Cardinal Characteristics in the Uncountable

1.-2.Vorlesungen

Introduction

The study of cardinal characteristics of the continuum is now a vast subject. Its focus is on the relationships among a wide array of naturally defined uncountable cardinal numbers that are at most $2^{\aleph_0}$, the size of the continuum. Some examples (which are defined below and whose generalisations play an important role in this course) are the dominating number $\mathfrak{d}$, the bounding number $\mathfrak{b}$, the almost disjointness number $\mathfrak{a}$, the splitting number $\mathfrak{s}$, the tower number $\mathfrak{t}$ and the pseudo-intersection number $\mathfrak{p}$. These are defined in terms of the existence of certain families of subsets of ω or of functions from ω to ω with specific properties.

The main tool in the study of cardinal characteristics of the continuum is the method of iterated forcing, especially the theory of *proper countable support iterations* and its variants. The study of higher cardinal characteristics provides a rich set of problems which demand a generalisation of the theory of properness to iterations with uncountable support.

Questions

For a fixed cardinal characteristic $\kappa \mapsto \chi(\kappa)$ we have the following general questions:

(a) (Basic question) For which κ, λ can we force $\chi(\kappa) = \lambda$ (preserving cardinals)?
(b) (Global behaviour) For which definable cardinal-valued functions F defined on the class of all regular cardinals can we force $\chi(\kappa) = F(\kappa)$ for all regular κ (preserving cardinals)?
(c) (Large cardinal context) Can we have $\chi(\kappa) > \kappa^+$ for a measurable κ? If so, what is the consistency strength of this inequality?
(d) (Internal consistency) Once some global result as in (b) above is achieved, can it also be shown to hold in some inner model, assuming the existence of large cardinals? If so, what large cardinals are needed for this internal consistency?

(e) (Transfer of open questions from the classical setting) Can the basic question be answered for $\chi(\kappa)$ for some uncountable κ when the answer is not known for the case $\kappa = \omega$?

Of course it is often important to consider the above questions not just for a single cardinal characteristic, but for several of them simultaneously, revealing relationships among them.

We turn now to some specific examples.

The characteristic $\mathfrak{c}(\kappa) = 2^\kappa$

The most familiar characteristic meaures the size of the powerset of κ. Recall the basic results about this dating from the time of the ancient Greeks:

Theorem 1 *(Solovay) Assume GCH, κ regular and cof(λ) $> \kappa$. Then we can force $2^\kappa = \lambda$, preserving cofinalities.*

Proof. Use Add(κ, λ), the forcing to add λ subsets of κ. Conditions are partial functions $f : \kappa \times \lambda \to 2$ of size less than κ, ordered by extension. The forcing is κ-closed and adds at least λ-many subsets of κ. Cofinalities are preserved thanks to the κ^+-cc:

Lemma 2 *Suppose that A is an antichain in $\mathbb{P}$ = Add(κ, λ). Then A has size at most κ.*

Proof of Lemma. We can assume that A is maximal. Let M be a sufficiently elementary submodel of the universe which contains A as an element, is κ-closed and has size κ. Then $A \cap M$ is also a maximal antichain, as the restriction $p \cap M$ of any condition p to M is an element of M which is compatible with an element of $A \cap M$, and this element of $A \cap M$ is also compatible with p. □ (*Lemma*)

So $\mathbb{P}$ has the κ^+-cc and cofinalities are preserved. And any subset of κ added by $\mathbb{P}$ has a name which is *nice*, i.e. of the form $\{(\alpha, p) \mid p \in A_\alpha\}$ where the A_α's are antichains, and using the fact that cof$(\lambda) > \kappa$ there are only λ-many such names in the ground model. It follows that $\mathbb{P}$ adds exactly λ-many subsets of κ. □

Easton went global:

Theorem 3 *(Easton) Assume GCH and suppose that $F : Reg \to Card$ is a definable function, where Reg denotes the class of infinite regular cardinals and Card denotes the class of arbitrary infinite cardinals. Suppose that cof($F(\kappa)$) $> \kappa$ for each κ and F is nondecreasing. Then we can force $2^\kappa = F(\kappa)$ for all regular κ, preserving cofinalities.*

Proof. We use the Easton product $\prod_\kappa^E \mathrm{Add}(\kappa, F(\kappa))$, which consists of all conditions p in $\prod_\kappa \mathrm{Add}(\kappa, F(\kappa))$ such that $\{\kappa \mid p(\kappa) \neq \emptyset\}$ has bounded intersection with each inaccessible cardinal. The key point is that for any regular γ the Easton product factors as $\prod_{\kappa \geq \gamma^+}^E \mathrm{Add}(\kappa, F(\kappa)) \times \prod_{\kappa \leq \gamma}^E \mathrm{Add}(\kappa, F(\kappa))$, the first factor is γ^+-closed and the second factor is γ^+-cc (even after adding a generic for the first factor). The proof of the γ^+-cc for the second factor is just like the proof that $\mathrm{Add}(\gamma, \lambda)$ is γ^+-cc for any λ. Given this it follows that for any regular γ if an ordinal has cofinality at least γ^+ in the ground model then it still does after forcing with the Easton product; this is enough to conclude that cofinalities are preserved.

One also has to worry that the forcing relation is definable (something not automatic for class forcing), i.e. that the class of (p, σ, τ) such that p forces $\sigma \in \tau$ and the class of (p, σ, τ) such that p forces $\sigma = \tau$ are both L-definable classes. But this is easy because p forces $\sigma \in \tau$ or $\sigma = \tau$ in $\mathbb{P}$ iff $p \restriction \gamma$ forces this in $\mathbb{P} \restriction \gamma$, where γ is greater than the ranks of σ, τ, by the factoring property.

The rest follows as in the proof of the previous theorem. □

Do we really need to use the Easton product $\prod_\kappa^E \mathrm{Add}(\kappa, F(\kappa))$ instead of the full product $\prod_\kappa \mathrm{Add}(\kappa, F(\kappa))$? The following shows that there are problems even with the full product $\prod_\kappa \mathrm{Add}(\kappa, 1)$. A cardinal κ is *Mahlo* if it is inaccessible and $\{\alpha < \kappa \mid \alpha \text{ is inaccessible}\}$ is stationary in κ.

Theorem 4 *Suppose G is $\mathbb{P}$-generic over L where $\mathbb{P}$ denotes the full product $\prod_\kappa Add(\kappa, 1)$ of κ-Cohen forcings for regular κ. Suppose that κ is L-regular. Then $(\kappa^+)^{L[G]} = (\kappa^+)^L$ iff κ is not Mahlo in L.*

Proof. Let $G = (G(\beta) \mid \beta \text{ regular})$ be P-generic. For each $\alpha < \kappa$ consider $A_\alpha \subseteq \kappa$ defined by: $\beta \in A_\alpha \longleftrightarrow \alpha \in G(\beta)$.

Claim. Suppose κ is Mahlo. Then $\{A_\alpha \mid \alpha < \kappa\} \subseteq L$ but for no $\gamma < (\kappa^+)^L$ do we have $\{A_\alpha \mid \alpha < \kappa\} \subseteq L_\gamma$.

Proof of Claim. For any $\alpha < \kappa$ and condition p, we can extend p to q so that $\alpha < \beta < \kappa, \beta$ regular $\longrightarrow p(\beta)$ has length greater than α. Thus A_α is forced to belong to L.

Given $\gamma < (\kappa^+)^L$ and a condition p, define $f(\beta) = \text{length } (p(\beta))$ for regular $\beta < \kappa$. As κ is Mahlo, f has stationary domain and hence by Fodor's Theorem we may choose $\alpha < \kappa$ such that length $(p(\beta))$ is less than α for stationary many regular $\beta < \kappa$. Then p can be extended so that A_α is guaranteed to be distinct from the κ-many subsets of κ in L_γ. $\square$ *(Claim)*

Thus κ^+ is collapsed if κ is Mahlo. If κ is a successor cardinal then κ^+ is not collapsed as we can factor $\mathbb{P}$ as $\mathbb{P}(\geq \kappa^+) \times \mathbb{P}(\leq \kappa)$ where the first factor is κ^+-closed and the second is κ^+-cc. so assume that κ is a non-Mahlo limit cardinal and choose a CUB $C \subseteq \kappa$ consisting of cardinals which are not inaccessible. Suppose that $\langle D_\alpha \mid \alpha \in C \rangle$ is a definable sequence of dense classes. Given p we can successively extend $p(\geq \alpha^+), \alpha \in C$ so that $\{q \leq p \mid q, p \text{ agree } \geq \alpha^+, q \in D_\alpha\}$ is predense $\leq p$. There is no difficulty in obtaining a condition at a limit stage less than κ precisely because conditions are trivial at limit points of C. Thus we have shown that $\mathbb{P}(< \kappa) \times \mathbb{P}(> \kappa)$ preserves κ^+ as κ-many dense classes can be simultaneously reduced to predense subsets of size $< \kappa$. Finally $\mathbb{P} \simeq \mathbb{P}(< \kappa) \times \mathbb{P}(> \kappa) \times \mathbb{P}(\kappa)$ and $\mathbb{P}(\kappa)$ preserves κ^+ as it has size κ. $\square$

Remark. Cofinality-preservation does hold for the *thin* product, defined like the full product but with the requirement that for inaccessible $\kappa, \{\alpha < \kappa \mid p(\alpha) \neq \emptyset\}$ is *nonstationary* in κ. It also holds for the full product of the κ-Cohen forcings for *successor* cardinals κ.

Now we turn to the large cardinal result, in its original form due to Silver.

Theorem 5 *(Silver) Assume GCH and κ supercompact. Then we can force $2^\kappa > \kappa^+$ keeping κ measurable and preserving cofinaliities.*

Proof. The strategy of the proof is to start with $j : V \to M$ that witnesses the κ^{++}-supercompactness of κ and then lift j to an embedding $j^* : V[G] \to M[G^*]$ where G is generic for a forcing $\mathbb{P}$ which adds κ^{++}-many subsets of κ.

But if we use the Easton product, as in Easton's theorem, then we have a problem lifting j, as if A is generic over V for $\text{Add}(\kappa, 1)$ (and the Easton product will add many such A's), then $j^*(A)$ should be generic over M for

$\mathrm{Add}(j(\kappa),1)^M$, but this is impossible as $j^*(A)\cap\kappa = A$ does not belong to M.

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So instead we use the reverse Easton iteration of the $\mathrm{Add}(\alpha,\alpha^{++})$ forcings for regular $\alpha\leq\kappa$ and for technical reasons we in fact restrict to inaccessible $\alpha\leq\kappa$. This is the iteration $(P_\gamma,Q_\gamma)\mid\gamma\leq\kappa)$ where Q_γ is trivial unless γ is inaccessible, in which case Q_γ is a P_γ-name for $\mathrm{Add}(\gamma,\gamma^{++})$. When defining P_γ we take direct limits for regular γ and inverse limits otherwise (the effect of this is to ensure Easton support). The facts that this reverse Easton iteration $\mathbb{P}$ preserves cofinalities and kills the GCH at κ are established in a way that is very similar to the Easton product case.

It remains to show that κ is still measurable if it was originally supercompact. Let $j:V\to M$ witness κ^{++}-supercompactness (i.e. κ is the critical point and M is closed under κ^{++}-sequences) and suppose that G is $\mathbb{P}$-generic. We claim that we can lift j to $j^*:V[G]\to M[G^*]$ where G^* is generic over M for M's version $\mathbb{P}^*$ of the reverse Easton iteration $\mathbb{P}$. It's enough to find such a G^* so that $j[G]\subseteq G^*$, for then we obtain a lifting by defining $j^*(\sigma^G)$ to be $j(\sigma)^{G^*}$. We assume that $j:V\to M$ is given by an ultrapower, i.e. that every element of M is of the form $j(f)(j[\kappa^{++}])$ for some f in V with domain $[\kappa^{++}]^{<\kappa}=P_\kappa\kappa^{++}$.

V and M have the same $\mathrm{Add}(\kappa,\kappa^{++})$ so we can take G^* to equal G up to and including stage κ. On the interval $[\kappa^+,j(\kappa))$ we have only κ^{+++}-many dense sets (they are all represented by functions f from $P_\kappa\kappa^{++}$ to the powerset of V_κ), the forcing in this interval is κ^{+++}- closed and M is closed under κ^{++}- sequences; it follows that we can build G^* on this interval inside $V[G]$. We can do the same at stage $j(\kappa)$ but we have to worry that $G^*(j(\kappa))$ contains the j^*-image of $G(\kappa)$; but the forcing $\mathrm{Add}(\kappa,\kappa^{++})$ has only κ^{++}-many conditions so using the closure of M under κ^{++}-sequences we can find a single "master" condition in the forcing at stage $j(\kappa)$ which ensures that the generic will obey this requirement. So we have succeeded in building the desired G^* and the proof is complete. $\Box$

Remark. In fact supercompactness is preserved in the previous theorem, as the above argument works with κ^{++} replaced by any regular cardinal greater then κ^+. In fact with a little more work ("partial master conditions"), it can be shown that even κ^+- supercompactness is sufficient.

Internal consistency

A statement is *internally consistent* if it holds in an inner model, assuming the existence of large cardinals. Typically to prove the internal consistency of a statement from large cardinals one forces the statement to hold over a suitable inner model and then shows that it is possible to construct a generic for that forcing.

Theorem 6 *(with Ondrejović) Suppose that $0^\#$ exists. Then there is an inner model in which GCH fails at all regular cardinals.*

Proof. To prove his result, Easton forced over a model of GCH with the Easton product of $\mathrm{Add}(\alpha, \alpha^{++})$, α regular, to obtain a (class-) generic extension where GCH fails at all regulars. As in Silver's theorem, the Easton product cannot be used here:

Lemma 7 *Suppose that $0^\#$ exists and let κ denote ω_1^V. Then there is no generic over L for $Add(\kappa, 1)$, i.e. no κ-Cohen set over L.*

Proof of Lemma. The proof will work when κ is any indiscernible of uncountable V-cofinality. For any limit indiscernible i, any constructible subset A of i can be written as $t(\alpha, i, \vec{j})$ where t is an L-definable function, $\alpha < i$ and $\vec{j}$ is a finite increasing sequence of indiscernibles greater than i (the choice of $\vec{j}$ does not matter as long as it has the right length). Suppose that G were κ-Cohen over L. Then for any L-definable t and any $\beta < \kappa$ there is a condition p in G which meets all dense sets of the form $t(\alpha, \kappa, \vec{j})$ for $\alpha < \beta$, as the collection of such dense sets is constructible of size less than κ. So build a sequence of conditions $p_0, p_1, \ldots$ in G and indiscernibles $\alpha_0 < \alpha_1 < \cdots < \kappa$ such that p_{n+1} has length less than α_{n+1} and meets all dense sets of the form $t_n(\alpha, \kappa, \vec{j})$ for $\alpha < \alpha_n$. If p is the limit of the p_n's then p is a condition which meets all dense sets of the form $t(\alpha, \kappa, \vec{j})$ for some $\alpha < \alpha_\omega =$ the sup of the α_n's. But then p meets every constructible dense set D for α_ω-Cohen forcing, as any such D is of the form $t(\alpha, \alpha_\omega, \vec{j}) = t(\alpha, \kappa, \vec{j}) \cap (\mathbb{P} \restriction \alpha_\omega)$ for some t, some $\alpha < \alpha_\omega$ and indiscernibles $\vec{j}$ greater than κ. Therefore p is α_ω-generic over L. But this contradicts the fact that p is a condition in $\mathrm{Add}(\kappa, 1)$ and therefore constructible. $\Box$

So instead we use the reverse Easton iteration; however, as $\mathrm{Add}(\alpha, \alpha^{++}) * \mathrm{Add}(\alpha^{+}, \alpha^{+++})$ collapses α^{++}, we in fact need a reverse Easton iteration of ω-products

$$\prod_n \mathrm{Add}(\omega_n, \omega_{n+2}) * \prod_n \mathrm{Add}(\omega_{\omega+n+1}, \omega_{\omega+n+3}) * \cdots .$$

To build a generic G for this forcing $\mathbb{P}$, we build a generic $G(\leq i)$ for $\mathbb{P}(\leq i)$, the first $i+1$ stages of this iteration, by induction on the indiscernible i. To handle limit indiscernibles of uncountable cofinality we need to ensure the coherence property: $\pi_{ij}[G(\leq i)] \subseteq G(\leq j)$ for indiscernibles $i < j$, where $\pi_{ij} : L \to L$ has critical point i and sends i to j. The key inductive step is to ensure that $\pi_{ii^*}[G(\leq i)] \subseteq G(\leq i^*)$, where i^* is the least indiscernible greater than the indiscernible i. This is equivalent to requiring $G(< i) \subseteq G(< i^*)$ and $\pi^*_{ii^*}[G(i)] \subseteq G(i^*)$, where $\pi^*_{ii^*} : L[G(< i)] \to L[G(< i^*)]$ is the canonical extension of $\pi_{ii^*} : L \to L$.

It is not difficult to construct in ω steps a $\mathbb{P}(\leq i^*)$-generic $G'(\leq i^*)$ such that $G'(< i^*)$ includes $G(< i)$. The key step is to *modify* $G'(i^*)$ to a $G(i^*)$ which contains $\pi^*_{ii^*}[G(i)]$. The latter modification is performed by changing values of $G'(i^*)$ on the range of $\pi^*_{ii^*}$ to make it agree with $\pi^*_{ii^*}[G(i)]$. A key lemma states that if x belongs to L and has L-cardinality at most i^* then $x \cap \mathrm{Range}(\pi_{ii^*})$ belongs to L and has L-cardinality at most i. This enables us to verify the genericity of the modified $G'(i^*)$, using the homogeneity of the forcing $\mathrm{Add}(i^*, (i^*)^{++})$. □

5.-6.Vorlesungen

The characteristics $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$

We consider the partial order of eventual dominance for functions $f : \kappa \to \kappa$. We write $f \leq^* g$ iff $f(\alpha) \leq g(\alpha)$ for sufficiently large $\alpha < \kappa$. A family of functions B is *unbounded* if there is no g such that $f \leq^* g$ for all $f \in B$ and is *dominating* if for every g there is $f \in B$ such that $g \leq^* f$. The characteristics $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$ are the least sizes of unbounded and dominating families, respectively.

Lemma 8 *Assume as usual that κ is regular. Then:*
(a) $\kappa^+ \leq \mathfrak{b}(\kappa)$ and $\mathfrak{b}(\kappa)$ is regular.

(b) $\mathfrak{b}(\kappa) \le$ *cof*$(\mathfrak{d}(\kappa))$.
(c) $\mathfrak{d}(\kappa) \le 2^\kappa$.
(d) *cof*$(2^\kappa) > \kappa$.

Proof. (a) Given functions $(f_i \mid i < \kappa)$ we can (eventually) dominate them all by f where $f(i) = \sup_{j<i} f_j(i)$. So $\mathfrak{b}(\kappa)$ is greater than κ. If B is unbounded of size $\mathfrak{b}(\kappa)$ and $\mathrm{cof}(\mathfrak{b}(\kappa)) = \gamma < \mathfrak{b}(\kappa)$ then write B as the union of smaller subsets B_i, $i < \gamma$ and for each $i < \gamma$ choose f_i which dominates all functions in B_i and f which dominates each f_i; it follows that f dominates all of B, contradiction.
(b) If D is dominating of size $\mathfrak{d}(\kappa)$ then write D as the union of smaller D_i, $i < \mathrm{cof}(\mathfrak{d}(\kappa))$ and for each i choose f_i not dominated by D_i; then the set of f_i's is unbounded.
(c) is trivial and (d) is König's theorem. □

Theorem 9 *(Cummings-Shelah) Assume GCH and κ regular. Let β, δ and μ be cardinals such that β is regular, cof$(\mu) > \kappa$ and $\kappa^+ \le \beta \le$ cof(δ), $\delta \le \mu$. Then in a cofinality-preserving forcing extension we have* $(\mathfrak{b}(\kappa), \mathfrak{d}(\kappa), 2^\kappa)$ *equals* (β, δ, μ).

To prove this we choose a wellfounded partial order $\mathbb{P} = \mathbb{P}(\beta, \delta, \mu)$ of size μ with $\mathfrak{b}(\mathbb{P}) = \beta$ and $\mathfrak{d}(\mathbb{P}) = \delta$ (where $\mathfrak{b}(\mathbb{P})$ and $\mathfrak{d}(\mathbb{P})$ are defined for partial orders $\mathbb{P}$ in the obvious way) and then force a cofinal embedding from $\mathbb{P}(\beta, \delta, \mu)$ into $({}^\kappa\kappa, \le^*)$ preserving $\mathfrak{b}(\mathbb{P})$ and $\mathfrak{d}(\mathbb{P})$.

We obtain $\mathbb{P}(\beta, \delta, \mu)$ as follows. First define $\mathbb{P}_0$ to be $[\delta]^{<\beta}$ ordered by $x \le y$ iff $x \subseteq y$. Then $\{\{\alpha\} \mid \alpha < \beta\}$ is unbounded and as β is regular, any set of size less than β is bounded. Also our hypotheses that GCH holds and δ has cofinality at least β imply that the entire partial order has size δ and it is easy to see that no set of size less than δ is dominating. So $\mathfrak{b}(\mathbb{P}_0) = \beta$ and $\mathfrak{d}(\mathbb{P}_0) = \delta$, as desired. Now $\mathbb{P}_0$ is not wellfounded but to fix that simply choose a wellfounded, cofinal suborder $\mathbb{Q}$, by enumerating elements $(q_i \mid i < \lambda)$ of $\mathbb{P}_0$ so that no q_i is below any of the earlier q_j's until a cofinal set is reached; as $q_i < q_j$ implies $i < j$ this cofinal set $\mathbb{Q}$ is wellfounded. And it is easy to check that $\mathfrak{b}, \mathfrak{d}$ don't change when passing to a cofinal subset. Finally to guarantee a partial order of size μ, stick a copy of $(\mu, <)$ at the bottom.

So what is left to show is the following.

Lemma 10 *(Main Lemma) Assume GCH, κ regular and $\mathbb{R}$ a wellfounded partial order with $\mathfrak{b}(\mathbb{R}) \geq \kappa^+$. Then there is a cofinality-preserving forcing $\mathbb{D}(\kappa, \mathbb{R})$ which preserves $\mathfrak{b}(\mathbb{R})$ and $\mathfrak{d}(\mathbb{R})$ and forces that $\mathbb{R}$ can be cofinally embedded into $({}^\kappa\kappa, \leq^*)$.*

To prove this we use a nonlinear iteration of *Hechler forcing* $\mathbb{D}(\kappa)$. A condition in $\mathbb{D}(\kappa)$ is a pair (s, F) where $s : |s| \to \kappa$ for some $|s| < \kappa$ and $F : \kappa \to \kappa$. Conditions are ordered by: $(s, F) \leq (t, G)$ iff:

s extends t
$F(\alpha) \geq G(\alpha)$ for all $\alpha < \kappa$
$s(\alpha) \geq G(\alpha)$ for $|t| \leq \alpha < |s|$.

The effect of $\mathbb{D}(\kappa)$ is to add a generic function $g : \kappa \to \kappa$ such that $f \leq^* g$ for all $f : \kappa \to \kappa$ in V. Clearly $\mathbb{D}(\kappa)$ is κ-closed and κ^+-cc.

Now we prove the Main Lemma. Let $\mathbb{R}$ be wellfounded with $\mathfrak{b}(\mathbb{R}) \geq \kappa^+$. Let $\mathbb{R}^+$ denote $\mathbb{R}$ with one new element at the top of $\mathbb{R}$ and by induction on $a \in \mathbb{R}^+$ we define a forcing $\mathbb{P}_a$. For any $a \in \mathbb{R}^+$ let $\mathbb{R}_a$ denote $\{b \in \mathbb{R} \mid b <_{\mathbb{R}^+} a\}$. Then the conditions in $\mathbb{P}_a$ will be functions with domain contained in $\mathbb{R}_a$ and it will be the case that $b <_{\mathbb{R}^+} a$ and $p \in \mathbb{P}_a$ implies $p \restriction \mathbb{R}_b \in \mathbb{P}_b$. To ease notation we denote $<_{\mathbb{R}^+}$ simply by $<$.

Suppose that $\mathbb{P}_b$ is defined for $b < a$. Then p is a condition in $\mathbb{P}_a$ iff:

(i) p is a function whose domain is a size $< \kappa$ subset of $\mathbb{R}_a$.
(ii) For b in the domain of p, $p(b)$ is of the form $(t, \dot{F})$ where $t : |t| \to \kappa$ for some $|t| < \kappa$ and $\dot{F}$ is a $\mathbb{P}_b$-name such that $\mathbb{P}_b \Vdash \dot{F} : \kappa \to \kappa$.

Extension of conditions is defined by $p \leq q$ iff:

(iii) The domain of p contains the domain of q.
(iv) Suppose $q(b) = (t, \dot{H})$ and $p(b) = (s, \dot{G})$; then s extends t and $p \restriction \mathbb{R}_b \Vdash_{\mathbb{P}_b}$ $|t| \leq \alpha < |s| \to s(\alpha) \geq \dot{H}(\alpha)$.
(v) $p \restriction \mathbb{R}_b \Vdash_{\mathbb{P}_b} \dot{G}(\alpha) \geq \dot{H}(\alpha)$ for each $\alpha < \kappa$.

The desired forcing $\mathbb{D}(\kappa, \mathbb{R})$ is $\mathbb{P}_a$ where a is the new element at the top of $\mathbb{R}^+$. It is clear that $\mathbb{D}(\kappa, \mathbb{R})$ is κ-closed. Also $\mathbb{D}(\kappa, \mathbb{R})$ is κ^+-cc, as any set X of conditions of size greater than κ contains a subset Y of size κ^+ such that the domains of the conditions in Y form a Δ-system with some root r; then we can choose $Z \subseteq Y$, also of size κ^+, so that $p(b) = (s_b, \dot{F}_p(b))$ for all b in r, where s_b is independent of the choice of p. But then any two conditions in Z are compatible, so X could not have been an antichain.

Also note that just as for linear iterations, $b < a$ implies that the function $p \mapsto p \restriction \mathbb{R}_b$ is a projection from $\mathbb{P}_a$ onto $\mathbb{P}_b$. In particular any sentence mentioning only $\mathbb{P}_b$-names which is forced by $p \restriction \mathbb{R}_b$ in $\mathbb{P}_b$ is also forced by p in $\mathbb{P}_a$.

Let G be $\mathbb{D}(\kappa, \mathbb{R})$-generic and for each $a \in \mathbb{R}$ let f_a^G be the union of the first coordinates of the $p(a)$ for $p \in G$. We claim that $a \mapsto f_a^G$ is a cofinal embedding from $\mathbb{R}$ into the $({}^\kappa\kappa, \leq^*)$ of $V[G]$.

Claim 1. If $a < b$ then $f_a^G <^* f_b^G$.

Proof. Given any condition p we can extend p to q so that $q(b) = (s, \dot{F})$ where $\dot{F}$ is a $\mathbb{P}_b$-name for a function which strictly dominates the canonical $\mathbb{P}_b$-name for f_a^G. Then q forces that f_b^G strictly dominates f_a^G on a final segment. □

Claim 2. If $a \not< b$ then $f_a^G \not<^* f_b^G$.

Proof. We may assume $a \not\leq b$. Let p be a condition, $\alpha_0 < \kappa$ and choose $\alpha < \kappa$ to be larger than α_0 as well as the domain of $p(a)_0$, where $p(a)_0$ is the first component of $p(a)$. Obtain an extension q of p by extending $p \restriction (\mathbb{R}_b \cup \{b\})$ so that $q(b)_0$ has length greater than α (if necessary). Then as $a \not\leq b$, $q(a)_0 = p(a)_0$ and is therefore undefined at α. Then extend q to r so that $r(a)_0$ takes a value at α which is greater than $q(b)_0(\alpha)$. So we have shown that for each $\alpha_0 < \kappa$ it is dense for p to force $f_a^G(\alpha) > f_b^G(\alpha)$ for some $\alpha > \alpha_0$ and therefore $f_a^G \not<^* f_b^G$. □

Claim 3. In $V[G]$ the functions f_a^G, $a \in \mathbb{R}$, are cofinal in $({}^\kappa\kappa, \leq^*)$.

Proof. Let $f : \kappa \to \kappa$ belong to $V[G]$. Then by the κ^+-cc there is $X \subseteq \mathbb{R}$ of size κ such that f has a name which only mentions conditions with domain contained in X. As we have assumed that $\mathfrak{b}(\mathbb{R})$ is greater than κ we may choose $a \in \mathbb{R}$ which is an upper bound of X, from which it follows that f, which has a $\mathbb{P}_a$-name, is dominated by f_a^G. □

Finally, note that by the κ^+-cc, every set of ordinals in $V[G]$ of size less than $\beta = \mathfrak{b}(\mathbb{R})$ is covered by a set of ordinals of size less than β, as we have assumed that β is greater than κ. From this it follows that $\mathfrak{b}(\mathbb{R}) = \mathfrak{b}(\mathbb{R})^{V[G]}$. Similarly, with β replaced by $\delta = \mathfrak{d}(\mathbb{R})$, we obtain $\mathfrak{d}(\mathbb{R}) = \mathfrak{d}(\mathbb{R})^{V[G]}$. $\square$

7.-8.Vorlesungen

Global Cummings-Shelah

We have shown that the triple $(\mathfrak{b}(\kappa), \mathfrak{d}(\kappa), 2^\kappa)$ for a regular κ can be anything that follows the basic rules: $\kappa < \mathfrak{b}(\kappa) \leq \mathrm{cof}(\mathfrak{d}(\kappa))$, $\mathfrak{d}(\kappa) \leq 2^\kappa$ and $\mathfrak{b}(\kappa)$ regular, $\mathrm{cof}(2^\kappa) > \kappa$. Cummings-Shelah also gave a global version of this result:

Theorem 11 *Assume GCH and suppose that $\kappa \mapsto (\beta(\kappa), \delta(\kappa), \mu(\kappa))$ obeys the basic rules for each regular κ and in addition $\kappa \mapsto \mu(\kappa)$ is nondecreasing. Then there is a cofinality-preserving forcing extension in which $(\mathfrak{b}(\kappa), \mathfrak{d}(\kappa), 2^\kappa) = (\beta(\kappa), \delta(\kappa), \mu(\kappa))$ for all regular κ.*

This is harder than it looks. The obvious approach is to use an Easton product of forcings to arrange $(\mathfrak{b}(\kappa), \mathfrak{d}(\kappa), 2^\kappa) = (\beta(\kappa), \delta(\kappa), \mu(\kappa))$ for the different κ's. But this does not give the desired result at κ when the Easton product restricted to κ is not κ-cc, i.e., when κ is either the successor of a singular or is a non-Mahlo inaccessible. One can show in these cases that the Easton product restricted to κ adds κ^+-many κ-Cohen sets, forcing the bounding number at κ to be κ^+. (In the other cases, i.e. κ equal to ω, the successor of a regular or Mahlo, we do get the desired result because a κ-cc forcing of size at most 2^κ will not affect the values of $\mathfrak{b}(\kappa), \mathfrak{d}(\kappa)$ and 2^κ.) And we can't use the full reverse Easton iteration as that may collapse cardinals.

To solve this problem, Cummings-Shelah used a thinned-out reverse Easton iteration due to Magidor-Shelah called *the tail iteration*. To illustrate this, suppose that λ is singular, we have defined the iteration $\mathbb{P}_\lambda$ below λ and we want to define $\mathbb{P}_\lambda * \mathbb{Q}$ to ensure the right values of $(\mathfrak{b}(\lambda^+), \mathfrak{d}(\lambda^+), 2^{\lambda^+})$. The naive thing is to define the non-linear Hechler iteration $\mathbb{Q}$ as before, by defining $\mathbb{Q}_a$ for $a \in \mathbb{R}^+ = \mathbb{R} \cup \{top\}$ by induction on a to consist of partial functions q with domain a size at most λ subset of $\mathbb{R}_a$ such that for each b, $q(b) = (s, \dot{F})$ where $s : |s| \to \lambda^+$ for some $|s| < \lambda^+$ and $\dot{F}$ is a $\mathbb{Q}_b$-name for a function from λ^+ to λ^+. As before this will add a function g_b which dominates all functions in $V^{\mathbb{Q}_b}$ for each b, but in fact we need to dominate all functions in $V^{\mathbb{P}_\lambda * \mathbb{Q}_b}$. So it is tempting to simply allow $\dot{F}$ to be a $\mathbb{P}_\lambda * \mathbb{Q}_b$-name; but for this to make sense we also need allow s to come from $V^{\mathbb{P}_\lambda}$, which may force us to collapse cardinals.

The solution is to arrange inductively that for each $\alpha < \lambda$, the forcing $\mathbb{P}_\lambda$ is equivalent to a product $\mathbb{P}_{\alpha+1} \times \mathbb{P}^\alpha \restriction (\alpha+1, \lambda)$ for each $\alpha < \lambda$ where $\mathbb{P}^\alpha \restriction (\alpha+1, \lambda)$ is an α^+-closed iteration. Then instead of taking all $V^{\mathbb{P}_\lambda * \mathbb{Q}_b}$-names $\dot{F}$ we only take those which are *symmetric*, which means that $\dot{F}$ is forced to be equal to a $\mathbb{P}^\alpha \restriction (\alpha+1, \lambda)$-name for each $\alpha < \lambda$. We then have:

Lemma 12 *For $b < a$ in $\mathbb{R}^+$, any function from λ^+ to λ^+ in $V^{\mathbb{P}_\lambda * \mathbb{Q}_b}$ is dominated by the generic function g_a added by $\mathbb{Q}_a$ over $V^{\mathbb{P}_\lambda}$.*

Proof. It suffices to show that any function added by $\mathbb{P}_\lambda * \mathbb{Q}_b$ is dominated by one with a symmetric name. Suppose that $\dot{F}$ is an arbitary name. Now for each regular $\alpha < \lambda$ we can factor $\mathbb{P}_\lambda * \mathbb{Q}_b$ as $(\mathbb{P}_{\alpha+1} \times \mathbb{P}^\alpha \restriction (\alpha+1, \lambda)) * \mathbb{Q}_b$ and thereby regard $\dot{F}$ as a $\mathbb{P}_{\alpha+1}$-name in $V^{\mathbb{P}^\alpha \restriction (\alpha+1,\lambda) * \mathbb{Q}_b}$. Define:

$$G_\alpha(\beta) = \sup\{\gamma \mid p \Vdash \dot{F}(\beta) = \gamma \text{ for some } p \in \mathbb{P}_{\alpha+1}\}.$$

and regard this as a $\mathbb{P}_\lambda * \mathbb{Q}_b$-name. Finally, define the symmetric name $\dot{G}$ by $\dot{G}(\beta) = \sup_{\alpha<\lambda} G_\alpha(\beta)$. $\square$

The rest of the proof that we get the right values of $\mathfrak{b}(\lambda^+), \mathfrak{d}(\lambda^+)$ and 2^{λ^+} is now as before. The inaccessible cases are handled similarly, also using symmetric names.

$\mathfrak{b}(\kappa)$, $\mathfrak{d}(\kappa)$, $\mathfrak{c}(\kappa)$ *and large cardinals*

Using a technique of Laver we can easily arrange to realise any allowable values of $\mathfrak{b}(\kappa)$, $\mathfrak{d}(\kappa)$ and $\mathfrak{c}(\kappa)$ preserving supercompactness.

Lemma 13 *(Laver function) Suppose that κ is supercompact. Then there is a function $f : \kappa \to V_\kappa$ such that for every set x and every cardinal $\lambda \geq \kappa$ such that $x \in H_{\lambda^+}$ there is a $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$, $M^\lambda \subseteq M$ and $j(f)(\kappa) = x$.*

Proof. Assume that the result fails. For each $f : \kappa \to V_\kappa$ let λ_f be the least cardinal $\lambda \geq \kappa$ such that for some $x \in H_{\lambda^+}$, (λ, x) witnesses that f is not a *Laver function* for κ, i.e. $j(f)(\kappa) \neq x$ for every $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and $M^\lambda \subseteq M$. Let ν be greater than all of the λ_f's and let $j : V \to M$ witness the ν-supercompactness of κ.

Inductively define $f : \kappa \to V_\kappa$ as follows: If $f \restriction \alpha$ is not a Laver function for α then let λ be least so that (λ, x) witnesses this for some $x \in H_{\lambda^+}$ and choose $f(\alpha)$ to be such an x; otherwise set $f(\alpha) = 0$.

Now consider $x = j(f)(\kappa)$. By the definition of f and the elementarity of j, (λ_f^M, x) witnesses the failure of f to be a Laver function in M. As $M^\nu \subseteq M$, $\lambda_f^M = \lambda_f$ and (λ_f, x) also witnesses the failure of f to be a Laver function in V. This is a contradiction, as $j(\kappa) > \lambda_f$ and $j(f)(\kappa) = x$. □

Theorem 14 *(Laver preparation) Assume GCH and κ supercompact. Then in a set-forcing extension, GCH holds at and above κ and κ is supercompact in any further κ-directed closed forcing extension.*

Proof. Let $f : \kappa \to V_\kappa$ be a Laver function. We say that an inaccessible $\alpha < \kappa$ is *closed under* f if the range of f on α is contained in V_α. Perform a reverse Easton iteration $\mathbb{P}$ of length κ where at each inaccessible stage α closed under f, one chooses $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{Q}_\alpha$ where $\mathbb{Q}_\alpha$ is the $\mathbb{P}_\alpha$-name $f(\alpha)$ in case this is a name for an α-directed closed forcimg; $\mathbb{Q}_\alpha$ is trivial otherwise. Let G be $\mathbb{P}$-generic. We claim that κ is supercompact after forcing over $V[G]$ with any κ-directed closed forcing. To see this, suppose that $\mathbb{P}$ forces that $\dot{\mathbb{Q}}$ is κ-directed closed and let λ be any regular cardinal big enough so that $\dot{\mathbb{Q}}$ belongs to $H(\lambda^+)$. As f is a Laver function we can choose a λ-supercompactness embedding $j : V \to M$ with critical point κ so that $j(f)(\kappa) = \dot{\mathbb{Q}}$. We may assume that j is given by an ultrapower, which means that $M = \{j(g)(j[\lambda]) \mid g : [\lambda]^{<\kappa} \to V,\ g \in V\}$. Let H be $\dot{\mathbb{Q}}^G$-generic over $V[G]$; we show that j can be lifted to a λ-supercompactness embedding $j^* : V[G * H] \to M[G^* * H^*]$ in $V[G * H]$.

We describe how to find $G^* * H^*$. At stage κ we must choose $G^*(\kappa)$ to be $j(f)(\kappa)^G = \dot{\mathbb{Q}}^G$-generic so we take $G^*(\kappa)$ to be H. Between stages κ and $j(\kappa)$ we can build $G^*(\kappa, j(\kappa))$ in $V[G * H]$ as the forcing in this interval has only $\text{card}({}^{[\lambda]^{<\kappa}}V_\kappa) = \lambda^+$-many maximal antichains, is λ^+-closed and $M[G * H]$ is closed under λ-sequences in $V[G*H]$. We can also build a generic H^* at stage

$j(\kappa)$ but have to in addition ensure that H^* contains $j'[H]$, the pointwise image of H under the lifting $j' : V[G] \to M[G^*]$ of j. Here we invoke the κ-directed closure of the forcing $\dot{\mathbb{Q}}^G$ which yields the $j(\kappa)$-directed closure of the forcing at stage $j(\kappa)$. We need only note that that $j'[H]$ is an element of $M[G^*]$ by λ-supercompactness and is a directed set of size less than $j(\kappa)$, enabling us to choose a "master condition" q^* below all conditions in $j'[H]$; by choosing H^* to contain q^* as an element we ensure that it contains $j'[H]$ as a subset, finishing the proof. $\square$

Remark. "κ-directed closed" cannot be replaced by "κ-closed" in the above as the natural κ-closed forcing that adds a κ-Kurepa tree destroys the measurability of κ.

9.-10.Vorlesungen

Returning to the cardinal charcteristics $\mathfrak{b}(\kappa)$, $\mathfrak{d}(\kappa)$ and $\mathfrak{c}(\kappa)$ we now obtain:

Corollary 15 *Suppose that GCH holds, κ is supercompact and $\kappa < \beta \leq$ cof(δ) $\leq \delta \leq \mu$ with β regular and cof(μ) $> \kappa$. Then there is a forcing extension preserving cofinalities $\geq \kappa$ in which κ remains supercompact and $(\mathfrak{b}(\kappa), \mathfrak{d}(\kappa), \mathfrak{c}(\kappa))$ equals (β, δ, μ).*

Proof. First force as in Laver preparation; this will preserve cofinalities $\geq \kappa$ and the GCH $\geq \kappa$. Then apply the Cummings-Shelah forcing. As the latter is κ-directed closed, the supercompactness is preserved. $\square$

Remark. "Preserving cofinalities $\geq \kappa$" can be replaced by "preserving all cofinalities" in the previous result. To see this, prepare as before using the Laver function f but only force at an inaccessible stage $\alpha < \kappa$ if $f(\alpha)$ is a Cummings-Shelah forcing to control $\mathfrak{b}(\alpha)$, $\mathfrak{d}(\alpha)$ and $\mathfrak{c}(\alpha)$. This will preserve cofinalities and be a sufficient preparation to ensure that forcing with Cummings-Shelah at κ will preserve supercompactness.

We now ask: Do we really need supercompactness to control the above characteristics at a measurable? It is known that more than a measurable is needed to violate GCH at a measurable; a degree of "hypermeasurability" is required. Suppose that $\kappa \leq \lambda$ are cardinals. We say that κ is λ-*hypermeasurable* if it is the critical point of $j : V \to M$ where $H(\lambda) \subseteq M$. If GCH holds, κ is λ-hypermeasurable and $\lambda > \kappa^+$ is regular then we can force $2^\kappa = \lambda$ without changing cofinalities and keeping κ measurable (this is nearly optimal). Can we also control the above characteristics from hypermeasurability assumptions?

We begin with separating $\mathfrak{d}(\kappa)$ from $\mathfrak{c}(\kappa)$ for a measurable κ.

Theorem 16 *Assume GCH, $\kappa^+ < \lambda$, λ regular and κ λ-hypermeasurable. Then in a cofinality-preserving forcing extension, $\mathfrak{d}(\kappa) = \kappa^+$ and $2^\kappa = \lambda$.*

Proof. We will use a variant of *κ-Sacks forcing* to increase 2^κ. The advantage of κ-Sacks over κ-Cohen is that it is *κ^κ bounding*, i.e. any function $f : \kappa \to \kappa$ that it adds is dominated by such a function in the ground model.

For inaccessible α let Sacks(α) denote the following forcing. A condition is a subset T of $2^{<\alpha}$ (= the set of functions from an ordinal less than α into 2) such that:

1. $s \in T$, $t \subseteq s \to t \in T$.
2. Each $s \in T$ has a proper extension in T.
3. If $s_0 \subseteq s_1 \subseteq \cdots$ is a sequence in T of length less than α then the union of the s_i's belongs to T.
4. Let Split(T) denote the set of s in T such that both $s * 0$ and $s * 1$ belong to T. Then for some (unique) closed, unbounded $C(T) \subseteq \alpha$, $\text{Split}(T) = \{s \in T \mid \text{length}(s) \in C(T)\}$.

Extension is defined by $S \leq T$ iff S is a subset of T. For $i < \alpha$, the *i-th splitting level of T*, $\text{Split}_i(T)$, is the set of s in T of length α_i, where $\alpha_0 < \alpha_1 < \cdots$ is the increasing enumeration of $C(T)$. Sacks(α) is an α-closed forcing of size α^+. This forcing also preserves α^+, as it obeys the following *α-fusion property.* For $\beta < \alpha$ we write $S \leq_\beta T$ iff $S \leq T$ and $\text{Split}_i(S) = \text{Split}_i(T)$ for $i < \beta$.

α-fusion: Suppose that $T_0 \geq T_1 \geq \cdots$ is a descending sequence in Sacks(α) of length α and suppose in addition that $T_{i+1} \leq_i T_i$ for each i less than α. Then the intersection of the T_i, $i < \alpha$, is a condition in Sacks(α).

α-fusion implies that α^+ is preserved, as given a condition T_0 and a name $\dot{f} : \alpha \to \text{Ord}$, one can build a sequence as in the hypothesis of α-fusion so that T_i forces $\dot{f}(i)$ to belong to a set of size at most $2^i = i^+$; then the intersection of the T_i's forces that the range of $\dot{f}$ is covered by a set of size α. From this we can conclude that Sacks(α) preserves α^+ and is α^α-bounding.

Now consider the effect of κ-Sacks forcing on large cardinals. Assume GCH and let $j : V \to M$ be an elementary embedding with critical point κ and for the sake of illustration assume that j is just a measure ultrapower embedding. Now as we have seen before we cannot just force to add a new subset A of κ whose proper initial segments are in V and expect to lift the embedding j; indeed if j^* were such a lifting then $j^*(A)$ has a proper initial segment not in M, namely A, and therefore by elementarity A must have a proper initial segment not in V. So if we want Sacks(κ) to preserve the measurability of κ then we have to "prepare" by first forcing with Sacks(α) for measure-one many $\alpha < \kappa$.

So consider the reverse Easton iteration $\mathbb{P} = (\mathbb{P}_\alpha, \mathbb{Q}_\alpha) \mid \alpha \leq \kappa)$ were at inaccessible stages $\alpha \leq \kappa$ one forces with $\mathbb{Q}_\alpha = \text{Sacks}(\alpha)$. Let G be $\mathbb{P}$-generic and we show that $j : V \to M$ can be lifted in $V[G]$ to $j^* : V[G] \to M[G^*]$ where G^* is $j(\mathbb{P}) = \mathbb{P}^*$-generic over M. As $\mathbb{P}$ and $\mathbb{P}^*$ agree upto and including stage κ we can take $G^*(\leq \kappa)$ to equal $G(\leq \kappa)$. Between κ and $j(\kappa)$ we need to hit $\text{card}(j(\kappa)) = \kappa^+$-many maximal antichains for a κ^+-closed forcing in a model closed under κ-sequences; this is easy. So we have $G^*(< j(\kappa))$. The remaining question is: how do we define the generic $G^*(j(\kappa))$ for Sacks($j(\kappa)$) (in the sense of the model $M[G^*(< j(\kappa))]$) so as to contain each condition $j^*(T)$ where T is a κ-Sacks tree in $G(\kappa)$?

Actually there are exactly two choices for $G^*(j(\kappa))$. Recall that the splt-ting levels of a κ-Sacks tree T form a club Split(T) in κ. And by a density argument, these clubs can be arbitrarily thin, i.e. any club in κ will contain some $C(T)$, $T \in G(\kappa)$, as a subset. It follows that if we intersect the trees $j^*(T)$ for $T \in G(\kappa)$ then we get a tree whose only splitting levels lie in the intersection of the $j(C)$, C a club in κ.

Lemma 17 *The intersection of the $j(C)$ for C a club in κ consists of just the single ordinal κ.*

Proof. Any ordinal $\gamma < j(\kappa)$ is of the form $j(f)(\kappa)$ where $f : \kappa \to \kappa$. Let C be the set of closure points of f. Then if γ is greater than κ, as γ is not a closure point of $j(f)$ it follows that γ does not belong to $j(C)$. If γ is less than κ then γ does not belong to the club C of ordinals between γ and κ nor to $j(C)$. Finally, κ does belong to $j(C)$ for any club C in κ as it is the sup of $C = j(C) \cap \kappa$ and therefore a limit point of $j(C)$. $\square$

So we see that the trees $j^*(T)$ for T in $G(\kappa)$ intersect to a *tuning fork*, i.e. the union of two cofinal branches through $2^{<j(\kappa)}$ which split at κ. It can be shown that each of these two branches $b_0.b_1$ is Sacks$(j(\kappa))$-generic over $M[G^*(< j(\kappa))]$ and therefore we get two liftings of $j^* : V[G(< \kappa)] \to M[G^*(< j(\kappa))]$ to $j^{**} : V[G(\leq \kappa)] \to M[G^*(\leq j(\kappa))]$, obtained by choosing $G^*(j(\kappa))$ to be the generic consisting of all $j(\kappa)$-Sacks trees having b_i as a branch for $i = 0, 1$.

Tuning forks are nice, but sometimes a bit distracting, especially when considering products or iterations of Sacks(κ), so we introduce a variant Sacks$^*(\kappa)$ which reduces the two possible choices of lifting to just one.

For inaccessible α a condition in Sacks$^*(\alpha)$ is a subset T of $2^{<\alpha}$ such that:

1. $s \in T$, $t \subseteq s \to t \in T$.
2. Each $s \in T$ has a proper extension in T.
3. If $s_0 \subseteq s_1 \subseteq \cdots$ is a sequence in T of length less than α then the union of the s_i's belongs to T.
4. Let Split(T) denote the set of s in T such that both $s * 0$ and $s * 1$ belong to T. Then for some (unique) closed, unbounded $C(T) \subseteq \alpha$, Split$(T) = \{s \in T \mid \text{length}(s) \in C(T)$ and length(s) is singular$\}$.

Extension is defined as before: $S \leq T$ iff S is a subset of T. Thus the difference between Sacks(α) and Sacks$^*(\alpha)$ (= "Sacks(α) with singular splitting") is that the splitting levels are the *singular* elements of some club, One still has α-closure and fusion, and the forcing is α^α-bounding. But when performing the lifting argument, the intersection of the $j(\kappa)$-trees $j^*(T)$ for T a κ-tree in $G(\kappa)$ is now a single cofinal branch through $2^{<j(\kappa)}$ as κ is regular and therefore there is no splitting at level κ.

Now we outline the proof of Theorem 16. For inaccessible α let Sacks$^*(\alpha, \alpha^{++})$ denote the product of α^{++} copies of Sacks$^*(\alpha)$ with size α support. Thus a condition is a sequence $\vec{T} = \langle T(i) \mid i < \alpha^{++} \rangle$ whose support Supp$(\vec{T}) = \{i \mid T(i) \neq 2^{<\alpha}\}$ has size at most α, ordered componentwise. This forcing is again α-closed, and preserves α^+ via a suitable version of α-fusion, which we now describe. For $\beta < \alpha$ and $X \subseteq \alpha^{++}$ of size less than α, we write $\vec{T}_0 \leq_{\beta,X} \vec{T}_1$

iff $\vec{T}_0 \leq \vec{T}_1$ (i.e., $T_0(i) \leq T_1(i)$ for each $i < \alpha^{++}$) and in addition, for i in X, $T_0(i) \leq_\beta T_1(i)$.

Generalised α-fusion: Suppose that $\vec{T}_0 \geq \vec{T}_1 \geq \cdots$ is a descending sequence in Sacks$^*(\alpha, \alpha^{++})$ of length α and suppose in addition that $\vec{T}_{i+1} \leq_{i,X_i} \vec{T}_i$ for each i less than α, where the X_i's form an increasing sequence of subsets of α^{++} of size less than α whose union is the union of the supports of the $\vec{T}_i$'s. Then the $\vec{T}_i$'s have a lower bound in Sacks$^*(\alpha, \alpha^{++})$ (obtained by taking intersections at each component, using α-fusion).

Again this implies that α^+ is preserved, as given a condition $\vec{T}_0$ and a name $\dot{f} : \alpha \to \mathrm{Ord}$ one can build a sequence as in the hypothesis of generalised α-fusion so that $\vec{T}_i$ forces $\dot{f}(i)$ to belong to a set of size at most $(2^i)^\gamma < \alpha$ for some $\gamma < \alpha$; then a lower bound of the $\vec{T}_i$'s forces that the range of $\dot{f}$ is covered by a set in the ground model of size at most α. This also shows that the forcing is α^α-bounding.

Using the GCH at α, a Δ system argument shows that Sacks$^*(\alpha, \alpha^{++})$ is α^{++}-cc and therefore preserves α^{++}.

11.-12.Vorlesungen

Now force over our ground model V with the reverse Easton iteration of Sacks$^*(\alpha, \alpha^{++})$ for α inaccessible, $\alpha \leq \kappa$. Let G denote the generic for the first κ stages of this iteration and g the generic for the κ-th stage. Thus g is generic over $V[G]$ for Sacks$^*(\kappa, \kappa^{++})$ as defined in $V[G]$.

We would like to find a suitable generic over M for M's version of the above iteration. As M contains $H(\kappa^{++})^V$, the first $\kappa+1$ stages of the M and V iterations are the same, so we may use $G * g$ as our generic over M for the first $\kappa+1$ stages of the M-iteration. Next we want a generic H over $M[G][g]$ for the M-iteration between κ and $j(\kappa)$; given this we obtain a lifting of $j : V \to M$ to an embedding $j^* : V[G] \to M[G][g][H]$. In the case of lifting a measure ultrapower after an iteration of Sacks(α) for $\alpha \leq \kappa$ we built such an H using the fact that the forcing for which it must be generic had κ^+-many maximal antichains and was κ^+-closed in a model closed under κ-sequences; in the present context the collections of antichains we need to hit is the κ^+-union of sets in $M[G][g]$, the forcing is κ^{+++}-closed and $M[G][g]$ is closed under κ-sequences in $V[G][g]$. So again it is easy to build the desired generic H inside $V[G][g]$.

The last step is to find a generic h over $M[G][g][H]$ for the $j(\kappa)$-th stage of the M-iteration, where we force with the $\text{Sacks}^*(j(\kappa), j(\kappa^{++}))$ of $M[G][g][H]$. As g is a set of conditions in $\text{Sacks}^*(\kappa, \kappa^{++})$ of $V[G]$, $j^*[g]$ consists of a set of conditions in $\text{Sacks}(j(\kappa), j(\kappa^{++}))$ of $M[G][g][H]$. Below is the "tuning fork" picture we would get if we had used ordinary $\text{Sacks}(\alpha)$ instead of $\text{Sacks}^*(\alpha)$:

Lemma 18 *For $\alpha < j(\kappa^{++})$ let t be the intersection of the trees $j^*(p)(\alpha)$, p in g. If α belongs to the range of j, then t is a $(\kappa, j(\kappa))$-*tuning fork*, i.e., a subtree of $2^{<j(\kappa)}$ which is the union of two cofinal branches which split at κ. If α does not belong to the range of j, then t consists of exactly one cofinal branch through $2^{<j(\kappa)}$.*

But as we have used $\text{Sacks}^*(\alpha)$, our lemma simplifies to:

Lemma 19 *For $\alpha < j(\kappa^{++})$ let t be the intersection of the trees $j^*(p)(\alpha)$, p in g. Then t consists of exactly one cofinal branch $x(\alpha)$ through $2^{<j(\kappa)}$.*

The final step is to verify that these cofinal branches yield a generic:

Lemma 20 *Let h consist of all conditions p in $\text{Sacks}(j(\kappa), j(\kappa^{++}))$ of $M[G][g][H]$ such that for each $\alpha < j(\kappa^{++})$, $x(\alpha)$ is contained in $p(\alpha)$. Then h is generic for $\text{Sacks}^*(j(\kappa), j(\kappa^{++}))$ of $M[G][g][H]$ over $M[G][g][H]$ and contains $j^*[g]$.*

Thus we can lift the embedding $j^* : V[G] \to M[G][g][H]$ to an embedding $j^{**} : V[G][g] \to M[G][g][H][h]$, and this lifting is definable in $V[G][g]$. So $V[G][g]$ is a model where κ is measurable and the GCH fails at κ. Moreover as $\text{Sacks}^*(\kappa, \kappa^{++})$ is κ^κ-bounding, we have $\mathfrak{d}(\kappa) = \kappa^+$ in this model. □

Pushing up $\mathfrak{d}(\kappa)$ at a measurable κ

So far we have seen that with Sacks forcing at κ we can enlarge 2^κ for a measurable κ without increasing $\mathfrak{d}(\kappa)$. We next show that we can enlarge both $\mathfrak{d}(\kappa)$ and 2^κ simultaneously.

Theorem 21 *Assume GCH. Suppose that $\kappa^{++} \le \mu$ with μ regular and κ μ-hypermeasurable. Then without changing cofinalities we can force $\mathfrak{d}(\kappa) = \kappa^{++}$ and $2^\kappa = \mu$ keeping κ measurable.*

To prove this we use an iteration of a suitable form of κ-Miller forcing.

Let κ be inaccessible. The forcing Miller(κ) consists of $<\kappa$-closed subtrees of the tree $\kappa_{\uparrow}^{<\kappa}$ of *increasing* sequences in $\kappa^{<\kappa}$ with the property that every splitting node is club-splitting, every node can be extended to a club-splitting node and the limit of club-splitting nodes is club-splitting. We also require *continuous club-splitting*, which means that if s is a limit of club-splting nodes then the club witnessing club-splitting for s is the intersection of the clubs witnessing club-splitting for the club-splitting proper initial segments of s. Conditions are ordered by inclusion.

Miller(κ) is κ-closed. For any condition p we let Split(p) be the set of nodes in p which split in p. There is a natural bijection π_p between Split(p) and the full tree $\kappa^{<\kappa}$ which preserves the lexicographical ordering of nodes. For $\alpha < \kappa$ we let $\text{Split}_\alpha(p)$ be the inverse image of $\alpha^{\leq\alpha}$ under this bijection.

The point of this definition of $\text{Split}_\alpha(p)$ is that it facilitates fusion: For $q \leq p \in$ Miller(κ) and $\alpha < \kappa$ the notation $q \leq_\alpha p$ means that $\text{Split}_\alpha(p) = \text{Split}_\alpha(q)$. A sequence $(p_\alpha \mid \alpha < \kappa)$ of conditions in Miller(κ) is called a *fusion sequence*, iff

(*i*) If $\alpha \leq \beta$, then $p_\beta \leq p_\alpha$.
(*ii*) $p_{\alpha+1} \leq_\alpha p_\alpha$.
(*iii*) $p_\delta = \cap_{\alpha<\delta} p_\alpha$ for limit $\delta < \kappa$.

Then we have:

Lemma 22 *Let $(p_\alpha \mid \alpha < \kappa)$ be a fusion sequence. Then $q = \bigcap_{\alpha<\kappa} p_\alpha \in$ Miller(κ) and $q \leq_\alpha p_\alpha$ for all $\alpha < \kappa$.*

It follows from fusion that the range of any function $f : \kappa \to \text{Ord}$ added by Miller(κ) is covered by a set in the ground model of size κ and therefore Miller(κ) preserves κ^+ (and as it is κ-closed of size κ^+ it preserves all cofinalities). But unlike Sacks(κ), Miller(κ) is not κ^κ-bounding: if $f : \kappa \to \kappa$ belongs to V and p is a κ-Miller condition then using club-splitting we can extend p to force that the generic $f^{\dot{G}} : \kappa \to \kappa$ (i.e. the intersection of the κ-Miller trees in the generic $\dot{G}$) is not everywhere dominated by f.

Next we look at the effect that κ-Miller forcing has on a measurable cardinal κ. Assume GCH and that $j : V \to M$ is a measure ultrapower witnessing the measurability of κ. Force with the reverse Easton iteration $\mathbb{P} = ((\mathbb{P}_\alpha, \mathbb{Q}_\alpha) \mid \alpha \leq \kappa)$ where at inaccessible $\alpha \leq \kappa$, Q_α is Miller(α). Cofinalities are preserved. We want to verify that κ remains measurable by lifting j to $j^* : V[G] \to M[G^*]$ where G^* is $\mathbb{P}^* = j(\mathbb{P})$-generic over M.

As in the case of iterated Sacks(α), we can easily define the required G^* below stage $j(\kappa)$ by copying G up to and including stage κ and building a generic $G^*(\kappa, j(\kappa))$ between stages κ and $j(\kappa)$. And we can also build a generic $G^*(j(\kappa))$ at stage $j(\kappa)$, but the key is to ensure that it will contain the image of $G(\kappa)$ under the lifting j' of j to $V[G(<\kappa)] \to M[G^*(<j(\kappa))]$.

As in the Sacks* case we need to consider the intersection of the Miller($j(\kappa)$)-trees $j'(p)$ for p in the Miller(κ)-generic $G(\kappa)$. Clearly this intersection contains $f^{G(\kappa)} : \kappa \to \kappa$, the κ-Miller generic function equal to the intersection of the trees in $G(\kappa)$. In the Sacks case we could then infer that the analogous intersection also contains both $f^{G(\kappa)} * 0$ and $f^{G(\kappa)} * 1$, giving rise to a tuning fork. In the present situation we need to know that the intersection of the $j(\kappa)$-trees $j'(p)$, $p \in G(\kappa)$ contains some extension $f^{G(\kappa)} * \alpha$ of $f^{G(\kappa)}$, else this intersection will not yield a cofinal branch through $j(\kappa)^{<j(\kappa)}$.

But thanks to continuous club-splitting, we know that for any condition p in Miller(κ) and any node s of $j'(p)$ of length κ, the set of α such that $s * \alpha$ belongs to $j'(p)$ is a club equal to the intersection of clubs of the form $j'(C)$ where C is a club in κ; all such $j'(C)$ contain κ as an element and therefore $s * \kappa$ belongs to $j'(p)$. Taking p to range over $G(\kappa)$ and s to be $f^{G(\kappa)}$ we see that $f^{G(\kappa)} * \kappa$ belongs to $j'(p)$ for all $p \in G(\kappa)$ and therefore the intersection of these $j'(p)$ does indeed contain a (unique) node of length $\kappa + 1$.

Now as in the Sacks case we can argue that there are no splits in the intersection of the $j'(p)$, $p \in G(\kappa)$, and this intersection is a single branch through $j(\kappa)^{<j(\kappa)}$ of length $j(\kappa)$. Indeed this branch is Miller($j(\kappa)$)-generic over $M[G^*(<j(\kappa))]$.

13.-14.Vorlesungen

κ-Miller products

Recall κ-Miller forcing for an inaccessible κ: Miller(κ) consists of $<\kappa$-closed subtrees of the tree $\kappa_{\uparrow}^{<\kappa}$ of increasing sequences in $\kappa^{<\kappa}$ with the property that every splitting node is club-splitting, every node can be extended to a club-splitting node and the limit of club-splitting nodes is club-splitting. We also require *continuous club-splitting*, which means that if s is a limit of club-spltting nodes then the club witnessing club-splitting for s is the intersection of the clubs witnessing club-splitting for the club-splitting proper initial segments of s. Conditions are ordered by inclusion.

Miller(κ) is κ-closed and there is a suitable form of κ-fusion: For any condition p we let Split(p) be the set of nodes in p which split in p. There is a natural bijection π_p between Split(p) and the full tree $\kappa^{<\kappa}$ which preserves the lexicographical ordering of nodes. For $\alpha < \kappa$ we let $\text{Split}_\alpha(p)$ be the inverse image of $\alpha^{\leq\alpha}$ under this bijection. For $q \leq p \in$ Miller(κ) and $\alpha < \kappa$ the notation $q \leq_\alpha p$ means that $\text{Split}_\alpha(p) = \text{Split}_\alpha(q)$. A sequence $(p_\alpha \mid \alpha < \kappa)$ of conditions in Miller(κ) is a *fusion sequence* iff it is decreasing, continuous at limits and $p_{\alpha+1} \leq_\alpha p_\alpha$ for each $\alpha < \kappa$. Fusion sequences have greatest lower bounds.

It follows from fusion that Miller(κ) preserves κ^+ (and indeed all cofinalities, assuming GCH in the ground model). Unlike Sacks(κ), Miller(κ) is not κ^κ-bounding. Thus our strategy to push up $\mathfrak{d}(\kappa)$ at a measurable κ will be to use a κ-support product of κ-Miller forcings. (We use κ-Miller instead of κ-Cohen to facilitate the preservation of measurability.)

At first this seems a bad idea, as it is known that when κ equals ω, the full support product of ω-many ω-Miller = Miller forcings collapses ω_1. In fact, this product includes the full support product of ω-many Cohen forcings and:

Proposition 23 *The full support product of ω-many Cohen forcings collapses the continuum to ω.*

Proof. Think of the Cohens as functions from ω to ω and write the generic as $(f_0.f_1,\ldots)$. Then for each finite k consider the function g_k which at argument n takes the value $f_n(f_{n-1}(\ldots(f_1(f_0(k)))\ldots))$ and define $h_k : \omega \to 2$ by $h_k(n) = 1$ iff $g_k(n)$ is even. Then any ground model function from ω to 2 is equal to some h_k. $\square$

However:

Proposition 24 *Assume GCH at κ and let κ be (uncountable and) inaccessible. Then the full support product of κ-many κ-Cohen forcings preserves cofinalities.*

Proof. It suffices to show that κ^+ is preserved. Let P be the fully-supported product of κ-many κ-Cohen forcings and let P_0 be the $< \kappa$-supported product of κ-many κ-Cohen forcings. Let p be a condition in P which forces that $\dot{f}$ is a function from κ into Ord.

Write $q \leq_\alpha p$ iff $q \leq p$ and $q = p$ on the first α components, i.e., if $p = (p(i)|i < \kappa)$, $q = (q(i)|i < \kappa)$ then $q(i) = p(i)$ for all $i < \alpha$. Obviously we have fusion for these relations: If $\vec{p} = (p_i|i < \kappa)$ is a fusion sequence ($i < j \rightarrow p_j \leq p_i$, $p_{i+1} \leq_i p_i$ for all i and glb's are taken at limits) then $\vec{p}$ has a greatest lower bound.

Now build a fusion sequence $\vec{p} == (p_i|i < \kappa)$ as follows. $p_0 = p$ and if p_i is defined then consider all all conditions s in P_0 with support contained in $i \times i$. There are fewer than κ-many such s by the inaccessibility of κ. Choose $p_{i+1} \leq_i p_i$ so that for each $j < i$ and each such s, if s is compatible with p_i then s has an extension s^* with support contained in $i \times \kappa$, such that $s^* \cup p_{i+1}$ forces a value for $\dot{f}(j)$. This easy to do using $< \kappa$ closure.

Let p^* be the result of the fusion. We claim that for any $j < \kappa$, p^* forces that $\dot{f}(j)$ takes one of the values that appeared in the above construction (there are only κ-many such values). Indeed, suppose that q extends p^* and forces a value for $\dot{f}(j)$. Choose $i < \kappa$ so that j is less than i and $s = q \restriction i$ has support contained in $i \times i$. This is possible as κ is uncountable. Clearly s is compatible with p_i so s has an extension s^* with support contained in $i \times \kappa$ such that $s^* \cup p_{i+1}$ forces a value for $\dot{f}(j)$, one of the values that appeared in the above construction. But $s^* \cup p_{i+1}$ is compatible with q so the value that q forces for $\dot{f}(j)$ must be the same as the value forced by $s^* \cup p_{i+1}$. $\square$

Remark. It is not hard to see that the above argument works for any uncountable κ provided $\diamondsuit_\kappa$ holds (for then at stage i we need only consider a single s with support contained in $i \times i$). It follows that under GCH the result holds for all uncountable κ with the possible exception of ω_1 when $\diamondsuit$ fails.

Thus in fact there is no obvious obstruction to cofinality-preservation for a κ-supported κ-Miller product for an (uncountable) inaccessinle κ and indeed:

Proposition 25 *Assume GCH at κ and above. Let κ be inaccessible and δ a cardinal of cofinality greater than κ. Then the κ-support product of δ-many copies of κ-Miller forcing preserves cofinalities and forces $\mathfrak{b}(\kappa) = \kappa^+ \leq \mathfrak{d}(\kappa) = \delta = 2^\kappa$.*

Proof. By a Δ-system argument, the forcing has the κ^{++}-cc so for cofinality-preservation one only needs to check that κ^+ is preserved. But any function $f : \kappa \to \mathrm{Ord}$ is added by a size κ subproduct of the full product and therefore it is sufficient to verify that the full support product of κ-many copies if κ-Miller forcing preserves κ^+. The proof of this is similar to the proof that the iteration preserves κ^+, making use of clubs as in the proof of the previous proposition. An easy counting argument yields $2^\kappa = \delta$.

Suppose that $\mathcal{F}$ is a family of fewer than δ functions from κ to κ in the generic extension $V[G]$. The $\mathcal{F}$ belongs to $V[G_0]$ for some restriction of G to fewer than δ of its components; let i be a component missing from G_0. The $G(i)$ is κ-Miller generic over $V[G_0]$ and therefore not dominated by any function in $V[G_0]$; it follows that $\mathcal{F}$ is not dominating, so $\mathfrak{d}(\kappa) = \delta$. It is also known that a κ-Miller product does not add a dominating function and therefore the functions of the ground model witness that $\mathfrak{b}(\kappa) = \kappa^+$. □

Finally, by combining the proofs of large cardinal preservation for (products of) κ-Sacks and κ-Miller forcing, we obtain:

Theorem 26 *Assume GCH. Suppose that κ is μ-hypermeasurable and $\kappa < \delta \leq \mu$ with δ and μ cardinals of cofinality greater than κ. Then without changing cofinalities we can force $\kappa^+ = \mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa) = \delta$ and $2^\kappa = \mu$ keeping κ measurable.*

Proof. Let $j : V \to M$ witness the μ-hypermeasurability of κ. First assume that for some functions $f_0, f_1 : \kappa \to \kappa$ we have $j(f_0)(\kappa) = \delta$ and $j(f_1)(\kappa) = \mu$ (we will show later how to arrange this). Now prepare with a reverse Easton iteration where at inaccessible stage $\alpha < \kappa$ we force with $(\alpha\text{-Miller})^{f_0(\alpha)} \times (\alpha\text{-Sacks})^{f_1(\alpha)}$, both products with size α support. At stage κ we then force with $(\kappa\text{-Miller})^\delta \times (\kappa\text{-Sacks})^\mu$. The fact that j can be lifted to $V[G]$ where G is generic is proved by combining the analogous proofs for the κ-Sacks and κ-Miller forcings. Note that $j(f_0)(\kappa) = \delta$ and $j(f_1)(\kappa) = \mu$ guarantee that the forcing on the M-side at stage κ is the same as the forcing on the V-side at stage κ.

As we have forced with a κ-Sacks product of size μ we get $2^\kappa = \mu$. Write the $(\kappa\text{-Miller})^\delta \times (\kappa\text{-Sacks})^\mu$-generic G as $G_0 \times G_1$. Then any function from κ to κ in $V[G]$ is dominated by one in $V[G_0]$, using fusion for the product together with the fact that $(\kappa\text{-Sacks})^\mu$ is κ^κ-bounding. It follows that in $V[G]$, $\mathfrak{d}(\kappa) \leq \delta$ and by the argument of Proposition 25 we get $\mathfrak{d}(\kappa) = \delta$.

Finally to arrange for the existence of the functions f_0, f_1 first perform a reverse Easton iteration where at inaccessible stage $\alpha \leq \kappa$ one adds a pair of α-Cohen functions from α to α. It is easy to lift the embedding and moreover we can choose the generic functions $j^*(f_0), j^*(f_1)$ at stage $j(\kappa)$ on the M-side to take the values δ, μ respectively at κ, as desired. $\square$

Pushing up $\mathfrak{b}(\kappa)$ keeping κ measurable without invoking supercompactness is difficult and it may be that core model theory implies that more than hypermeasurability is required.

On the almost disjointness number at κ

For a regular cardinal κ, $\mathfrak{a}(\kappa)$ denotes the least size of a maximal almost disjoint family of subsets of κ, where we always assume that our families have size at least κ, their elements have size κ and "almost disjoint" means disjoint with fewer than κ exceptions. Clearly $\kappa^+ \leq \mathfrak{a}(\kappa) \leq 2^\kappa$.

Proposition 27 *$\mathfrak{b}(\kappa) \leq \mathfrak{a}(\kappa)$.*

Proof. Using a bijection between $\kappa \times \kappa$ and κ we can choose a MAD family $(A_i \mid i < \mathfrak{a}(\kappa))$ of almost disjoint subsets of $\kappa \times \kappa$. Moreover we can assune that the A_i for $i < \kappa$ are disjoint with union all of $\kappa \times \kappa$; then with a suitable permutation of $\kappa \times \kappa$ we can arrange that A_i is just the i-th column $\{i\} \times \kappa$ for $i < \kappa$.

Now for each $j \geq \kappa$ there is a function f_j so that the intersection of A_j with $A_i =$ the i-th column is contained in $\{i\} \times f_j(i)$. If $\mathfrak{a}(\kappa)$ were less than $\mathfrak{b}(\kappa)$ then we could choose an f eventually dominating each f_j; but then the graph of f is almost disjoint from each A_i, contradicting maximality. $\square$

In the case $\kappa = \omega$, $\mathfrak{a} < \mathfrak{d}$ is possible: Kunen showed that it holds after adding $\aleph_2$ (or more) Cohen reals. Shelah obtained the consistency of $\mathfrak{d} < \mathfrak{a}$ but in his model $\mathfrak{d}$ is greater than ω_1. This led Roitman to ask if $\omega_1 = \mathfrak{d} < \mathfrak{a}$ is possible and this was recently verified by Mildenberger.

Surprisingly:

Theorem 28 *(Blass-Hyttinen-Zhang) For uncountable κ, if $\mathfrak{d}(\kappa)$ equals κ^+ then so does $\mathfrak{a}(\kappa)$.*

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Proof. The assumption $\mathfrak{d}(\kappa) = \kappa^+$ implies that we can choose a $<^*$-increasing and cofinal sequence $(f_i \mid i < \kappa^+)$ of increasing functions from $\kappa \to \kappa$ where $<^*$ denotes strict dominance on a final segment.

We build a family $(S_i \mid i < \kappa^+)$ of subsets of κ by induction on i, using the f_i's; our desired MAD family will consist of the S_i for $i < \kappa$ together with the unbounded S_i for i of cofinality ω. To facilitate the construction we use bijections $G_i : \kappa \to i$ for $i \in (\kappa, \kappa^+)$ as well as a club-guessing sequence:

Fact. We can assign cofinal ω-sequences C_i to $i \in (\kappa, \kappa^+)$ of cofinality ω such that for any club C in κ^+ there is some i with $C_i \subseteq C$.

Let the S_i for $i < \kappa$ partition κ into unbounded pieces. For $i \in (\kappa, \kappa^+)$ of cofinality ω we proceed as follows. Let D_i be the club of $\alpha < \kappa$ such that for each $j \in C_i$, the range of G_j on α equals the intersection of j with the range of G_i on α. Then we take S_i to consist of all $\gamma < \kappa$ such that for $j < i$ if γ is greater than f_i applied to the least element of D_i greater than $G_i^{-1}(j)$ then γ does not belong to S_j. Clearly S_i is almost disjoint from each S_j, $j < i$; we need to show that the unbounded S_i's form a maximal AD family.

Suppose that there were an unbounded subset X of κ which is almost disjoint from each S_i. Define a continuous sequence $\kappa = \alpha_0 < \alpha_1 < \cdots$ of length κ^+ as follows. If α_j is defined then for $\beta < \kappa$ let $h_{j+1}(\beta)$ be the sup of $X \cap S_{G_{\alpha_j}(\beta)}$ and choose α_{j+1} to be a cofinality ω ordinal greater than α_j so that $h_{j+1} <^* f_{\alpha_{j+1}}$.

Apply club-guessing to the club C consisting of the α_j's to obtain a cofinality ω ordinal α such that $C_\alpha \subseteq C$. We now shrink X. First remove all elements which belong to S_α. Next fix a $\delta < \kappa$ such that whenever α_i belongs to C_α, f_α strictly dominates h_{i+1} everywhere above δ. Then for each β in C_α remove all elements of X that belong to some S_j with $j < \beta$ and $G_\beta^{-1}(j) < \delta$.

We have removed only fewer than κ elements from X; let γ be an element of X that remains.

Since γ remains, it does not belong to S_α. So for some $j < \alpha$ we have $\gamma \in S_j$ and f_α applied to the least element of D_α greater than $G_\alpha^{-1}(j)$ is less than γ.

Fix an i such that $j < \alpha_i \in C_\alpha$. Since $j < \alpha_i$ and $\gamma \in S_j$, the fact that γ remains implies that $G_{\alpha_i}^{-1}(j) \geq \delta$. By the choice of δ we get $h_{i+1}(G_{\alpha_i}^{-1}(j)) < f_\alpha(G_{\alpha_i}^{-1}(j))$. And by definition of h_{i+1}, $h_{i+1}(G_{\alpha_i}^{-1}(j))$ is at least all elements of $X \cap S_j$ and in particular at least γ; so $\gamma < f_\alpha(G_{\alpha_i}^{-1}(j))$. On the other hand, because γ does not belong to S_α, we know that f_α applied to the least element of D_α greater than $G_\alpha^{-1}(j)$ is less than γ. As f_α is increasing, the least element β of D_α greater than $G_\alpha^{-1}(j)$ is less than $G_{\alpha_i}^{-1}(j)$. So β is an element of D_α such that $G_\alpha^{-1}(j)$ is smaller than β but $G_{\alpha_i}^{-1}(j)$ is greater than β; this contradicts the definition of D_α. □

As mentioned earlier, Kunen showed that $\mathfrak{a} < \mathfrak{d}$ is possible: the proof generalises nicely to all regular κ.

Theorem 29 *Assume GCH and let κ be regular, μ a cardinal of cofinality greater than κ. Then after adding μ-many κ-Cohen reals (via a product with supports of size $< \kappa$) one has $\mathfrak{a}(\kappa) = \kappa^+ \leq \mu = \mathfrak{d}(\kappa) = 2^\kappa$.*

Proof. The fact that $\mu = \mathfrak{d}(\kappa) = 2^\kappa$ in the extension is standard and uses the fact that κ-Cohen forcing is not κ^κ-bounding. And to show that $\mathfrak{a}(\kappa) = \kappa^+$ in the extension it suffices to build a MAD family $\mathcal{F}$ of subsets of κ in V that stays maximal after adding just one κ-Cohen set, as any subset of κ added by a κ-Cohen product is added by a single κ-Cohen forcing. We say that $\mathcal{F}$ is *κ-Cohen indestructible.*

We build $\mathcal{F} = \{A_i \mid i < \kappa^+\}$ inductively. Let $(\sigma_i \mid \kappa \leq i < \kappa^+)$ be a list of all nice κ-Cohen names for subsets of κ. Let $\{A_i \mid i < \kappa\}$ be any partition of κ into disjoint unbounded pieces. Suppose that A_j has been defined for $j < i$ where $\kappa \leq i$ and we now define A_i. Let $((p_k, j_k, \alpha_k) \mid k < \kappa)$ enumerate all triples (p, j, α) where p is a κ-Cohen condition, j is less than i and α is less than κ. For each $k < \kappa$, if p_k forces that σ_i is bounded or has unbounded intersection with A_{j_l} for some $l < k$ then ignore k. Otherwise choose an extension of p_k which forces a specific bound on all of the $\sigma_i \cap A_{j_l}$, $l < k$

and a specific ordinal β_k which is greater than α_k as well as this bound to belong to σ_i; put β_k into A_i. Note that β_k does not belong to A_{j_l} for $l < k$ and therefore $A_i = \{\beta_k \mid k < \kappa\}$ is almost disjoint from each A_j, $j < i$.

We now verify that $\mathcal{F}$ is MAD after adding a κ-Cohen set. Suppose that σ is a nice κ-Cohen name for a subset of κ and p is a condition forcing that σ is unbounded but almost disjoint from each A_i, $i < \kappa^+$. Choose i so that $\sigma = \sigma_i$. Then when forming A_i we considered triples of the form (q, j, α) where q is a κ-Cohen condition, j is less than i and α is less than κ. In case q extends p, q does not force that $\sigma = \sigma_i$ is bounded or has unbounded intersection with A_j, so we ensured that q had an extension forcing a specific ordinal greater than α to belong to σ, which we then put into A_i. Thus for each $\alpha < \kappa$ it is dense below p to force that an ordinal greater than α belongs to $\sigma \cap A_i$, which by genericity means that p forces that σ has unbounded intersection with A_i, contradiction. □

Remark. Assuming GCH it is also possible to construct MAD families at κ which stay maximal after forcing with a κ-Sacks product (of any size). However when κ is uncountable this follows from the Blass-Hyttinen-Zhang theorem, as a κ-Sacks product is κ^κ-bounding.

17.-18.Vorlesungen

The ultrafilter number $\mathfrak{u}(\kappa)$

An ultrafilter on a regular κ is *uniform* if all of its elements have size κ. The characteristic $\mathfrak{u}(\kappa)$ is the smallest size of a base for a uniform ultrafilter on κ.

Theorem 30 *(Garti-Shelah) Assume GCH, κ is supercompact and cof(γ) $> \kappa$. Then in a cofinality-preserving forcing extension, $\mathfrak{u}(\kappa) = \kappa^+$ and $2^\kappa = \gamma$.*

Proof. κ-Mathias filter forcing $\mathbb{M}(\kappa, F)$ for a filter F on κ is defined as follows: Conditions are pairs (s, X) where s is a bounded subset of κ and X belongs to F; $(t, Y) \leq (s, X)$ iff t end-extends s, Y is contained in X and $t \setminus s$ is contained in X. This forcing is κ-closed and κ^+-cc. We will only use it in case F is in fact a uniform and normal ultrafilter on κ.

Now perform an iteration $(\mathbb{P}_i, Q_i) \mid i < \gamma^+)$ where at stage i, Q_i is the sum of all forcings $\mathbb{M}(\kappa, U)$ for U a normal uniform ultrafilter on κ. (The supercompactness of κ will enable us to ensure that there are such ultrafilters at each stage $i < \gamma^+$.) The sum means that a condition is either trivial at i or chooses an ultrafilter U_i as well as a condition in the associated $\mathbb{M}(\kappa, U_i)$. The generic will provide $\mathbb{M}(\kappa, U_i)$-generics x_i for "generically chosen" ultrafilters U_i. We require that the choices of ultrafilters U_i are made with supports of size less than γ^+ and that the choices of conditions in the associated $\mathbb{M}(\kappa, U_i)$'s are made with supports of size less than κ. For $\alpha < \gamma^+$ it is dense in $\mathbb{P}_\alpha$ to specify the U_i for all $i < \alpha$ and below any condition in this dense set the forcing $\mathbb{P}_\alpha$ is an iteration with supports of size less than κ of κ-centered forcings and therefore κ^+-cc. As $\mathbb{P}$ (and each $\mathbb{P}_\alpha$) is κ-directed closed it follows that $\mathbb{P}_\alpha$ preserves cofinalities.

Lemma 31 *Suppose that $p \in \mathbb{P} = \mathbb{P}_{\gamma^+}$ forces that $\dot{U}$ is an ultrafilter on κ. Then for some $\alpha < \gamma^+$ there is an extension q of p forcing that $\dot{U}_\alpha$ (the canonical name for the ultrafilter chosen by the generic at stage α) is the intersection of $\dot{U}$ with the model $V[\dot{G}_\alpha]$. Even more, we can ensure that q forces this for a set S of $\alpha < \gamma^+$ of ordertype κ^+ and that the intersection of $\dot{U}$ with the model $V[\dot{G}_{\sup S}]$ belongs to this model.*

Proof. Assume that $p \in \mathbb{P}_{\alpha_0}$ is *full*, meaning that it specifies U_i for each $i < \alpha_0$. Then $\mathbb{P}_{\alpha_0} \restriction p$ is κ^+-cc with a dense subset of size at most γ, so γ-many $\mathbb{P}_{\alpha_0} \restriction p$-names suffice to name each subset of κ added by $\mathbb{P}_{\alpha_0} \restriction p$. Each condition has its "ultrafilter part" where U_i's are specified and its "Mathias part", where conditions in the $\mathbb{M}(\kappa, U_i)$'s are specified. Extend $p = p_0$ on its ultrafilter part to a full p_1 (leaving the Mathias part unchanged) so that p_1 together with some Mathias part decides whether or not the first name σ_0 for a subset of κ belongs to $\dot{U}$. Then extend p_1 on its ultrafilter part to a full p_2 tö make the same decision with a Mathias part that is incompatible with the previously chosen Mathias part. Continue extending on the ultrafilter part deciding whether or not σ_0 belongs to $\dot{U}$ with an antichain of different Mathias parts until a maximal antichain is reached (after fewer than κ^+ steps). If the resulting condition is called q_1 and has length (support) $\alpha_1 < \gamma^+$ then the set of conditions in $\mathbb{P}_{\alpha_1} \restriction q_1$ which decide whether or not σ_0 belongs to $\dot{U}$ is predense in $\mathbb{P} \restriction q_1$. Repeat this γ times for each possible $\mathbb{P}_{\alpha_0} \restriction p$-name for a subset of κ until reaching a q_2 with the same property for all such names. Finally, repeat all of this for $\mathbb{P}_{\alpha_2} \restriction q_2$-names for subsets of κ arriving at a condition q_3, for $\mathbb{P}_{\alpha_3} \restriction q_3$-names for subsets of κ arriving at a condition q_4 and so on for κ^+ steps. If q is the final condition of length α then for any

$\mathbb{P}_\alpha \restriction q$-name σ for a subset of κ the set of conditions in $\mathbb{P}_\alpha \restriction q$ which decide whether or not σ belongs to $\dot{U}$ is predense in $\mathbb{P} \restriction q$. It follows that if $G = G_\mathbb{P}$ is generic containing q then $\dot{U}^G$ intersect $V[G_\alpha]$ is determined by G_α and therefore is a normal ultrafilter U_α on κ in $V[G_\alpha]$. Now extend q once more to length $\alpha + 1$ by choosing $\dot{U}_\alpha$ to be a name for U_α and q is as desired. For the "even more" conclusion, simply repeat the above κ^+ times. (The property that $\dot{U}^G$ intersect $V[G_\alpha]$ is determined by G_α is inherited at the supremum using the fact that this supremum has cofinality greater than κ.) □

Let $j : V \to M$ be an ultrapower embedding witnessing the γ^+-supercompactness of κ and as before we may assume that we have a function $f : \kappa \to \kappa$ such that $j(f)(\kappa) = \gamma^+$. Then prepare below κ with the reverse Easton iteration $\mathbb{Q}$ where at inaccessible stages α closed under f, Q_α is the length $f(\alpha)$ iteration of α-Mathias filter forcings for generically chosen normal ultrafilters on α. Then stage κ of the iteration $j(\mathbb{Q})$ is the forcing $\mathbb{P}$ above (as defined in the model $V^\mathbb{Q}$). Let $G_\mathbb{Q}$ be $\mathbb{Q}$-generic over V and let $G_\mathbb{P}$ be $\mathbb{P}$-generic over $V[G_\mathbb{Q}]$. Then we can lift $j : V \to M$ to $j_0^* : V[G_\mathbb{Q}] \to M[G_\mathbb{Q}][G_\mathbb{P}][H]$ where $G_\mathbb{Q} * G_\mathbb{P} * H$ is generic over M for $\mathbb{Q}^* = j(\mathbb{Q})$, and then lift again to $j^* : V[G_\mathbb{Q}][G_\mathbb{P}] \to M[G_\mathbb{Q}][G_\mathbb{P}][H][G_{\mathbb{P}^*}]$ where $\mathbb{P}^* = j_0^*(\mathbb{P})$. The first lifting uses the fact that (by preparation) stage κ of $j(\mathbb{Q})$ is indeed the forcing $\mathbb{P}$, enabling us to copy the generic $G_\mathbb{P}$ to the M-side and that we can build the generic H simply listing the relevant maximal antichains in a γ^{++}-sequence and using the γ^{++}-closure of the forcing. The second lifting is a matter of constructing $G_{\mathbb{P}^*}$ using a similar listing of maximal antichains but this time starting below a (master) condition in $\mathbb{P}^*$ which extends every condition of the form $j_0^*(p)$ for $p \in G_\mathbb{P}$. The obvious choice for the master condition is the greatest lower bound p_0^* of $j_0^*[G_\mathbb{P}]$. This condition has support $j[\gamma^+]$ and for each $i < \gamma^+$ chooses the filter-name $\dot{U}^*_{j(i)}$ to be $j_0^*(\dot{U}_i)$ as well as a $j(\kappa)$-Mathias name with first component x_i, the κ-Mathias generic added by $G_\mathbb{P}$ at stage i of the iteration $\mathbb{P}$. However we choose the stronger master condition p^* defined as follows. For $i < \gamma^+$ let $G_{\mathbb{P}_i}$ denote the restriction of the generic $G_\mathbb{P}$ to $\mathbb{P}_i$.

$(*)$ If $i < \gamma^+$ and for each $A \in U_i$ there is a $\mathbb{P}_i$-name σ such that $A = \sigma^{G_{\mathbb{P}_i}}$ and a condition $p \in G_{\mathbb{P}_i}$ such that $j_0^*(p)$ forces that κ belongs to $j_0^*(\sigma)$, then $p^*(j(i))$ is obtained from $p_0^*(j(i))$ by replacing the first component x_i of its $j(\kappa)$-Mathias name by $x_i \cup \{\kappa\}$; otherwise p^* is the same as p_0^*.

Lemma 32 *(a) p^* as defined above is an extension of p_0^* therefore serves as*

a master condition.
(b) If G_{P^*} is chosen to contain p^*, j^* is the resulting lifting of j_0^* and U is the normal ultrafilter on κ derived from j^* then whenever U_i is contained in U we have that x_i is a member of U.

Proof. (a) For $i < \gamma^+$ let p_i^* be defined like p^* but only replacing $x_{j(k)}$ by $x_{j(k)} \cup \{\kappa\}$ for (appropriate) $k < i$. We show by induction on i that p_i^* extends each $j_0^*(p)$, $p \in G_{\mathbb{P}}$ and therefore extends p_0^*. The base case and limit cases are trivial. Suppose that the claim holds for i and we wish to verify it for $i+1$; without loss of generality we may assume that the hypothesis of $(*)$ is met for i. Let $G_{\mathbb{P}^*_{j(i)}}$ be any generic containing $p_i^* \restriction j(i)$ and extend it to a generic $G_{\mathbb{P}^*}$ containing p_i^*. By induction $G_{\mathbb{P}^*}$ also contains p_0^* and therefore yields a lifting j^* of j_0^*. Any $p \in G_{\mathbb{P}}$ can be extended in $G_{\mathbb{P}}$ so that the κ-Mathias condition it specifies at i is of the form (s, A) where s is an initial segment of x_i and A belongs to U_i. Apply $(*)$ at i to infer that $A = \sigma^{G_{\mathbb{P}_i}}$ where $j_0^*(q)$ for some q in $G_{\mathbb{P}_i}$ forces that κ belongs to $j_0^*(\sigma)$. But $j_0^*(q)$ belongs to $G_{\mathbb{P}^*_{j(i)}}$ (as p_0^* belongs to $G_{\mathbb{P}^*}$) and therefore κ does belong to $j_0^*(\sigma)^{G_{\mathbb{P}^*_{j(i)}}} = j^*(A)$. It follows that the $j(\kappa)$-Mathias condition specified by $p_{i+1}^*(j(i))^{G_{\mathbb{P}^*_{j(i)}}}$ with first component $x_i \cup \{\kappa\}$ does extend $(x_i, j^*(A)) = (x_i, j_0^*(\sigma)^{G_{\mathbb{P}^*_{j(i)}}}) \leq (s, j_0^*(\sigma)^{G_{\mathbb{P}^*_{j(i)}}})$. We have shown that $p_i^* \restriction j(i)$ forces $p_{i+1}^*(j(i))$ to extend $(s, j_0^*(\sigma)) = j_0^*(p)(j(i))$ and therefore p_{i+1}^* extends $j_0^*(p)$.
(b) If U_i is contained in U then then κ belongs to $j^*(A)$ for all $A \in U_i$ which implies that the hypothesis of $(*)$ is satisfied at i. It then follows that κ belongs to $j^*(x_i)$ and therefore x_i belongs to U. □

Now we finish the proof. To ease notation write G for $G_{\mathbb{P}}$ and G_i for $G_{\mathbb{P}_i}$, the restriction of G to $\mathbb{P}_i$. By the lemma there is a subset S of γ^+ of ordertype κ^+ such that for i in $S \cup \{\sup S\}$, the restriction of U, the measure on κ derived from j^*, to the model $V[G_i]$ belongs to $V[G_i]$ and if i belongs to S then this is the ultrafilter U_i^G chosen by G at stage i. Moreover by our choice of master condition the κ-Mathias generics x_i^G chosen by G at stages $i \in S$ belong to U. Now let α be the supremum of S and consider the model $V[G_\alpha]$. Then the restriction of U to this model is a normal ultrafilter in this model which is generated by the $x_i^G \in U$ for $i \in S$ and therefore $\mathfrak{u}(\kappa)$ is κ^+ in this model. Also we may assume that α is greater than γ and therefore we have $2^\kappa = \gamma$ in this model as well. □

19.Vorlesung

The *tower number* $\mathfrak{t}(\kappa)$ for a regular κ is the least length of a *tower*, i.e. a sequence $(A_i \mid i < \lambda)$ of unbounded subsets of κ such that $i < j \to A_j \subseteq^* A_i$ and $\mathrm{cof}(\lambda) \geq \kappa$, with no *pseudo-intersection*, i.e. no $A \subseteq^* A_i$ for all $i < \lambda$.

By an easy diagonalisation, $\mathfrak{t}(\kappa)$ is at least κ^+.

Proposition 33 $\mathfrak{t}(\kappa) \leq \mathfrak{b}(\kappa)$.

Proof. We just prove this for uncountable κ. Choose a $<^*$-increasing and $<^*$-cofinal sequence $(f_i \mid i < \mathfrak{b}(\kappa))$. For $i < \mathfrak{b}(\kappa)$ let C_i be the club of ordinals closed under f_i. By thinning out $(f_i \mid i < \mathfrak{b}(\kappa))$ we can assume that the C_i's are $\subseteq^*$-decreasing. If the C_i's had a pseudo-intersection A then consider $f : \kappa \to \kappa$ which increasingly enumerates the successor points of A. Then for each i, $f_i <^* f$ as for large enough $\alpha < \kappa$, $\alpha < f(\alpha)$ is a closure point of f_i and therefore $f_i(\alpha) < f(\alpha)$. This contradicts the assumption that $(f_i \mid i < \mathfrak{b}(\kappa))$ is $<^*$-cofinal. $\square$

Theorem 34 *(Shelah-Spasojević) Assume κ is ω or $\diamondsuit_\kappa$ holds. Then $\mathfrak{t}(\kappa)$ is at most the least λ such that $2^\kappa < 2^\lambda$.*

This is easy when κ is ω: Suppose $\kappa < \mathfrak{t} = \mathfrak{t}(\omega)$. Define A_σ for σ a $0,1$-sequence of length at most κ by induction on the length of σ as follows: If A_σ is defined then $A_{\sigma*0}, A_{\sigma*1}$ split A_σ into two infinite pieces, and for σ of limit length use the fact that this length is less than $\mathfrak{t}$ to choose a pseudo-intersection. We have shown that 2^κ, the number of such σ's, is at most 2^ω. The proof for uncountable κ using $\diamondsuit_\kappa$ is harder.

Using Theorem 34 we prove:

Theorem 35 *(Shelah-Spasojević) Assume GCH and $\kappa \leq \lambda \leq \mu \leq \theta$ are regular. Then in a cofinality-preserving extension by a κ-closed forcing, $\mathfrak{t}(\kappa) = \lambda \leq \mathfrak{b}(\kappa) = \mu \leq 2^\kappa = \theta$.*

Proof. Begin by adding a κ-Cohen set. This forces a $\diamondsuit_\kappa$ sequence, which will remain a $\diamondsuit_\kappa$ sequence after any further κ-closed forcing. Then add θ^+ subsets of λ using a λ-Cohen product. Now perform an iteration of length $\theta \cdot \mu < \theta^+$ with $< \kappa$-support as follows:

At stages not of the form $\theta \cdot \xi$, $\xi < \mu$, towers of length less than λ are killed; via bookkeeping, all such towers added by the iteration will be killed at some stage, yielding $\mathfrak{t}(\kappa) \geq \lambda$. To kill a tower $(A_i \mid i < \eta)$ we use the forcing whose conditions are pairs (a, x) where a is a bounded subset of κ and x is a subset of η of size less than κ; (b, y) extends (a, x) iff b contains a, y contains x and for i in x, $b \setminus a$ is contained in A_i. This adds a pseudo-intersection for $(A_i \mid i < \eta)$ and is both κ-centered and κ-closed.

At stages of the form $\theta{\cdot}\xi$, $\xi < \mu$, a dominating function $f_\xi : \kappa \to \kappa$ is added by κ-Hechler forcing, which is also κ-centered and κ-closed. At the end of the iteration we have $\mathfrak{b}(\kappa) = \mu$. Cofinalities are preserved by standard arguments. Finally, note that at the end of the iteration we also have $2^\kappa = 2^\eta = \theta$ for $\kappa \leq \eta < \lambda$ as well as $\diamondsuit_\kappa$, so by Theorem 34 we get $\mathfrak{t}(\kappa) \leq \lambda$. $\square$