\alpha, \beta < \omega_1$, $\beta \neq 0$. Moreover there are reals R such that $I^R = I_{\alpha,\beta}$, for any $\alpha, \beta < \omega_1$, $\beta \neq 0$ (where I^R denotes the Silver indiscernibles for $L[R]$).

Solution to the Genericity Problem

Theorem 4.21. (Jensen) There is a real $R <_L 0^{\#}$ that is not set-generic over L.

Proof. Take $R \in L[0^{\#}]$ to result from applying the proof of the Coding Theorem to the ground model $\langle L, \emptyset\rangle$, obtaining a generic G coded by R. Note that in $L[G] = L[R]$ there are $P(\kappa^+)$-generic sets for each infinite successor L-cardinal κ^+, where $P(\kappa^+) = \kappa^+$-Colen forcing. In a P-generic extension of L, where $P \in L$, there can be no κ^+-Cohen set where $\kappa = L$-cardinality (P). So $L[R]$ is not a set-generic extension of L. □

Note also that R as in the previous Theorem can be chosen to preserve both L-cofinalities and indiscernibles.

The other two Solovay problems, the Π^1_2-Singleton and Admissibility Spectrum problems, also have positive solutions via further elaborations on the coding method.

4.5 More about $0^{\#}$

So far we have examined the following topics, using the indicated techniques.

1. Constructibility: Fine structure theory, developed to study the generalised Suslin problem.

2. Set Forcing over L: Iterated forcing with finite and countable support, developed to study the Suslin problem and the Borel conjecture.

3. Class Forcing over L: The coding method, developed to study genericity over L.

Next we came to $0^{\#}$, which we introduced to organise the study of class forcing. Soon we will generalise $0^{\#}$ to a "# operation", which will lead us to inner models for large cardinals. But first we take a closer look at the motivation for introducing $0^{\#}$ in the first place.

It will be convenient to work not with the usual theory ZFC, but with an appropriate class theory. This allows us to discuss classes which are not necessarily definable. For an inner model M, a class A *belongs to* M iff $A \cap x$ belongs to M for every set x in M.

$V = L$ is not a theorem of class theory: The forcing method allows us to consistently enlarge L to models $L[G] \neq L$ where G is a class that is P-generic over L for some *L-forcing* P, i.e., some partial ordering P that belongs to L.

Assume that generic extensions of L do exist, and let us see what implications this has for the nature of the set-theoretic universe.

Definition. A class C of ordinals is CUB (closed and unbounded) iff it is a proper class of ordinals which contains all of its limit points. A class X of ordinals is *large* iff it contains a CUB subclass.

Largeness is not absolute: It is possible that a class X belonging to L is not large but becomes large after expanding the universe by forcing.

Definition. A class X is *potentially large* iff it is large in a generic extension of the universe.

Can the universe be *CUB-complete over* L in the sense that every class which belongs to L and is potentially large is already large? Yes, if $0^\#$ exists, as then each class of ordinals belonging to L either contains or is disjoint from a CUB class. We now show that this in fact leads to a characterisation of $0^\#$.

Theorem 1. There exists a sequence X_n, $n \in \omega$ of classes such that:
1. Each X_n belongs to L and indeed the relation "α belongs to X_n" is definable in L.
2. $X_n \supseteq X_{n+1}$ for each n and each X_n is potentially large.
3. If each X_n is large then $0^\#$ exists and therefore the universe is CUB-complete over L.

Thus we have the following picture: Let n be least so that X_n is not large, if such a finite n exists, and $n = \infty$ otherwise. If n is finite then n can be increased by going to a generic extension of the universe, further increased by going to a further generic extension, and so on. The only alternative is that the universe is CUB-complete over L, i.e., that $0^\#$ exists.

Proof of Theorem 1. We show the following:

$(*)$ There exists an L-definable function $n : L\text{-Singulars} \to \omega$ such that if M is an inner model, $0^\# \notin M$:
(a) For some k, $M \vDash \{\alpha \mid n(\alpha) \leq k\}$ is stationary.
(b) For each k there is a generic extension of M in which $0^\#$ does not exist and $\{\alpha \mid n(\alpha) \leq k\}$ is non-stationary.

"Stationary in M" means "intersects every CUB class which belongs to M".

We define $n(\alpha)$. Let $\langle C_\alpha \mid \alpha\ L\text{-singular}\rangle$ be an L-definable $\Box$-sequence: C_α is closed unbounded in α, ordertype $C_\alpha < \alpha$ and $\bar\alpha \in \lim C_\alpha \to C_{\bar\alpha} = C_\alpha \cap \bar\alpha$. Let ot C_α denote the ordertype of C_α. If ot C_α is L-regular then $n(\alpha) = 0$. Otherwise $n(\alpha) = n(\text{ot } C_\alpha) + 1$.

(a) is clear, as otherwise (using the fact that we are working in a sufficiently strong class theory) there is a closed unbounded $C \subseteq L$-regulars amenable to M, contradicting the Covering Theorem and the hypothesis that $0^\#$ does not belong to M.

Now we prove (b). Fix $n \in \omega$. In M let P consist of closed, bounded $p \subseteq$ ORD such that $\alpha \in p \to \alpha$ L-regular or $n(\alpha) \geq n+1$, ordered by $p \leq q$ iff p end extends q.

We claim that P is ∞-distributive in M. Suppose that $p \in P$ and $\langle D_\alpha \mid \alpha < \kappa\rangle$ is a definable sequence of open dense subclasses of P, κ regular. We wish to find $q \leq p$, $q \in D_\alpha$ for all $\alpha < \kappa$. Let C be the class of all strong limit cardinals β such that $D_\alpha \cap V_\beta$ is dense in $P \cap V_\beta$ for all $\alpha < \kappa$, a closed unbounded class of ordinals. It suffices to show that $C \cap \{\beta \mid n(\beta) \geq n+1\}$ has a closed subset of ordertype $\kappa+1$, for then p can be successively extended $\kappa+1$ times meeting the D_α's, to conditions with maxima in $\{\beta \mid n(\beta) \geq n+1\}$; the final condition (at stage κ) extends p and meets each D_α.

Lemma. Suppose $m > k$, $\alpha > \omega$ is regular and C is a closed set of ordertype $\alpha^{+m}+1$, consisting of ordinals greater than α^{+m} (where $\alpha^{+0} = \alpha$, $\alpha^{+(p+1)} = (\alpha^{+p})^+$). Then $C \cap \{\beta \mid n(\beta) \geq k\}$ has a closed subset of ordertype $\alpha^{+(m-k-1)}+1$.

Proof of Lemma. We shall use the following easy consequence of the Covering Theorem.

If $0^\#$ does not exist, $\beta \geq \omega_2$ and $\mathrm{cof}\,\beta < \mathrm{Card}\,\beta$ then β is singular in L.

We prove the lemma by induction on k. Suppose $k = 0$. Let β be the $(\alpha^{+(m-1)}+1)$st element of C. Then β is L-singular since it is at least ω_2 and its cofinality $(= \alpha^{+(m-1)})$ is less than its cardinality $(\geq \alpha^{+m})$. Similarly, each element of $\mathrm{Lim}(C \cap \beta)$ is L-singular and therefore $\mathrm{Lim}(C \cap \beta)$ is a closed subset of $C \cap \{\beta \mid n(\beta) \geq 0\}$ of ordertype $\alpha^{+(m-1)}+1$, as desired.

Suppose that the lemma holds for k and let $m+1 > k+1$, C a closed set of ordertype $\alpha^{+(m+1)}+1$ consisting of ordinals greater than $\alpha^{+(m+1)}$. Let β be the $(\alpha^{+m}+\alpha^{+m}+1)$st element of C. β is L-singular as it is at least ω_2 and its cofinality is less than its cardinality; so C_β is defined. Let $\bar\beta$ be the $(\alpha^{+m}+1)$st element of C. Then $\bar C = \{\mathrm{ot}\ C_\gamma \mid \gamma \in C \cap \mathrm{Lim}\, C_\beta \cap [\bar\beta, \beta]\}$ is a closed set of ordertype $\alpha^{+m}+1$ consisting of ordinals greater than α^{+m}. By induction there is a closed $\bar D$ contained in $\bar C \cap \{\gamma \mid n(\gamma) \geq k\}$ of ordertype $\alpha^{+(m-k-1)}+1$. But then $D = \{\gamma \in C \cap \mathrm{Lim}\, C_\beta \mid \mathrm{ot}\ C_\gamma \in \bar D\}$ is a closed subset of $C \cap \{\gamma \mid n(\gamma) \geq k+1\}$ of ordertype $\alpha^{+(m-k-1)}+1$. As $m-k-1 = (m+1)-(k+1)-1$ we are done.

By the lemma, $C \cap \{\beta \mid n(\beta) \geq n+1\}$ has arbitrary long closed subsets for any n, for any closed unbounded $C \subseteq$ ORD. It follows that P is ∞-distributive. Now to prove (b), we apply the forcing P to M, producing C witnessing the nonstationarity of $\{\alpha \mid n(\alpha) \leq n\}$. Of course this will not produce $0^{\#}$ as no sets are added. This completes the proof of $(*)$. Theorem 1 now follows, as we may take X_n to be $\{\alpha \mid$ Either α is regular in L or $n(\alpha) \geq n\}$.

We conjecture another way to obtain $0^{\#}$ through forcing, motivated by the following result.

Theorem 2. Assume slightly more than class theory (precisely: ORD is $\omega+\omega$-Erdős, defined below). If $0^{\#}$ exists, P is L-definable without parameters and there exists a P-generic, then there exists a P-generic definable in $L[0^{\#}]$. If P is L-definable with parameters and there exists a P-generic, then there exists a P-generic definable in a set-generic extension of $L[0^{\#}]$ (indeed, in any extension of $L[0^{\#}]$ in which those parameters are countable).

Thus the inner model $L[0^{\#}]$ is *saturated* with respect to L-definable forcings. If $0^{\#}$ exists, then it can be shown that any inner model which is saturated in this sense must contain $L[0^{\#}]$. (Reason: The L-definable forcing to add a CUB subclass to X_n using constructible, closed subsets of X_n has a generic in $L[0^{\#}]$. Thus if $0^{\#}$ exists, each X_n has a CUB subclass in any inner model which is saturated in the above sense, and therefore this inner model contains $0^{\#}$.) A stronger claim would be the following.

Conjecture. If the universe is saturated with respect to L-definable forcings in the sense of Theorem 2 then $0^{\#}$ exists.

To prove this Conjecture it would suffice to show that for X_n as in Theorem 1, not only does X_n contain a CUB subclass in a generic extension of the universe, but this can be accomplished via an L-definable forcing. This is reminiscent of the following.

Theorem (Baumgartner). Suppose that X is a constructible subset of ω_1 and X is stationary. Then there is a *constructible* set-forcing P which adds a CUB subset to X.

The P in this Theorem is a forcing which adds a CUB subset to X using "finite conditions". Is there a version of this result with ω_1 replaced by ORD, X replaced by X_n from Theorem 1?

We now turn to the proof of Theorem 2.

Definition. Let $\mathcal{A} = \langle V, \in, \ldots \rangle$ be a structure for a countable language. $I \subseteq$ ORD is a *good set of indiscernibles* for $\mathcal{A}$ iff $\gamma \in I \longrightarrow I - \gamma$ is a set of indiscernibles for $\langle \mathcal{A}, \alpha \rangle_{\alpha < \gamma}$.

Definition. ORD is α-Erdős iff whenever $\mathcal{A} = \langle V, \in, \ldots \rangle$ is a structure for a countable language, and C is CUB there exists $I \subseteq C$, ot $I = \alpha$ such that I is a good set of indiscernibles for $\mathcal{A}$.

The proof of Theorem 2 makes use of periodic subclasses of the Silver indiscernibles.

Definition. Let $I = \langle i_\gamma \mid \gamma \in \mathrm{ORD} \rangle$ be the increasing enumeration of the Silver indiscernibles. For any ordinals λ_0 and λ ($\lambda > 0$) define $I_{\lambda_0, \lambda} = \{ i_\alpha \mid \alpha$ of the form $\lambda_0 + \lambda \cdot \beta$, $\beta \in \mathrm{ORD}\}$. An L-definable forcing P is λ_0, λ*-periodic* iff in a set-generic extension of V, there is a P-generic G such that $I_{\lambda_0,\lambda}$ is a class of indiscernibles for $\langle L[G], \in, G \rangle$.

Fact. If P is λ_0, λ-periodic then P has a generic in a set-generic extension of $L[0^\#]$.

Proof Sketch. Assume that P is L-definable without parameters. Consider a set-generic extension M of $L[0^\#]$ in which λ_0 and λ are countable. Build a tree in M, a branch through which produces a generic G_0 for $P \cap L_{i_{\lambda_0 + \lambda \cdot \omega}}$ relative to which $I_{\lambda_0, \lambda} \cap i_{\lambda_0 + \lambda \cdot \omega}$ is a good set of indiscernibles. As P is λ_0, λ-periodic, this tree has a branch, therefore a branch in M, and the resulting G_0 can be "stretched" to a generic for P. If P is L-definable with parameters, then we require that those parameters be countable in M.

Proof of Theorem 2. Fix a P-generic G and assume that P is L-definable without parameters. We shall construct another P-generic G^* such that for some λ_0 and λ, $I_{\lambda_0,\lambda}$ is a class of indiscernibles for $\langle L[G^*], \in, G^* \rangle$. Let X be a good set of indiscernibles for $\langle L[0^\#, G], \in, G \rangle$ of ordertype $\omega + \omega$ such that $\alpha \in X \longrightarrow \langle L_\alpha[0^\#, G], G \rangle$ is an elementary submodel of $\langle L[0^\#, G], G \rangle$. (We refer to this last condition as the "stability" of α relative to $0^\#, G$.)

Select a canonical enumeration of the L-definable open dense subclasses of P : Thus let $\langle D_n | n \in \omega \rangle$ be a sequence of predicates such that each $D_n(x, \alpha_1 \ldots \alpha_n)$ is definable over L, $\{x \in L \mid D_n(x, \alpha_1 \ldots \alpha_n)\}$ is an open dense subclass of P for each $\alpha_1 < \ldots < \alpha_n$ in ORD and every L-definable open dense subclass of P is of this form for some n, for some $\alpha_1 < \ldots < \alpha_n$ in I. We may also assume that $\{\langle n, x, \vec{\alpha}\rangle | D_n(x, \vec{\alpha})\}$ is definable in L relative to a satisfaction predicate for L. For $\alpha_1 < \ldots < \alpha_n$ in ORD we abuse notation and write $D(\alpha_1 \ldots \alpha_n)$ for $\{x \in L | D_n(x, \alpha_1, \ldots \alpha_n)\}$. Also let $D^*(\alpha_1 \ldots \alpha_n) = \cap\{D(\vec{\beta}) | \vec{\beta} \subseteq \vec{\alpha}\}$.

Now we construct an ω-sequence of terms with Silver indiscernible parameters which we will use to define G^*.

For $j_0 \in X$ choose the least $t_{j_0}(\vec{k}_0(j_0), j_0, \vec{k}_1(j_0))$ in $D(j_0) \cap G$, where t_{j_0} is a Skolem term for $L, \vec{k}_0(j_0) < j_0 < \vec{k}_1(j_0)$ is an increasing sequence of Silver indiscernibles. By the good-indiscernibility of X, $t_{j_0} = t_0$, $\vec{k}_0(j_0) = \vec{k}_0$ are fixed. Thus we can write $t_0(\vec{k}_0, j_0, \vec{k}_1(j_0)) \in D(j_0) \cap G$ for $j_0 \in X$. By the stability relative to $0^{\#}, G$ of the elements of X we have: $j_0 < j_1$ in $X \longrightarrow \vec{k}_1(j_0) < j_1$.

Next for $j_0 < j_1$ in X choose the least $t_{j_0,j_1}(\vec{k}^1_0(j_0, j_1), j_0, \vec{k}^1_1(j_0, j_1), j_1, \vec{k}^1_2(j_0, j_1))$ in $D^*(\vec{k}_0, j_0, \vec{k}_1(j_0), j_1, \vec{k}_1(j_1)) \cap G$. By the good-indiscernibility of X we can write the above term with Silver indiscernible parameters as $t_1(\vec{k}^1_0, j_0, \vec{k}^1_1(j_0), j_1, \vec{k}^1_2(j_0, j_1))$. However, we want to argue that $\vec{k}^1_2(j_0, j_1)$ can be chosen independently of j_0. To arrange this, first note that $t_{j_0,j_1}(\vec{k}^1_0(j_0, j_1), j_0, \vec{k}^1_1(j_0, j_1), j_1, \vec{k}^1_2(j_0, j_1))$ $= t_{j_0,j_1}(\vec{k}^1_0(j_0, j_1), j_0, \vec{k}^1_1(j_0, j_1), j_1, \vec{k}^1_{2,0}(j_0, j_1), \vec{\infty})$ where the latter is independent of the choice of the Silver indiscernibles $\vec{\infty}$ above $\vec{k}^1_{2,0}(j_0, j_1)$ and where $(\vec{k}^1_0(j_0, j_1), \vec{k}^1_1(j_0, j_1), \vec{k}^1_{2,0}(j_0, j_1))$ is the least sequence of *ordinals* such that this term with parameters belongs to $D^*(\vec{k}_0, j_0, \vec{k}_1(j_0), j_1, \vec{k}_1(j_1)) \cap G \cap L_{\min \vec{\infty}}$. By the good-indiscernibility of X we can write this as $t_1(\vec{k}^1_0, j_0, \vec{k}^1_1(j_0), j_1, \vec{k}^1_{2,0}(j_0, j_1), \vec{\infty})$. Note that $(\vec{k}^1_0, \vec{k}^1_1(j_0), \vec{k}^1_{2,0}(j_0, j_1))$ is definable in $\langle L[G], G\rangle$ from $\vec{\infty}, \vec{k}_0, j_0, \vec{k}_1(j_0), j_1, \vec{k}_1(j_1)$ and therefore $\vec{k}^1_{2,0}(j_0, j_1)$ is definable in $\langle L[G], G\rangle$ from $\vec{\infty}, \vec{k}_1(j_1)$ and parameters $\leq j_1$.

Lemma. $\vec{k}^1_{2,0}(j_0, j_1)$ is independent of j_0.

Proof. Enumerate the first $\omega + 1$ elements of X in increasing order as $j_0 < j_1 < \ldots < j = (\omega + 1)$st element of X and for any m, n let $\vec{k}(j_n, j)$ (m) denote the m-th element of $\vec{k}^1_{2,0}(j_n, j)$. If the Lemma fails then for some fixed m, $\vec{k}(j_0, j)(m) < \vec{k}(j_1, j)(m) < \ldots$ forms an increasing ω-sequence of Silver indiscernibles with supremum $\ell \in I$. By the remark immediately preceding this Lemma, ℓ has cofinality $\leq j$ in $L[G]$. By Covering between L and $L[G]$, ℓ

has cofinality $< (j^+$ in $L[G])$ in L. This contradicts the following.

Claim. j^+ in $L[G] = j^+$ in L.

Proof of Claim. If not then in $L[G]$ there is a CUB $C \subseteq j$ such that C is almost contained in each CUB constructible $D \subseteq j$. But $I \cap j$ is the intersection of countably many such D and therefore as j is regular (in $L[G, 0^\#]$) we get that C is almost contained in I; so $0^\#$ belongs to $L[G]$, contradiction. This proves the Claim and hence the Lemma.

Thus we can write $t_1(\vec{k}_0^1, j_0, \vec{k}_1^1(j_0), j_1, \vec{k}_2^1(j_1)) \in D^*(\vec{k}_0, j_0, \vec{k}_1(j_0), j_1, \vec{k}_1(j_1)) \cap G$ for $j_0 < j_1$ in X. By modifying the term t_1 we may assume that $\vec{k}_1^1(j_0) = \vec{k}_2^1(j_0)$ for $j_0 \neq \min(X)$. Also we can assume that $\vec{k}_0 \subseteq \vec{k}_0^1, \vec{k}_1(j_0) \subseteq \vec{k}_1^1(j_0)$ for $j_0 \in X$ and moreover that the structure $\langle \vec{k}_1^1(j_0), < \rangle$ with a unary predicate for $\vec{k}_1(j_0)$ has isomorphism type independent of $j_0 \in X$.

We obtain t_2 in a similar way: thus,

$$t_2(\vec{k}_0^2, j_0, \vec{k}_1^2(j_0), j_1, \vec{k}_1^2(j_1), j_2, \vec{k}_1^2(j_2)) \in D^*(\vec{k}_0^1, j_0, \vec{k}_1^1(j_0), j_1, \vec{k}_1^1(j_1), j_2, \vec{k}_1^1(j_2)) \cap G$$

for $j_0 < j_1 < j_2$ in X and $\vec{k}_0^1 \subseteq \vec{k}_0^2, \vec{k}_1^1(j_0) \subseteq \vec{k}_1^2(j_0)$, $\langle \vec{k}_1^2(j_0), < \rangle$ with unary predicates for $\vec{k}_1^1(j_0), \vec{k}_1(j_0)$ has isomorphism type independent of j_0. Continue in this way to define $t_n(\vec{k}_0^n, j_0, \vec{k}_1^n(j_0), \ldots, j_n, \vec{k}_1^n(j_n))$ for each n and for $j_0 < \ldots < j_n$ in X. (The analogous version of the Lemma uses the first $\omega + n$ elements of X.)

Let $i_{\lambda_0} = \min X$ and $\lambda =$ ordertype $(\bigcup_n \vec{k}_1^n(j_0))$ for $j_0 \in X$, an ordinal independent of the choice of j_0.

We may assume that λ is a limit ordinal and in a generic extension where λ_0 is countable we may arrange that $\bigcup_n \vec{k}_0^n = I \cap i_{\lambda_0}$. Also note that $I - i_{\lambda_0}$ is a class of indiscernibles for L. Now in $V[g]$, where g is a Lévy collapse of i_{λ_0} to ω, carry out the above construction, arranging that $\bigcup_n \vec{k}_0^n = i_{\lambda_0}$. For any Silver indiscernible i_δ define $\vec{k}_1^n(i_\delta) \subseteq I \cap (i_\delta, i_{\delta+\lambda})$ so that $\langle I \cap (i_\delta, i_{\delta+\lambda}), < \rangle$ with a predicate for $\vec{k}_1^n(i_\delta)$ is isomorphic to $\left\langle \bigcup_n \vec{k}_1^n(j_0), < \right\rangle$ with a predicate for $\vec{k}_1^n(j_0)$, for $i_{\lambda_0} < j_0 \in X$. Define:

$G^* = \{p \in P \mid p$ is extended by some $t_n(\vec{k}_0^n, i_{\lambda_1}, \vec{k}_1^n(i_{\lambda_1}), \ldots i_{\lambda_n}, \vec{k}_1^n(i_{\lambda_n}))$ where $\lambda_0 \leq \lambda_1 < \ldots < \lambda_n$ are of the form $\lambda_0 + \lambda \cdot \alpha, \alpha \in \mathrm{ORD}\}$.

Using the indiscernibility of $I - i_{\lambda_0}$ we see that G^* is compatible and meets every L-definable open dense class on P. Thus P is λ_0, λ-periodic. This proves Theorem 2.

5. Extender Models

If we are willing to accept the existence of $0^\#$, then we surely should also admit the existence of $0^{\#\#}$, which relates to the model $L[0^\#]$ in the same way as $0^\#$ relates to L. Indeed, through iteration of a suitable "# operation", we are led to models much larger than L, which can satisfy strong large cardinal axioms. These are called *extender models.*

How do we define a # operation? We have said that the existence of $0^\#$ is equivalent to the non-rigidity of L, i.e., to the existence of a nontrivial elementary embedding from L to itself. Let us use this as a basis for generalisation. Suppose that M is an inner model satisfying ZFC and let $\pi : M \to M$ be a nontrivial elementary embedding from M to itself.

Theorem 1. (Kunen) Suppose $\pi : M \to M$ is a nontrivial elementary embedding. Then for some $x \in M$, $\pi \restriction x$ does not belong to M.

Proof of Theorem. First we prove the following Lemma.

Lemma 2. Let λ be an infinite cardinal such that $2^\lambda = \lambda^{\aleph_0}$. There exists a function $F : \lambda^\omega \to \lambda$ such that whenever A is a subset of λ of cardinality λ and $\gamma < \lambda$, there exists some $s \in A^\omega$ such that $F(s) = \gamma$.

Proof of Lemma 2. Let $\langle (A_\alpha, \gamma_\alpha) \mid \alpha < 2^\lambda \rangle$ be an enumeration of all pairs (A, γ) where $\gamma < \lambda$ and A is a subset of λ of cardinality λ. By induction on $\alpha < 2^\lambda$ choose $s_\alpha \in \lambda^\omega$ so that $s_\alpha \in A_\alpha^\omega$ and $s_\alpha \neq s_\beta$ for $\beta < \alpha$. Define $F(s)$ to be γ_α if $s = s_\alpha$ for some α, $F(s) = 0$ otherwise. This function F has the desired property, proving Lemma 2.

Our next lemma lists some general facts about elementary embeddings between inner models.

Lemma 3. Suppose that $\pi : M \to N$ is an elementary embedding of inner models with critical point κ. Then κ is a regular cardinal of M and $H^M_{\kappa^+}$ is contained in N. If H^N_κ is contained in M then κ is inaccessible in M.

Proof of Lemma 3. If κ were singular in M then choose $\gamma < \kappa$ and a cofinal $f : \gamma \to \kappa$ in M. Then $\pi(f)$ is cofinal in $\pi(\kappa)$, but $\pi(f) = f$ so κ cannot be the critical point of π. If X is a subset of κ in M, then $X = \pi(X) \cap \kappa$ so

X belongs to N. As any element of $H^M_{\kappa^+}$ is coded by a subset of κ in M, it follows that N contains each element of $H^M_{\kappa^+}$ as well.

Suppose that H^N_κ is contained in M. If κ is not inaccessible in M then choose $\gamma < \kappa$ and a surjection $f : 2^\gamma \to \kappa$ in M. Then $\pi(f)$ is a surjection from 2^γ of N onto $\pi(\kappa)$. By hypothesis 2^γ of $N = 2^\gamma$ of M and therefore $\kappa = \text{Range } f = \text{Range } \pi(f) = \pi(\kappa)$, a contradiction. This proves Lemma 3.

Now to prove the Theorem: Let κ be the critical point of π and define $\kappa_0 = \kappa$, $\kappa_{n+1} = \pi(\kappa_n)$. Let λ be the limit of the κ_n's. We shall show that $\pi \restriction \lambda$ does not belong to M. Otherwise, λ has cofinality ω in M and therefore $\pi(\lambda) = \lambda$. Also by Lemma 3 κ (and therefore each κ_n) is a strong limit cardinal of M. Therefore λ is a strong limit cardinal of cofinality $\aleph_0$ in M and hence $2^\lambda = \lambda^{\aleph_0}$ in M. Let S be the range of π on λ, a subset of λ of M-cardinality λ.

By Lemma 2 (applied in M) there is a function $F : \lambda^\omega \to \lambda$ in M such that M satisfies: The range of F on A^ω is all of λ, for each $A \subseteq \lambda$ of cardinality λ. By the elementarity of π and the fact that $\pi(\lambda) = \lambda$, the same holds for $\pi(F)$. Applying this to the set $S = \text{Range } (\pi \restriction \lambda)$ we obtain $s \in S^\omega$ in M such that $\pi(F)(s) = \kappa$. But s belongs to the range of π, as it equals $\pi(t)$, where $t(n) = \pi^{-1}(s(n))$ for each n; it follows that κ belongs to the range of π. This contradicts the fact that κ is the critical point of π. $\square$

How small can x satisfying the previous Theorem be? Let κ be the critical point of π. Of course $\pi \restriction H^M_\kappa$ belongs to M, as this is just the identity on H^M_κ. And if x belongs to $H^M_{\kappa^+}$ then again $\pi \restriction x$ belongs to M, as if $f : x \to \kappa$ is an injection, we have:

$$y \in x \to \pi(y) = \pi(f^{-1}(f(y))) = \pi(f^{-1})(\pi(f(y))) = \pi(f)^{-1}(f(y)).$$

So the least natural candidate for x satisfying the Theorem is $H^M_{\kappa^+}$, where κ is the critical point of π.

Embeddings $\pi : M \to M$ where $\pi \restriction H^M_{\kappa^+}$ belongs to M give rise to very strong large cardinal properties.

Definition. κ is *measurable* iff κ is the critical point of a $\pi : V \to M$. κ is *α-strong* iff κ is the critical point of a $\pi : V \to M$ such that $\alpha \leq \pi(\kappa)$ and every bounded subset of α belongs to M. κ is *strong* iff κ is α-strong for every α. If $f : \kappa \to \kappa$ then κ is *f-strong* iff κ is the critical point of a $\pi : V \to M$ such that every bounded subset of $\pi(f)(\kappa)$ belongs to M. δ

is *Woodin* iff for each $f : \delta \to \delta$ there is a $\kappa < \delta$ closed under f which is $f \restriction \kappa$-strong. κ is *superstrong* iff κ is the critical point of a $\pi : V \to M$ such that every bounded subset of $\pi(\kappa)$ belongs to M.

To analyse these properties we introduce the notion of *extender*.

Definition. The *extender derived from* $\pi : M \to N$ (where M, N are inner models of ZFC) is the restriction $E_\pi = \pi \restriction H^M_{\kappa(\pi)^+}$, where $\kappa(\pi)$ is the critical point of π. An *extender on* M is an extender derived from some embedding $\pi : M \to N$. A *# for* M is an extender derived from some $\pi : M \to N$ where $H^M_{\pi(\kappa(\pi))} = H^N_{\pi(\kappa(\pi))}$.

Thus extenders on M are restrictions of embeddings of M. Conversely, each extender on M gives rise to a canonical embedding of M, of which it is a restriction.

Theorem 4. Suppose that E is an extender on M, with critical point κ. Then there is a unique $\pi_E : M \to N_E$ such that E is the extender derived from π_E and every element of N_E is of the form $\pi_E(f)(a)$ for some $f : H^M_\kappa \to M$ in M and a in $H^{N_E}_{\pi_E(\kappa)}$.

Proof (The Ultrapower Construction). Consider pairs (f, a) where $f : H^M_\kappa \to M$ belongs to M and $a \in E(H^M_\kappa)$. We say that (f, a) and (g, b) are *equivalent*, written $(f, a) =^* (g, b)$, iff (a, b) belongs to $E(\{(u, v) \in H^M_\kappa \mid f(u) = g(v)\})$ and (f, a) *belongs to* (g, b), written $(f, a) \in^* (g, b)$, iff (a, b) belongs to $E(\{(u, v) \in H^M_\kappa \mid f(u) \in g(v)\})$. We write $[f, a]$ for the $=^*$ equivalence class of (f, a) and define N^* to be the structure whose universe consists of these $=^*$ equivalence classes, together with the (induced) relation $\in^*$ on these equivalence classes. By a straightforward induction (using the axiom of choice in M for the quantifier case) we have: $\langle N^*, \in^* \rangle \vDash \varphi([f_1, a_1], \ldots, [f_n, a_n])$ iff $(a_1, \ldots, a_n)$ belongs to $E(\{(u_1, \ldots, u_n) \in H^M_\kappa \mid M \vDash \varphi(f_1(u_1), \ldots, f_n(u_n))\})$. Using this we obtain an elementary embedding $\pi^*_E : M \to N^*$ defined by $\pi^*_E(x) = [f_x, 0]$, where f_x is the function on H^M_κ with constant value x.

Now suppose that E is derived from $\pi : M \to N$. Then $\langle N^*, \in^* \rangle \vDash \varphi([f_1, a_1], \ldots, [f_n, a_n])$ iff $(a_1, \ldots, a_n)$ belongs to $E(\{(u_1, \ldots, u_n) \in H^M_\kappa \mid M \vDash \varphi(f_1(u_1), \ldots, f_n(u_n))\})$ iff $(a_1, \ldots, a_n)$ belongs to $\{(v_1, \ldots, v_n) \in H^N_{\pi(\kappa)} \mid N \vDash \varphi(\pi(f_1)(v_1), \ldots, \pi(f_n)(v_n))\}$, so we get an elementary embedding $k^* : \langle N^*, \in^* \rangle \to \langle N, \in \rangle$ defined by $k^*([f, a]) = \pi(f)(a)$. The embedding π is the composition $k^* \circ \pi^*_E$. Note that the range of k^* includes all of $H^N_{E(\kappa)}$ since

$H^N_{E(\kappa)} = E(H^M_\kappa)$ and for each $a \in E(H^M_\kappa)$, $k^*([\mathrm{id}, a]) = a$ (where id is the identity on H^M_κ); also the range of k^* includes $E(\kappa) = k^*([f_\kappa, 0])$.

A consequence of the existence of $k^* : \langle N^*, \in^*\rangle \to \langle N, \in\rangle$ is that $\langle N^*, \in^*\rangle$ is extensional, well-founded, set-like (i.e. for any $[f, a] \in N^*$ some set provides representatives to all of the equivalence classes $[g, b] \in^* [f, a]$) and therefore isomorphic to a transitive structure $\langle N_E, \in\rangle$. Write $i : \langle N^*, \in^*\rangle \simeq \langle N, \in\rangle$. Then define $k = k^* \circ i^{-1}$ and $\pi_E = i \circ \pi^*_E$. Then $\pi = k \circ \pi_E$ and as Range $k =$ Range k^* includes $H^N_{E(\kappa)} \cup \{E(\kappa)\}$, it follows that k^{-1} is the identity on $H^N_{E(\kappa)} \cup \{E(\kappa)\}$ and therefore k is the identity on $H^{N_E}_{E(\kappa)^+}$ (which is contained in, but not necessarily equal to $H^N_{E(\kappa)^+}$). For $x \in H^M_{\kappa^+}$ we have $\pi_E(x) \in H^{N_E}_{E(\kappa)^+}$ and therefore $\pi(x) = k \circ \pi_E(x) = \pi_E(x)$. It follows that the extender derived from π_E is the same as that derived from π, namely E. As each element of the range of k is of the form $\pi(f)(a)$ for some $f : H^M_\kappa \to M$ in M and some a in $H^N_{E(\kappa)}$, it follows that each element of N_E is of the form $k^{-1} \circ \pi(f)(a) = \pi_E(f)(a)$ for some $f : H^M_\kappa \to M$ in M and a in $H^N_{E(\kappa)} = H^{N_E}_{E(\kappa)}$. So $\pi_E : M \to N_E$ has the desired properties. The uniqueness of π_E is clear, as if we began with an embedding $\pi : M \to N$ also satisfying the desired properties, the above construction produces $k : N_E \to N$ with $\pi = k \circ \pi_E$, k equal to the identity on $H^{N_E}_{E(\kappa)}$ and therefore as each element of N is of the form $\pi(f)(a) = k \circ \pi_E(f)(k(a)) = k(\pi_E(f)(a))$ for some $a \in H^{N_E}_{E(\kappa)}$ it follows that k is onto, and therefore the identity. $\square$

Remarks. (a) We write N_E as Ult (M, E). It follows from the ultrapower construction that the notion of extender is first-order. Indeed, E is an extender on M iff E is an elementary embedding $E : H^M_{\kappa^+} \to N_0 = \cup(\text{Range } E)$ with critical point κ and the structure $\langle N^*, \in^*\rangle$ resulting from the ultrapower construction using E and M is well-founded.
(b) Note that if E is an extender on M with critical point κ then for any ordinal α, $\pi_E(\alpha)$ has cardinality at most that of (α^{H_κ} of M) $\times$ ($H_{\pi_E(\kappa)}$ of Ult (M, E)), as each ordinal less than $\pi_E(\alpha)$ is represented in Ult (M, E) by a pair (f, a) where $f : H^M_\kappa \to \alpha$ belongs to M and a belongs to $H_{\pi_E(\kappa)}$ of Ult (M, E). Also $\pi_E(\alpha) = \cup \pi_E[\alpha]$ whenever α has M-cofinality greater than the M-cardinality of H^M_κ. It follows that if $\alpha > \pi_E(\kappa)$ is a strong limit cardinal of M-cofinality greater than $2^{<\kappa}$ then α is a fixed point of π_E.
(c) If α is inaccessible in M then $E \in H^M_\alpha$ is an extender on M iff E is an extender on H^M_α: If N^* from the ultrapower construction is not well-founded, then this is witnessed by a sequence $[f_{n+1}, a_{n+1}] \in^* [f_n, a_n]$, $n \in \omega$. Such a

witness exists not only in M, but also in H_α^M, as we may assume that the functions f_n take ordinal values less than α.

Theorem 5. If κ is measurable, α-strong, strong, f-strong, Woodin or superstrong, respectively then this is witnessed by embeddings of the form π_E for some extender E on V. Thus these properties are first-order.

Proof. If π witnesses the α-strength of κ then so does π_E, where E is derived from π, since by definition α must be less than or equal to $\pi(\kappa)$. The same holds for f-strength, as $\pi(f)(\kappa)$ is less than $\pi(\kappa)$ for $f : \kappa \to \kappa$. As measurability, strength, Woodinness and superstrength can be defined in terms of α-strength and f-strength, these properties are all witnessed by embeddings of the form π_E and therefore are first-order since the notion of extender is first-order. □

For our next result it will be useful to consider the following variant of the ultrapower construction: Suppose that $\pi : M \to N$ has critical point κ and α is a cardinal of N, $\kappa < \alpha \leq \pi(\kappa)$. Then define N_α^* just like N^*, but only using pairs (f, a) where a belongs to H_α^N. We obtain a well-founded, set-like and extensional structure $\langle N_\alpha^*, \in^* \rangle$, isomorphic to a transitive class $N_{E,\alpha}$, with canonical embeddings $\pi_{E,\alpha} : M \to N_{E,\alpha}$ and $k_\alpha : N_{E,\alpha} \to N$, $k_\alpha = \text{id}$ on H_α^N. Thus if $\pi : V \to N$ witnesses the α-strength of κ, so does $\pi_{E,\alpha}$. We define the *cutback of E to α*, written $E \downarrow \alpha$, to be the extender derived from the embedding $\pi_{E,\alpha}$. As each ordinal less than $(E \downarrow \alpha)(\kappa)$ is represented by a pair (f, a) where $f : H_\kappa^M \to \kappa$, $a \in H_\alpha^N$, it follows that $(E \downarrow \alpha)(\kappa)$ has cardinality $2^{<\alpha}$ and therefore so does $E \downarrow \alpha$. The *true length* of E is the least α such that $E \downarrow \alpha = E$.

Let us say that a property $P(\kappa)$ is *stronger* than a property $Q(\kappa)$ iff the existence of a κ satisfying $P(\kappa)$ implies the existence of a transitive set which is a ZFC-model and in which there is a $\bar{\kappa}$ satisfying $Q(\bar{\kappa})$.

Theorem 6. Superstrength is stronger than Woodinness, Woodinness is stronger than strength and strength is stronger than measurability.

Proof. Suppose that κ is superstrong via the embedding π and let E be the extender derived from π. We claim that κ is witnessed to be Woodin using extenders in H_κ, and therefore is Woodin in the ZFC-model $H_{\pi(\kappa)}$. If not, pick a function $f : \kappa \to \kappa$ such that no $\bar{\kappa} < \kappa$ closed under f is witnessed to be $f \restriction \bar{\kappa}$-strong by an extender in H_κ. Then κ is not witnessed to be f-strong in $M = \text{Ult }(V, E)$ using extenders in $H_{\pi(\kappa)}$. But by superstrength $E \downarrow \alpha$ belongs to M for each α less than $\pi(\kappa)$, and in particular for $\alpha = \pi(f)(\kappa)^+$. Thus $E \downarrow \pi(f)(\kappa)^+$ witnesses the f-strength of κ in M, contradiction.

Suppose that δ is Woodin. We claim that some $\kappa < \delta$ is strong in the ZFC-model H_δ. If not, then define $f(\kappa) = (2^{<g(\kappa)})^+$ where $g(\kappa)$ is least so that κ is not $g(\kappa)$-strong in H_δ, and therefore not $g(\kappa)$-strong in V. Apply the Woodinness of δ to obtain κ closed under f which is $f \restriction \kappa$-strong, via an embedding $\pi : V \to M$. By elementarity κ is not $\pi(g)(\kappa)$-strong in M. But if E is the extender derived from π, we have $E \downarrow (\pi(g)(\kappa)) \in M$, witnessing the $\pi(g)(\kappa)$-strength of κ in M, contradiction.

Suppose that κ is strong. We claim that there is a measurable cardinal less than κ and therefore the ZFC-model H_κ satisfies that there exists a measurable cardinal. Suppose not and let $\pi : V \to M$ witness the $(2^\kappa)^+$-strength of κ. Then κ is not measurable in M. But if E is the extender derived from π, we have $E \downarrow \kappa^+ \in M$, witnessing the measurability of κ in M, contradiction. $\square$

Let us now return to our discussion of Kunen's Theorem. Suppose that $\pi : M \to M$ and let E be the extender derived from π. Then $\pi_E : M \to \text{Ult }(M, E)$ has the property that M and $\text{Ult }(M, E)$ have the same bounded subsets of $E(\kappa)$, where κ is the critical point of π. Thus if E belonged to M, we would have a superstrong cardinal in M. The same applies to any #-embedding for M, i.e., to any embedding $\pi : M \to N$ such that M, N have the same bounded subsets of $\pi(\kappa)$, $\kappa =$ the critical point of π. Therefore:

M has a superstrong cardinal iff M contains a sharp for itself.

Superstrength is essentially the strongest large cardinal property which can be witnessed by an extender embedding. Indeed, if E is an extender on V and $\pi_E : V \to \text{Ult }(V, E) = M$ is the resulting ultrapower embedding, then E cannot belong to M, as E maps κ^+ cofinally into $E(\kappa)^+$ of M, where κ is the critical point of E. As E belongs to $H_{\pi(\kappa)^+}$ it follows that whereas M might contain all bounded subsets of $\pi(\kappa)$, it cannot contain all subsets of $\pi(\kappa)$, as some of them code E itself, which does not belong to M.

#-Iterations

Gödel's universe L provides a canonical inner model of V which satisfies not only ZFC, but also GCH. Is there a similar result for the theory ZFC + there exists a superstrong cardinal?

Inner Model Conjecture. Suppose that there is a superstrong cardinal. Then there is an inner model satisfying ZFC + GCH in which there is a superstrong cardinal.

We might try to prove this conjecture as follows. Assume that there is a well-ordering of the universe. (If there is no such well-ordering, then we can easily add one by forcing without creating new sets.) Let κ be superstrong in V, witnessed by the embedding $\pi : V \longrightarrow M$; thus E_π is a # for V. Set $M_0 = L$. Now $\pi \restriction M_0$ maps M_0 to M_0 and therefore M_0 has a #. Let $M_1 = M_0[E_0]$ where E_0 is the least sharp for M_0. Inductively, if M_i has been defined then $\pi \restriction M_i$ maps M_i into $N_i = \pi(M_i) = \bigcup \text{Range } \pi \restriction M_i$. Perhaps $\pi \restriction M_i$ witnesses that M_i has a sharp. Define M_{i+1} to be $M_i[E_i]$ where E_i is the least sharp for M_i. Take an appropriate limit at limit stages. Then at some least stage ∞, M_∞ must contain a sharp for itself, and therefore a superstrong cardinal. If we can arrange that each M_i satisfy the GCH, then we have established the Inner Model Conjecture.

There are many difficulties with the above sketch. As a start, we assume somewhat more than a superstrong and carry out the above construction, ignoring the important problem of ensuring the GCH.

Definition. Suppose that A is a class. We say that κ is *A-strong* iff for each cardinal α there exists $\pi : V \longrightarrow M$ with critical point κ such that $\alpha \leq \pi(\kappa)$ and $A \cap H_\alpha = \pi(A) \cap H_\alpha$ $(= \pi(A \cap H_\kappa) \cap H_\alpha)$. κ is *A-superstrong* iff for each cardinal α there exists $\pi : V \longrightarrow M$ with critical point κ such that $\alpha \leq \pi(\kappa)$ and $A \cap H_{\pi(\kappa)} = \pi(A) \cap H_{\pi(\kappa)}$ $(= \pi(A \cap H_\kappa) \cap H_{\pi(\kappa)})$. V is *super-Woodin* iff for each class A there is a κ which is A-superstrong.

κ is *supersolid* iff there is $\pi : V \longrightarrow M$ with critical point κ such that M contains all bounded subsets of $\pi(\kappa)$ and in addition $\pi(\kappa)$ is regular. κ is *hyperstrong* iff there is $\pi : V \longrightarrow M$ with critical point κ such that M contains all subsets of $\pi(\kappa)$.

Proposition 7. Hyperstrength > Supersolidity > Super-Woodinness.

Proof Sketch. Suppose that κ is hyperstrong, witnessed by $\pi : V \to M$. As κ is Mahlo, it follows that $\pi(\kappa)$ is also Mahlo, as $\pi(\kappa)$ is Mahlo in M and M contains all subsets of $\pi(\kappa)$. We claim that some $\bar{\kappa} < \kappa$ is supersolid in V_κ. Otherwise κ is not supersolid in $V^M_{\pi(\kappa)} = V_{\pi(\kappa)}$. But as κ is Mahlo in M, there exists a regular $\delta < \pi(\kappa)$ such that only ordinals less than δ are represented in Ult (V, E_π) by a pair (f, a) where a belongs to V_δ. It follows that $E_\pi \downarrow \delta$ is an extender sending κ to δ, witnessing the supersolidity of κ in $V_{\pi(\kappa)}$, contradiction.

Suppose that κ is supersolid, witnessed by the embedding $\pi : V \to M$. Then we claim that V_κ is super-Woodin (with respect to all subsets of V_κ). If not, choose $A \subset V_\kappa$ such that no $\bar{\kappa} < \kappa$ is A-superstrong in V_κ. Then κ is not $\pi(A)$-superstrong in $V^M_{\pi(\kappa)} = V_{\pi(\kappa)}$. But for CUB-many $\delta < \pi(\kappa)$, $E_\delta = E_\pi \downarrow \delta$ witnesses the superstrength of κ and moreover $E_\delta(A \cap \kappa) = E_\delta(\pi(A) \cap \kappa) = \pi(A) \cap E_\delta(\kappa)$ so in fact the E_δ's witnesses the $\pi(A)$-superstrength of κ in $V_{\pi(\kappa)}$, contradiction. □

We also need one more simple fact. If $\pi : M \to N$ then we define the *critical image of* π, written crim π, to be $\pi(\kappa)$, where κ is the critical point of π. If E is the extender derived from π we also write crim E for crim π.

Proposition 8. Suppose that V is super-Woodin. Then for every class A and every CUB class of ordinals C there exists a $\kappa \in C$ which is witnessed to be A-superstrong via embeddings with critical image in C.

Proof. Suppose that κ is A, C-superstrong. For any cardinal α choose $\pi : V \to M$ with critical point κ such that $\alpha \leq \pi(\kappa)$ and $A \cap H_\pi(\kappa), C \cap \pi(\kappa)$ are the same as $\pi(A) \cap H_\pi(\kappa), \pi(C) \cap \pi(\kappa)$. We may also assume that some element of C lies between κ and $\pi(\kappa)$ for each such π. But then $C \cap \kappa$ must be unbounded in κ and therefore κ belongs to C. Also $\pi(C) \cap \pi(\kappa) = C \cap \pi(\kappa)$ is unbounded in $\pi(\kappa)$ so $\pi(\kappa)$ belongs to C. □

Assume now that V is super-Woodin. Then every inner model M has a #: Choose $\pi : V \to N$ with critical point κ such that $M \cap H_{\pi(\kappa)}$ agrees with $\pi(M) \cap H_{\pi(\kappa)}$. Then $\pi \restriction M : M \to \pi(M)$ provides a # for M. The fact that both crit π and crim π can be chosen from any CUB class of ordinals will be used a bit later.

As suggested above, a # *iteration* is, roughly speaking, a sequence $M_0, M_1, \ldots$ of inner models where:

$M_0 = L$
$M_{i+1} = M_i[E_i]$, where E_i is a # for M_i
$M_\lambda =$ the limit of $\langle M_i \mid i < \lambda\rangle$ for limit λ.

The type of model that arises through such an iteration is called an *extender model* and is of the form $L[E]$ where $E = \langle E_\alpha \mid \alpha \in \mathrm{ORD}\rangle$ is a sequence of extenders (for appropriate models). For any extender E we define the *index of* E, written ind E, to be $\cup(\text{Range } E \cap \mathrm{ORD})$.

Definition. An *extender sequence* is a sequence $E = \langle E_\nu \mid \nu \in \mathrm{ORD}\rangle$ such that for all ν, E_ν is either empty or an extender on $L[E \restriction \nu]$ such that:
1. $\nu = \text{ind } E_\nu$.
2. Let κ be the critical point of E_ν. The E_ν *preserves* $E \restriction \nu$: if π is the canonical embedding $L[E \restriction \nu] \to \text{Ult } (L[E \restriction \nu], E_\nu]$ then $E \restriction \nu = \pi(E \restriction \nu) \restriction \nu$.
An *extender model* is a structure $L^E = \langle L[E], E\rangle$ where E is an extender sequence.

To carry out our inductive definition of extender models we must consider well-orderings of length greater than ORD. To formalise this it is convenient to work in a strengthened class theory. We shall assume:

$(*)$ There is a relation E on V such that $V^* = \langle V, E\rangle$ is a model of ZFC^- with an element isomorphic to $\langle V, \in\rangle$.

Fix such a V^* and let $<$ be the well-ordering of its ordinals. An element of the field of $<$ is called a *hyperordinal* and we order hyperordinals using $<$. Then our desired definition of the extender models M_i is by induction on the hyperordinal i. Fix also a well-ordering of V belonging to V^* to be used in this induction.

We explain now what is done at successor stages. Suppose that $M = L^E$ is an extender model and suppose that F is an extender on M with critical point κ, $\nu = \text{ind } F$, $F[E \restriction \kappa] = E \restriction F(\kappa)$, $H^M_\kappa \subseteq L[E \restriction \kappa]$ and $H^M_{F(\kappa)} \subseteq L[E \restriction F(\kappa)]$. Then $M[F]$ is defined to be the extender model with extender sequence E^*, where $E^*_\sigma = E_\sigma$ for $\sigma < F(\kappa)$, $E^*_\sigma = F[E \restriction (\kappa^+ \text{ of } M)]_\sigma$ for $F(\kappa) \le \sigma < \nu$, $E^*_\nu = F$ and $E^*_\sigma = \emptyset$ for $\nu < \sigma$.

At limit stages we have:

Definition. Suppose that $\langle M_i \mid i < \lambda\rangle$ is a sequence of extender models with corresponding extender sequences E_i, $i < \lambda$. Then the *limit* of the M_i's, $\lim\langle M_i \mid i < \lambda\rangle$, is the extender model L^E where E is defined by: $E_\nu = \emptyset$ unless $E_i \restriction \nu + 1 = E_j \restriction \nu + 1$ for all sufficiently large $i, j < \lambda$, in which case E_ν is the common value of $(E_i)_\nu$ for sufficiently large $i < \lambda$.

Theorem 9. Assume that V is super-Woodin and isomorphic to an element of some ZFC$^-$ model $V^* = \langle V, E\rangle$. Fix a well-ordering of V in V^* and let $<$ denote the well-ordering of the ordinals of V^*, called *hyperordinals*. Define a sequence of extender models $M_i = L^{E_i}$, i a hyperordinal, as follows:

$M_0 = L$, with $(E_0)_\nu = \emptyset$ for all ν

$M_{i+1} = M_i[F_i]$ where F_i is the least # for M_i such that if κ_i is the critical point of F_i then $F_i(E_i \restriction \kappa_i) = E_i \restriction F_i(\kappa_i)$, $H^{M_i}_{\kappa_i} \subseteq L[E_i \restriction \kappa_i]$ and $H^{M_i}_{F_i(\kappa_i)} \subseteq L[E_i \restriction F_i(\kappa_i)]$

$M_\lambda = \lim\langle M_i \mid i < \lambda\rangle$ for limit hyperordinals λ.

Then M_i is defined for all i and for some hyperordinal ∞, $M_\infty \vDash$ There is a superstrong cardinal.

Proof. Suppose that M_i is defined and we wish to show that F_i exists. By Proposition 8 there exists an embedding $\pi : V \to M$ with critical point κ such that $\pi(E_i \restriction \kappa) = E_i \restriction \pi(\kappa)$, $H^{M_i}_\kappa \subseteq L[E_i \restriction \kappa]$ and $H^{M_i}_{\pi(\kappa)} \subseteq L[E_i \restriction \pi(\kappa)]$. Then E_π is a candidate for F_i.

Now suppose that no M_i has a superstrong cardinal.

We show by induction that for each ordinal α, crim $F_i \geq \alpha$ for sufficiently large i (where crim $F_i = F_i(\kappa_i)$, the critical image of F_i). Suppose that this is true for α and we wish to show that crim $F_i > \alpha$ for sufficiently large i. Choose i_0 so that crim $F_i \geq \alpha$ for $i \geq i_0$. If crim $F_i > \alpha$ for all $i \geq i_0$ then we are done. Otherwise choose $i_1 \geq i_0$ such that crim $F_{i_1} = \alpha$. We claim that crim $F_j >$ ind F_{i_1} for all $j > i_1$. Otherwise let $i_2 > i_1$ be least so that crim $F_{i_2} \leq$ ind F_{i_1}. Then F_{i_1} belongs to the model M_{i_2} and therefore ind $F_{i_1} < \alpha^+$ of M_{i_2}. It follows that crim F_{i_2} is at most α as it is an inaccessible cardinal of M_{i_2}. By choice of i_0, in fact crim F_{i_2} equals α. Now by definition, $H^{M_{i_2}}_\alpha$ is contained in $L[E_{i_2} \restriction \alpha]$ which equals $L[E_{i_1} \restriction \alpha]$ as all F_i, $i_1 \leq i \leq i_2$, preserve E_i up to α and therefore $E_i \restriction \alpha$ does not change between i_1 and i_2. Thus M_{i_2} contains the extender F_{i_1} and all bounded subsets of $\alpha =$ crim F_{i_1} belong to Ult (M_{i_2}, F_{i_1}). (The latter is well-founded as it embeds into Ult $(V, F^*_{i_1})$, where F_{i_1} is the restriction to M_{i_1} of the V-extender $F^*_{i_1}$.) Thus F_{i_1} witnesses the superstrength of its critical point κ_{i_1} in M_{i_2}, contrary to hypothesis.

As crim F_i is eventually at least α for any ordinal α, it follows that F_i is eventually undefined, in contradiction to the first paragraph of the proof. □

How can we ensure that the inner models M_i of the previous theorem satisfy GCH? A natural way is to enforce the *Gödel property*:

Gödel Property. Suppose that $V = L^E$ is an extender model. If X is a subset of the infinite cardinal κ then X belongs to $L^E_{\kappa^+}$.

In our inductive definition of the M_i's, we begin with $M_0 = L$, which satisfies the Gödel property by virtue of Gödel's proof of the GCH in L. Now suppose that M_i satisfies the Gödel property and we wish to maintain this property when defining M_{i+1}. Write $M_{i+1} = M_i[F_i]$, where F_i has index ν_i. As F_i can be coded by a subset of crim $F_i = F_i(\text{crit } F_i)$, it follows by Gödel's argument that the Gödel property holds for $\kappa \geq$ crim F_i. Also if κ is less than crim F_i and X is a subset of κ in M_i, then since the Gödel property holds for X in M_i it also holds for X in M_{i+1}, as M_{i+1} agrees with M_i up to crim F_i and κ^+ of M_{i+1} is at least κ^+ of M_i. Now let X be a bounded subset of crim F_i in $M_{i+1} - M_i$ and choose ν such that X belongs to $L^{E_{i+1}}_\nu$. Then ν is at least ν_i. For simplicity let us assume that $X = R$ is a subset of ω. We would like to guarantee the Gödel property for R in M_{i+1}. Of course the difficulty is that R does not appear until stage ν, which may be uncountable in M_{i+1}.

So consider $H =$ the Skolem hull of $\{R\}$ in $L^{E_{i+1}}_\nu$. Then H is countable and R belongs to the countable structure $\overline{M}_{i+1} =$ the transitive collapse of H. We replace M_{i+1} by M^*_{i+1}, the extender model obtained by adding empty levels to the top of $\overline{M}_{i+1}$. We have the Gödel property for M^*_{i+1}, as this model is of the form $L[S]$ for some real S.

But we must show that this new inductive definition converges to a model of a superstrong. For this purpose we need to know that when "hulling down" to form $\overline{M}_{i+1}$, not too much information is lost. In particular we would like to know that we still have all the reals present in M_i. The crucial property needed is the following:

$(*)$ Suppose that H is a countable elementary submodel of $N = L^{E_{i+1}}_\nu$. Then the reals in H form an initial segment of the reals in N.

Currently, the only known technique for proving $(*)$ is to use the theory of *iteration* and *comparison*. Suppose that $M = L^E_\alpha$ is an initial segment of an extender model. We can use our earlier ultrapower construction to form Ult (M, F) whenever F is an extender on M (i.e., derived from some embedding $M \to N$). By taking repeated ultrapowers and taking direct limits at limit stages, we from an *iteration* of M. In an iteration, we allow ourselves to *truncate* the current iterate M_i of M, by replacing M_i by one of its proper initial segments. We consider *iterable* M, which give rise to well-founded iterates, with only finitely many truncations occuring in any iteration. When iterating the reals do not change, except when truncation occurs, when a final segment of the reals may be lost.

Comparison says the following:

Comparison. Suppose that M, N are iterable. Then either some iterate M^* of M without truncations is an initial segment of an iterate N^* of N, or vice-versa.

It follows from Comparison that if M, N are iterable then either the reals of M form an initial segment of the reals of N, or vice-versa. This yields $(*)$: Let $\overline{H}$ = the transitive collapse of H. Then either the reals of $\overline{H}$ form an initial segment of the reals of $N = L^{E_{i+1}}_\nu$ or vice-versa. But as the reals of $\overline{H}$ equal the reals of $H \subseteq N$, the former must hold, as desired.

Unfortunately obtaining iterable extender models is problematic and has not yet been carried out to the level of a superstrong cardinal. One does have the desired theory at the level of a Woodin cardinal, and therefore the inner model conjecture has been proven, if one replaces "superstrong" by "Woodin" in its statement.

$\square$ *in Extender Models*

For an extender model L^E to serve as a good analogue of L in the large cardinal context, it would be uesful to know that not only GCH, but also Jensen's $\square$ principle holds in such a model. Schimmerling and Zeman established this result for the extender models which are known to satisfy GCH. We now discuss some of the ideas behind this proof.

Suppose that $V = L^E$ is an extender model and fix an uncountable cardinal κ. We would like to prove the following version of $\square$ for ordinals between κ and κ^+:

$\square_\kappa$. There exists $\langle C_\nu \mid \kappa < \nu < \kappa^+,\ \nu \text{ limit}\rangle$ such that C_ν is CUB in ν, C_ν has ordertype $\leq \kappa$ and $\bar\nu \in \operatorname{Lim} C_\nu \to C_{\bar\nu} = C_\nu \cap \bar\nu$.

Let's begin by recalling some of the ideas behind the proof of this principle in L. It suffices to define the C_ν for limit ordinals $\nu < \kappa^+$ such that $L_\nu \vDash \kappa$ is the largest cardinal, as the set of such ν forms a CUB subset of κ^+. Let ν be such a limit ordinal and choose $\beta(\nu) \geq \nu$ least so that $\nu \subseteq$ the Σ_n Skolem hull of $\kappa \cup \{p\}$ in $L_{\beta(\nu)}$ for some n and some parameter $p \in L_{\beta(\nu)}$. We assume that $n = 1$, $\beta(\nu)$ is a limit ordinal and that $p = p(\nu)$ is least in the maximum-difference ordering $\leq^*$ of finite sets of ordinals: $p \leq^* q$ iff the largest ordinal in $(q - p) \cup (p - q)$ belongs to q. For any $\alpha < \kappa$ we consider $H_\alpha =$ the Σ_1 Skolem hull of $\alpha \cup \{p(\nu)\}$ in $L_{\beta(\nu)}$, as well as $\beta_\alpha = \sup(H_\alpha \cap \beta(\nu))$ and $\nu_\alpha = \sup(H_\alpha \cap \nu)$. The rough idea is to define C_ν to be the collection of those ν_α which are less than ν.

One must verify coherence: $\bar\nu \in \operatorname{Lim} C_\nu \to C_{\bar\nu} = C_\nu \cap \bar\nu$. Choose α so that $\bar\nu = \nu_\alpha$. The key step is to show:

$L_{\beta(\bar\nu)}$ is the transitive collapse of $H =$ the Σ_1 Skolem hull of $\bar\nu \cup \{p(\nu)\}$ in L_{β_α}.

Then one also verifies that $p(\bar\nu)$ is the image of $p(\nu)$ under the collapsing map, and ultimately that coherence holds. In the L-context we necessarily have that H is isomorphic to $L_{\bar\beta}$ for some $\bar\beta$, by condensation, and then one can argue that $\bar\beta$ equals $\beta(\bar\nu)$. The difficulty in the L^E-context is that H need not be of the form $L^E_{\bar\beta}$, in cases where $E_{\beta(\nu)}$ is nonempty.

Let us take a closer look at this last point. Suppose that $F = E_{\beta(\nu)}$ is an extender with critical point $\gamma < \kappa$. Thus F is a function from (H_{γ^+} of $L^E_{\beta(\nu)}) = H_{\gamma^+}$ cofinally into $L^E_{\beta(\nu)}$. Typically, $\beta_\alpha < \beta(\nu)$ and therefore $F \cap L^E_{\beta_\alpha}$ is only partially defined on H_{γ^+}. This means that the transitive collapse of H is a structure $L^{\bar E}_{\bar\beta}$ with a function $\bar F = \bar E_{\bar\beta}$ at the top which is not an extender, but a function mapping a *proper* subset of H_{γ^+} cofinally into $L^{\bar E}_{\bar\beta}$. Such a function is called an *extender fragment*, as its domain is smaller than it should be.

This suggests that when defining $C_{\bar\nu}$ we should not use the usual collapsing structure $L^E_{\beta(\bar\nu)}$, but rather an appropriate "fragment structure" associated to $\bar\nu$, in order to obtain coherence. But what fragment structure do we choose? It turns out that all candidates $L^{\bar E}_{\bar\beta}$ for the fragment structure have the following property: The ultrapower of $L^{\bar{E}}_{\bar{\beta}}$ by the extender fragment $\bar{E}_{\bar{\beta}}$ is equal to the usual collapsing structure $L^{E}_{\beta(\bar{\nu})}$. Thus we are led to an analysis of the different ways in which the usual collapsing structure $L^{E}_{\beta(\bar{\nu})}$ can be obtained via a fragment ultrapower.

The possible fragments are parametrised by pairs (μ, q), where μ is a cardinal less than κ and q is an initial segment of the standard parameter $p = p(\bar{\nu})$. (Thus we view p as a finite set of ordinals, and q is of the form $p \cap \delta$ for some ordinal δ.) A fragment is associated to (μ, q) provided $X(\mu, q) =$ the Σ_1 Skolem hull of $\mu \cup \{p - q\}$ in $L^{E}_{\beta(\bar{\nu})}$ does not intersect the interval $[\mu, \max q]$; in this case the associated fragment is $\pi \restriction (H_{\mu^+}$ of $\bar{X}(\mu, q))$ where $\bar{X}(\mu, q)$ is the transitive collapse of $X(\mu, q)$ and π is the inverse to the collapsing map. We are interested in *strong fragments* which have the additional property that $X(\mu, q)$ and the same hull with $p - q$ replaced by p have the same subsets of μ after transitive collapse.

For any q we consider $D(q) = \{\mu \mid (\mu, q)$ gives rise to a strong fragment$\}$. This is closed and bounded in κ. Finally consider the smallest initial segment $q = q(\bar{\nu})$ of p such that $D(q)$ is nonempty and the largest element $\mu = \mu(\bar{\nu})$ of the associated $D(q)$. The desired fragment is the one associated to this pair $(\mu(\bar{\nu}), q(\bar{\nu}))$. Schimmerling-Zeman prove coherence for this choice of fragment.

6. Set Forcing over Extender Models

Singular Cardinals

First we recall the following consequence of Jensen's Covering Theorem.

Theorem 1. Suppose that $0^\#$ does not exist. Then the GCH holds at all singular strong limit cardinals.

Proof. Suppose that κ is a singular strong limit cardinal and let λ be the cofinality of κ. Then $2^\kappa = \kappa^\lambda$. As every subset of κ of cardinality λ is contained in a constructible one of cardinality at most λ^+, κ^λ is at most $\kappa^+ \cdot 2^{\lambda^+} = \kappa^+$. $\square$

We will show that the conclusion of this theorem can be violated if we assume the existence of large cardinals. First we need a method for making a cardinal singular without collapsing it.

Prikry Forcing

Theorem 2. Let κ be a measurable cardinal. Then there is a generic extension in which the cofinality of κ is ω and no cardinals are collapsed. Moreover, no bounded subsets of κ are added.

Proof. Let U be the ultrafilter on κ derived from an embedding $\pi : V \to M$ with critical point κ. I.e., U is the collection of subsets of κ defined by

$A \in U$ iff $\kappa \in \pi(A)$.

Let P consist of all pairs $p = (s, A)$ where s is a finite subset of κ and A belongs to U. Extension is defined by

$(s, A) \leq (t, B)$ iff
(i) t is an initial segment of s
(ii) A is a subset of B
(iii) $s - t \subseteq B$.

Two conditions with the same first component are compatible, since U is a filter. Thus P has the κ^+-cc and therefore all cardinals greater than κ are preserved by P.

Let G be P-generic. Then κ has cofinality ω in $V[G]$, as $\bigcup\{s \mid (s, A) \in G$ for some $A\}$ is an unbounded subset of κ of ordertype ω.

It remains only to show that P does not add new bounded subsets of κ. The proof is based on two lemmas.

Lemma 3. Suppose that $f : [\kappa]^{<\omega} \to 2$, where $[\kappa]^{<\omega}$ denotes the set of finite subsets of κ. Then there exists $A \in U$ such that for each n, f is constant on $[A]^n$, the set of subsets of A of size n.

Proof of Lemma 3. First note that U is a *normal* ultrafilter, i.e., if A_i belongs to U for each $i < \kappa$ then so does $\Delta\{A_i \mid i < \kappa\} = \{i < \kappa \mid i \in A_j \text{ for all } j < i\}$. This is because by hypothesis κ belongs to $\pi(A_i) = (\pi(\langle A_i \mid i < \kappa\rangle))_i$ for each $i < \kappa$, and therefore κ belongs to $\Delta\, \pi(\{A_i \mid i < \kappa\}) = \pi(\Delta\{A_i \mid i < \kappa\})$.

Now by induction on n we show that there exists $A \in U$ such that f is constant on $[A]^n$. This is clear for $n = 1$, since U is an ultrafilter. If it holds for n then for each $i < \kappa$ choose $A_i \in U$ such that f_i is constant on $[A_i]^n$, where f_i is defined on $\kappa - (i + 1)$ by $f_i(\alpha_1, \ldots, \alpha_n) = f(i, \alpha_1, \ldots, \alpha_n)$. Let $A^* = \Delta\{A_i \mid i < \kappa\}$ and choose $A \subseteq A^*$ in U such that f_i has the same constant value on $[A_i]^n$ for each $i \in A$. Then f is constant on $[A]^{n+1}$, completing the induction.

Now by intersecting sets $A_n \in U$ such that f is constant on $[A_n]^n$ for each n, we get the desired set $A \in U$. □

Lemma 4. Let φ be a sentence of the forcing language and (s_0, A_0) a condition. Then there exists $A \subseteq A_0$ in U such that (s_0, A) decides φ.

Proof of Lemma 4. We may assume that $\min A_0 > \max s_0$. Let S^+ be the set of $s \in [A_0]^{<\omega}$ such that $(s_0 \cup s, A) \Vdash \varphi$ for some $A \subseteq A_0$ and S^- the set of $s \in [A_0]^{<\omega}$ such that $(s_0 \cup s, A) \Vdash\sim \varphi$ for some $A \subseteq A_0$. By Lemma 3, choose $A \in U$ such that for each n, either $[A]^n \subseteq S^+$, $[A]^n \subseteq S^-$ or $[A]^n$ is disjoint from $S^+ \cup S^-$.

We claim that (s_0, A) decides φ. If not then there are $(s_0 \cup s, B), (s_0 \cup t, C)$ extending (s_0, A) which force $\varphi, \sim \varphi$, respectively. We may assume that s and t have the same length n. But then $s \in S^+$, $t \in S^-$ and therefore $[A]^n$ intersects both S^+ and S^-, contrary to the choice of A. □

Now suppose that $(s, A) \Vdash \sigma$ is a subset of $\lambda < \kappa$. For each $i < \lambda$, choose $A_i \subseteq A$ such that (s, A_i) decides the sentence $i \in \sigma$. Then (s, A^*) forces that

σ is in the ground model, where $A^* = \bigcap_i A_i$. This completes the proof of Theorem 2. $\square$

Our strategy to obtain a singular strong limit cardinal where GCH fails is now as follows. We will show that the GCH can fail at a measurable cardinal κ. Then by applying Prikry forcing to κ, we obtain a singular strong limit cardinal of cofinality ω where the GCH fails.

First we state a general lemma which can be used to extend an embedding $k : M \to N$ to a generic extension of M.

Proposition 5. Let $k : M \to N$ be an elementary embedding between ZFC models, $P \in M$, G P-generic over M and let H be $k(P)$-generic over N. If $k[G] \subseteq H$ then there exists $k^* : M[G] \to N[H]$ extending k such that $k(G) = H$.

Proof. If $k[G] \subseteq H$ define k^* by $k^*(\sigma^G) = k(\sigma)^H$. This is well-defined, as if $\sigma^G = \tau^G$ then there is $p \in G$ such that $p \Vdash \sigma = \tau$ and therefore $k(p) \Vdash k(\sigma) = k(\tau)$, yielding $k(\sigma)^H = k(\tau)^H$, since $k(p)$ belongs to H. Similarly k^* is elementary. As k sends a standard P-name for an element of M to a standard $k(P)$-name for the image of that element, it follows that k^* extends k. Similarly, as k sends a standard P-name for the generic G to a standard $k(P)$-name for the generic H, we get $k(G) = H$. $\square$

Theorem 6. Suppose that it is consistent to have GCH and a cardinal κ which is κ^{++}-strong. Then it is consistent to have the GCH fail at a measurable cardinal.

Proof. Suppose that V satisfies GCH and has a κ which is κ^{++}-strong, witnessed by an embedding $\pi : V \to M$. We may assume that M is the ultrapower of V by an extender E with critical point κ, and that E equals $E \downarrow \kappa^{++}$. Thus every element of M is of the form $\pi(f)(a)$, where $f : H_\kappa \to V$ and $a \in H_{\kappa^{++}}$. Note that M is closed under κ-sequences, as if for each $i < \kappa$, $m_i \in M$ is represented as $\pi(f_i)(a_i)$, we can represent $\langle m_i \mid i < \kappa\rangle$ by $\pi(f)(a)$ where $f(\langle x_j \mid j < i\rangle) = \langle f_j(x_j) \mid j < i\rangle$ and $a = \langle a_i \mid i < \kappa\rangle$. Also since $H_{\kappa^{++}}$ is contained in M, we have that $\kappa^{++} = \kappa^{++}$ of M.

Let U be $E \downarrow \kappa^+$, the "measure" derived from E, and $\pi_U : V \to N = \text{Ult}\ (V, U)$ the ultrapower embedding given by U. Using the same argument as used above for M, N is closed under κ-sequences. Also $\pi = k \circ \pi_U$ where $k : N \to M$ is given by $k(\pi_U(f)(a)) = \pi(f)(a)$ for $a \in H_{\kappa^+}$.

Let λ be the κ^{++} of N. Since the GCH holds at κ, $\lambda < \kappa^{++} = \kappa^{++}$ of M. It follows that λ is the critical point of $k : N \to M$. Every element of M is of the form $\pi(f)(a)$ for some $a \in H_{\kappa^{++}}$. Now $\pi(f)(a) = (k(\pi_U(f)) \restriction H_{\kappa^{++}})(a) = k(\pi_U(f) \restriction H^N_\lambda)(a)$. Therefore every element of M is of the form $k(g)(a)$ for some $a \in H_{\kappa^{++}}$ and some $g \in N$ whose domain has N-cardinality λ.

Our goal is to extend the embedding π to a generic extension of V in which $2^\kappa = \kappa^{++}$ and in which this extension of π is definable. We shall first show how to extend π to the natural reverse Easton extension of V in which $2^\kappa = \kappa^{++}$, and then extend this embedding once more to a further generic extension in which this second extension of π is definable.

Let $P = P_{\kappa+1}$ be the reverse Easton iteration where at stage $\alpha \leq \kappa$, $P_{\alpha+1} = P_\alpha * Q_\alpha$, where Q_α is trivial unless α is inaccessible, in which case $Q_\alpha = \text{Add}\ (\alpha, \alpha^{++})^{V[G_\alpha]}$, the forcing to add α^{++} subsets to α with conditions of size less than α. We use Easton supports, taking direct limits at inaccessibles and inverse limits elsewhere.

Let G be P_κ-generic over V and let g be Q_κ-generic over $V[G]$. Our first step is to extend π and π_U to $V[G]$. Thus we must choose generics for $\pi(P_\kappa) = \pi(P)_{\pi(\kappa)}$ and $\pi_U(P_\kappa) = \pi_U(P)_{\pi_U(\kappa)}$ containing $\pi[G]$ and $\pi_U[G]$, respectively. Up to stage κ, the iterations P, $\pi_U(P)$ and $\pi(P)$ are all the same: $P_\kappa = \pi_U(P)_\kappa = \pi(P)_\kappa$.

Lemma 7. $\pi(P)_{\kappa+1} = P_{\kappa+1}$.

Proof. This is because V and M have the same $H_{\kappa^{++}}$. □

Lemma 8. $\pi_U(P)_{\kappa+1} = P_\kappa * Q^*_\kappa$ where Q^*_κ is the Add (κ, λ) of $V[G]$.

Proof. Clearly Q^*_κ is the Add (κ, λ) of $N[G]$. But V and N have the same H_{κ^+} and λ has cofinality greater than κ; therefore N, V have the same size $< \kappa$ subsets of λ and $N[G]$, $V[G]$ have the same size $< \kappa$ subsets of λ. □

Let g_0 equal $g \cap Q^*_\kappa$. Then g_0 is Q^*_κ-generic over $V[G]$ and therefore also over $N[G]$. As N is closed under κ-sequences and $P_\kappa * Q^*_\kappa$ has the κ^+-cc, it follows that $V[G][g_0] \vDash N[G][g_0]$ is closed under κ-sequences (since every κ-sequence in $V[G][g_0]$ has a name in V of size κ).

Let $R_0 = P^N_{\kappa+1,\pi_U(\kappa)}$ be the factor forcing to prolong $G * g_0$ to a generic for $\pi_U(P_\kappa)$. We may build H_0 in $V[G][g_0]$ which is R_0-generic over $N[G][g_0]$, using the fact that R_0 is κ^+-closed in $N[G][g_0]$, has cardinality κ^+ in $V[G][g_0]$ and $V[G][g_0] \vDash N[G][g_0]$ is closed under κ-sequences.

Since k has critical point $\lambda > \kappa$, $k[G] = G$ and we can lift $k : N \to M$ to $k : N[G] \to M[G]$. Also, $k[g_0] = g_0 \subseteq g$ and so we may lift again to get $k : N[G][g_0] \to M[G][g]$.

Let $R = P^M_{\kappa+1,\pi(\kappa)}$. We claim that $H = \{r \in R \mid k(r_0) \leq r$ for some $r_0 \in H_0\}$ is R-generic over $M[G][g]$. To see this, note that each open dense $D \subseteq R$ in $M[G][g]$ is of the form $k(f)(a)$ for some $f \in N[G][g_0]$ with domain of size λ. We may assume that $f(x)$ is open dense on R_0 for each $x \in \text{Dom } f$, and since R_0 is λ^+-closed in $N[G][g_0]$, we may choose $r_0 \in H_0$ belonging to each $f(x)$, $x \in \text{Dom } f$. It follows that $k(r_0) \in H$ belongs to D.

Thus we have now extended the original π, π_U and k to embeddings $\pi : V[G] \to M[G][g][H]$, $\pi_U : V[G] \to N[G][g_0][H_0]$ and $k : N[G][g_0][H_0] \to M[G][g_0][H]$ so that $\pi = k \circ \pi_U$. These embeddings are definable in $V[G][g]$.

Now we try to lift π to $V[G][g]$. Let $S_0 = \pi_U(Q_\kappa) = \text{Add } (\pi_U(\kappa), \pi_U(\kappa)^{++})$ of $N[G][g_0][H_0]$. Notice that S_0 has cardinality κ^{++}, so we cannot choose an S_0-generic the way we chose an R_0-generic. Instead we must force over $V[G][g]$ with S_0.

Lemma 9. S_0 is κ^+-closed and κ^{++}-*Knaster* in $V[G][g_0]$. (P is κ-*Knaster* iff for every κ-sequence of conditions $\langle p_\alpha \mid \alpha < \kappa\rangle$ there is an unbounded $X \subseteq \kappa$ such that p_α, p_β are compatible for all $\alpha, \beta \in X$.)

Proof. κ^+-closure follows because $V[G][g_0] \vDash N[G][g_0][H_0]$ is closed under κ-sequences and $N[G][g_0][H_0] \vDash S_0$ is κ^+-closed. Let $\langle p_\alpha \mid \alpha < \kappa^{++}\rangle$ be a sequence of conditions in S_0 and represent p_α as $\pi_U(f_\alpha)(a_\alpha)$ where $a_\alpha \in H_{\kappa^+}$ and $f_\alpha : H_\kappa \to Q_\kappa$, $f_\alpha \in V[G]$. Then for some unbounded $X \subseteq \kappa^{++}$, $f_\alpha(y), f_\beta(y)$ are compatible for all $y \in H_\kappa$ and $a_\alpha = a_\beta$, for all $\alpha, \beta \in X$. It follows that p_α, p_β are compatible for all $\alpha, \beta \in X$. □

Lemma 10. S_0 is κ^+-distributive and κ^{++}-cc in $V[G][g]$.

Proof. $V[G][g]$ is a generic extension of $V[G][g_0]$ via a forcing which is isomorphic to Q_κ, which is κ^+-cc in $V[G][g_0]$. As S_0 is κ^+-closed in $V[G][g_0]$, it follows that it is κ^+-distributive in $V[G][g]$.

The product of a κ^{++}-Knaster forcing and a κ^{++}-cc forcing is κ^{++}-cc. So as S_0 is κ^{++}-Knaster in $V[G][g_0]$ and Q_κ is κ^{++}-cc (in fact κ^+-cc) in $V[G][g_0]$ it follows that $S_0 \times Q_\kappa$ is κ^{++}-cc in $V[G][g_0]$. Thus since $V[G][g]$ is a generic extension of $V[G][g_0]$ via a forcing isomorphic to Q_κ, it follows that S_0 is κ^{++}-cc in $V[G][g]$. □

By Lemma 10, if we force with S_0 over $V[G][g]$ we preserve cardinals. Let h_0 be S_0-generic over $V[G][g]$.

Just as we could obtain an R-generic (over $M[G][g]$) $H = \{r \in R \mid k(r_0) \leq r$ for some $r_0 \in H_0\}$, we can obtain an S-generic (over $M[G][g][H]$) $h = \{s \in S \mid k(s_0) \leq s$ for some $s_0 \in h_0\}$, where $S = \pi(Q_\kappa) = \text{Add }(\pi(\kappa), \pi(\kappa)^{++})$ of $M[G][g][H]$. Our wish is to extend π to an embedding from $V[G][g]$ to $M[G][g][H][h]$, but we have to first guarantee that $\pi[g] \subseteq h$.

Let $f = \cup g : \kappa \times \kappa^{++} \to 2$ be the function corresponding to the generic g. Then $\cup\pi[g]$ is the function $f^* : \kappa \times \pi[\kappa^{++}] \to 2$ defined by $f^*(\alpha, \pi(\beta)) = f(\alpha, \beta)$. We have to modify h to h^* so that each q^* in h^* is compatible with f^*.

For any $q \in h$ let q^* be defined by altering q on Dom $q \cap (\kappa \times \pi[\kappa^{++}])$ to agree with f^*. We claim that q^* belongs to $M[G][g][H]$, and therefore belongs to S. Write $q = \pi(f)(a)$ where a belongs to $H_{\kappa^{++}}$ and $f : H_\kappa \to Q_\kappa$ belongs to $V[G]$. (Of course H_κ, denotes the H_κ of $V[G]$ and Q_κ denotes the Add (κ, κ^{++}) of $V[G]$.) If $(\alpha, \pi(\beta))$ belongs to Dom q then (α, β) belongs to Dom $f(x)$ for some $x \in H_\kappa$, so $\{(\alpha, \beta) \mid (\alpha, \pi(\beta)) \in \text{Dom } q\}$ is contained in $Z_0 = \bigcup_x \text{Dom } f(x) \in V[G]$. As Z_0 has size at most κ and P_κ is κ-cc, there is $Z \in V$, $Z_0 \subseteq Z \subseteq \kappa \times \kappa^{++}$ of size at most κ. Then Z belongs to M and $\pi \restriction Z$ also belongs to M. Using $q, g, \pi \restriction Z$ we can define q^*, and therefore q^* belongs to $M[G][g][H]$.

Lemma 11. $h^* = \{q^* \mid q \in h\}$ is S-generic over $M[G][g][H]$.

Proof. Suppose that D is open dense on S, $D \in M[G][g][H]$. For any $q \in S$ define $N(q)$ to be the set of $r \in S$ with the same domain as q which disagree with q on a set of size at most κ. Then $E = \{q \mid N(q) \subseteq D\}$ is dense on S, using the $\pi(\kappa)$-closure of S. Choose q in $E \cap h$. Then q^* belongs to $N(q)$ and therefore to D. It follows that h^* intersects D. □

As $\pi[g] \subseteq h^*$ we may lift π to an embedding $V[G][g] \to M[G][g][H][h^*]$. And as before, by taking $I = \{p \mid \pi(p_0) \leq p$ for some $p_0 \in h_0\}$, we obtain a $\pi(S_0)$-generic over $M[G][g][H][h^*]$, and therefore an embedding

$V[G][g][h_0] \to M[G][g][H][h^*][I]$ which is definable in $V[G][g][h_0]$, as desired. $\Box$

Remark. The hypothesis of Theorem 6 can be weakened slightly. The above proof only needed an embedding $\pi : V \to M$ with critical point κ, where GCH holds in V, M is closed under κ-sequences and for some function f, $\pi(f)(\kappa) = \kappa^{++}$. ($f$ does not have to be the function $f(\alpha) = \alpha^{++}$.) Gitik showed that this weaker statement is equiconsistent with the statement that for some κ and every $A \subseteq \kappa^+$, there is an embedding $\pi : V \to M$ with critical point κ such that A belongs to M, and also equiconsistent with the statement that GCH fails at a measurable cardinal.

Regularity Properties

We show that if there is a Woodin limit of Woodin cardinals then every set of reals in $L(R)$, the smallest inner model containing all the reals, is Lebesgue measurable. The *consistency* of the latter statement is rather weak, following from the consistency of the existence of an inaccessible cardinal:

Theorem 12. (Solovay) Suppose that δ is inaccessible and G is generic for $\text{Coll}(\omega, <\delta)$, the forcing with finite conditions that makes each ordinal less than δ countable. Then in $V[G]$, every set of reals in $L(R)$ is Lebesgue measurable.

We shall show that if there is a Woodin limit of Woodin cardinals, then $L(R)$ elementarily embeds into $L(R)^{V[G]}$ where G is as in Theorem 12. It follows that every set of reals in $L(R)$ is Lebesgue measurable.

Definition. Suppose that $c \subseteq \mathcal{P}(u)$. Then c is *CUB on u* iff c consists of the universes of all substructures of a fixed structure for a countable language with universe u. $s \subseteq \mathcal{P}(u)$ is *stationary on u* iff $s \cap c \neq \emptyset$ for each CUB $c \subseteq \mathcal{P}(u)$, i.e., iff every structure for a countable language with universe u has a substructure with universe in s. The *stationary tower forcing* $Q = Q_\delta$, where δ is an inaccessible cardinal, consists of all pairs (u, s) where $u \in V_\delta$ is transitive and $s \subseteq \mathcal{P}_\omega(u) = \{x \subseteq u \mid x \text{ is countable}\}$ is stationary on u, ordered by:

$(u, s) \leq (v, t)$ iff $u \supseteq v$ and $s \restriction v = \{x \cap v \mid x \in s\} \subseteq t$.

If G is Q-generic then G assigns an ultrafilter G_u on $\mathcal{P}_\omega(u)$ to each u: $G_u = \{s \mid (u,s) \in G\}$.

The main fact that we need to prove is that if δ is Woodin and G is Q-generic then δ is a cardinal of $V[G]$. It turns out that every ordinal less than δ is countable in $V[G]$ and therefore it will be enough to show that δ has uncountable cofinality in $V[G]$.

The following is a version of Fodor's Lemma in this context.

Fodor's Lemma. Suppose that s is stationary on u and $f : s \to u$ such that $f(x) \in x$ for each $x \in s$. Then there is a stationary $s' \subseteq s$ such that f is constant on s'.

Proof. If not, then for each $v \in u$ choose a c_v which is CUB on u such that $f(x) \neq v$ whenever $x \in c$. Let c be the diagonal intersection of the c_v: $c = \{x \mid x \in c_v \text{ for all } v \in x\}$. Then c is CUB, so by the stationarity of s there is $x \in s \cap c$. But this means that $f(x) \neq v$ for all $v \in x$, contradicting the hypothesis on f. □

Recall that $\Delta \subseteq Q$ is *predense below* $p \in Q$ iff every extension of p is compatible with an element of Δ. To show that δ has uncountable cofinality in $V[G]$, it suffices to show:

$(*)$ If Δ_i, $i \in \omega$ are dense on Q and $p \in Q$ then there exists $\kappa < \delta$ and $q \leq p$ such that $\Delta_i \cap Q_\kappa$ is predense below q for each i, where $Q_\kappa = Q \cap V_\kappa$.

Write $p = (u_p, s_p)$. We say that a set x *captures* Δ iff there is $(u,s) \in x \cap \Delta$ such that $x \cap u \in s$. A condition p *captures* Δ iff every $x \in s_p$ captures Δ.

Proposition 13. If q captures Δ then Δ is predense below q.

Proof. Suppose that $r \leq q$. Then for each $x \in s_r$, $x \cap u_q \in s_q$ and hence $x \cap u_q$ captures Δ. For each such x choose $(u_x, s_x) \in x \cap u_q \cap \Delta$ such that $x \cap u_q \cap u_x \in s_x$. As u_q is transitive, $x \cap u_q \cap u_x = x \cap u_x$ for such x. By Fodor there is a fixed (u,s) and a stationary $s' \subseteq s_r$ such that for all $x \in s'$: $(u,s) \in x \cap \Delta$ and $x \cap u \in s$. Thus (u_r, s') extends both r and $(u,s) \in \Delta$, so r is compatible with an element of Δ. □

Thus to obtain $(*)$ it suffices show that there is $q \le p$ which captures each $\Delta_i \cap Q_\kappa$, $i \in \omega$. For this we need that each $\Delta_i \cap Q_\kappa$ can be captured by many sets in V_κ, in the following sense.

Definition. $\Delta \cap Q_\kappa$ is *semiproper* iff for CUB-many countable $x \subseteq V_{\kappa+1}$ there is a countable $z \in V_\kappa$ such that:

(i) z captures $\Delta \cap Q_\kappa$: There is $(u, s) \in \Delta \cap z$ such that $z \cap u \in s$.
(ii) z *end-extends* $x \cap V_\kappa$, i.e., $x \cap V_\kappa = z \cap V_\alpha$ for some $\alpha < \kappa$.
(iii) z is *x-closed*: If $c \in x$ is CUB on V_κ then z belongs to c.

Proposition 14. Suppose that $p \in Q_\kappa$ and each $\Delta_i \cap Q_\kappa$ is semiproper. Then there exists $q \le p$ such that q captures each $\Delta_i \cap Q_\kappa$.

Proof. Set $q = (V_\kappa, t)$ where $t = \{x \subseteq V_\kappa \mid x \cap u_p \in s_p$ and x captures each $\Delta_i \cap Q_\kappa\}$. It suffices to show that t is stationary.

For each i let c_i be CUB on $V_{\kappa+1}$ witnessing that $\Delta_i \cap V_\kappa$ is semiproper (i.e, the CUB set of countable x in the statement of semiproperness can be chosen to be the countable elements of c_i). Let c be the intersection of the c_i.

Now suppose that b is CUB on V_κ and we show that t has an element which belongs to b. Choose $x_0 \in c$ such that $x_0 \cap u_p \in s_p$; this is possible as s_p is stationary and therefore intersects $c \restriction u_p = \{x \cap u_p \mid x \in c\}$. Also require that p, b and $c \restriction V_\kappa$ belong to x_0.

Let $z_0 = x_0 \cap V_\kappa$. Applying the semiproperness of $\Delta_0 \cap Q_\kappa$, choose z_1 end-extending z_0 to capture $\Delta_0 \cap Q_\kappa$ and to be x_0-closed. As $c \restriction V_\kappa \in x_0$ and z_1 is x_0-closed, it follows that $z_1 \in c \restriction V_\kappa$ and therefore $z_1 = x_1 \cap V_\kappa$ for some $x_1 \in c$. Similarly, choose z_2 end-extending z_1 to capture $\Delta_1 \cap Q_\kappa$ and to be x_1-closed; then $z_2 = x_2 \cap V_\kappa$ for some $x_2 \in c$. Continue, getting $z_0 \subseteq z_1 \subseteq \cdots$ with union z. Note that $z \cap u_p = z_0 \cap u_p \in s_p$ since $p \in z_0$ and z end-extends z_0. Thus z belongs to t, and since b belongs to x_0 and z is x_0-closed, z belongs to b, as desired. □

Recall that $\kappa < \delta$ is *A-strong below δ* iff for each $\alpha < \delta$ there exists $\pi : V \longrightarrow M$ with critical point κ such that $\pi(\kappa) > \alpha$, $V_\alpha \subseteq M$ and $\pi(A) \cap V_\alpha = A \cap V_\alpha$. δ is Woodin iff for any $A \subseteq V_\delta$ there exists a $\kappa < \delta$ which is A-strong below δ.

Lemma 15. Let $\kappa < \alpha < \delta$, α inaccessible, $\pi : V \to M$ with critical point κ, $\pi(\kappa) > \alpha$, $V_\alpha \subseteq M$, $\pi(\Delta) \cap V_\alpha = \Delta \cap V_\alpha$ where $\Delta \cap Q_\alpha$ is predense on $Q_\alpha = Q \cap V_\alpha$. Then $\Delta \cap Q_\kappa$ is semiproper.

Proof. If not, then $s = \{x \subseteq V_{\kappa+1} \mid$ There is no x-closed $z \in V_\kappa$ such that z end extends $x \cap V_\kappa$ and captures $\Delta\}$ is stationary, and therefore $(V_{\kappa+1}, s)$ is a condition in Q_α. Choose $(u, s') \in \Delta \cap V_\alpha$ compatible with $(V_{\kappa+1}, s)$. We can choose $(V_\beta, t) \leq (u, s'), (V_{\kappa+1}, s)$ where β is less than α and $(u, s') \in V_\beta$. Now choose $x \in t$ to contain (u, s') and to belong to $\pi(c) \restriction V_\beta$ for each $c \in x \cap V_{\kappa+1}$ which is CUB on V_κ. Thus $x = x^* \cap V_\beta$ for some $x^* \in \pi(V_\kappa)$ which is $\pi(x \cap V_{\kappa+1})$-closed. Since $(V_\beta, t) \leq (V_{\kappa+1}, s)$, it follows that $x \cap V_{\kappa+1} \in s$. Applying π, we have that $\pi(x \cap V_{\kappa+1}) \in \pi(s)$. Using the definition of $\pi(s)$, there is no $\pi(x \cap V_{\kappa+1})$-closed $z \in \pi(V_\kappa)$ such that z end extends $\pi(x \cap V_\kappa) = x \cap V_\kappa$ and captures $\pi(\Delta)$. But consider $z = x^* \in \pi(V_\kappa)$. x^* is $\pi(x \cap V_{\kappa+1})$-closed and end-extends $x \cap V_\kappa$. Also x^* captures $\pi(\Delta)$, since x contains $(u, s') \in \Delta \cap V_\alpha = \pi(\Delta) \cap V_\alpha$ and $x^* \cap u = x \cap u \in s'$ since $(V_\beta, t) \leq (u, s')$. This is a contradiction. □

Corollary 16. Suppose that δ is Woodin. Then δ has uncountable cofinality in $V[G]$ for Q-generic G.

Proof. By Proposition 14 it suffices to show that if $p \in Q$ and Δ_i, $i \in \omega$ are dense on Q then there exists $q \leq p$ and $\kappa < \delta$ such that $q \in Q_\kappa$ and each $\Delta_i \cap Q_\kappa$ is semiproper. To prove this, apply the Woodinness of δ to obtain $\kappa < \delta$ such that $p \in Q_\kappa$ and κ is A-strong below δ, where A is the join of the Δ_i's. Then apply Lemma 15 where $\alpha < \delta$ is chosen so that each $\Delta_i \cap Q_\alpha$ is predense on Q_α. □

Suppose that G is Q-generic. Then we can form an ultrapower Ult (V, G) as follows:

$D = \{(u, f) \mid f : \mathcal{P}(u) \to V,\ f \in V\}$
$(u, f) \sim (v, g)$ iff $\{x \subseteq u \cup v \mid f(x \cap u) = g(x \cap v)\} \in G_{u \cup v}$.
$(u, f) E (v, g)$ iff $\{x \subseteq u \cup v \mid f(x \cap u) \in g(x \cap v)\} \in G_{u \cup v}$.
Ult (V, G) has universe $D/\sim$ and membership relation E on the $\sim$-equivalence classes $[u, f]$.

We have: Ult $(V, G) \models \varphi([u_1, f_1], \ldots, [u_n, f_n])$ iff $\{x \subseteq u \mid V \models \varphi(f_1(x \cap u_1), \ldots, f_n(x \cap u_n))\} \in G_u$, where $u = \cup_i u_i$. Thus we get an elementary embedding $\sigma : V \to$ Ult (V, G) defined by: $\sigma(x) = [\emptyset, c_x]$ where $c_x(\emptyset) = x$.

Assume that δ is Woodin and that G is Q-generic over V.

Lemma 17. (a) Identify the well-founded part of Ult (V, G) with its transitive collapse. Then every element of $V_\delta[G]$ belongs to Ult (V, G) and is countable in Ult (V, G). All reals of $V[G]$ belong to Ult (V, G).
(b) In fact, Ult (V, G) is well-founded.

Proof. (a) Suppose that $u \in V_\delta$ is transitive. By Fodor's Lemma, $[u, \mathrm{id}]$ represents $\sigma[u]$ in Ult (V, G), and therefore u, the transitive collapse of $\sigma[u]$, belongs to Ult (V, G). Thus $V_\delta \subseteq$ Ult (V, G). Also, as $V \vDash x$ is countable, for each $x \in G_u$, we have that Ult $(V, G) \vDash [u, \mathrm{id}] = \sigma[u]$ is countable and therefore u, the transitive collapse of $\sigma[u]$, is also countable in Ult (V, G). Finally, $s \in G_u$ iff $[u, \mathrm{id}] E \sigma(s)$, so as $\sigma \restriction \mathcal{PP}(u)$ belongs to Ult (V, G), it follows that G_u belongs to Ult (V, G). Thus $V_\delta[G] \subseteq$ Ult (V, G). As every ordinal less than δ is countable in Ult (V, G) and hence in $V[G]$, δ is regular in $V[G]$. Thus every real in $V[G]$ belongs to $V_\delta[G]$ and therefore to Ult (V, G).
(b) Suppose that $\langle \tau_n \mid n \in \omega \rangle$ is forced by some condition in Q to be a descending sequence of ordinals in Ult (V, G). For simplicity, we assume that this condition is the weakest condition of Q. Then for each n, the set of (u, s) such that for some $f^n_{(u,s)} : \mathcal{P}_\omega(u) \to \mathrm{ORD}$ in V, $(u, s) \Vdash \tau_n = [u, f^n_{(u,s)}]$ is dense on Q. Choose $q \in Q$ which captures each Δ_n. For each $x \in s_q$ and each n, choose some $(u, s) \in \Delta_n \cap x$ such that $x \cap u \in s$ and set $f^n(x) = f^n_{(u,s)}(x \cap u)$. We claim that for each n, $f^{n+1}(x) \in f^n(x)$ for CUB-many $x \in s_q$. Otherwise, $t = \{x \in s_q \mid f^{n+1}(x) \notin f^n(x)\}$ is stationary, by Fodor we can choose $(u_n, s_n) \in \Delta_n$, $(u_{n+1}, s_{n+1}) \in \Delta_{n+1}$ such that $x \cap u_n \in s_n$, $x \cap u_{n+1} \in s_{n+1}$ and $f^n(x) = f^n_{(u_n,s_n)}(x \cap u_n)$, $f^{n+1}(x) = f^{n+1}_{(u_{n+1},s_{n+1})}(x \cap u_{n+1})$ for all x in a stationary $t' \subseteq t$ and then $(u_q, t') \Vdash \sim (\tau_{n+1} = [u_{n+1}, f^{n+1}_{(u_{n+1},s_{n+1})}]\ E\ [u_n, f^n_{(u_n,s_n)}] = \tau_n)$, contradicting our hypothesis about the τ_n's. If $x \in s_q$ belongs to the intersection of the CUB sets witnessing $f^{n+1}(x) < f^n(x)$, then we get an infinite descending sequence of ordinals, contradiction. □

To prove that every set of reals in $L(R)$ is Lebesgue measurable, we only need one more fact.

Lemma 18. Suppose that δ is not only Woodin but also the limit of Woodin cardinals. Suppose that G is Q_δ-generic. Then every real in $V[G]$ is generic over V for a forcing of size less than δ.

Proof. For any inaccessible $\kappa < \delta$ define:

$t = \{x \subseteq V_{\kappa+1} \mid x$ is countable and captures all predense $\Delta \subseteq Q_\kappa$ in $x\}$.

Assuming that κ is Woodin, we show that t is stationary. Let c be CUB on $V_{\kappa+1}$ and assume that for $x \in c$, $x \cap V_\alpha \in b$ whenever $\alpha, b \in x$, b is CUB on V_α. The latter condition is a CUB condition, so can be assumed without loss of generality. Now let x_0 be an arbitrary element of c. If x_0 belongs to t then we are done. Otherwise choose a predense $\Delta_0 \subseteq Q_\kappa$ in x_0 not captured by x_0 and choose $\gamma < \delta$ in x_0 which is Δ_0-strong below κ. Then $\Delta_0 \cap Q_\gamma$ is semiproper so for CUB-many countable $x \subseteq V_{\gamma+1}$ we may choose $z \in V_\gamma$ end-extending $x \cap V_\gamma$ which captures $\Delta_0 \cap Q_\gamma$ and is x-closed. There is such a CUB collection of x's in x_0 so it follows that such a z exists for $x = x_0 \cap V_{\gamma+1}$, by our assumption about c. Choose such a z_1 and let x_1 be the smallest element of c such that $z_1 \cup x_0 \subseteq x_1$. Then $x_1 \cap V_\gamma = z_1$, using the x_0-closure of z_1. If x_1 belongs to t then we are done. Otherwise repeat the above for some predense $\Delta_1 \subseteq Q_\kappa$ in x_1 not captured by x_1, producing z_2 and x_2. We can continue in this way, arranging that every predense $\Delta \subseteq Q_\kappa$ in $\cup\{x_i \mid i \in \omega\}$ is considered, resulting in $x = \cup\{x_i \mid i \in \omega\}$ such that $x \in t$.

Thus if κ is Woodin, $(V_{\kappa+1}, t)$ is a condition, where t is defined as above. Similarly, for any $(u, s) \in Q_\kappa$, $\{x \in t \mid x \cap u \in s\}$ is stationary and therefore (u, s) is compatible with $(V_{\kappa+1}, t)$.

Now we claim that $G \cap Q_\kappa$ is Q_κ-generic for all Q-generic G containing $(V_{\kappa+1}, t)$. It suffices to show that each $q \leq (V_{\kappa+1}, t)$ is compatible with each predense $\Delta \subseteq Q_\kappa$. To see this, consider $s' = \{x \in s_q \mid \Delta \in x\}$ and form the condition $q' = (u_q, s') \leq q$. As $q' \leq (V_{\kappa+1}, t)$, $x \cap V_{\kappa+1} \in t$ for each $x \in s'$, and in particular $x \cap V_{\kappa+1}$ captures Δ for each $x \in s'$. Thus q' captures Δ and therefore is compatible with a condition in Δ. It follows that q is compatible with a condition in Δ, as desired.

Thus if δ is a limit of Woodin cardinals, $\{p \in Q_\delta \mid$ For some $\kappa < \delta$, $p \Vdash G \cap V_\kappa$ is Q_κ-generic$\}$ is dense. Thus $G \cap V_\kappa$ is Q_κ-generic for unboundedly many $\kappa < \delta$, proving that every element of $V_\delta[G]$, and hence any real in $V[G]$, is generic over V for a forcing of size less than δ. $\square$

Theorem 19. Suppose that δ is a Woodin limit of Woodin cardinals. Then there exists an elementary embedding $L(R) \to L(R)^{V[H]}$ where H is V-generic for $\mathrm{Coll}(\omega, < \delta)$. Therefore, every set of reals in $L(R)$ is Lebesgue measurable.

Proof. Let G be Q-generic over V, where $Q = Q_\delta$, the stationary tower forcing. Then there is an elementary embedding $V \to$ Ult (V, G) where $\delta = \omega_1$ of $V[G]$ and Ult (V, G), $V[G]$ have the same reals. Also every real in $V[G]$ belongs to a generic extension of V by a forcing of size less than δ. Now in a generic extension of $V[G]$ in which δ is countable, we can define a sequence $G_0 \subseteq G_1 \subseteq \cdots$ of length ω where $G_n \in V[G]$ is generic over V for Coll$(\omega, < \delta_n)$, the δ_n's form a cofinal, increasing ω-sequence of V-inaccessibles less than δ and each real in $V[G]$ belongs to some $V[G_n]$. If H is the union of the G_n's then $V[H]$ is generic over V for Coll$(\omega, < \delta)$ and the reals of $V[H]$ are precisely the reals of $V[G]$.

Thus we have an elementary embedding $L(R) \to (L(R)$ of Ult $(V, G)) = (L(R)$ of $V[G]) = (L(R)$ of $V[H])$, where H is Coll$(\omega, < \delta)$-generic over V. By Theorem 12, every set of reals in $(L(R)$ of $V[H])$ is Lebesgue measurable and therefore this also holds for $L(R)$. $\square$.