HOD and the Stable Core

1.-2.Vorlesungen

Gödel invented not one, but two definable inner models satisfying ZFC: L and HOD.

HOD is the inner model of hereditarily ordinal-definable sets. A set is *ordinal-definable* if it is definable with ordinal parameters. Gödel observed:

Lemma 1 *The class of ordinal-definable sets is definable as a class, without parameters.*

Proof. Use Lévy reflection: If a set is definable from ordinals then it is definable in some V_α from ordinals less than α. Conversely, if a set is definable in some V_α from ordinals less than α it is definable in V from those ordinals less than α together with α. □

A set is *hereditarily* ordinal-definable if not only it but also every element of its transitive closure is ordinal-definable. Clearly this is also a definable notion.

Proposition 2 *The class HOD of all hereditarily ordinal-definable sets is an inner model of ZFC.*

Proof. Obviously HOD is a transitive class containing all ordinals. It is straightforward to check each of the ZF axioms: For example, if x is in HOD then for any formula φ with parameters from HOD, $y = \{a \in x \mid \mathrm{HOD} \vDash \varphi(a)\}$ is ordinal-definable since x is ordinal-definable and since HOD is a definable class, and the property "$\mathrm{HOD} \vDash \varphi(a)$" is an ordinal-definable property of a. As y is a subset of x its transitive closure is a subset of the transitive closure of x and therefore consists of ordinal-definable sets, as by assumption x is hereditarily ordinal-definable. If x is hereditarily ordinal-definable then its HOD-powerset $\mathcal{P}(x) \cap \mathrm{HOD}$ is ordinal-definable as HOD is a definable class, and the transitive closure of the latter consists of itself together with the union of the transitive closures of elements of HOD; it follows that $\mathcal{P}(x) \cap \mathrm{HOD}$ is hereditarily ordinal-definable.

For the Axiom of Choice note the following: We can wellorder the elements of HOD as follows: To each x in HOD assign the least ordinal $\alpha(x)$ such that x

is ordinal-definable in $V_{\alpha(x)}$ and the least finite sequence $\vec{\beta}(x)$ of ordinals less than $\alpha(x)$ such that x is definable in $V_{\alpha(x)}$ with parameters $\vec{\beta}(x)$. Then say that x is less than y iff $(\alpha(x), \vec{\beta}(x))$ is less than $(\alpha(y), \vec{\beta}(y))$ as finite sequences of ordinals. This wellorder is definable and therefore when restricted to a set in HOD is an element of HOD. So in HOD, every set can be wellordered. $\Box$

Note something interesting in the last argument above: It does not show that HOD carries a HOD-definable wellorder. Indeed this need not be the case.

Fact 1. $V =$ HOD iff there is a definable wellorder of V (of length Ord, without loss of generality). For, the above shows that HOD has a V-definable wellorder and conversely, if V has a definable wellorder then every set is definable from its position in the wellorder, which is an ordinal. It follows that to get an example where HOD has no HOD-definable wellorder it suffices to get an example where the HOD of HOD is smaller than HOD.

Fact 2. Say that a forcing $\mathbb{P}$ is *forcing-homogeneous* iff whenever G is $\mathbb{P}$-generic and p is a condition in $\mathbb{P}$ there is another $\mathbb{P}$-generic G_p such that $V[G] = V[G_p]$ and p belongs to G_p. If this is the case then the HOD of $V[G]$ is contained in the V for each $\mathbb{P}$-generic G.

Proof of Fact 2. If x is an OD set of ordinals in $V[G]$ then choose a formula φ with ordinal parameters such that α belongs to x iff $V[G] \vDash \varphi(\alpha)$. But by forcing-homogeneity this is equivalent to saying that $p \Vdash \varphi(\alpha)$ for all $p \in \mathbb{P}$, and therefore x belongs to V. $\Box$

Proposition 3 *The HOD of HOD can be smaller than HOD.*

Proof. Add a Cohen real x to L. Then pull a MacAloon: Kill GCH at ω_n iff n belongs to x. The HOD of the resulting model $L[x, G]$ contains x but is also contained in $L[x]$ by forcing-homgeneity. So the HOD of $L[x, G]$ is $L[x]$. But the HOD of $L[x]$ is L, again by forcing-homogeneity. $\Box$

Another way in which HOD is inferior to L is with regard to cardinal exponentiation.

Proposition 4 *Consistently, CH can fail in HOD.*

Proof. Start with L and add ω_2 reals without collapsing cardinals; the result is a model $L[X]$ where X is a subset of ω_2 in which CH fails. Now force further in the style of MacAloon: For each ordinal α in X force $2^{\aleph_{\alpha+1}} = \aleph_{\alpha+3}$ by a full-support product. Cardinals are preserved. Now look at the HOD of the resulting model $L[X][G]$: As X can be read off from the GCH pattern it belongs to the HOD of $L[X][G]$ and therefore CH fails in it. $\Box$

So HOD loses the beauty contest when competing with L. But the attraction of HOD is that it is "close to V" in ways in which L might not be

Here are four ways in which HOD could be "close" to V:

Genericity: V is a generic extension of HOD.

Weak Covering: For arbitrarily large cardinals α, $\alpha^+ = (\alpha^+$ of HOD).

Rigidity: There is no nontrivial elementary embedding from HOD to HOD.

Large Cardinal Witnessing: Any large cardinal property witnessed in V is witnessed in HOD.

We begin with *Genericity.*

Genericity

A striking result of Vopenka is the following:

Theorem 5 *(Vopenka) Any set of ordinals is set-generic over HOD.*

Proof. Suppose that X is a subset of κ.

Consider formulas $\varphi = \varphi(v)$ of the form "$V_\alpha \models \psi(v)$" where $\kappa < \alpha$, ψ has ordinal parameters less than α and v is a free variable. We interpret φ as $\mathcal{S}_\varphi = \{x \subseteq \kappa \mid \varphi(x) \text{ is true}\}$. Order these formulas by $\varphi \leq \psi$ iff $\mathcal{S}_\varphi \subseteq \mathcal{S}_\psi$. The induced equivalence relation has only set-many (indeed 2^{2^κ}-many) equivalence classes. So we can thin out our partial order to a set-sized partial order $\mathcal{P}$ in HOD. Discard the equivalence class of the formulas φ for which $\mathcal{S}_\varphi$ is empty.

Now to our given $X \subseteq \kappa$ associate $G_X = \{\varphi \in \mathcal{P} \mid \varphi(X) \text{ is true}\}$. This is a compatible, upwards-closed subclass of $\mathcal{P}$. But in fact G_X is $\mathcal{P}$-generic over HOD because if A is a maximal antichain on $\mathcal{P}$ in HOD then the union of the $\mathcal{S}_\varphi$, $\varphi \in A$ must be all of $\mathcal{P}(\kappa)$ (else the complement of this union would produce a formula incompatible with every formula in A). Clearly X belongs to $L[G_X] \subseteq \mathrm{HOD}[G_X]$ as α belongs to X iff the formula $V_{\kappa+1} \models \alpha \in v$ belongs to G_X. □

The Vopenka result is *local*: it talks only about sets of ordinals. It is natural to look for a *global* result, stating that the entire universe V is generic over HOD. The *Stable Core* was invented to accomplish this.

3.Vorlesung

Let C denotes the closed unbounded class of all infinite cardinals β such that $H(\alpha)$ has size less than β whenever α is an infinite cardinal less than β. (The limit cardinals in C are exactly the strong limit cardinals and under GCH, C consists of all infinite cardinals.) For a finite $n > 0$, *α is n-Admissible* iff α is a limit point of C and $(H(\alpha), H \restriction \alpha)$ satisfies Σ_n replacement, where $H \restriction \alpha$ is $(H(\beta) \mid \beta < \alpha)$. For $\alpha < \beta$ limit points of C, *α is n-Stable in β* iff $(H(\alpha), H \restriction \alpha)$ is Σ_n elementary in $(H(\beta), H \restriction \beta)$.

Then the *Stability predicate* S consists of all triples (α, β, n) such that α is n-Stable in β and β is n-Admissible.

Theorem 6 *V is generic over $(L[S], S)$ for an $(L[S], S)$-definable forcing. The same is true with $(L[S], S)$ replaced by $(M[S], S)$ for any definable inner model M.*

The proof of Theorem 7 comes in two parts. First we show that V can be written as $L[F]$ where F is a function from the ordinals to 2 which "preserves" the Stability predicate S, in the sense that for (α, β, n) in S, α is n-Stable in β relative to F. Then we use this function to prove the genericity of V over $M[S]$ for any definable inner model M.

I'll start with the second part, which is easier.

V is generic over the Stability predicate

Now fix a function $F : \mathrm{Ord} \to 2$ which codes V and "preserves" the Stability predicate, i.e.

1. $V = L[F]$, (V, F) satisfies replacement with a predicate for F.
2. If $0 < n < \omega$, α is n-Stable in β and β is n-Admissible, then α is n-Stable in β relative to F.

We devise a forcing Q definable over $(L[S], S)$ such that for some Q-generic G, $V = L[S, G] = L[G]$ and G is definable over (V, F).

The language $\mathcal{L}$ is defined inductively as follows, where $\dot{F}$ is a unary function symbol.

1. For each ordinal α, "$\dot{F}(\alpha) = 0$" and "$\dot{F}(\alpha) = 1$" are sentences of $\mathcal{L}$.
2. If Φ is a set of sentences of $\mathcal{L}$ and Φ belongs to $L[S]$, then $\bigwedge \Phi$ and $\bigvee \Phi$ are sentences of $\mathcal{L}$.

A sentence φ of $\mathcal{L}$ is *valid* iff it is true when the symbol $\dot{F}$ is replaced by any function that belongs to a set-generic extension of $L[S]$. This notion is $L[S]$-definable and moreover if φ is a sentence of $L[S]$ and M is any outer model of $L[S]$, then φ is valid in $L[S]$ iff it is valid in M[1].

Now let T consist of all sentences of $\mathcal{L}$ of the form

$$\bigwedge(\Phi \cap H(\alpha)) \to \bigwedge(\Phi \cap H(\beta)),$$

where for some $\alpha < \beta$ and $1 < n < \omega$ we have:

(a) Φ is Σ_n definable over $H(\beta) \cap L[S]$ using parameters from $H(\alpha) \cap L[S]$.
(b) α is n-Stable in β (in V) and β is n-Admissible (in V).

Note that (a) implies that Φ is Σ_n definable over $(H(\beta), H \restriction \beta)$ (using parameters from the $H(\alpha)$ of V). It follows that the sentences in T are true

[1]Indeed, if there is a function witnessing the non-validity of φ in a set-generic extension of M then we may assume that this generic extension is $M[G]$ where G is generic for a Lévy collapse making φ countable; then $L[S][G]$ also has a witness to the non-validity of φ, by Lévy absoluteness. Conversely, if the non-validity of φ is witnessed in a set-generic extension of $L[S]$ then this will happen in $L[S][G]$ where G is Lévy collapse generic over $L[S]$. Choose a condition in the Lévy collapse forcing this and H containing this condition which is Lévy collapse generic over M; then the non-validity of φ is witnessed in $M[H]$, a set-generic extension of M.

when $\dot{F}$ is interpreted as F. Also note that T is $(L[S], S)$ definable, as (b) is expressed by the Stability predicate S.

4.-5.Vorlesungen

Theorem 7 *V is generic over $(L[S], S)$ for an $(L[S], S)$-definable forcing. The same is true with $(L[S], S)$ replaced by $(M[S], S)$ for any definable inner model M.*

The desired forcing Q consists of all sentences φ of $\mathcal{L}$ which are consistent with T, in the sense that for no subset T_0 of T is the sentence $\bigwedge T_0 \rightarrow\sim \varphi$ valid. The sentences in Q are ordered by: $\varphi \leq \psi$ iff T implies $\varphi \rightarrow \psi$.

Lemma 8 *Q has the Ord-chain condition, i.e., any $(L[S], S)$-definable maximal antichain in Q is a set.*

Proof. Suppose that A is an $(L[S], S)$-definable maximal antichain and consider $\Phi = \{\sim \varphi \mid \varphi \in A\}$. Then Φ is also $(L[S], S)$-definable. Choose n so that Φ is Σ_n-definable over $(L[S], S)$ and choose α to be n-Stable in Ord and large enough so that $H(\alpha) \cap L[S]$ contains the parameters in the definition of Φ. Then T together with $\Phi \cap H(\alpha)$ implies $\Phi \cap H(\beta)$ for all β greater than α which are n-Stable in Ord and since there are arbitrarily large such β and such β are n-Admissible, T together with $\Phi \cap H(\alpha)$ implies all of Φ. It follows that A equals $A \cap H(\alpha)$: Otherwise let φ belong to $A \setminus H(\alpha)$. As $\sim \varphi$ belongs to Φ it is implied by T together with $\Phi \cap H(\alpha)$. But as A is an antichain, T together with φ implies $\Phi \cap H(\alpha)$ and therefore T together with φ implies $\sim \varphi$, contradicting the fact that φ belongs to Q. $\square$

Now it is easy to see that $V = L[F] = L[G]$ where G is Q-generic over $(L[S], S)$: Let G consist of all sentences in Q which are true when $\dot{F}$ is interpreted as F. It is obvious that G intersects all maximal antichains of Q which are sets in $L[S]$, as if the set A is an antichain missed by G then $\bigwedge\{\sim \varphi \mid \varphi \in A\}$ is consistent with T and witnesses the failure of A to be maximal. By Lemma 8 this gives full genericity over $(L[S], S)$.

The above argument was carried out for the ground model $L[S]$. But the same argument can be used for any ground model $M[S]$ provided M is a definable inner model; simply replace n by $n - k - 1$ in (a) above, where M

is Σ_k-definable. This completes the second part of the proof of Theorem 7. I'll now turn to the first part.

Forcing a Stability-preserving predicate

Our aim is to force a function F from the ordinals to 2 which codes V (i.e., $V = L[F]$) and which obeys the following.

$(*)$ Suppose that $0 < n < \omega$ and α is n-Stable in β and β is n-Admissible. Then α is n-Stable in β *relative to* F: $(H(\alpha), H \restriction \alpha, F \restriction \alpha)$ is Σ_n elementary in $(H(\beta), H \restriction \beta, F \restriction \beta)$.

To this end we define by induction on $\beta \in C$ a collection $P(\beta)$ of functions from β to 2.

If β is the least element of C then $P(\beta)$ consists of all functions $p : \beta \to 2$. If β is the C-successor to $\alpha \in C$, then $P(\beta)$ consists of all functions $p : \beta \to 2$ such that $p \restriction \alpha$ belongs to $P(\alpha)$.

Suppose now that β is a limit point of C. Let $P(< \beta)$ denote the union of the $P(\alpha)$, $\alpha \in C \cap \beta$, ordered by extension. Assuming *extendibility for* $P(< \beta)$, i.e. the statement that for $\alpha_0 < \alpha_1 < \beta$ in C, each q_0 in $P(\alpha_0)$ can be extended to some q_1 in $P(\alpha_1)$, this forcing adds a generic function which we denote by $\dot{f} : \beta \to 2$. We say that $p : \beta \to 2$ is *n-generic for* $P(< \beta)$ iff $G(p) = \{p \restriction \alpha \mid \alpha \in C \cap \beta\}$ contains a condition which *strongly decides* each $\Pi_n(H(\beta), H \restriction \beta, \dot{f})$ sentence $\varphi = \forall x \psi$, i.e. meets every dense subset of $P(< \beta)$ of the form $\{q \in P(< \beta) \mid q \Vdash \varphi \text{ or } q \Vvdash\sim \varphi\}$, where "$q \Vvdash\sim \varphi$" ($q$ *strongly forces* $\sim \varphi$) means that q forces a counterexample $\sim \psi(x)$ to φ. We define $P(\beta)$ to consist of all $p : \beta \to 2$ which are n-generic for $P(< \beta)$ for all n such that β is n-Admissible.

Let P be the union of all of the $P(\beta)$'s, ordered by extension.

Lemma 9 *Assume Extendibility for P. Suppose that G is P-generic over V and let F be the union of the functions in G. Then $V = L[F]$ and $(*)$ holds for F. Moreover, V satisfies replacement with F as an additional predicate.*

Proof. Extendibility implies that it is dense to code any set of ordinals into the P-generic function F, from which it follows that V is contained in $L[F]$.

As $F \restriction \alpha$ belongs to V for each $\alpha \in C$ it also follows that $L[F]$ is contained in V and therefore $L[F]$ equals V.

Suppose that $0 < n < \omega$, α is n-Stable in β and β is n-Admissible. The relation $q \Vdash \varphi$ for q in $P(< \beta)$ and $\Pi_1(H(\beta), H \restriction \beta, \dot{f})$ sentences φ with parameters from $H(\beta)$ is Π_1 over $(H(\beta), H \restriction \beta)$: $q \Vdash \varphi$ iff for all $r \leq q$ and $\gamma \in C$ with $\gamma \leq \mathrm{Dom}\ (r)$ and $H(\gamma)$ containing the parameters in φ, $(H(\gamma), H \restriction \gamma, r) \vDash \varphi$. It then follows by induction on $n \geq 1$ that the relation $q \Vdash \varphi$ for q in $P(< \beta)$ and $\Pi_n(H(\beta), H \restriction \beta, \dot{f})$ sentences φ with parameters from $H(\beta)$ is Π_n over $(H(\beta), H \restriction \beta)$ (and the same for α). As $F \restriction \alpha$ is n-generic for $P(< \alpha)$, it follows that any true $\Pi_n(H(\alpha), H \restriction \alpha, F \restriction \alpha)$ sentence φ with parameters from $H(\alpha)$ is forced by some condition $F \restriction \alpha_0$, $\alpha_0 \in C \cap \alpha$. But then as α is n-Stable in β, $F \restriction \alpha_0$ also forces "φ holds in $(H(\beta), H \restriction \beta, \dot{f} \restriction \beta)$"; by the n-genericity of $F \restriction \beta$ (which holds due to the n-Admissibility of β), it follows that φ holds in $(H(\beta), H \restriction \beta, \dot{f} \restriction \beta)$ when $\dot{f} \restriction \beta$ is interpreted as the real $F \restriction \beta$. Thus we have proved that α is n-Stable in β relative to F.

To verify replacement relative to F, we need only observe that the above implies that for each n, if α is n-Stable in Ord (i.e., $(H(\alpha), H \restriction \alpha)$ is Σ_n elementary in (V, H)) then it remains so relative to F. $\square$

We now turn to extendibility for P.

Lemma 10 *Suppose that $\alpha < \beta$ belong to C and p belongs to $P(\alpha)$. Then p has an extension q in $P(\beta)$.*

Proof. By induction on β. The statement is immediate by induction if β is not a limit point of C.

Suppose that β is a limit point of C but is not 1-Admissible. Then there is a closed unbounded subset D of $C \cap \beta$ of ordertype less than β whose intersection with each of its limit points $\gamma < \beta$ is Δ_1 definable over $(H(\gamma), H \restriction \gamma)$. We can assume that both α and the ordertype of D are less than the minimum of D. Now enumerate D as $\beta_0 < \beta_1 < \cdots$ and using the induction hypothesis, successively extend p to $q_0 \subseteq q_1 \subseteq \cdots$ with q_i in $P(\beta_i)$, taking unions at limits. Note that for limit i, q_i is indeed a condition because β_i is not 1-Admissible. The union of the q_i's is the desired extension of p in $P(\beta)$.

Next suppose that β is n-Admissible but not $n+1$-Admissible for some finite $n > 0$:

If β is a limit of n-Stables (i.e., the set of $\alpha < \beta$ which are n-Stable in β is cofinal in β), then proceed as in the previous paragraph: Choose a closed unbounded subset D of $C \cap \beta$ of ordertype less than β consisting of n-Stables in β, whose intersection with each of its limit points $\gamma < \beta$ is Δ_{n+1} definable over $(H(\gamma), H \restriction \gamma)$. Assume that both α and the ordertype of D are less than the minimum of D, enumerate D as $\beta_0 < \beta_1 < \cdots$ and using the induction hypothesis, successively extend p to $q_0 \subseteq q_1 \subseteq \cdots$ with q_i in $P(\beta_i)$, taking unions at limits. For limit i, q_i is indeed a condition because β_i is not $n+1$-Admissible and as it is a limit of n-Stables, q_i is n-generic for $P(< \beta_i)$. The union of the q_i's is the desired extension of p in $P(\beta)$.

If β is not a limit of n-Stables then β must have cofinality ω (else by n-Admissibility, we could find cofinally many n-Stables in β using the fact that β has uncountable cofinality). It suffices to show that any condition q in $P(< \beta)$ can be extended to strongly decide each of fewer than β-many Π_n sentences with parameters from $H(\beta)$ (given this, we can extend p in ω steps to a condition in $P(\beta)$ which is n-generic). To show this, let $(\varphi_i \mid i < \delta)$ enumerate the given collection of Π_n sentences and if $n > 1$, let D consist of all γ which are limits of $(n-1)$-Stables in β and large enough so that $H(\gamma)$ contains both q and this enumeration. (If $n = 1$ then let D consist of all γ which are limit points of C and large enough so that $H(\gamma)$ contains both q and this enumeration.) Now extend q successively to elements q_i of $P(\gamma_i)$, where $\gamma_{i+1} \geq \gamma_i$ is the least element of D so that either q_i forces φ_i or q_{i+1} strongly forces $\psi_i =$ the negation of φ_i (i.e. with corresponding witness to the Σ_n sentence ψ_i), taking unions at limits. For limit i, q_i is a condition as γ_i is not n-Admissible but (in case $n > 1$) is a limit of $(n-1)$-Stables. (The failure of γ_i to be n-Admissible uses the fact that the set of $j < i$ such that q_{j+1} forces the negation of φ_j can be treated as a parameter in $H(\gamma_i)$.) As β is n-Admissible, this construction results in a sequence of q_i's of length δ, whose union it the desired extension of q deciding all of the given Π_n sentences.

Finally, suppose that β is n-Admissible for every finite n. Choose C to be closed unbounded in β so that any $\gamma < \beta$ which is a limit point of C is a limit of n-Stables for every n. (Note that we may choose C to be any cofinal ω-sequence if β has cofinality ω.) Assume that α is less than the least element of C and enumerate C as $\beta_0 < \beta_1 < \cdots$. Then successively extend p to $q_0 \subseteq q_1 \subseteq \cdots$ with q_i in $P(\beta_i)$, taking unions at limits, and note that for limit i, q_i is a condition because its n-genericity follows from the fact that β_i is a limit of n-Stables. The union of the q_i's is the desired q. □

This completes the proof of Theorem 7.

Remark. Instead of considering just Π_n sentences but more generally $\Sigma_0\Pi_n$ sentences, i.e. sentences of the form $\exists x \in x_0 \varphi$ where φ is Π_n, with a correspondingly stronger notion of *n-generic*, one can in fact further require that F preserve all n-Admissibles. This stronger notion of n-genericity requires that for n-Admissible β and $\Sigma_0\Pi_n$ sentences of the above form, one meets all dense sets of the form $\{p \in P(<\beta) \mid p \Vdash \varphi(x)$ for some $x \in x_0$ or $p \Vdash \forall x \in x_0 (H(\alpha), H \restriction \alpha, \dot{f}) \vDash \sim \varphi(x)$ for some α which is $(n-1)$-Stable in $\beta\}$. This is enough to ensure the preservation of n-Admissibility. And in the problematic case where β is n-admissible but not the limit of n-Stables, one can indeed build n-generic conditions in this stronger sense by maximising those $x \in x_i$ for which $\sim \varphi_i(x)$ holds for each of a given set of $\Sigma_0\Pi_n$ sentences $\exists x \in x_i \varphi_i$, in the sense that in chosen wellorders $<_i$ of the x_i's, those x's include an initial segment of $<_i$ which is as long as possible.

The resulting F will then preserve not only instances of n-Stability but also the entire Stability Predicate S.

In general the inner model $L[S]$ may be strictly smaller than HOD: Using Jensen coding methods one can produce a real R that is not set-generic over L such that the Stability Predicate of $L[R]$ is the same as the Stability Predicate of L. By Vopenka's Theorem, the Stable Core of $L[R]$ is a proper inner model of HOD. Also note that the Stable Core is more absolute than HOD, as no forcing of size less than $\beth_\omega$ will affect the Stability Predicate.

In fact the entire V is generic over HOD. To explain this I need to introduce the *Stability Predicate.*

Let SL denote the class of strong limit cardinals

α in SL is *n-Admissible* if $(H(\alpha), \mathrm{SL} \cap \alpha)$ satisfies Σ_n Replacement

For $\alpha < \beta$ in SL, α is *n-Stable in β* if $(H(\alpha), \mathrm{SL} \cap \alpha)$ is Σ_n-elementary in $(H(\beta), \mathrm{SL} \cap \beta)$

The *Stability Predicate* S consists of all triples (α, β, n) where α is n-stable in β and β is n-Admissible.

The Stability Predicate S is definable and therefore (HOD, S) is a model of ZFC.

Theorem 11 *V is generic over (HOD, S).*

The forcing used to prove this is definable over (HOD, S).
I strongly doubt that S is HOD-definable in general or even that V must be generic over HOD for a HOD-definable forcing.

Actually V is generic over the inner model $(L[S], S)$ called the *Stable Core* which can be strictly smaller than HOD.
The Stable Core is a very useful tool for understanding HOD.

One of the milestones of core model theory is:

Weak Covering at Singulars: If α is a singular cardinal then $\alpha^+ = (\alpha^+$ of "the core model")

Does one have Weak Covering for HOD? Unfortunately:

Theorem 12 *(Cummings, me, Golshani) It is consistent that $\alpha^+ > (\alpha^+$ of $HOD)$ for every infinite cardinal α.*

But there may be more to the story.

If we want to have a supercompact cardinal κ as well then the best we can get is $\alpha^+ > (\alpha^+$ of HOD) for a club of $\alpha < \kappa$. And Woodin has conjectured that one cannot improve this to all $\alpha < \kappa$.

An easy Corollary of the genericity of V over the Stable Core $\mathbb{S} = (L[S], S)$ is the following.

Proposition 13 *There is no nontrivial V-definable elementary embedding from $\mathbb{S}$ to itself.*

Proof. Let V be generic over $\mathbb{S}$ via the definable forcing $\mathbb{P}$. Then fix a formula φ that (with parameters) defines in V a nontrivial embedding from $\mathbb{S}$ to itself, and let α be the least ordinal which some condition in $\mathbb{P}$ forces to be the critical point of such an embedding. But the ordinal α is $\mathbb{S}$-definable and therefore cannot be the critical point of any embedding from $\mathbb{S}$ to itself, contradiction. $\square$

It follows that also (HOD, S) is rigid for V-definable embeddings.

But what about embeddings which are not V-definable?
The previous proof took advantage of the fact that the embeddings were just as definable as they were elementary.

The Enriched Stable Core

I won't bore you with the definition, but there is a richer form S^* of the Stability Predicate S which is also first-order definable and can be used to get a better rigidity result.

For simplicity, work in Morse-Kelley and note that this theory is strong enough to build an "L-hierarchy" over V and therefore a notion of *V-constructible class.*

Theorem 14 *The Enriched Stable Core $(L[S^*], S^*)$ is rigid for V-constructible embeddings and therefore so is (HOD, S^*).*

But it is still unknown whether HOD without the predicate S^* is rigid with respect to V-constructible embeddings.
The long-standing open conjecture is that HOD is in fact rigid for arbitrary embeddings.

Here the news is pretty bad.

Theorem 15 *(Cheng, me and Hamkins) It is consistent that there are supercompacts but none in HOD. One can even add to this that no supercompact is weakly compact in HOD.*

But this doesn't quite end the story:

1. Maybe if there are very large cardinals like extendibles then there must be supercompacts in HOD. In other words it could be that HOD does witness large cardinal properties, but with a certain loss of strength.

2. Maybe the Stable Core does a better job of *Large Cardinal Witnessing*. If so, then the Stable Core is a better choice of "canonical" inner model than HOD.

We looked at four ways that HOD could be "close" to V (of course they all hold if $V = \mathrm{HOD}$!):

Genericity: V is a generic extension of HOD
Holds with HOD replaced by (HOD, S); open otherwise.

Weak Covering: For arbitrarily large cardinals α,
$\alpha^+ = (\alpha^+ \text{ of HOD})$
Fails but maybe holds if there are supercompacts.

Rigidity: There is no nontrivial elementary embedding from HOD to HOD
Holds with HOD replaced by (HOD, S^*) for V-constructible embeddings, open otherwise.

Large Cardinal Witnessing: Any large cardinal property witnessed in V is witnessed in HOD
Fails, but maybe holds allowing a drop in strength from V to HOD.

HOD is better than L because it is closer to V.

But L satisfies GCH!

Perhaps the biggest challenge is to get models that have the advantages of both:

HUGE *Open Problem:* Is there an inner model that satisfies any of *Genericity*, *Weak Covering*, *Rigidity*, or *Large Cardinal Witnessing* and also satisfies GCH?

If you can do that then you can have my job.

The End

Gödel's universe HOD of hereditarily ordinal definable sets is of central interest in set theory. Like L it provides a definable inner model which satisfies AC, but unlike L it is in some ways "close to V". The property of an inner model M being "close to V" can be described in a number of ways:

Genericity: V is a generic extension of M
Weak Covering: For arbitrarily large cardinals α, $\alpha^+ = (\alpha^+ \text{ of } M)$
Rigidity: There is no nontrivial elementary embedding from M to M
Large Cardinal Witnessing: Any large cardinal property witnessed in V is witnessed in M

To what extent do these properties hold when M equals HOD? A new inner model called the *Stable Core* is a useful tool for studying this question. The *Stability Predicate* S consists of those triples (n, α, β) where V_α is elementary in V_β for Σ_n formulas. (The actual definition is only slightly different than this; one uses $H(\alpha)$ instead of V_α and restricts to "nice" α's; see my BSL paper on the Stable Core.) In this case we say that "α is n-Stable in β". The *Stable Core* is the model $(L[S], S)$ where S is the Stability Predicate.

Regarding *Genericity* we have:

Theorem. V is generic over the Stable Core $(L[S], S)$ for a class-forcing which is definable over the Stable Core. The same holds with the Stable Core replaced by (HOD, S).

This fits nicely with a theorem of Vopenka which states that each set is set-generic over HOD. We therefore have that V is *locally class-generic* over (HOD, S) in the sense that it is class-generic with every set being set-generic. (Vopenka's theorem does not apply to the Stable Core, as is shown in my BSL paper.)

Questions 1. Must S be definable over HOD? Must V be generic over HOD for a HOD-definable forcing?

I conjecture that the answer to both of these questions is "no".

What about *Weak Covering* for HOD? The first result is negative:

Theorem. (Cummings, SDF, Golshani) It is consistent that α^+ is greater than α^+ of HOD for all cardinals α.

But there may be more to the story. If we want to have a supercompact cardinal κ as well then the best we can get is $\alpha^+ > (\alpha^+$ of HOD) for a club of $\alpha < \kappa$ and not for all $\alpha < \kappa$.

Question 2. Suppose that there is a supercompact cardinal. Must there be a cardinal $\alpha < \kappa$ such that α^+ equals (α^+ of HOD)?

I turn now to *Rigidity*. Using the above result about genericity over the Stable Core one easily derives the following

Corollary: (HOD, S) is rigid for V-definable embeddings (i.e. no elementary $j : (\mathrm{HOD}, S) \to (\mathrm{HOD}, S)$ is V-definable other than the identity).

To do better we need the *Enriched Stable Core*. This is defined using the *Enriched Stability Predicate* S^*; for the details look at my paper on the Enriched Stable Core.

Work now in Morse-Kelley, a theory strong enough to build an "L-hierarchy" over V and therefore a notion of *V-constructible class*.

Theorem. The Enriched Stable Core $(L[S^*], S^*)$ is rigid for V-constructible embeddings and therefore so is (HOD, S^*).

Although this is good progress, the following remains open:

Question 3. Is HOD (with no predicate) rigid for V-constructible embeddings? Is HOD rigid for arbitrary embeddings?

The popular belief is that the answer to these long-standing open questions is "yes".

With regard to *Large Cardinal Witnessing*, the initial results are negative:

Theorem 16 *(Cheng, SDF, Hamkins) It is consistent that there are supercompacts but none in HOD. One can add to this that no supercompact is even weakly compact in HOD.*

But again this doesn't quite end the story:

Question 4. Does the existence of a very large cardinal, like an extendible cardinal, imply that there are supercompacts in HOD?

In other words it could be that HOD does witness large cardinal properties, but with a certain loss of strength.

It is much harder to "trick" the Stable Core than it is to "trick" HOD (in the sense of the previous Theorem). The reason is that it is much easier to make sets ordinal-definable than it is to code them into the Stability Predicate. In fact this suggests that all of our properties (other than *Weak Covering*) should also be considered for the Stable Core and not just for HOD, as the Stable Core may exhibit better properties than HOD.

Questions 5. Is V generic over the "pure" Stable Core $L[S]$ without the predicate S? Is the Stable Core (or its Enriched version) rigid for arbitrary embeddings? If there is a supercompact must there be one in the Stable Core (without further large cardinal assumptions)?

The deepest question about the Stable Core is also the most exciting: Is it possible to produce a version of it which obeys the GCH? Core model theory is able to do this up to the level of Woodin cardinals; what can be done at the level of supercompacts?

Question 6. Is there a definable inner model (perhaps a variant of the Stable Core) which satisfies GCH and over which V is generic? More generally, is there a definable inner model M with the GCH for which one has *Genericity, Weak Covering, Rigidity* or *Large Cardinal Witnessing*?

The above is perhaps the simplest version of the *inner model problem.*

Returning briefly to criteria in the Hyperuniverse Programme, we remark that HOD and the Stable Core can be naturally introduced into the study of maximality. Indeed, the violation of $V = L$ that results from maximality can be interpreted as saying that "not every set is predicatively-definable from ordinals". Similarly a violation of the axiom $V =$ HOD could be expressed by "not every set is definable from ordinals". In light of Vopenka's work and my work on the Stable Core, we must accept that every set is set-generic over HOD and the universe is class-generic over the Stable Core. Therefore maximality criteria which imply that V is not generic over the Stable Core or that some sets are not set-generic over HOD are inconsistent. But it is consistent to assert that the entire universe V is not set-generic over the Stable Core and that there are sets which are not set-generic over the Stable Core and there may be maximality criteria which imply these statements. Further investigation of the Stability predicate will enable us to formulate such criteria and facilitate their synthesis with other desirable criteria of maximality.

6.-7.Vorlesungen

Rigidity and the Enriched Stable Core

As V is generic over the Stable Core SC $= (L[S], S)$ (where S is the Stability predicate) for a definable forcing whose definable antichains are sets, we obtain as a consequence:

Corollary 17 *Any V-definable club contains an SC-definable club. And SC is rigid for V-definable embeddings, i.e., there is no V-definable elementary embedding of SC to itself other than the identity.*

Proof: The statement about clubs follows immediately from the fact that V is generic over SC for a definable forcing whose definable antichains are sets.

We give two proofs of rigidity for V-definable embeddings, as both are useful for generalisations.

First proof.

Suppose that V is $\mathbb{P}$-generic over SC for the SC-definable forcing $\mathbb{P}$ and that there were an elementary (equivalently Σ_1-elementary) embedding of SC to itself which is Σ_n-definable over V. Let κ be the least ordinal which is forced to be the critical point of such a Σ_n-definable embedding by some condition in $\mathbb{P}$. Then κ is SC-definable and therefore cannot be moved by any elementary embedding from SC to itself, contradiction.

Second proof.

We first claim that there is an SC-definable $\diamondsuit$ sequence for SC that concentrates on ordinals of cofinality ω and guesses SC-definable classes on SC-definably stationary classes. More precisely, there is an SC-definable sequence $(X_\alpha \mid \alpha$ of SC-cofinality $\omega)$ such that $X_\alpha \subseteq \alpha$ for each α and whenever X is an SC-definable class of ordinals and C is an SC-definable club there is α in C such that $X \cap \alpha = X_\alpha$. To see this, define X_α inductively as follows: Let n be least so that some pair (X_α, C_α) is Σ_n-definable over $(L_\alpha[S], S \cap L_\alpha[S])$ and such that $X_\alpha \subseteq \alpha$, C_α is club in α and $X_\alpha \cap \bar\alpha \neq X_{\bar\alpha}$ for all $\bar\alpha \in C_\alpha$. If α does not have SC-cofinality ω or if there is no such pair then we set $X_\alpha = \emptyset$; otherwise we let (X_α, C_α) be the least such pair (where Σ_n sets are ordered by the formulas which define them and for a fixed formula by the parameters used). We claim that the sequence $(X_\alpha \mid \alpha$ of SC-cofinality $\omega)$ is as desired. If not, let n be least so that some $X \subseteq \mathrm{Ord}$ which is Σ_n-definable over SC is not guessed correctly anywhere on some Σ_n-definable club $C \subseteq \mathrm{Ord}$; fix the least such pair (X, C) and notice that by reflection there is an α of SC-cofinality ω such that $X \cap \alpha = X_\alpha$, $C \cap \alpha = C_\alpha$. But this is a contradiction because α belongs to C.

Now use the $\diamondsuit$-sequence to produce an SC-definable partition $(X_i \mid i \in \mathrm{Ord})$ of the ordinals of SC-cofinality ω into pieces which are SC-definably stationary (i.e. which intersect each SC-definable club). Suppose that $j : \mathrm{SC} \to \mathrm{SC}$ were elementary with critical point κ with j definable in V. Now $C = \{\alpha \mid j[\alpha] \subseteq \alpha\}$ is a V-definable club and therefore contains an SC-definable club; it follows that there is an ordinal α of SC-cofinality ω in $j((X_i \mid i \in \mathrm{Ord}))_\kappa$ such that $j[\alpha] \subseteq \alpha$ and therefore $j(\alpha) = \alpha$. But then as $j(\alpha)$ belongs to $j((X_i \mid i \in \mathrm{Ord}))_i$ for some $i < j(\kappa)$ it follows that α belongs to X_i for some $i < \kappa$ and therefore $j(\alpha) = \alpha$ belongs to $j((X_i \mid i \in \mathrm{Ord}))_i$ for some $i < \kappa$; this contradicts the fact that $j((X_i \mid i \in \mathrm{Ord}))$ is a partition into disjoint pieces. $\square$

But what about embeddings that are not V-definable?

From now on we work in class theory, whose models look like $(V, \mathcal{C})$ where V consists of the sets and $\mathcal{C}$ consists of the classes. A reformulation of the previous Corollary is:

Corollary 18 *Suppose that $(V, \mathcal{C})$ is the least model of Gödel-Bernays built over V (i.e., $\mathcal{C}$ consists only of the V-definable classes). Also let $(L[S], \mathcal{C}^S)$ be the least model of Gödel-Bernays built over $L[S]$ which has S as a class (i.e.*

$\mathcal{C}^S$ consists only of the SC-definable classes). Then any club in $\mathcal{C}$ contains a club in $\mathcal{C}^S$ and SC is rigid for embeddings in $\mathcal{C}$.

To obtain rigidity of the Stable Core in larger models of class theory we put more information into the Stability Predicate.

The Enriched Stable Core

We define the enriched stability predicate S^* as follows. For β in C, $i < \beta^+$ of $L(H(\beta))$ and $0 < n < \omega$ we say that β is (i,n)-*Admissible* iff β is a limit point of C and β is $\Sigma_n(L_i(H(\beta)), H\restriction\beta)$-regular. If $\alpha < \beta$ are both limit points of C, $i < \beta^+$ of $L(H(\beta))$ and $0 < n$ then we say that α is (i,n)-*Stable in* β iff $H_n^{(\beta,i)}(\alpha) \cap H(\beta) = H(\alpha)$, where $H_n^{(\beta,i)}(\alpha)$ is the least $H \prec_{\Sigma_n} (L_i(H(\beta)), H\restriction\beta)$ containing α as a subset.

Note that α is $(0,n)$-Stable in β (β is $(0,n)$-Admissible) iff α is n-Stable in β (β is n-Admissible) via the earlier definition. We set:

$S^* = \{(\alpha, \beta, i, n) \mid \alpha$ is (i,n)-stable in β and β is (i,n)-Admissible$\}$.

$\mathrm{SC}^* = (L[S^*], S^*)$, the *Enriched Stable Core.*

Our aim now is to show that the Enriched Stable Core is rigid for embeddings which are V-*constructible* and not necessarily V-definable. This notion can be defined in a weak theory like Gödel-Bernays, but for simplicity I'll just discuss it here in the context of set-sized *β-models of Morse-Kelley.* Recall that a model $(V, \mathcal{C})$ of Gödel-Bernays is a model of Morse-Kelley if it satisfies class-comprehension for formulas which quantify over classes. It is a *β-model of Morse-Kelley* if in addition any linear order in $\mathcal{C}$ which is wellfounded in $(V, \mathcal{C})$ is truly wellfounded (in the ambient universe in which $(V, \mathcal{C})$ exists as a set-sized structure).

Definition 19 *Let $(V, \mathcal{C})$ be a β-model of Morse-Kelley. A class in $\mathcal{C}$ is V-*constructible *(in $(V, \mathcal{C})$) if it belongs to $L_\alpha(V)$ for some ordinal α which is the ordertype of a wellorder in $\mathcal{C}$.*

Remark. In a paper with Carolin Antos we observed that if $(V, \mathcal{C})$ is a β-model of Morse-Kelley which in addition satisfies *Class Bounding* ($\forall x \exists Y \varphi(x, Y) \to \exists Y \forall x \varphi(x, (Y)_x)$) then $\mathcal{C}$ consists of the subsets of V in a transitive model

M^+ of ZFC$^-$ (= ZFC without powerset but still with Σ_n bounding for every n). Then the V-constructible classes are just those which belong to the L of M^+.

Lemma 20 *(Main Lemma) Assume that $(V, \mathcal{C})$ is a β-model of Morse-Kelley in which every class is V-constructible. Then there is an SC^*-definable class forcing $\mathbb{P}^*$ which adds a function from Ord to 2 such that for $\mathbb{P}^*$-generic $F^* : Ord \to 2$, $(V[F^*], \mathcal{C}[F^*])$*[2] *is a model of Morse-Kelley minus Power, V is a definable inner model of $(L[F^*], F^*)$ and F^* preserves*[3] *the enriched stability predicate S^*.*

Before proving the Main Lemma we describe its implications for the rigidity of HOD.

Theorem 21 *In a β-model $(V, \mathcal{C})$ of Morse-Kelley let $\mathcal{C}^*$ consist of the $(L[S^*], S^*)$-constructible classes, where S^* is the enriched stability predicate. Then $(L[S^*], \mathcal{C}^*)$ has an outer model $(L[F^*], \mathcal{C}^*[F^*])$ of Morse-Kelley minus Power which is generic over $(L[S^*], \mathcal{C}^*)$ for an SC^*-definable forcing which is ∞-cc (i.e. whose antichains in $\mathcal{C}^*$ are sets) such that V is a definable inner model of $(L[F^*], F^*)$. Moreover SC^* is rigid in $\mathcal{C}^*[F^*]$.*

Corollary 22 *In any β-model $(V, \mathcal{C})$ of Morse-Kelley, any V-constructible club contains an $(L[S^*], S^*)$-constructible club and $SC^* = (L[S^*], S^*)$ is rigid for V-constructible embeddings. It follows that also (HOD, S^*) is rigid for V-constructible embeddings.*

Proof of Corollary 22 from Theorem 21. It suffices to show that any V-constructible class belongs to the $\mathcal{C}^*[F^*]$ of Theorem 21. Any such class belongs to a model A_V of KP+ "every set is constructible from V" which is an end-extension of V. Then A_V has an inner model $A_{L[S^*]}$ with the same properties, replacing V by $L[S^*]$. As V is a definable inner model of $(L[F^*], F^*)$ it follows that A_V is contained in $A_{L[S^*]}[F^*]$. But any class in $A_{L[S^*]}$ is $L[S^*]$-constructible and so the classes of $A_{L[S^*]}[F^*]$ belong to $\mathcal{C}^*[F^*]$. $\square$

[2] $\mathcal{C}[F^*]$ consists of those classes which are definable in $(V[F^*], X, F^*)$ for some $X \in \mathcal{C}$.

[3] I.e. for any $\alpha < \beta$, $i < \beta^+$ of $L(H(\beta))$ and $0 < n$, if α is (i, n)-Stable in β and β is (i, n)-Admissible then α is (i, n)-Stable in β and β is (i, n)-Admissible relative to F^*. But as Power Set fails it is important to distinguish between $H(\beta)[F^*]$ and $H(\beta)^{L[F^*]}$; indeed the latter may fail to exist.

Proof of Theorem 21 from the Main Lemma. For the first conclusion of the theorem of course we take F^* to be as in the Main Lemma and need to define an ∞-cc SC*-definable forcing Q^* for which F^* is generic. In analogy to the case of the (unenriched) Stable Core we build the forcing Q^* out of quantifier-free infinitary sentences which belong to $L[S^*]$. Such sentences are obtained by closing the atomic sentences "$\dot{F}(\alpha) = 0$", "$\dot{F}(\alpha) = 1$" under infinitary conjunctions and disjunctions in $L[S^*]$. We let $\mathcal{L}^*$ denote the collection of such sentences which are consistent, i.e., which are true for some interpretation of $\dot{F}$ in a set-generic extension of $L[S^*]$; this notion of consistency is definable in $L[S^*]$.

Now we introduce a certain theory T^*, consisting of sentences of $\mathcal{L}^*$. For each $\alpha < \beta$, $i < \beta^+$ of $L(H(\beta))$ and $n > 0$ such that α is (i, n)-Stable in β and β is (i, n)-Admissible, and each set Φ of sentences of $\mathcal{L}^* \cap H(\beta)$ which is Σ_n-definable over $(L_i(H(\beta)^{L[S^*]}, H \restriction \beta)$ with parameters β, p with p in $H(\alpha)^{L[S^*]}$, we put the sentence

$$\bigwedge(\Phi \cap H(\alpha)) \rightarrow \bigwedge \Phi$$

into T. The forcing Q^* consists of all sentences φ of $\mathcal{L}^*$ which are consistent with T (i.e. $\bigwedge(T_0 \cup \{\varphi\})$ is consistent for each $T_0 \subseteq T$, $T_0 \in L[S^*]$). We order Q^* by $\varphi \leq \psi$ iff $\varphi \wedge \sim \psi$ is not consistent with T.

The sentences in T are all true when $\dot{F}$ is interpreted as F^*, thanks to the fact that F^* preserves instances of (i, n)-Stability.

Fact 1. The forcing Q^* is ∞-cc in $\mathcal{C}^*$.

Proof of Fact 1. Let A be a maximal antichain on Q^* which is $L[S^*]$-constructible and choose a wellorder $<$, club D, parameter p and $n > 0$ that witness the $L[S^*]$-constructibility of A. Let α be the least element of D; we claim that $A = A \cap H(\alpha)$ and therefore A is a set in $L[S^*]$. Indeed, for any β in D, the axioms of T yield $\bigwedge(A \cap H(\alpha)) \rightarrow \bigwedge(A \cap H(\beta))$ by virtue of the (i, n)-Stability of α in β where $i = \text{ot}\,(< \restriction \beta)$. As A is an antichain, $A \cap H(\alpha)$ must equal all of A for each β in D and as D is unbounded, $A \cap H(\alpha)$ equals all of A. $\Box$ (*Fact 1*)

Let G^* consist of all sentences of $\mathcal{L}^*$ which are true when $\dot{F}$ is interpreted as F^*. Clearly G^* intersects each maximal antichain A of Q^* which is a set in $L[S^*]$ as otherwise $\bigwedge_{\varphi \in A} \sim \varphi$ would be a sentence consistent with T (and therefore in $\mathcal{L}^*$) violating the pmaximality of A. But by *Fact 1*, all antichains of Q^* in $\mathcal{C}^*$ are sets in $L[S^*]$ and so G^* is fully Q^*-generic over $(L[S^*], \mathcal{C}^*)$. This establishes the first conclusion of the theorem.

For the second conclusion we give two proofs. The first is simpler, but appears to need Morse-Kelley in $(V, \mathcal{C})$ as the background theory.

First proof.

Suppose that $j : \mathrm{SC}^* \to \mathrm{SC}^*$ is not the identity and j belongs to $\mathcal{C}^*[F^*]$. Assuming Morse-Kelley in $(V, \mathcal{C})$ (and therefore Morse-Kelley minus Power in $(L[F^*], \mathcal{C}^*[F^*])$), we show that j can be extended to $j^* : (L[S^*], \mathcal{C}^*) \to (L[S^*], \mathcal{C}^*)$. Indeed, for each ordinal α, each class $X \in \mathcal{C}^*$ which codes a sequence of classes $(X_i \mid i \in \mathrm{Ord})$ and each $i \in \mathrm{Ord}$ let $H(\alpha, X, i)$ consist of all elements of the structure $(L[S^*], \{X_i \mid i \in \mathrm{Ord}\})$ which are definable with parameters from $\alpha \cup \{i\}$. We write $(\beta, Y, j) > (\alpha, X, i)$ iff $\beta > \alpha$, $X = Y_k$ for some k and $i < \beta$; this implies that $H(\beta, Y, j)$ contains $H(\alpha, X, i)$ as a substructure. The structures $H(\alpha, X, i)$ ordered by $<$ form a direct system which is isomorphic to a direct system whose elements and maps belong to $L[S^*]$. We can apply j to this system Π to obtain a system $j(\Pi)$ whose limit is isomorphic to $(L[S^*], \mathcal{C}^*)$, using the fact that $\mathcal{C}^*$ consists only of the SC^*-constructible classes. This yields an elementary embedding $j^* : (L[S^*], \mathcal{C}^*) \to (L[S^*], \mathcal{C}^*)$ as desired.

But now we can proceed as in the first proof of Corollary 17: The embedding j^* is definable over $(L[F^*], \mathcal{C}^*[F^*])$ and therefore generic over $(L[S^*], \mathcal{C}^*)$ for an ∞-cc definable forcing. The least ordinal forced by some condition in this forcing to be the critical point of such an embedding is $(L[S^*], \mathcal{C}^*)$-definable and therefore cannot be moved by such an embedding, a contradiction.

Second proof.

We only assume that $(V, \mathcal{C})$ models Gödel-Bernays, and need two facts.

Fact 2. There is an $L[S^*]$-definable $\diamondsuit$-sequence $(S_\alpha \mid \alpha \in \mathrm{Ord})$ for $(L[S^*], \mathcal{C}^*)$ which concentrates on strong limit cardinals of cofinality ω of $L[S^*]$; i.e., if X belongs to $\mathcal{C}^*$ and D is a club in $\mathcal{C}^*$ then there is a strong limit cardinal α of cofinality ω of $L[S^*]$ such that $X \cap \alpha = S_\alpha$.

Proof of Fact 2. Let S_α be empty if α is not a limit point of C which in addition is a strong limit cardinal of cofinality ω of $L[S^*]$. Otherwise, assuming that S_β is defined for $\beta < \alpha$ we take (S_α, C_α) to be the least pair in $L(H(\alpha)^{L[S^*]})$ such that C_α is closed unbounded in α and $S_\alpha \cap \bar{\alpha} \neq S_{\bar{\alpha}}$ for $\bar{\alpha}$ in C_α, if it exists, $(\emptyset, \emptyset)$ otherwise. (Note that even though α has cofinality ω, we can still talk about closed unbounded subsets of α, which indeed may appear at a level of $L(H(\alpha)^{L[S^*]})$ before it is recognised that α is singular.) Suppose that the resulting sequence is not the desired $\diamondsuit$-sequence and let (S, D) in $\mathcal{C}^*$ be a counterexample, i.e., D is a club and for limit points α of C which are strong limit cardinals of cofinality ω of $L[S^*]$ in D, $S \cap \alpha \neq S_\alpha$. Then for each α in D (which is a limit point of C and a strong limit cardinal of cofinality ω of $L[S^*]$), the pair (S_α, C_α) was chosen as the least pair such that $S_\alpha \cap \bar{\alpha} \neq S_{\bar{\alpha}}$ for $\bar{\alpha}$ in C_α. But this choice of S_α is Σ_1-definable in $L_{\mathrm{ot}\,(<\restriction\alpha)}(H(\alpha)^{L[S^*]})$ for a club E of α's, where $<$, E belong to $\mathcal{C}^*$ and witness the $L[S^*]$-constructibility of (S, D), and by elementarity, $S_\beta \cap \alpha = S_\alpha$, $C_\beta \cap \alpha = C_\alpha$ for $\alpha < \beta$ in E. This is a constradiction as we can choose $\alpha < \beta$ in $E \cap D$ to be limit points of C which are strong limit cardinals of cofinality ω of $L[S^*]$, yielding $S_\beta \cap \alpha = S_\alpha$ with α in C_β. $\square$ (*Fact 2.*)

Fact 3. Any club in $\mathcal{C}^*[F^*]$ contains a club in $\mathcal{C}^*$.

Proof of Fact 3. This is because by *Fact 1*, $(L[F^*], \mathcal{C}^*[F^*])$ is an ∞-cc generic extension of $(L[S^*], \mathcal{C}^*)$. $\square$ (*Fact 3.*)

Now for the rigidity of SC* in $\mathcal{C}^*[F^*]$ we argue as before as follows. Using *Fact 2* we can obtain an SC*-definable partition $(T_\alpha \mid \alpha \in \mathrm{Ord})$ of the ordinals of cofinality ω into pieces which are $\mathcal{C}^*$-stationary, i.e., which intersect any club in $\mathcal{C}^*$. By *Fact 3* any club in $\mathcal{C}^*[F^*]$ contains a club in $\mathcal{C}^*$. But now there can be no nontrivial elementary embedding $j : \mathrm{SC}^* \to \mathrm{SC}^*$ in $\mathcal{C}^*[F^*]$: otherwise we can choose α in $j((S_\alpha \mid \alpha \in \mathrm{Ord}))_\kappa$ to be a fixed point of j and derive the contradiction that α belongs to both $j((S_\alpha \mid \alpha \in \mathrm{Ord}))_\kappa$ as well as to $j((S_\alpha \mid \alpha \in \mathrm{Ord}))_\gamma$ for some $\gamma < \kappa$. $\square$

Proof of the Main Lemma. The desired forcing $\mathbb{P}^*$ is the final stage $\mathbb{Q}^*_\infty$ of a *finite support* iteration $(\mathbb{P}^*_\beta, \mathbb{Q}^*_\beta \mid \beta \in C \cup \{\infty\})$. The β-th stage $\mathbb{Q}^*_\beta$ of the iteration will add a function $p^* : \beta \to 2$. If $\beta = \omega$ is the minimum of C then $\mathbb{Q}^*_\beta$ is the atomic forcing whose conditions are functions $p^* : \omega \to 2$. If β is a successor point of C and β_0 is its C-predecessor then $\mathbb{Q}_\beta$ is an atomic forcing, whose conditions consist of all $p^* : \beta \to 2$ in $V[G^*_{\beta_0}, G^*(\beta_0)]$ such that $p^* \restriction \beta_0$

is $\mathbb{Q}^*_{\beta_0}$-generic over $V[G^*_{\beta_0}]$ (where G^*_α, $G^*(\alpha)$ denote the generics for $\mathbb{P}^*_\alpha$, $\mathbb{Q}^*_\alpha$ respectively for each α in C); we also require that $p^*\restriction[\beta_0,\beta)$ belong to V, $p^*(\beta_0)=1$ and $p^*(2\gamma)=0$ for all γ in (β_0,β). (These latter requirements ensure that both V and C are definable over $(L[F^*],F^*)$ when $F^*:\mathrm{Ord}\to 2$ is $\mathbb{P}^*$-generic.)

Suppose that β is a limit point of C. Let $\mathbb{Q}^{*,0}_\beta$ denote the set (or class if $\beta=\infty$) of all $p^*:\alpha\to 2$ in $V[G^*_\alpha,G^*(\alpha)]$ where $\alpha\in C\cap\beta$ and $p^*\restriction\alpha$ is $\mathbb{Q}^*_\alpha$-generic over $V[G^*_\alpha]$; $\mathbb{Q}^{*,0}_\beta$ is ordered by extension. If β is regular in $L(H(\beta))$ or $\beta=\infty$ then $\mathbb{Q}^*_\beta$ is equal to $\mathbb{Q}^{*,0}_\beta$. Otherwise, proceed as follows. We say that $p^*:\beta\to 2$ is (i,n)*-generic for* $\mathbb{Q}^{*,0}_\beta$ iff $G^*(p^*)=\{p^*\restriction\alpha\mid\alpha\in C\cap\beta\}$ meets every dense subset of $\mathbb{Q}^{*,0}_\beta$ of the form $\{q^*\in\mathbb{Q}^{*,0}_\beta\mid q^*\Vdash\varphi \text{ or } q^*\Vdash\sim\varphi\}$, where φ is a $\Pi_n(L_i(H(\beta)),C\cap\beta,\dot f)$ sentence with parameters from $L(H(\beta))$ ($\dot f$ denotes the generic function with domain β). Then we take $\mathbb{Q}^*_\beta$ to be the atomic forcing whose conditions are functions $p^*:\beta\to 2$ in $V[G^*_\beta]$ which are (i,n)-generic for $\mathbb{Q}^{*,0}_\beta$ for the (fewer than β^+ of $L(H(\beta))$-many) (i,n) such that β is (i,n)-Admissible.

Lemma 23 *Suppose that β belongs to C and β is either a successor point of C or not regular in $L(H(\beta))$. Then in $V[G^*_\beta]$, each p^* in $\mathbb{Q}^{*,0}_\beta$ has an extension in $\mathbb{Q}^*_\beta$.*

Proof. By induction on β. Suppose that β is a successor point of C and let β_0 be its C-predecessor. If $\beta_0=\omega$ is the minimum of C then it is easy to extend any element of $\mathbb{Q}^*_{\beta_0}$ to an element of $\mathbb{Q}^*_\beta$. If β_0 is a successor point of C or not regular in $L(H(\beta_0))$ then by induction, in $V[G^*_{\beta_0}]$ each p^* in $\mathbb{Q}^{*,0}_{\beta_0}$ has an extension p^{**} in $\mathbb{Q}^*_{\beta_0}$; it is then easy to extend p^{**} further to an element of $\mathbb{Q}^*_\beta$. If β_0 is a limit point of C and is regular in $L(H(\beta_0))$ then by induction any p^* in $\mathbb{Q}^{*,0}_{\beta_0}$ has extensions in $\mathbb{Q}^*_\gamma$ for arbitrarily large $\gamma\in C\cap\beta_0$; it follows that any $\mathbb{Q}^*_{\beta_0}$-generic p^{**} has domain β_0 and it then follows that each p^* in $\mathbb{Q}^{*,0}_{\beta_0}$ can be extended to some $\mathbb{Q}^*_{\beta_0}$-generic p^{**} in $V[G^*_\beta]$ (the forcing $\mathbb{Q}^*_{\beta_0}$ is homogeneous). It is then easy to extend p^{**} further to an element of $\mathbb{Q}^*_\beta$ in $V[G^*_\beta]$.

Suppose that β is a limit point of C and is not regular in $L(H(\beta))$. Let $(i,n+1)$ be least so that β is not $(i,n+1)$-Admissible. First suppose that $n=0$. If $i=0$ then β is not 1-Admissible and there is a closed unbounded subset D of $C \cap \beta$ of ordertype less than β whose successor points γ are not regular in $L(H(\gamma))$ and whose intersection with each of its limit points $\gamma < \beta$ is Δ_1 definable over $(H(\gamma), C \cap \gamma)$. Given $\alpha \in C \cap \beta$ and a p^* in $\mathbb{Q}^{*,0}_\beta$ that we want to extend into $\mathbb{Q}^*_\beta$, we can assume that both α and the ordertype of D are less than the minimum of D. Now enumerate D as $\beta_0 < \beta_1 < \cdots$ and using the induction hypothesis, successively extend p^* to $q^*_0 \subseteq q^*_1 \subseteq \cdots$ with q^*_j in $\mathbb{Q}^*_{\beta_j}$, taking unions at limits. Note that for limit j, q^*_j is indeed a condition because β_j is not 1-Admissible. The union of the q_j's is the desired extension of p^* in $\mathbb{Q}^*_\beta$. If $i = i_0 + 1$ is a successor ordinal then we instead choose D to be a closed unbounded subset of $C \cap \beta$ of ordertype less than β whose successor points γ are not regular in $L(H(\gamma))$ and such that for limit points $\gamma < \beta$ of D, $D \cap \gamma$ is Δ_1 definable over the transitive collapse of $H^{\beta,i_0}_\omega(\gamma) =$ the Σ_ω Skolem hull of γ in $(L_{i_0}(H(\beta)), C \cap \beta)$ and this hull contains no ordinals between γ and β. Again we make successive extensions of p^* to $q^*_0 \subseteq q^*_1 \subseteq \cdots$ with q^*_j in $\mathbb{Q}^*_{\beta_j}$, taking unions at limits, where the β_j's increasingly enumerate D. We get a condition at limit stages using the fact that γ is not Σ_1-regular over the transitive collapse of $H^{\beta,i_0}_\omega(\gamma)$ when it is a limit point of D (and using reflection to infer that the associated limit q^*_j is indeed sufficiently generic for the forcing $\mathbb{Q}^{*,0}_\gamma$).

Now suppose that n is greater than 0.

If β is a limit of α which are (i,n)-Stable in β, then proceed as in the previous paragraph: Choose a closed unbounded subset D of $C \cap \beta$ of ordertype less than β consisting of α which are (i,n)-Stable in β, whose successor points γ are not regular in $L(H(\gamma))$ and whose intersection with each of its limit points $\gamma < \beta$ is Δ_{n+1} definable over the transitive collapse of $H^{(i,\beta)}_n(\gamma)$. Assume that the ordertype of D as well as the domain of the given $p^* \in \mathbb{Q}^{*,0}_\beta$ that we wish to extend are less than the minimum of D, enumerate D as $\beta_0 < \beta_1 < \cdots$ and using the induction hypothesis, successively extend p to $q_0 \subseteq q_1 \subseteq \cdots$ with q_j in $\mathbb{Q}^*_{\beta_j}$, taking unions at limits. For limit j, q_j is indeed a condition because β_j is not $(\bar{i}, n+1)$-Admissible, where $\bar{i}$ is the height of the transitive collapse of $H^{(i,\beta)}_n(\beta_j)$, and as it is a limit of ordinals which are (i,n)-Stable in β, q_j is $(\bar{i},n)$-generic for $\mathbb{Q}^{*,0}_{\beta_j}$. The union of the q_j's is the desired extension of p^* in $\mathbb{Q}^*_\beta$.

If β is not a limit of α which are (i,n)-Stable in β then β must have cofinality ω (else by (i,n)-Admissibility, we could find cofinally many (i,n)-

Stables in β). It suffices to show that any condition p^* in $\mathbb{Q}^{*,0}_\beta$ can be extended to decide (i.e. force or force the negation of) each of fewer than β-many $\Pi_n(L_i(H(\beta)), C\cap\beta)$ sentences with parameters from $H(\beta)$ (given this, we can extend p^* in ω steps to a condition in $\mathbb{Q}^*_\beta$ which is (i,n)-generic for $\mathbb{P}^*_\beta$). To show this, let $(\varphi_i \mid i<\delta)$ enumerate the given collection of $\Pi_n(L_i(H(\beta)), C\cap\beta)$ sentences and if $n>1$, let D consist of all γ which are limits of $(i,n-1)$-Stables in β and large enough so that $H(\gamma)$ contains both p^* and this enumeration. (If $n=1$ then let D consist of all γ which are limit points of C and large enough so that $H(\gamma)$ contains both p^* and this enumeration.) Now extend p^* successively to elements q_j of $\mathbb{Q}^*_{\gamma_j}$, where $\gamma_{j+1}\geq\gamma_j$ is the least element γ of D so that γ is not regular in $L(H(\gamma))$ and either q_j forces φ_j or q_{j+1} forces $\psi_j =$ the negation of φ_j (with corresponding witness to the Σ_n sentence ψ_j), taking unions at limits. For limit j, q_j is a condition as γ_j is not $(\bar{i},n)$-Admissible but (in case $n>1$) is a limit of $(\bar{i},n-1)$-Stables, where $\bar{i}$ is the height of the transitive collapse of $H^{(i,\beta)}_{n-1}(\gamma_j)$. (The failure of γ_j to be $(\bar{i},n)$-Admissible uses the fact that the set of $j_0<j$ such that q_{j_0+1} forces the negation of φ_{j_0} can be treated as a parameter in $H(\gamma_j)$.) As β is (i,n)-Admissible, this construction results in a sequence of q_j's of length δ, whose union is the desired extension of p^* deciding all of the given $\Pi_n(L_i(H(\beta)), C\cap\beta)$ sentences. □

Lemma 24 *Suppose that G^* is $\mathbb{Q}^*_\infty$-generic where $\mathbb{Q}^*_\infty$ is the class of $p^*:\alpha\to 2$ in $V[G^*_\infty]$ such that α belongs to C and p^* is $\mathbb{Q}^*_\alpha$-generic. Let $F^*: Ord\to 2$ be the union of the functions in G^*. Then V is a definable inner model of $L[F^*]$ and for any $\alpha<\beta$, $i<\beta^+$ of $L(H(\beta))$ and $0<n<\omega$, if α is (i,n)-Stable in β and β is (i,n)-Admissible then α is (i,n)-Stable in β relative to F^*.*

Proof. It is easy to define V from F^* as from F^* we can first identify the elements of C and then V consists of those sets coded by F^* restricted to some adjacent interval of C. Suppose that α is (i,n)-Stable in β and β is (i,n)-Admissible. Then by the definition of $\mathbb{Q}^*_\infty$, $F^*\restriction\beta$ is (i,n)-generic for $\mathbb{Q}^{*,0}_\beta$ and $F^*\restriction\alpha$ is $(\bar{i},n)$-generic for $\mathbb{P}^*_\alpha$ where $H^{(i,\beta)}_n(\alpha)$ has transitive collapse of height $\bar{i}$, as α is $(\bar{i},n)$-Admissible. But as the forcing relation for Π_n formulas is Π_n-definable, this implies that α is (i,n)-Stable in β relative to F^*, as desired. □

Now notice that since we iterate with finite support, the forcing $\mathbb{P}^*_\infty$ is ∞-cc, i.e., all antichains for this forcing which belong to $\mathcal{C}$ are sets in V.

It follows that Gödel-Bernays minus Power is preserved. This completes the proof of the Main Lemma and therefore of Theorem 21. □

Open question. Can one prove in Morse-Kelley (or even in Gödel-Bernays) that HOD is relatively rigid for arbitrary class embeddings?

We'll show next that HOD is indeed relatively rigid for arbitrary class embeddings, if one assumes *vertical maximality*, a strengthening of Morse-Kelley.

8.-9.Vorlesungen

Relative Rigidity, #-generation and the Indiscernible Core

Our aim now is to prove the following.

Theorem 25 *(Relative Rigidity Theorem) Suppose that $(V, \mathcal{C})$ is a #-generated model of Gödel-Bernays. Then for some V-definable class of ordinals $\mathcal{I}$, $(HOD, \mathcal{I})$ is rigid for arbitrary embeddings in $\mathcal{C}$.*

The conclusion is that there is no elementary embedding from (HOD, E) to itself in $\mathcal{C}$ other than the identity. The main thing we need to explain is the hypothesis of *#-generation.* This notion first arose in work of mine with Honzik where we arrived at this notion as the ultimate form of "vertical maximality" for the universe of sets. The notion applies equally well to models of class theory.

Reflection with indiscernibles – vertical maximality

Let us extrapolate from the usual reflection and see where it takes us. It is natural to strengthen the reflection of individual first-order properties from V to some V_α to the simultaneous reflection of all first-order properties of V to some V_α, even with parameters from V_α. Thus V_α is an elementary submodel of V. Repeating this process suggests that in fact there should be an increasing, continuous sequence of ordinals $(\kappa_i \mid i < \infty)$ such that the models $(V_{\kappa_i} \mid i < \infty)$ form a continuous chain $V_{\kappa_0} \prec V_{\kappa_1} \prec \cdots$ of elementary submodels of V whose union is all of V (where ∞ denotes the ordinal height of the universe V).

But the fact that for a closed unbounded class of κ's in V, V_κ can be "lengthened" to an elementary extension (namely V) of which it is a rank initial segment suggests via reflection that V itself should also have such a lengthening V^*. But this is clearly not the end of the story, because we can also infer that there should in fact be a continuous increasing sequence of such lengthenings $V = V_{\kappa_\infty} \prec V^*_{\kappa_{\infty+1}} \prec V^*_{\kappa_{\infty+2}} \prec \cdots$ of length the ordinals. For ease of notation, let us drop the *'s and write W_{κ_i} instead of $V^*_{\kappa_i}$ for $\infty < i$ and instead of V_{κ_i} for $i \leq \infty$. Thus V equals W_∞.

But which tower $V = W_{\kappa_\infty} \prec W_{\kappa_{\infty+1}} \prec W_{\kappa_{\infty+2}} \prec \cdots$ of lengthenings of V should we consider? Can we make the choice of this tower "canonical"?

Consider the entire sequence $W_{\kappa_0} \prec W_{\kappa_1} \prec \cdots \prec V = W_{\kappa_\infty} \prec W_{\kappa_{\infty+1}} \prec W_{\kappa_{\infty+2}} \prec \cdots$. The intuition is that all of these models resemble each other in the sense that they share the same first-order properties. Indeed by virtue of the fact that they form an elementary chain, these models all satisfy the same first-order sentences. But again in the spirit of "resemblance", it should be the case that any two pairs $(W_{\kappa_{i_1}}, W_{\kappa_{i_0}})$, $(W_{\kappa_{j_1}}, W_{\kappa_{j_0}})$ (with $i_0 < i_1$ and $j_0 < j_1$) satisfy the same first-order sentences, even allowing parameters which belong to both $W_{\kappa_{i_0}}$ and $W_{\kappa_{j_0}}$. Generalising this to triples, quadruples and n-tuples in general we arrive at the following situation:

$(*)$ Our approximation V to the universe should occur in a continuous elementary chain $W_{\kappa_0} \prec W_{\kappa_1} \prec \cdots \prec V = W_{\kappa_\infty} \prec W_{\kappa_{\infty+1}} \prec W_{\kappa_{\infty+2}} \prec \cdots$ of length the ordinals, where the models W_{κ_i} form a *strongly-indiscernible chain* in the sense that for any n and any two increasing n-tuples $\vec{i} = i_0 < i_1 < \cdots < i_{n-1}$, $\vec{j} = j_0 < j_1 < \cdots < j_{n-1}$, the structures $W_{\vec{i}} = (W_{\kappa_{i_{n-1}}}, W_{\kappa_{i_{n-2}}}, \cdots, W_{\kappa_{i_0}})$ and $W_{\vec{j}}$ (defined analagously) satisfy the same first-order sentences, allowing parameters from $W_{\kappa_{i_0}} \cap W_{\kappa_{j_0}}$.

But this is again not the whole story, as we would want to impose higher-order indiscernibility on our chain of models. For example, consider the pair of models $W_{\kappa_0} = V_{\kappa_0}$, $W_{\kappa_1} = V_{\kappa_1}$. Surely we would want that these models satisfy the same second-order sentences; equivalently, we would want $H(\kappa_0^+)^V$ and $H(\kappa_1^+)^V$ to satisfy the same first-order sentences. But as with the pair $H(\kappa_0)^V$, $H(\kappa_1)^V$ we would want $H(\kappa_0^+)^V$, $H(\kappa_1^+)^V$ to satisfy the same first-order sentences *with parameters*. How can we formulate this? For example, consider κ_0, a parameter in $H(\kappa_0^+)^V$ that is second-order with respect to $H(\kappa_0)^V$; we cannot simply require $H(\kappa_0^+)^V \vDash \varphi(\kappa_0)$ iff $H(\kappa_1^+)^V \vDash \varphi(\kappa_0)$, as

κ_0 is the largest cardinal in $H(\kappa_0^+)^V$ but not in $H(\kappa_1^+)^V$. Instead we need to replace the occurence of κ_0 on the left side with a "corresponding" parameter on the right side, namely κ_1, resulting in the natural requirement $H(\kappa_0^+)^V \vDash \varphi(\kappa_0)$ iff $H(\kappa_1^+)^V \vDash \varphi(\kappa_1)$. More generally, we should be able to replace each parameter in $H(\kappa_0^+)^V$ by a "corresponding" element of $H(\kappa_1^+)^V$ and conversely, it should be the case that, to the maximum extent possible, all elements of $H(\kappa_1^+)^V$ are the result of such a replacement. This also should be possible for $H(\kappa_0^{++})^V$, $H(\kappa_0^{+++})^V$, ... and with the pair κ_0, κ_1 replaced by any pair κ_i, κ_j with $i < j$.

It is natural to solve this parameter problem using embeddings, as in the last subsection. But the difference here is that there is no assumption that these embeddings are internal to V; they need only exist in the "real universe", outside of V. In this way we will arrive at a principle compatible with $V = L$ in which the choice of embeddings is indeed "canonical".

Thus we are led to the following.

Definition 26 *Let V be a transitive set-size model of ZFC of ordinal height ∞. We say that V is* indiscernibly-generated *iff there exists a continuous sequence $\kappa_0 < \kappa_1 < \ldots$ of length the ordinals such that $\kappa_\infty = \infty$ and commuting elementary embeddings $\pi_{ij} : V \to V$ where π_{ij} has critical point κ_i and sends κ_i to κ_j. Moreover, for any $i \leq j$, any element of V is first-order definable in V from elements of the range of π_{ij} together with κ_k's for k in the interval $[i, j)$.*

The last clause in the above definition formulates the idea that to the maximum extent possible, elements of V are in the range of the embedding π_{ij} for each $i \leq j$; notice that the interval $[\kappa_i, \kappa_j)$ is disjoint from this range, but by allowing the κ_k's in this interval as parameters, we can first-order definably recover everything.

Indiscernible-generation as formulated in the above definition does indeed give us our advertised higher-order indiscernibility: For example, in the notation of the definition, if $\vec{i} = i_0 < i_1 < \ldots < i_{n-1}$ and $\vec{j} = j_0 < j_1 < \ldots < j_{n-1}$ with $i_0 \leq j_0$, and $x_k \in H(\kappa_{i_0}^+)^V$ for $k < n$ then the structure $V_{\vec{i}}^+ = (H(\kappa_{i_{n-1}}^+)^V, H(\kappa_{i_{n-2}}^+)^V, \cdots, H(\kappa_{i_0}^+)^V)$ satisfies a sentence with parameters $(\pi_{i_0,i_{n-1}}(x_{n-1}), \ldots, \pi_{i_0,i_0}(x_0))$ iff $V_{\vec{j}}^+$ satisfies the same sentence with corresponding parameters $(\pi_{i_0,j_{n-1}}(x_{n-1}), \ldots, \pi_{i_0,j_0}(x_0))$. There is a similar statment with V^+ replaced by higher-order structures $V^{+\alpha}$ for arbitrary α.

Indiscernible-generation has a clearer formulation in terms of #-*generation*, which we explain next.

Definition 27 *A structure $N = (N, U)$ is called a* sharp with critical point *κ, or just a #, if the following hold:*

- *N is a model of ZFC$^-$ (ZFC minus powerset) in which κ is the largest cardinal and κ is strongly inaccessible.*
- *(N, U) is amenable (i.e. $x \cap U \in N$ for any $x \in N$).*
- *U is a normal measure on κ in (N, U).*
- *N is iterable, i.e., all of the successive iterated ultrapowers starting with (N, U) are well-founded, yielding iterates (N_i, U_i) and Σ_1 elementary iteration maps $\pi_{ij} : N_i \to N_j$ where $(N, U) = (N_0, U_0)$.*

We will use the convention that κ_i denotes the the largest cardinal of the i-th iterate N_i.

If N is a # and λ is a limit ordinal then $\mathrm{LP}(N_\lambda)$ denotes the union of the $(V_{\kappa_i})^{N_i}$'s for $i < \lambda$. (LP stands for "lower part".) $\mathrm{LP}(N_\infty)$ is a model of ZFC.

Definition 28 *We say that a transitive model V of ZFC is* #-generated *iff for some sharp $N = (N, U)$ with iteration $N = N_0 \to N_1 \to \cdots$, V equals LP(N_∞) where ∞ denotes the ordinal height of V.*

Fact. The following are equivalent for transitive set-size models V of ZFC:
(a) V is indiscernibly-generated.
(b) V is #-generated.

Proof. The last clause in the definition of indiscernible-generation ensures that the embeddings π_{ij} in that definition in fact arise from iterated ultrapowers of the embedding π_{01}, itself an ultrapower by the measure U_0 on κ_0 given by $X \in U_0$ iff $\pi_{01}(X)$ contains κ_0 as an element. Conversely, if (N, U) generates V then the chain of embeddings given by iteration of (N, U) witnesses that V is indiscernibly-generated. □

#-generation fulfills our requirements for vertical maximality, with powerful consequences for reflection. L is #-generated iff $0^{\#}$ exists, so this principle is compatible with $V = L$. If V is #-generated via (N, U) then there are embeddings witnessing indiscernible-generation for V which are canonically-definable through iteration of (N, U). Although the choice of # that generates V is not in general unique, it can be taken as a fixed parameter in the canonical definition of these embeddings. Moreover, #-generation evidently provides the maximum amount of vertical reflection: If V is generated by (N, U) as $\mathrm{LP}(N_\infty)$ where ∞ is the ordinal height of V, and x is any parameter in a further iterate $V^* = N_{\infty^*}$ of (N, U), then any first-order property $\varphi(V, x)$ that holds in V^* reflects to $\varphi(V_{\kappa_i}, \bar{x})$ in N_j for all sufficiently large $i < j < \infty$, where $\pi_{j,\infty^*}(\bar{x}) = x$. This implies any known form of vertical reflection and summarizes the amount of reflection one has in L under the assumption that $0^{\#}$ exists, the maximum amount of reflection in L.

Thus #-generation tells us what lengthenings of V to look at, namely the initial segments of V^* where V^* is obtained by further iteration of a # that generates V. And it fully realises the idea that M should look exactly like closed unboundedly many of its rank initial segments as well as its "canonical" lengthenings of arbitrary ordinal height.

In summary, we believe that #-generation is the correct formalization of the principle of (vertical) reflection (R), and we shall refer to #-generated models as being *vertically maximal*.

Finally we apply #-generation to models of class theory. We say that such a model $(V, \mathcal{C})$ is *#-generated* if there is a # $N = (N, U)$ with iteration $N = N_0 \to N_1 \to \cdots$ such that V equals $\mathrm{LP}(N_\infty)$ and $\mathcal{C}$ consists of the subsets of V in $N_{\infty+1}$, where ∞ denotes the ordinal height of V.

Thus not only has the ordinal height of V been maximised, but so has the collection of its classes, both via the principle of #-generation.

On the methodology of set theory

Set theory is usually about first-order properties of V and the consequences of certains first-order theories for such properties. But when studying the rigidity problem for HOD we see that we are led to second-order properties and the need to work in class theory. This raises the obvious question of what form of class theory to adopt. Ideally one would like to derive consequences from the weakest possible theory, namely Gödel-Bernays, but this proves to not always be possible. Indeed, as we will see with regard to the rigidity problem for HOD it is advantageous to work in a context that is not even expressible within a second-order theory of classes, i.e. in the context of #-generation.

This is not special to class theory. Indeed, it is natural to also impose non first-order hypotheses when studying first-order features of V. Ideally we would like to verify first-order consequences of first-order theories by showing that any countable transitive model of a well-chosen first-order theory obeys relevant first-order statements. But we can naturally take the position that not all countable transitive models of the given first-order theory are relevant and impose further non first-order restrictions on such models such as #-generation, our formulation of vertical maximality. Of course one can then only make conclusions about #-generated models, but it is not unreasonable to claim that these are the most interesting models anyway.

This is the spirit behind the *Relative Rigidity Theorem*: HOD is relatively rigid in all #-generated models of class theory. If one accepts #-generation as a valid maximality principle then this is sufficient to establish the relative rigidity of HOD as a true statement of set theory.

The indiscernible core (postponed)

Large cardinal witnessing

We now move on to the question of witnessing large cardinal properties in HOD. This material comes from a paper of mine with Joel Hamkins and Cheng Yong.

Question.

1. To what extent must a large cardinal in V exhibit its large cardinal properties in HOD?

2. To what extent does the existence of large cardinals in V imply the existence of large cardinals in HOD?

For large cardinal concepts beyond the weakest notions the answers are generally negative. For example, a supercompact need not be even weak compact in HOD. There can be a proper class of supercompacts with no supercompact in HOD. But there are some positive results: Any ω-Erdős is also ω-Erdős in HOD and below an ω-Erdős there are many weak compacts. So a measurable in V gives a weak compact in HOD. Woodin has claimed that a supercompact gives a measurable in HOD.

Theorem 29 *If κ is a supercompact cardinal, then there is a forcing extension in which κ remains supercompact, but is not weakly compact in HOD.*

Proof. Suppose that κ is a supercompact cardinal and assume GCH. By Laver we may assume that the supercompactness of κ is preserved by $< \kappa$-directed forcing. We now force over V to add a κ-Cohen set g. By a result of Kunen we may factor this forcing as $\mathrm{Add}(\kappa, 1) \cong \mathbb{S} * \mathbb{T}$, first adding a weakly homogeneous κ-Suslin tree $\mathbb{T}$ and then forcing with the tree:

Lemma 30 (Kunen) *If κ is inaccessible, then there is a $< \kappa$-strategically closed notion of forcing $\mathbb{S}$ of size κ such that forcing with $\mathbb{S}$ adds a weakly homogeneous κ-Suslin tree T and the combined forcing $\mathbb{S}*\mathbb{T}$ is forcing-equivalent to the forcing $Add(\kappa, 1)$ to add a Cohen subset of κ.*

The key point is that $\mathbb{S} * \mathbb{T}$ has a $< \kappa$-closed dense subset of size κ, and all such (nontrivial) forcing is equivalent to $\mathrm{Add}(\kappa, 1)$.

A forcing notion $\mathbb{Q}$ is *weakly homogeneous*, if for any two conditions $p, q \in \mathbb{Q}$, there is an automorphism π of $\mathbb{Q}$ for which $\pi(q)$ and p are compatible. It follows that if φ is a statement in the forcing language involving only ground model names $\check{x}$, and some condition p forces φ, then every condition forces φ, since otherwise some q forces $\neg\varphi$, but in this case $\pi(q)$ will also force $\neg\varphi$, which is impossible if it is compatible with p.

Lemma 31 (Folklore) *If $\mathbb{Q}$ is a weakly homogenenous notion of forcing and $G \subseteq \mathbb{Q}$ is V-generic, then $HOD^{V[G]} \subseteq HOD(\mathbb{Q})^V$. In particular, if $\mathbb{Q}$ is also ordinal definable in V, then $HOD^{V[G]} \subseteq HOD^V$.*

The point is that if $A \subseteq \mathrm{Ord}$ is defined in $V[G]$ by $\alpha \in A \iff \varphi(\alpha, \beta)$, then we can define A in the ground model as $\{\alpha \mid 1 \Vdash \varphi(\check{\alpha}, \check{\beta})\}$, since conditions in a weakly homogeneous forcing cannot force different outcomes for assertions about ground model names.

Returning to the proof, we see that the extension $V[g]$ can be viewed as $V[T][b]$, where T is the generic κ-Suslin tree that is added and b is the $V[T]$-generic branch through T. Let $\mathbb{R}$ be the forcing in $V[T]$ to code T into the GCH (or $\diamond^*$) pattern on the next κ many regular cardinals above κ. Suppose that $H \subseteq \mathbb{R}$ is $V[T]$-generic (which is equivalent to assuming that H is $V[g]$-generic). We may view the extension $V[g][H]$ as $V[T][H][b]$. Since $\mathbb{R}$ is $< \kappa$-directed closed in $V[T]$, it is also $< \kappa$-directed closed in $V[g]$, since any subset of $\mathbb{R}$ of size less than κ in $V[g]$ is in $V[T]$. Therefore the forcing $\mathrm{Add}(\kappa, 1) * \mathbb{R}$ is $< \kappa$-directed closed in V, and so our indestructibility assumption on κ ensures that κ is supercompact in $V[g][H]$.

We claim that κ is not weakly compact in $\mathrm{HOD}^{V[g][H]}$. The tree T is in $\mathrm{HOD}^{V[g][H]}$, since we explicitly forced to encode it. But since $V[g][H] = V[T][H][b]$ and T is weakly homogeneous in $V[T]$ and hence in $V[T][H]$, it follows by Lemma 31 that $\mathrm{HOD}^{V[g][H]} \subseteq V[T][H]$. Since the forcing to add H adds no new subsets of T, it follows that T has no cofinal branches in $V[T][H]$ and hence none in $\mathrm{HOD}^{V[g][H]}$, and so the tree property fails for κ there, which means that κ is not weakly compact in $\mathrm{HOD}^{V[g][H]}$, as desired. $\Box$

Theorem 32 *There is a class forcing notion $\mathbb{P}$ forcing that*

1. *All measurable cardinals of the ground model are preserved and no new measurable cardinals are created.*
2. *There are no measurable cardinals in the HOD of the extension.*
3. *The measurable cardinals of the ground model are not weakly compact in the HOD of the extension.*

Proof. We'll assume that there is a proper class of measurable cardinals in the ground model V. Let $\bar{V} = V[F]$ be an extension so that the GCH holds at every inaccessible cardinal in $\bar{V}$ and every set of ordinals of $\bar{V}$ is coded into the GCH (or $\diamond^*$) pattern at, say, the triple successors δ^{+++} of the $\beth$-fixed points $\delta = \beth_\delta$, and furthermore that every measurable cardinal κ is indestructible by forcing over $\bar{V}$ with $\mathrm{Add}(\kappa, 1)$. Now, in $\bar{V}$, for each measurable cardinal κ, consider the forcing $\mathrm{Add}(\kappa, 1) * \dot{R}(\kappa)$, where as previously we factor $\mathrm{Add}(\kappa, 1)$

into two steps $\mathbb{S}_\kappa * \dot{T}_\kappa$, which first adds a homogeneous κ-Suslin tree T_κ and then forces with it, and then $\dot{R}(\kappa)$ is the forcing in $\overline{V}[T_\kappa]$ that codes the tree T_κ into the GCH pattern at the next κ many regular cardinals above κ, starting above κ^{+++}. Thus, $\mathrm{Add}(\kappa,1) * \dot{R}(\kappa) \simeq \mathbb{S}_\kappa * (\dot{R}(\kappa) \times \dot{T}_\kappa)$. Let $\mathbb{P} = \prod_\kappa(\mathrm{Add}(\kappa,1) * \dot{R}(\kappa))$ be the Easton-support product of these forcing notions, taken over all measurable cardinals κ, and let $G \subseteq \mathbb{P}$ be $\bar{V}$-generic. Our final model is $\bar{V}[G]$, which we shall now argue is as desired.

First, we claim that every measurable cardinal κ is preserved to $\bar{V}[G]$. Since the forcing above κ is $\leq \kappa$-closed, and the forcing $\mathbb{R}(\kappa)$ is $\leq \kappa$-distributive after adding the Cohen subset to κ, it suffices to argue that κ is measurable in the extension $\bar{V}[G_\kappa][g_\kappa]$, where G_κ performs the forcing at measurable cardinals below κ and $g_\kappa \subseteq \kappa$ is the Cohen subset of κ added by $\mathrm{Add}(\kappa,1)$ on coordinate κ. By our indestructibility assumption on κ in $\bar{V}$, we know that κ remains measurable in $V_1 = \bar{V}[g_\kappa]$, and so it suffices to argue merely that the forcing $\mathbb{P}_\kappa$ preserves the measurability of κ, forcing over V_1. And this can be done in the style of the Kunen-Paris theorem. Namely, fix in V_1 any normal ultrapower embedding $j : V_1 \to M$ for which κ is not measurable in M, and consider $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa \times \mathbb{P}_{\kappa,j(\kappa)}$, where $\mathbb{P}_{\kappa,j(\kappa)}$ is the rest of the product forcing from stage κ up to $j(\kappa)$ in M. Since κ is not measurable in M, there is no forcing at coordinate κ, and so $\mathbb{P}_{\kappa,j(\kappa)}$ is $\leq \kappa$-closed in M. Since $M^\kappa \subseteq M$ in V_1 and $\mathbb{P}_{\kappa,j(\kappa)}$ has $|j(2^\kappa)|^{V_1} = \kappa^+$ many dense subsets in M, we may construct by diagonalization in V_1 an M-generic filter $G_{\kappa,j(\kappa)} \subseteq \mathbb{P}_{\kappa,j(\kappa)}$. It follows that $G_\kappa \times G_{\kappa,j(\kappa)} \subseteq j(\mathbb{P}_\kappa)$ is M-generic, and so we may lift the embedding to $j : V_1[G_\kappa] \to M[j(G_\kappa)]$, where $j(G_\kappa) = G_\kappa \times G_{\kappa,j(\kappa)}$, thereby witnessing that κ is measurable in $V_1[G_\kappa]$ and hence also in $\bar{V}[G]$, as desired.

Next, the combined forcing $\bar{\mathbb{P}} * \mathbb{P}$ admits a closure point and therefore creates no new measurable cardinals. In particular, the three models $V \subseteq \bar{V} \subseteq \bar{V}[G]$ have the same measurable cardinals. The same reasoning shows that the large cardinals in $\bar{V}[\vec{T}]$, where $\vec{T}$ is the sequence of Suslin trees T_κ added by $\mathbb{S}_\kappa$ at coordinate κ, are also large in $\bar{V}$.

Finally, we claim that $\mathrm{HOD}^{\bar{V}[G]}$ has no measurable cardinals. To see this, we argue that $\mathrm{HOD}^{\bar{V}[G]} = \bar{V}[\vec{T}]$. The forward inclusion is a consequence of the fact that $\prod_\kappa(\mathbb{R}(\kappa) \times T_\kappa)$ is weakly homogeneous in $\bar{V}[\vec{T}]$, since the coding forcing is weakly homogeneous and the trees themselves are weakly homogeneous, and so $\mathrm{HOD}^{\bar{V}[G]} \subseteq \bar{V}[\vec{T}]$ by Lemma 31. Conversely, note that the coding performed by F at the triple successors of the $\beth$-fixed points is preserved to $\bar{V}[G]$, since the forcing to add G does not interfere with that coding and furthermore all $\beth$-fixed points are preserved from $\bar{V}$ to $\bar{V}[G]$. It follows that every set of ordinals in $\bar{V}$ is coded into the GCH or $\diamond^*$ pattern on such cardinals in $\bar{V}[G]$, and this implies $\bar{V} \subseteq \mathrm{HOD}^{\bar{V}[G]}$. Further, the trees T_κ themselves are coded into the GCH or $\diamond^*$ pattern on the next κ-many regular cardinals of $\bar{V}[G]$, and so $\vec{T}$ is definable in $\bar{V}[G]$. Thus, $\bar{V}[\vec{T}] \subseteq \mathrm{HOD}^{\bar{V}[G]}$, and we conclude $\mathrm{HOD}^{\bar{V}[G]} = \bar{V}[\vec{T}]$.

It remains to see that there are no measurable cardinals in $\bar{V}[\vec{T}]$. Suppose that κ is a measurable cardinal in $\bar{V}[\vec{T}]$. By our remark two paragraphs above, it follows that κ was measurable in $\bar{V}$ and therefore is one of the coordinates at which forcing is performed. In particular, at stage κ we added the κ-Suslin tree T_κ, which is Suslin in $\bar{V}[T_\kappa]$. The forcing above κ cannot affect whether T_κ is κ-Suslin, since it adds no new subsets to κ. The forcing $\prod_{\delta<\kappa} \mathbb{S}_\delta$ that adds the trees T_δ at measurable cardinals $\delta < \kappa$ is absolutely κ-c.c. (meaning it remains κ-c.c. in the forcing extension), and such forcing cannot add a κ-branch through a κ-Suslin tree. Thus, the tree property fails for κ in $\bar{V}[\vec{T}]$ and in particular, κ is not measurable there. So there are no measurable cardinals in $\bar{V}[\vec{T}]$, and consequently no measurable cardinals in $\mathrm{HOD}^{\bar{V}[G]}$, as desired. $\square$

An argument similar to the above shows:

Theorem 33 *There is a class forcing notion $\mathbb{P}$ forcing that*

1. *All supercompact cardinals of the ground model are preserved and no new supercompact cardinals are created.*

2. *There are no supercompact cardinals in the HOD of the extension.*

3. *The supercompact cardinals of the ground model are not weakly compact in the HOD of the extension.*

One may also ensure that the GCH holds in the extension and its HOD.

Theorem 34 *It is possible to have a strong cardinal but no measurable in HOD.*

We have shown that large cardinals can be large in V but small in HOD. A dual question is to ask for cardinals that are large in HOD, but not in V. A result of mine with Cummings and Golshani is that one can make $(\alpha^+)^{HOD} < \alpha^+$ for all infinite cardinals α. Gitik has claimed that one can improve this to get all regular cardinals inaccessible in HOD.

One could argue that the divergence between large cardinal properties in V and HOD is not really a problem about HOD but about the possibility of resurrecting largeness over a model where largeness has been lost with a homogeneous forcing.

10.-11.Vorlesungen

Relative Rigidity, #-generation and the Indiscernible Core

Our aim now is to prove the following.

Theorem 35 *(Relative Rigidity Theorem) Suppose that $(V, \mathcal{C})$ is a #-generated model of Gödel-Bernays. Then for some V-definable class of ordinals $\mathcal{I}$, $(HOD, \mathcal{I})$ is rigid for arbitrary embeddings in $\mathcal{C}$.*

The conclusion is that there is no elementary embedding from $(\mathrm{HOD}, \mathcal{I})$ to itself in $\mathcal{C}$ other than the identity. Recall that a model $(V, \mathcal{C})$ of class theory is *#-generated* if there is a # $N = (N, U)$ with iteration $N = N_0 \to N_1 \to \cdots$ such that V equals $\mathrm{LP}(N_\infty)$ and $\mathcal{C}$ consists of the subsets of V in $N_{\infty+1}$, where ∞ denotes the ordinal height of V.

The indiscernible core (still postponed)

Woodin's work on HOD

Since I'm stuck on the indiscneribble core I'll turn to something else, namely Woodin's work, which may give me some new ideas. Woodin studies HOD in the context of very large cardinals (extendibles) and makes a conjecture in this context.

First recall a result of Cummings, Golshani and myself (which I stated but didn't prove):

Theorem 36 *Assume GCH and κ supercompact. Then there is a generic extension in which κ remains measurable and for all infinite cardinals $\alpha < \kappa$, α^+ is greater than $(\alpha^+)^{HOD}$. In particular the V_κ of the generic extension is a model of "α^+ is greater than $(\alpha^+)^{HOD}$ for all infinite cardinals α".*

In the generic extension referred to in this theorem, κ loses its supercompactness and many successor cardinals are also successor cardinals in HOD. Contrast this with the following result of Woodin.

Theorem 37 *(Woodin) Suppose that δ is extendible. Then either α^+ equals α^+ of HOD for all singular cardinals $\alpha > \delta$ or all large enough regular cardinals are measurable in HOD.*

Theorem 48 shows that Woodin's theorem really needs the existence of extendibles. So extendibles have an impact on the relationsip between V and HOD. Recall that δ is *extendible* if for all $\eta > \delta$ there is an elementary embedding $j : V_{\eta+1} \to V_{j(\eta)+1}$ with critical point δ, sending δ above η. Extendibles are supercompact with many supercompacts below them.

Woodin proves his theorem using the concept of *weak extender model for δ supercompact*. An inner model N qualifies for this if for every $\gamma > \delta$ there is a supercompactness measure U on $P_\delta\gamma$ such that $N \cap P_\delta\gamma \in U$ and $U \cap N \in N$.

Lemma 38 *(a) If N is a weak extender model for δ supercompact and $\gamma > \delta$ is regular in N then card(γ) = cof(γ).*
(b) If N is a weak extender model for δ supercompact and $\gamma > \delta$ is a singular cardinal then γ is singular in N and γ^+ equals γ^+ of N.

Proof. By hypothesis we have for each $\mu > \delta$ a supercompactness measure U_μ on $P_\delta\mu$ such that $N \cap P_\delta\mu$ belongs to U_μ and $U_\mu \cap N$ belongs to N.

We first show that N has the δ-covering property, i.e. every set of ordinals of size less than δ is covered by such a set in N. Suppose that x is a set of ordinals of size less than δ and choose $\mu > \delta$ so that x is a subset of μ. By the fineness and δ-completeness of U_μ we know that the set of y in $P_\delta\mu$ containing x belongs to U_μ. As $N \cap P_\delta\mu$ also belongs to U_μ we can choose y in N of size less than δ containing x, as desired.

Now let $\gamma > \delta$ be regular in N. As N has δ-covering we know that $\mathrm{cof}(\gamma)$ is at least δ.

By a theorem of Solovay there is an $X \subseteq P_\delta\gamma$ in N such that the sup function is injective on X and X belongs to $U = U_\gamma$. Fix a club $D \subseteq \gamma$ of ordertype $\mathrm{cof}(\gamma)$ and set $A = \{y \in P_\delta\gamma \mid \sup(y) \in D\}$. Let j be the ultrapower embedding given by U; then $\beta = \sup j[\gamma]$ belongs to $j(D)$ as $j(D)$ is a club in $j(\gamma) > \sup j[\gamma] = \sup j[D]$ and $j[D] \subseteq j(D)$. As $U = \{B \mid j[\gamma] \in j(B)\}$ it follows that A belongs to U. As U is fine we know that $\bigcup\{y \in X \cap A \mid \sup(y) \in D\}$ is all of γ. But as the sup function is injective on X, we see that the cardinality of γ is at most $\mathrm{card}(\delta) \cdot \mathrm{card}(D) = \mathrm{card}(\delta) \cdot \mathrm{cof}(\gamma)$. As $\mathrm{cof}(\gamma)$ is at least δ, we get $\mathrm{card}(\gamma) \leq \mathrm{cof}(\gamma)$, as desired.

(b) follows readily from (a). □

Lemma 39 *Suppose that there is a proper class of regular cardinals which are not measurable in HOD and δ is extendible. Then HOD is a weak extender model for δ supercompact.*

Woodin's Theorem 37 now follows: If HOD is a weak extender model for δ supercompact, then by Lemma 38 γ is singular in HOD and $\gamma^+ = \gamma^+$ of HOD for all singular cardinals $\gamma > \delta$. Otherwise by Lemma 39, all large enough regular cardinals are measurable in HOD.

Proof of Lemma 39. Given $\zeta > \delta$ we want to find a ζ-supercompactness measure U on $P_\delta\zeta$ with $U \cap \mathrm{HOD}$ in HOD and $P_\delta\zeta \cap \mathrm{HOD}$ in U. Pick $\zeta < \gamma < \lambda < \eta$ such that $2^\zeta < \gamma$, $(\mathrm{card}(V_\gamma))^{\mathrm{HOD}} = \gamma$, $\lambda > 2^\gamma$ is regular but not measurable in HOD and $\mathrm{HOD}^{V_\eta} = \mathrm{HOD} \cap V_\eta$. As δ is extendible there is an embedding $j : V_{\eta+1} \to V_{j(\eta)+1}$ with critical point δ, sending δ above η.

We claim that $j[\gamma]$ belongs to the HOD of $V_{j(\eta)}$ and therefore to HOD.

First note that as λ is not measurable in HOD and $2^\gamma < \lambda$, there is a partition $(S_\alpha \mid \alpha < \gamma)$ of $\mathrm{Cof}(\omega) \cap \lambda$ into stationary sets such that $(S_\alpha \mid \alpha < \gamma)$ belongs to HOD. Otherwise, one argues that there is a stationary $S \subseteq \mathrm{Cof}(\omega) \cap \lambda$ in HOD which cannot be partitioned into two disjoint stationary sets in HOD, giving that the club filter on S restricted to HOD witnesses the measurability of λ in HOD.

As the HOD of V_η equals $\mathrm{HOD} \cap V_\eta$, $(S_\alpha \mid \alpha < \gamma)$ belongs to the HOD of V_η. By elementarity we have

$$(S^*_\alpha \mid \alpha < j(\gamma)) = j((S_\alpha \mid \alpha < \gamma)) \in \mathrm{HOD}^{V_{j(\eta)}}.$$

Set $\beta_\lambda = \sup j[\lambda]$. Then by an observation of Solovay,

$$j[\gamma] = \{\alpha < \sup j[\gamma] \mid S^*_\alpha \cap \beta_\lambda \text{ is stationary in } \beta_\lambda\}.$$

So $j[\gamma]$ is in the HOD of $V_{j(\eta)}$, as claimed.

Now let U be the ultrafilter on $P_\delta\zeta$ derived from j. Then $P_\delta\zeta \cap \mathrm{HOD}$ is in U since $j[\zeta]$ is in the HOD of $V_{j(\eta)}$ which equals $j(\mathrm{HOD} \cap V_\eta)$. And $U \cap \mathrm{HOD}$ is in HOD as $j \restriction (V_\gamma \cap \mathrm{HOD})$ belongs to HOD and $\gamma > 2^\zeta$. $\square$

12.-13.Vorlesungen

A little remark about Reinhardt cardinals without choice

Recall the following theorem of Kunen.

Theorem 40 *Suppose that $(V, \mathcal{C})$ is a model of Gödel-Bernays with the Axiom of Choice. Then $(V, \mathcal{C})$ is* rigid*: There is no elementary embedding $j : V \to V$ in $\mathcal{C}$ other than the identity.*

A *Reinhardt cardinal* is the critical point of an elementary embedding from V to V; Kunen's theorem says that there are no Reinhardt cardinals, assuming AC. But it does not rule out their existence if we drop AC.

We say that a model $(V, \mathcal{C})$ of class theory is *extendible* if there is an elementary embedding $\pi : (V, \mathcal{C}) \to (V^*, \mathcal{C}^*)$ which is the identity on V such that $\mathcal{C} = \mathcal{P}(V) \cap V^*$ (and V^* is wellfounded). The existence of extendible models of Gödel-Bernays follows from the existence of a weakly compact cardinal, so is not very strong.

Lemma 41 *Suppose that $(V, \mathcal{C})$ is extendible, witnessed by $\pi : (V, \mathcal{C}) \to (V^*, \mathcal{C}^*)$, and let U be the ultrafilter on $\mathcal{C}$ defined by $X \in U$ iff $Ord(V) \in \pi(X)$. Suppose that there exists a $j : V \to V$ in $\mathcal{C}$. Then there exists such a j whose set of fixed points does not belong to U.*

Proof. Choose $j : V \to V$ in $\mathcal{C}$ with least possible critical point. Now suppose that the set of fixed points of j belongs to U and let j^* be $\pi(j)$. By the elementarity of π, j^* is an elementary embedding from V^* to V^* extending j. As the set of fixed points of j belongs to U, $\mathrm{Ord}(V)$ is a fixed point of j^*. As $V = V^*_{\mathrm{Ord}(V)}$ and $\mathcal{C} = \mathcal{P}^{V^*}(V)$ are definable in V^* from the parameter $\mathrm{Ord}(V)$, $(V, \mathcal{C})$ is also a fixed point of j^*. Thus j^* restricted to $(V, \mathcal{C})$ is an elementary embedding from $(V, \mathcal{C})$ to $(V, \mathcal{C})$ which agrees with j on V.

Now we derive a contradiction. Let α be the critical point of j, the least possible critical point of an elemenetary embedding $V \to V$ in $\mathcal{C}$. Then α is definable in $(V, \mathcal{C})$. As j^* is elementary from $(V, \mathcal{C})$ to $(V, \mathcal{C})$, $j^*(\alpha)$ must equal α; but j^* extends j so $j^*(\alpha) = j(\alpha) > \alpha$. $\square$

The above did not use AC. So to rule out Reinhardt cardinals without AC for extendible models of class theory, it suffices to rule out elementary embeddings $j : V \to V$ with few fixed points.

The enriched stable core revisited

I'll give a simplified proof of the following application of the *enriched stable core.*

Theorem 42 *(Rigidity for V-constructible embeddings) Suppose that $(V, \mathcal{C})$ is an extendible model of Gödel-Bernays and all classes in $\mathcal{C}$ are V-constructible. Then for some V-definable class $\mathcal{S}^{**}$ of ordinals, $\mathbb{S}^{**} = (L[\mathcal{S}^{**}], \mathcal{S}^{**})$ is rigid for embeddings in $\mathcal{C}$.*

Corollary 43 *If $(V, \mathcal{C})$ is extendible and all classes in $\mathcal{C}$ are V-constructible then there is a V-definable class $\mathcal{S}^{**}$ of ordinals such that $(HOD, \mathcal{S}^{**})$ is rigid in $\mathcal{C}$.*

Fix a witness $\pi : (V, \mathcal{C}) \to (V^*, \mathcal{C}^*)$ to the extendibility of $(V, \mathcal{C})$. Then to say that all classes in $\mathcal{C}$ are V-constructible simply means that $\mathcal{C} = \mathcal{P}(V) \cap V^* = \mathcal{P}(V) \cap L(V)^{V^*}$. (I.e., all subsets of V in V^* in fact belong to the $L(V)$ of V^*.)

The above results were proved earlier using the enriched stability predicate $\mathcal{S}^*$, with a rather elaborate argument. I'll now give a much simpler argument, using a "semi-enriched" version $\mathcal{S}^{**}$ of $\mathcal{S}^*$.

In V define $\mathcal{S}^{**}$ to consist of all triples (α, β, i) where $\alpha < \beta$ are inaccessibles, $i < (\beta^+)^{L(H(\beta))}$ and α is (β, i)-*stable*. The latter means that $H^{\beta,i}(\alpha) \cap H(\beta) = H(\alpha)$, where $H^{\beta,i}(\alpha)$ is the least elementary submodel of $L_i(H(\beta))$ containing α as a subset.

Again referring to our witness $\pi : (V, \mathcal{C}) \to (V^*, \mathcal{C}^*)$ to the extendibility of $(V, \mathcal{C})$, note that $\mathrm{Ord}(V)$ is an inaccessible cardinal in V^* and in particular inaccessible in $(L(V))^{V^*}$, which we simply denote by $L^*(V)$. Also, any club $C \subseteq \mathrm{Ord}(V)$ in V^* contains an inaccessible of V. We let V^+ denote $H(\mathrm{Ord}(V)^+)^{V^*} = L_{Ord(V)^+}(V)^{V^*}$ and $L^*(V)$ denote $L(V)^{V^*}$. Thus $\mathcal{C} = \mathcal{P}(V) \cap V^+$ and $L((V, \mathcal{C}))^{V^*} = L^*(V)$.

To establish the rigidity of $\mathbb{S}^{**} = (L[\mathcal{S}^{**}], \mathcal{S}^{**})$ in $(V, \mathcal{C})$ we aim to show that $L^*(V)$, and therefore $\mathcal{C}$, is included in $L^*(V)[A] = (L[A])^{L^*(V)[A]}$ where $A \subseteq \mathrm{Ord}(V)$ is in turn generic over $(L(\mathcal{S}^{**}))^{V^*}$ for an $\mathbb{S}^{**}$-definable forcing. Denote $L^*(V)[A]$ simply by $L^*[A]$ and $(L(\mathcal{S}^{**}))^{V^*}$ simply by $L^*(\mathcal{S}^{**})$. Then as any $j : \mathbb{S}^{**} \to \mathbb{S}^{**}$ can be lifted to some $j^* : L^*(\mathcal{S}^{**}) \to L^*(\mathcal{S}^{**})$, the existence of a nontrivial such j in $\mathcal{C}$ will lead to a contradiction.

As in the case of the stability and enriched stability predicates, the genericity of A over $L^*(\mathcal{S}^{**})$ requires us to ensure that A preserve the stability relationships specified by the semi-enriched stability predicate $\mathcal{S}^{**}$. This will be accomplished using a reverse Easton iteration.

The forcing to add an $\mathcal{S}^{**}$-preserving $A \subseteq Ord(V)$ that codes V

We define a reverse Easton iteration $\mathbb{P}$ of length $\mathrm{Ord}(V)+1$. The iteration is nontrivial only at stages where are inaccessible in V^* (i.e. either inaccessible in V or equal to the last stage $\mathrm{Ord}(V)$). Let β be such a stage. The forcing $\mathbb{P}(\beta)$ consists of β-Cohen conditions p such that for all inaccessible $\alpha < \beta$, $p \restriction \alpha$ is $\mathbb{P}(\alpha)$-generic.

Lemma 44 *For any inaccessible $\beta \leq Ord(V)$, the forcing $\mathbb{P}(\beta)$ satisfies extendibility: Any condition in $\mathbb{P}(\beta)$ can be extended to any larger length less than β. Moreover $\mathbb{P}(\beta)$ is homogeneous.*

Proof. By induction on β. If β is a limit of inaccessibles then the result follows immediately by induction. If β is the least inaccessible then $\mathbb{P}(\beta)$ is just β-Cohen forcing, so the result is clear. Otherwise let α be the supremum of the inaccessibles less than β.

Suppose that α is inaccessible. By induction $\mathbb{P}(\alpha)$ is homogeneous and from this it follows that any condition in $\mathbb{P}(< \alpha)$ can be extended to a $\mathbb{P}(\alpha)$-generic (by modifying the $\mathbb{P}(\alpha)$-generic added at stage α to agree with this condition). Extending past α is easy as above α we just have β-Cohen forcing. Similarly we obtain the homogeneity of $\mathbb{P}(\beta)$.

Suppose that α is singular. Then to extend a condition in $\mathbb{P}(< \alpha)$ to length α we apply induction to extend it first to have length above the cofinality of α and then successively to conditions of length inside a club in α of ordertype less than α; at limit stages we still have a condition as the associated length is singular. Again extending past α is easy because above α we just have β-Cohen foricng. Similarly we obtain theh homogeneity of $\mathbb{P}(\beta)$. $\square$

Lemma 45 *The forcing $\mathbb{P}$ preserves cofinalities.*

We won't prove this as it follows using standard arguments.

Lemma 46 *Suppose that $\alpha < \beta \leq Ord(V)$ are inaccessible in V^*, $i < (\beta^+)^{L(H(\beta))}$ and α is (β, i)-stable. Then α is (β, i)-stable relative to A: $H^{A,\beta,i}(\alpha) \cap H(\beta)^{L[A]} = H(\alpha)^{L[A]}$, where $H^{A,\beta,i}(\alpha)$ is the least elementary submodel of $(L_i(H(\beta)^{L[A]}), A \cap \beta)$ containing α as a subset.*

Proof. Let $\pi : L_{\bar{i}}(H(\alpha)) \to L_i(H(\beta))$ be the inverse of the transitive collapse of $H^{\beta,i}(\alpha)$. Then $A \cap \alpha$ is generic for $\mathbb{P}(\alpha)$, $A \cap \beta$ is generic for $\mathbb{P}(\beta)$ and π sends $\mathbb{P}(\alpha)$ to $\mathbb{P}(\beta)$. It follows that π is elementary relative to A, as if a sentence holds of $A \cap \alpha$ there is a condition in $\mathbb{P}(\alpha)$ belonging to the generics determined by both $A \cap \alpha$ and $A \cap \beta$ which forces it in both $\mathbb{P}(\alpha)$ and $\mathbb{P}(\beta)$. $\square$

Corollary 47 *A is generic over $L^*(\mathcal{S}^{**})$ for an $\mathbb{S}^{**}$-definable forcing which is Ord(V)-cc in $L^*(\mathcal{S}^{**})$. It follows that any club in $\mathcal{C}$ (i.e any V-constructible club) contains a club in $L^*(\mathcal{S}^{**})$ (i.e. an $L[\mathcal{S}^{**}]$-constructible club).*

This Corollary is proved like the analogous claims regarding the stable and enriched stable cores: The forcing for which A is generic consists of infinitary propositional sentences of $L[\mathcal{S}^{**}]$ which are consistent with the theory T that documents the (β, i)-stability of α (relative to $\mathcal{S}^{**}$) whenever (α, β, i) belongs to $\mathcal{S}^{**}$. The forcing is Ord(V)-cc in $L^*(\mathcal{S}^{**})$ and as any maximal antichain of size less than Ord(V) is clearly met by (the set of sentences true about) A, we infer the genericity of A over $L^*(\mathcal{S}^{**})$ for this forcing. The last statement also follows from the $\mathrm{Ord}(V)$-cc of the forcing. $\Box$

We now easily infer the rigidity of $\mathbb{S}^{**}$ for embeddings in $\mathcal{C}$: Any such embedding belongs to a generic extension of $\mathbb{S}^{**}$ via an $\mathbb{S}^{**}$-definable forcing. But now consider the least ordinal forced to be the critical point of such a generic embedding; it is $L^*(\mathcal{S}^{**})$-definable and therefore cannot serve as the critical point of any embedding from $L^*(\mathcal{S}^{**})$ to itself. But as $\mathrm{Ord}(V)$ is inaccessible in $L^*(\mathcal{S}^{**})$, any embedding from $\mathbb{S}^{**}$ to itself extends to an embedding of $L^*(\mathcal{S}^{**})$ to itself, yielding the desired contradiction. (The same argument was used in our disucssion of the enriched stable core. An alternative second argument can be given, using the fact that the set of fixed points of any embedding in $\mathcal{C}$ is an ω-club which contains an ω-club in $L^*(\mathcal{S}^{**})$, together with the fact that $\diamondsuit_{\mathrm{Ord}(V)}$ holds in $L^*(\mathcal{S}^{**})$.

14.Vorlesung

Weak Covering for HOD

Yet another interpretation of the "closeness" of HOD to V is that α^+ of HOD equals α^+ of V for many cardinals α. This is for example the case if V does not contain $0^\sharp$ (using $L \subseteq$ HOD) or if V does not contain an inner model with a Woodin cardinal (using $K \subseteq$ HOD, where K denotes the core model for a Woodin cardinal).

The next result shows that we can't hope to approximate the cardinals of V by those of (inner models of) HOD in general:

Theorem 48 *Suppose GCH holds and κ is a supercompact cardinal. Then there is a generic extension V^* of V in which κ remains inaccessible and for all infinite cardinals $\alpha < \kappa$, $(\alpha^+)^{HOD} < \alpha^+$. In particular $W = V_\kappa^{V^*}$ is a model of ZFC in which for all infinite cardinals α, $(\alpha^+)^{HOD} < \alpha^+$.*

The proof also shows that the supercompactness of κ can be preserved, provided we weaken the conclusion to: For a club of cardinals $\alpha < \kappa$, $(\alpha^+)^{HOD} < \alpha^+$.

The proof of Theorem 48 makes use of supercompact Radin forcing, which we now introduce.

Measure sequences

Let $P_\kappa(\lambda) = \{x \subseteq \lambda : \text{ot }(x) < \kappa,\ x \cap \kappa \in \kappa\}$, and for $x \in P_\kappa(\lambda)$ set $\kappa_x = x \cap \kappa$ and $\lambda_x = \text{ot }(x)$.

Given $x, y \in P_\kappa(\lambda)$ we define the relation $x \prec y$ by

$$x \prec y \Leftrightarrow x \subseteq y \text{ and ot }(x) < y \cap \kappa.$$

For infinite cardinals $\kappa < \lambda$, let $S(\kappa, \lambda)$ be the set of sequences w such that $\text{lh}(w) < \kappa$, $w(0) \in P_\kappa(\lambda)$ and $w(\alpha) \in V_\kappa$ for $0 < \alpha < \text{lh}(w)$.

Given $\kappa < \lambda$ and $j : V \to M$ witnessing that κ is λ-supercompact, we generate a sequence as follows:

- $u_j(0) = j[\lambda]$.
- For $\alpha > 0$, $u_j(\alpha) = \{X \subseteq S(\kappa, \lambda) : u_j \upharpoonright \alpha \in j(X)\}$.

$u_j(\alpha)$ is defined as long as $u_j \upharpoonright \alpha \in M$. We denote the least α such that $u_j \upharpoonright \alpha \notin M$ by $\text{lh}(u_j)$.

So $u_j(1) = \{X \subseteq S(\kappa, \lambda) : u_j \upharpoonright 1 = \{\langle 0, j[\lambda]\rangle\} \in j(X)\}$ is defined and can be identified with the measure U on $P_\kappa(\lambda)$ derived from j. And if defined, $u_j(2) = \{X \subseteq S(\kappa, \lambda) : u_j \upharpoonright 2 = \{\langle 0, j[\lambda]\rangle, \langle 1, u_j(1)\rangle\} \in j(X)\}$ can be identified with a measure on $P_\kappa(\lambda) \times V_\kappa$ concentrating on pairs (x, u_x) where x belongs to $P_\kappa(\lambda)$ and u_x is a measure on $P_{\kappa_x}(\lambda_x)$. This is because in M, if x denotes $j[\lambda]$ then $x \cap j(\kappa) = \kappa$ and $\text{ot }(x) = \lambda$, so $u_j(1)$ can indeed be identified with a measure on the $P_{x \cap j(\kappa)}(\text{ot }(x))$ of M.

Similarly, if defined, $u_j(3)$ can be identified with a measure on $P_\kappa(\lambda) \times V_\kappa \times V_\kappa$ concentrating on triples (x, u_x^0, u_x^1) where x belongs to $P_\kappa(\lambda)$, u_x^0 is a measure on $P_{\kappa_x}(\lambda_x)$ and u_x^1 is a measure on pairs (y, u_y^0) where y belongs to $P_{\kappa_x}(\lambda_x)$ and u_y^0 is a measure on $P_{y \cap \kappa_x}(\text{ot }(y))$.

15.Vorlesung

Measure sequences

Let $P_\kappa(\lambda) = \{x \subseteq \lambda : \text{ot }(x) < \kappa,\ x \cap \kappa \in \kappa\}$, and for $x \in P_\kappa(\lambda)$ set $\kappa_x = x \cap \kappa$ and $\lambda_x = \text{ot }(x)$.

Given $x, y \in P_\kappa(\lambda)$ we define the relation $x \prec y$ by

$$x \prec y \Leftrightarrow x \subseteq y \text{ and ot } (x) < y \cap \kappa.$$

For infinite cardinals $\kappa < \lambda$, let $S(\kappa, \lambda)$ be the set of sequences w such that $\mathrm{lh}(w) < \kappa$, $w(0) \in P_\kappa(\lambda)$ and $w(\alpha) \in V_\kappa$ for $0 < \alpha < \mathrm{lh}(w)$.

Given $\kappa < \lambda$ and $j : V \to M$ witnessing that κ is λ-supercompact, we generate a sequence as follows:

- $u_j(0) = j[\lambda]$.
- For $\alpha > 0$, $u_j(\alpha) = \{X \subseteq S(\kappa, \lambda) : u_j \restriction \alpha \in j(X)\}$.

$u_j(\alpha)$ is defined as long as $u_j \restriction \alpha \in M$. We denote the least α such that $u_j \restriction \alpha \notin M$ by $\mathrm{lh}(u_j)$.

So $u_j(1) = \{X \subseteq S(\kappa, \lambda) : u_j \restriction 1 = \{\langle 0, j[\lambda]\rangle\} \in j(X)\}$ is defined and can be identified with the measure U on $P_\kappa(\lambda)$ derived from j. And if defined, $u_j(2) = \{X \subseteq S(\kappa, \lambda) : u_j \restriction 2 = \{\langle 0, j[\lambda]\rangle, \langle 1, u_j(1)\rangle\} \in j(X)\}$ can be identified with a measure on $P_\kappa(\lambda) \times V_\kappa$ concentrating on pairs (x, u_x) where x belongs to $P_\kappa(\lambda)$ and u_x is a measure on $P_{\kappa_x}(\lambda_x)$. This is because in M, if x denotes $j[\lambda]$ then $x \cap j(\kappa) = \kappa$ and ot $(x) = \lambda$, so $u_j(1)$ can indeed be identified with a measure on the $P_{x \cap j(\kappa)}(\text{ot } (x))$ of M.

Similarly, if defined, $u_j(3)$ can be identified with a measure on $P_\kappa(\lambda) \times V_\kappa \times V_\kappa$ concentrating on triples (x, u_x^0, u_x^1) where x belongs to $P_\kappa(\lambda)$, u_x^0 is a measure on $P_{\kappa_x}(\lambda_x)$ and u_x^1 is a measure on pairs (y, u_y^0) where y belongs to $P_{\kappa_x}(\lambda_x)$ and u_y^0 is a measure on $P_{y \cap \kappa_x}(\text{ot } (y))$.

We say that u is a (κ, λ)-*measure sequence* if $u(0)$ is a set of ordinals of order type λ, κ is the least ordinal not in $u(0)$ and $u(\alpha)$ is a measure on $S(\kappa, \lambda)$ for each $0 < \alpha < \mathrm{lh}(u)$, In this case we write (κ_u, λ_u) for (κ, λ).

Given a (κ, λ)-measure sequence u, say that *j is a constructing embedding for u* if j witnesses that κ is λ-supercompact and for all α with $0 < \alpha < \mathrm{lh}(u)$ we have that $u_j(\alpha)$ is defined with $u_j(\alpha) = u(\alpha)$. Note that possibly $u(0) \neq u_j(0)$ (as $j[\lambda]$ is not uniquely determined by the measure j induces on $P_\kappa(\lambda)$).

We define a the class $\mathcal{U}_\infty$ of *good measure sequences* as follows:

- $\mathcal{U}_0$ is the class of u such that u is a (κ_u, λ_u)-measure sequence with a constructing embedding for some (κ_u, λ_u).
- $\mathcal{U}_{n+1} = \{u \in \mathcal{U}_n : \text{for all nonzero } \alpha < \mathrm{lh}(u),\ u(\alpha) \text{ concentrates on } \mathcal{U}_n\}$.
- $\mathcal{U}_\infty = \bigcap_{n<\omega} \mathcal{U}_n$.

Note that if $u \in \mathcal{U}_\infty$, then it follows from the countable completeness of the measures in u that every measure in u concentrates on $\mathcal{U}_\infty$.

Given a $u \in \mathcal{U}_\infty$, α is called a *weak repeat point* for u if for all $X \in u(\alpha)$ there exists $\beta < \alpha$ such that $X \in u(\beta)$. A (κ, λ)-measure sequence of length $(2^{\lambda^{<\kappa}})^+$ contains a weak repeat point, as the measures $u(\alpha)$, $\alpha > 0$ live on $P_\kappa(\lambda)$ and the latter has only $2^{\lambda^{<\kappa}}$ many subsets. It follows that if GCH holds and κ is κ^{++}-supercompact then there is a (κ, κ^+)-measure sequence u with a weak repeat point, as a witness j to κ^{++}-supercompactness constructs a (κ, κ^+)-measure sequence of length κ^{+3}. The following lemma (whose proof is not difficult) says that with a bit more supercompactness we can also require that u be good (i.e. belongs to $\mathcal{U}_\infty$).

Lemma 49 *Let GCH hold and let $j : V \to M$ witness that κ is κ^{+4}-supercompact. Then j constructs a (κ, κ^+)-measure sequence u such that $u \in \mathcal{U}_\infty$ and u has a weak repeat point.*

The importance of measure sequences with weak repeat points is that Radin forcings defined using them preserve a degree of supercompactness.

Supercompact Radin forcing

A *good pair* is a pair (u, A) where $u \in \mathcal{U}_\infty$, $A \subseteq \mathcal{U}_\infty$, $A \subseteq S(\kappa_u, \lambda_u)$ and A is *u-large*, that is $A \in u(\alpha)$ for $0 < \alpha < \mathrm{lh}(u)$.

We define, for each $u \in \mathcal{U}_\infty$, the corresponding *supercompact Radin forcing* $\mathbb{R}_u$.

A condition in $\mathbb{R}_u$ is a finite sequence

$$p = \langle (u^0, A^0), \ldots, (u^i, A^i), \ldots, (u^n, A^n) \rangle$$

where:

1. $u^n = u$.

2. Each (u^i, A^i) is a good pair.

3. $u^i(0) \in P_{\kappa_{u^n}}(\lambda_{u^n})$, for $i < n$.

4. $u^i(0) \prec u^{i+1}(0)$, for $i < n-1$.

(Note that in the last item above we can't write $i \leq n-1$ as $u^n(0)$ is not an element of $P_{\kappa_{u^n}}(\lambda_{u^n})$ but rather a set of ordinals of ordertype λ_{u^n} that may fail to contain $u^{n-1}(0)$ as a subset.)

Given $p \in \mathbb{R}_u$, $p = \langle (u^0, A^0), \ldots, (u^i, A^i), \ldots, (u^n = u, A^n) \rangle$ and $v \in \mathcal{U}_\infty$, we say v *appears* in p, if $v = u^i$ for some $i < n$.

Given $u \in \mathcal{U}_\infty$, let $\pi_u : u(0) \to \lambda_u$ be the collapse map. Given $v \in \mathcal{U}_\infty$ with $v(0) \prec u(0)$, let $\pi_{v,u} : \lambda_v \to \lambda_u$ be defined by $\pi_{v,u} = \pi_u \circ \pi_v^{-1}$ and let $\pi_u(v)$ be obtained from v by replacing $v(0)$ by $\pi_u[v(0)]$.

We now define the notion of extension.

Let
$$p = \langle (u^0, A^0), \ldots, (u^i, A^i), \ldots, (u^n = u, A^n) \rangle$$
and
$$q = \langle (v^0, B^0), \ldots, (v^i, B^i), \ldots, (v^m = u, B^m) \rangle$$
be in $\mathbb{R}_u$. Then $q \leq p$ (q is an extension of p) iff:

1. There exist natural numbers $i_0 < \ldots < i_n = m$ such that $v^{i_k} = u^k$ and $B^{i_k} \subseteq A^k$.

2. If j is such that $0 \leq j \leq m$ and $j \notin \{i_0, \ldots, i_n\}$, and i is least such that $\kappa_{v^j} < \kappa_{u^i}$, then:

   - If $i = n$, then $v^j \in A^n$ and for all $x \in B^j$, $\pi_{v^j}^{-1}(x) \in A^n$, where $\pi_{v^j}^{-1}(x) = \langle \pi_{v^j}^{-1}[x(0)] \rangle^\frown \langle x(\alpha) : 0 < \alpha < \mathrm{lh}(x) \rangle$.
   - If $i < n$, then $v^j(0) \prec u^i(0)$, $\pi_{u^i}(v^j) \in A^i$ and for all $x \in B^j$, $\pi_{v^j,u^i}(x) \in A^i$, where $\pi_{v^j,u^i}(x) = \langle \pi_{v^j,u^i}[x(0)] \rangle^\frown \langle x(\alpha) : 0 < \alpha < \mathrm{lh}(x) \rangle$.

We also define $q \leq^* p$ (q is a direct or a Prikry extension of p) iff $q \leq p$ and $m = n$.

Let $u \in \mathcal{U}_\infty$ be a (κ_u, κ_u^+)-measure sequence. Below are the basic facts about the forcing $\mathbb{R}_u$.

Theorem 50 *Let G be $\mathbb{R}_u$-generic over V. The following hold in $V[G]$:*

1. *Let $C = \{v(0) : v$ appears in some $p \in G\}$. Then C is a $\prec$-increasing and continuous sequence in $P_{\kappa_u}(\kappa_u^+)$ of order type $\leq \kappa_u$. Furthermore if $lh(u) \geq \kappa_u$, then ot $(C) = \kappa_u$.*
2. *$\kappa_u^+ = \bigcup C$, in particular κ_u^+ is collapsed.*
3. *$\kappa_C = \{\kappa_{v(0)} : v(0) \in C\}$ is a club in κ_u.*
4. *$(\mathbb{R}_u, \leq)$ satisfies the $\kappa_u^{++} - c.c.$.*
5. *$(\mathbb{R}_u, \leq, \leq^*)$ satisfies the Prikry property: Given any $b \in ro(\mathbb{R}_u)$ and any condition $p \in \mathbb{R}_u$ there exists $q \leq^* p$ which decides b.*

We now give a factorization property of $\mathbb{R}_u$.

Theorem 51 *Suppose that*

$$p = \langle (u^0, A^0), \ldots, (u^i, A^i), \ldots, (u^n = u, A^n) \rangle \in \mathbb{R}_u$$

and $m < n$. Let

$$p^{>m} = \langle (u^{m+1}, A^{m+1}), \ldots, (u^n = u, A^n) \rangle$$

and

$$p^{\leq m} = \langle (v^0, B^0), \ldots, (v^{m-1}, B^{m-1}), (u^m, \pi_{u^m}[A^m]) \rangle$$

where for $i < m, v^i = \pi_{u^m}(u^i)$ and $B^i = \pi_{u^i, u^m}[A^i]$.

Then $p^{\leq m} \in \mathbb{R}_{u^m}, p^{>m} \in \mathbb{R}_u$ and there exists

$$i : \mathbb{R}_u/p \to \mathbb{R}_{u^m}/p^{\leq m} \times \mathbb{R}_u/p^{>m}$$

which is an isomorphism with respect to both $\leq$ and $\leq^*$.

Theorem 52 *Let G be $\mathbb{R}_u$-generic. Let $\vec{v} = \langle v_i : i < ot\,(C)\rangle$ enumerate $\{v : v$ appears in some $p \in G\}$. Then:*

1. $V[G] = V[\vec{v}]$.
2. *For every limit ordinal $j < ot\,(C), \langle v_i^* : i < j\rangle$ is $\mathbb{R}_{v_j}$-generic over V, where $v_i^* = \pi_{v_j}(v_i)$, and $\vec{v} \restriction [j, ot\,(C))$ is $\mathbb{R}_u$-generic over $V[\langle v_i^* : i < j\rangle]$.*
3. *For every $\gamma < \kappa$ and $A \subseteq \gamma$ with $A \in V[\vec{v}]$, $A \in V[\langle v_i^* : i < j\rangle]$. where $j < ot\,(C)$ is the least ordinal such that $\gamma < \kappa_{v_j}$.*

Theorem 53 *Let G be $\mathbb{R}_u$-generic, and let λ be a cardinal of V with $\lambda < \kappa_u$. Then λ is collapsed in $V[G]$ if and only if $\lambda = (\kappa^+_{v(0)})^V$ for some v which is a $\prec$-limit element of C.*

Theorem 54 *Let $j : V \to M$ witness κ is κ^{+4}-supercompact and let $v \in \mathcal{U}_\infty$ be a (κ, κ^+)-measure sequence constructed from j which has a weak repeat point α. Let $u = v \restriction \alpha$ and let G be $\mathbb{R}_u$-generic over V. Then in $V[G]$, κ remains λ-supercompact, where $\lambda = (\kappa^{+4})^V = (\kappa^{+3})^{V[G]}$.*

Projected forcing

We define a "projected supercompact Radin forcing" in such a way that the resulting quotient forcing is sufficiently homogeneous.

Suppose (u, A) is a good pair. Let $\pi(u, A) = (\pi(u), \pi(A))$ where

- $\pi(u) = \kappa_u{}^\frown u \restriction [1, \mathrm{lh}(u))$.
- $\pi(A) = \{\pi(v) : v \in A\}$.

Also let $\mathcal{U}^\pi_\infty = \{\pi(u) : u \in \mathcal{U}_\infty\}$. For $u \in \mathcal{U}_\infty$ set $\pi(u(\alpha))$ = the Rudin-Keisler projection of $u(\alpha)$, for $\alpha > 0$. I.e., a set is large for $\pi(u(\alpha))$ iff its inverse image under π is large for $u(\alpha)$. Note that $\pi(u(\alpha)) \neq \pi(u)(\alpha)$.

A *good pair for projected forcing* is a pair (u, A) where $u \in \mathcal{U}^\pi_\infty$, $A \subseteq \mathcal{U}^\pi_\infty$, $A \subseteq V_{\kappa_u}$ and A is of measure one for all $\pi(u(\alpha)), 0 < \alpha < \mathrm{lh}(u)$.

Remark. If (u, A) is a good pair, then $(\pi(u), \pi(A))$ is a good pair for projected forcing.

Given $u \in \mathcal{U}^\pi_\infty$, we define the projected forcing $\mathbb{R}^\pi_u$.

A condition in $\mathbb{R}^\pi_u$ is a finite sequence

$$p = \langle (u^0, A^0), \ldots, (u^i, A^i), \ldots, (u^n, A^n) \rangle$$

where:

1. $u^n = u$.
2. Each (u^i, A^i) is a good pair for projected forcing.
3. $\kappa_{u^i} < \kappa_{u^{i+1}}$, for all $i < n$.

And we define the extension relation:

Let

$$p = \langle (u^0, A^0), \ldots, (u^i, A^i), \ldots, (u^n = u, A^n) \rangle$$

and

$$q = \langle (v^0, B^0), \ldots, (v^i, B^i), \ldots, (v^m = u, B^m) \rangle$$

be in $\mathbb{R}^\pi_u$. Then $q \leq p$ (q is an extension of p) iff:

1. There exists an increasing sequence of natural numbers $i_0 < \ldots < i_n = m$ such that $v^{i_k} = u^k$ and $B^{i_k} \subseteq A^k$.
2. If j is such that $0 \leq j \leq m$ and $j \notin \{i_0, \ldots, i_n\}$, and if i is least such that $\kappa_{v^j} < \kappa_{u^i}$, then $v^j \in A^i$ and $B^j \subseteq A^i$.

We also define $q \leq^* p$ (q is a direct or a Prikry extension of p) iff $q \leq p$ and $m = n$.

It is easy to see that $(\mathbb{R}^\pi_u, \leq)$ satisfies the $\kappa^+_u - c.c.$.

Theorem 55 *Let G be $\mathbb{R}^\pi_u$-generic over V, and let*

$$C = \{\kappa_v : v \textit{ appears in some } p \in G\}.$$

Then C is a club of κ_u. Furthermore if $lh(u) \geq \kappa_u$, then $ot\,(C) = \kappa_u$,

As before we have a factorization property for $\mathbb{R}^\pi_u$.

Theorem 56 *Suppose that*

$$p = \langle (u^0, A^0), \ldots, (u^i, A^i), \ldots, (u^n = u, A^n) \rangle \in \mathbb{R}^\pi_u$$

and $m < n$. Let

$$p^{>m} = \langle (u^{m+1}, A^{m+1}), \ldots, (u^n = u, A^n) \rangle$$

and

$$p^{\leq m} = \langle (u^0, A^0), \ldots, (u^m, A^m) \rangle.$$

Then $p^{\leq m} \in \mathbb{R}^\pi_{u^m}, p^{>m} \in \mathbb{R}^\pi_u$ and there exists

$$i : \mathbb{R}^\pi_u / p \to \mathbb{R}^\pi_{u^m}/p^{\leq m} \times \mathbb{R}^\pi_u / p^{>m}$$

which is an isomorphism with respect to both $\leq$ and $\leq^*$.

Weak projection

Suppose that $u \in \mathcal{U}_\infty$ and consider the forcing notions $\mathbb{R}_u$ and $\mathbb{R}^\pi_{\pi(u)}$. We define a map $\pi : \mathbb{R}_u \to \mathbb{R}^\pi_{\pi(u)}$ in the natural way by

$$\pi(\langle (u^0, A^0), \ldots, (u^i, A^i), \ldots, (u^n, A^n) \rangle) = \langle \pi(u^0, A^0), \ldots, \pi(u^i, A^i), \ldots, \pi(u^n, A^n) \rangle$$

In general π is not a projection map, but we show that it has a weaker property, introduced by Foreman and Woodin.

Definition 57 $\pi : \mathbb{Q} \to \mathbb{P}$ *is called a* weak projection *if*

1. $\pi(1_{\mathbb{Q}}) = 1_{\mathbb{P}}$.
2. π *is order preserving.*
3. *For all $p \in \mathbb{Q}$ there is $p^* \leq p$ such that for all $q \leq \pi(p^*)$ there exists $r \leq p$ such that $\pi(r) \leq q$.*

The next lemma shows that $\pi : \mathbb{Q} \to \mathbb{P}$ being a weak projection is sufficient to imply that $\mathbb{Q}$-generics yield $\mathbb{P}$-generics.

Lemma 58 *Suppose $\pi : \mathbb{Q} \to \mathbb{P}$ is a weak projection and H is $\mathbb{Q}$-generic over V. Let G be the filter generated by $\pi[H]$. Then G is $\mathbb{P}$-generic over V.*

Proof. Let D be a dense open subset of $\mathbb{P}$. Let $E = \pi^{-1}[D]$. Note that E is dense in $\mathbb{Q}$: Let $p \in \mathbb{Q}$ and let $p^* \leq p$ be as in 5.1(3). Let $q \leq \pi(p^*)$ be such that $q \in D$ and let $r \leq p$ be such that $\pi(r) \leq q$. Then $\pi(r) \in D$, hence $r \in E$.

Now let $p \in H \cap E$. Then $\pi(p) \in G \cap D$, and hence $G \cap D \neq \emptyset$. $\square$

Let us now consider $\pi : \mathbb{R}_u \to \mathbb{R}^\pi_{\pi(u)}$.

Theorem 59 *$\pi : \mathbb{R}_u \to \mathbb{R}^\pi_{\pi(u)}$ is a weak projection, in fact for all $p \in \mathbb{R}_u$ there is $p^* \leq^* p$ such that for all $q \leq \pi(p^*)$ we can find $r \leq p$ such that $\pi(r) \leq^* q$.*

By weak projection we can find $1^* \leq^* 1_{\mathbb{R}_u}$ such that for all $q^* \leq \pi(1^*)$ there exists $r \in \mathbb{R}_u$ such that $\pi(r) \leq q^*$. In fact we may choose 1^* to be $1_{\mathbb{R}_u}$. It follows that $\pi[\mathbb{R}_u]$ is dense in $\mathbb{R}^\pi_{\pi(u)}$. Also by weak projection and the Prikry property for $\mathbb{R}_u$ we have that $\pi[\mathbb{R}_u]$ satisfies the Prikry property.

By Theorem 59, $\pi[\mathbb{R}_u]$ is in fact $\leq^*$-dense in $\mathbb{R}^\pi_{\pi(u)}$, so we get:

Corollary 60 *$\mathbb{R}^\pi_{\pi(u)}$ satisfies the Prikry property.*

Theorem 61 *Let G be $\mathbb{R}^\pi_u$-generic over V, and let C be as in Theorem 55. Also let $\vec{v} = \langle v_i : i < ot\ (C)\rangle$ enumerate $\{v : v \text{ appears in some } p \in G\}$ such that for $i < j < ot\ (C)$, $\kappa_{v_i} < \kappa_{v_j}$. Then:*

1. *$V[G] = V[\vec{v}]$.*
2. *For every limit ordinal $j < ot\ (C)$, $\vec{v} \restriction j$ is $\mathbb{R}_{v_j}$-generic over V, and $\vec{v} \restriction [j, ot\ (C))$ is $\mathbb{R}_u$-generic over $V[\vec{v} \restriction j]$.*

Theorem 62 *Suppose $\gamma < \kappa$, $A \subseteq \gamma$, $A \in V[\vec{v}]$. Let $i < ot\ (C)$ be the least ordinal such that $\gamma < \kappa_{v_i}$. Then $A \in V[\vec{v} \restriction j]$.*

Theorem 63 *Suppose that $u \in \mathcal{U}^\pi_\infty$ and let G be $\mathbb{R}^\pi_u$-generic over V. Then V and $V[G]$ have the same cardinals.*

The homogeneity property

Suppose that $u \in \mathcal{U}_\infty$ is a (κ, κ^+)-measure sequence.

Theorem 64 *For all $p, q \in \mathbb{R}_u$, if $\pi(p) = \pi(q)$, then there exists q^* compatible with q such that $\mathbb{R}_u/p \simeq \mathbb{R}_u/q^*$.*

Corollary 65 *(Weak homogeneity). Suppose $p, q \in \mathbb{R}_u$ and $\pi(p) = \pi(q)$. If $p \Vdash \phi(\alpha, \vec{\gamma})$, where $\alpha, \vec{\gamma}$ are ordinals, then it is not the case that $q \Vdash \neg\phi(\alpha, \vec{\gamma})$.*

It follows that:

Corollary 66 *Suppose that G is $\mathbb{R}_u$-generic and let G^π be the filter generated by $\pi[G]$ Then:*

1. *G^π is $\mathbb{R}^\pi_{\pi(u)}$-generic over V.*
2. *$HOD^{V[G]} \subseteq V[G^\pi]$.*

Now we prove the main theorem. Suppose κ is a κ^{+4}-supercompact cardinal and let $j : V \to M$ witness this. Let $v \in \mathcal{U}_\infty$ be a (κ, κ^+)-measure sequence constructed from j which has a weak repeat point α and let $u = v \upharpoonright \alpha$. Consider the forcing notions $\mathbb{R}_u$ and $\mathbb{R}^\pi_{\pi(u)}$. Let G be $\mathbb{R}_u$-generic over V and let G^π be the filter generated by $\pi[G]$. In summary we showed:

- κ remains λ-supercompact in $V[G]$, where $\lambda = (\kappa^{+4})^V = (\kappa^{+3})^{V[G]}$.
- κ remains λ-supercompact in $V[G^\pi]$, where $\lambda = (\kappa^{+4})^V = (\kappa^{+4})^{V[G^\pi]}$.
- There exists a club $C \in V[G^\pi]$ of κ such that for every limit point α of C we have $(\alpha^+)^{V[G^\pi]} = (\alpha^+)^V < (\alpha^+)^{V[G]}$.

By part 2 of Corollary 66 we have $HOD^{V[G]} \subseteq V[G^\pi]$, in particular for every limit point α of C we have

$$(\alpha^+)^{HOD^{V[G]}} \leq (\alpha^+)^{V[G^\pi]} = (\alpha^+)^V < (\alpha^+)^{V[G]}.$$

Claim. Let $\langle \kappa_i : i < \kappa \rangle$ be an increasing enumeration of C. Working in $V[G]$, let $\mathbb{Q}$ be the reverse Easton iteration for collapsing each κ_{i+1} to κ_i^+ for each each $i < \kappa$, and let H be $\mathbb{Q}$-generic over $V[G]$. Clearly

1. $CARD^{V[G*H]} \cap (\kappa_0, \kappa) = \{\kappa_i^+ : i < \kappa\} \cup \{\kappa_i : i < \kappa,\ i \text{ is a limit ordinal}\}$.
2. κ remains inaccessible in $V[G * H]$.

It also follows from a result of Dobrinen-Friedman that $\mathbb{Q}$ is *cone homogeneous*, that is for all $p, q \in \mathbb{Q}$ there are $p^* \leq p, q^* \leq q$ and an isomorphism $\phi : \mathbb{Q}/p^* \to \mathbb{Q}/q^*$. Hence we have

$$HOD^{V[G*H]} \subseteq HOD^{V[G]}.$$

Finally force with $\mathbb{P} = Col(\aleph_0, \kappa_0)^{V[G*H]}$ over $V[G*H]$ and let K be $\mathbb{P}$-generic over $V[G*H]$. It is now easily seen that κ remains inaccessible in $V[G*H*K]$ and by homogeneity of $\mathbb{P}$

$$HOD^{V[G*H*K]} \subseteq HOD^{V[G*H]}.$$

Hence

$$HOD^{V[G*H*K]} \subseteq HOD^{V[G*H]} \subseteq HOD^{V[G]} \subseteq V[G^\pi].$$

Thus for all infinite cardinals $\alpha < \kappa$ of $V[G * H * K]$ we have

$$(\alpha^+)^{HOD^{V[G*H*K]}} \leq (\alpha^+)^{V[G^\pi]} = (\alpha^+)^V < (\alpha^+)^{V[G*H*K]}.$$

Let $V^* = V[G * H * K]$. Then V^* is the required model and the theorem follows. $\Box$

Remark. (a) In fact we can show that κ remains measurable.
(b) If we start with a supercompact cardinal κ, then our proof shows that we can preserve the supercompactness of κ and have $(\alpha^+)^{HOD} < \alpha^+$ for a closed unbounded set of $\alpha < \kappa$.

Gitik has claimed an improvement: It is consistent that all regular uncountable cardinals are inaccessible cardinal in HOD. Is it consistent that κ is supercompact and $(\alpha^+)^{HOD} < \alpha^+$ for all cardinals $\alpha < \kappa$? This is open (and Woodin has conjectured a negative answer).