Invariant Descriptive Set Theory

1.-2.Vorlesungen

Introduction

Gao's book is a clear exposition of the major results concerning the structure of the class of analytic equivalence relations on the reals under Borel reducibility. As isomorphism on the countable models of a sentence of $L_{\omega_1\omega}$ is an analytic equivalence relation, this theory has important connections with infinitary model theory. But even beyond isomorphism of countable models there are important classification problems which can also be studied in this way, such as notions of equivalence for metric spaces, homeomorphisms of the unit square, unitary operators, complex manifolds, knots and indecomposable continua.

In this course I have a rather specific goal, which is to see which aspects of the classical theory of Gao's book can be extended from the Baire space ω^ω to generalised Baire space κ^κ, where κ is a regular uncountable cardinal. (We make the additional assumption that $\kappa^{<\kappa}$ has size κ, which is necessary for a good notion of Borel set for κ^κ.) As a result, I will not emphasize certain concepts, such as that of a *Polish space*, which are central to the classical theory but clearly have no analogue for κ^κ as there are no metrics on such a space. On the other hand a few of the basic results about Baire category extend nicely. Sometimes an important result from the classical theory is only consistent for generalised Baire space (such as the Burgess dichotomy), and sometimes it is just false (such as the Glimm-Effros dichotomy). Nevertheless I will prove a number of results that do not generalise for the sake of gaining a better understanding of the differences between the classical and generalised theories.

Baire Category

Let κ be an infinite cardinal satisfying $\kappa^{<\kappa} = \kappa$. We allow the case $\kappa = \omega$. The Baire space κ^κ consists of functions $\eta : \kappa \to \kappa$ using basic open sets

$$U_\sigma = \{\eta \mid \eta \text{ extends } \sigma\}$$

where σ belongs to $\kappa^{<\kappa}$ (i.e., σ is a function froms some ordinal less than κ into κ). We can think of the points in κ^κ as the cofinal branches through the tree obtained by ordering the elements of $\kappa^{<\kappa}$ by extension. A point worth noting is that only if κ is ω or inaccessible will the levels of this tree have size less than κ; for example, if $\kappa = \omega_1$ (and CH holds) then the infinite levels of this tree have size ω_1.

The Borel sets for κ^κ are obtained by closing the basic open sets under complements and unions of size κ. Note that this class contains all open sets, thanks to the assumption $\kappa^{<\kappa} = \kappa$. (Indeed without this assumption, open sets need not be Borel.)

A subset of κ^κ is *nowhere dense* if its closure contains no nonempty open set and is *meager* if it is the union of κ-many nowhere dense sets. It has the *Baire property (or property of Baire)* if its symmetric difference from some open set is meager.

The entire space is not meager, thanks to:

Theorem 1 *(Baire Category Theorem) The intersection of κ-many open dense sets is dense.*

Proof. Suppose that D_i, $i < \kappa$ are open dense and let U_σ be a basic open set. Build a κ-sequence $\sigma = \sigma_0 \subseteq \sigma_1 \subseteq \cdots$ where $U_{\sigma_{i+1}}$ is contained in D_i and $\sigma_\lambda = \cup_{i<\lambda}\sigma_i$ for limit $\lambda < \kappa$. This is possible as each D_i is open dense. Then $\eta = \cup_{i<\kappa}\sigma_i$ belongs to each D_i. □

Theorem 2 *Borel sets have the Baire property.*

Proof. It suffices to show that the collection of sets which have the Baire property contains the basic open sets and is closed under complements and size κ unions. The fact that it contains the basic open sets is trivial and as any closed set differs by a meager set from its interior, it is also closed under complements. The case of κ-unions follows from the fact that the union of κ-many meager sets is meager. □

Note that in L there is a Δ^1_1 wellorder of the subsets of κ for uncountable κ; from this it is easy to construct a Δ^1_1 set without the property of Baire and therefore in L there are Δ^1_1 sets which are not Borel. Actually this last fact is true in general (we prove this below).

An important result in the classical theory (I'll give a modern proof below) is that analytic sets have the Baire property. This however is false for uncountable κ:

Theorem 3 *(Halko-Shelah) Suppose $\kappa > \omega$ and let X be the set of $\eta \in \kappa^\kappa$ such that $\eta(i) = 0$ for all i in some closed unbounded subset of κ: Then X does not have the property of Baire.*

Proof. Otherwise choose a basic open set U_σ on which X is either meager or comeager (i.e. either $X \cap U_\sigma$ is meager or $U_\sigma \setminus X$ is meager). Suppose that it is comeager on U_σ and choose sets D_i, $i < \kappa$ which are open dense subsets of U_σ with intersection contained in X. But we can build a sequence $\sigma = \sigma_0 \subseteq \sigma_1 \subseteq \cdots$ so that $U_{\sigma_{i+1}}$ is contained in D_i and for limit λ, σ_λ is an extension of $\cup_{i<\lambda}\sigma_i$ with value 1 at λ. Then the union of the σ_i, $i < \kappa$, clearly does not belong to X but does belong to each D_i, $i < \kappa$, contradiction. If we instead require $\sigma_\lambda = 0$ for limit λ then we obtain something in X belonging to each D_i, verifying that X is not meager on U_σ. □

For $\kappa = \omega$ we have:

Theorem 4 *(a) Suppose that $X \subseteq \omega^\omega$ is Σ^1_1. Then X has the property of Baire.*
(b) It is consistent that some Δ^1_2 set does not have the property of Baire.

Proof. (a) It suffices to show that any basic open set U_σ has a basic open subset U_τ on which X is either meager or comeager (i.e., either $X \cap U_\tau$ is meager or $U_\tau \setminus X$ is meager). For given this, the union of the basic open sets on which X is either meager or comeager is open dense and therefore has meager complement, so X differs by a meager set from the union of the basic open sets on which it is comeager.

Let M be a countable transitive model of ZFC$^-$ containing the defining parameter for X. Let φ be a Σ^1_1 formula defining X and suppose that g is Cohen generic over M and belongs to U_σ. If $\varphi(g)$ is true then it is also true in $M[g]$ by Σ^1_1 absoluteness; then we can choose a Cohen condition τ extending σ forcing $\varphi(\dot{g})$ and any Cohen generic h over M extending τ will satisfiy φ in $M[h]$ and therefore in V. But the set of reals extending τ which are Cohen over M is comeager in U_τ and so X is comeager on U_τ. If $\varphi(g)$ is false then by the same argument (using just the easy direction of Σ^1_1 absoluteness) we get that X is meager on some U_τ contained in U_σ.
(b) In Gödel's model there is a Δ^1_2 wellorder of the reals and using this one can construct a Δ^1_2 set without the Baire property. □

Notice the use of Σ^1_1 absoluteness in the proof of (a) above. The reason the proof does not generalise to κ is due to this failure of absoluteness; indeed there could be a set in $M[g]$ which does not contain a club in $M[g]$ but does in V (where M now has size κ and g is generic for κ-Cohen over M).

However for uncountable κ it is consistent that Δ^1_1 sets have the Baire property. Note that as the collection of Δ^1_1 sets contains all basic open sets and is closed under κ-unions and complements, it follows that every Borel set is Δ^1_1. As mentioned above, the converse only holds if κ equals ω:

Theorem 5 *(a) (Luzin Separation) Suppose $\kappa = \omega$. If X, Y are disjoint Σ^1_1 sets then there is a Borel set B which contains X and is disjoint from Y. Therefore every Δ^1_1 set is Borel.*
(b) Suppose that κ is uncountable. Then not every Δ^1_1 set is Borel.

Proof. (a) We say that X, Y are *Borel separable* if the conclusion of (a) holds. Note that if $X = \cup_{i<\kappa} X_i$ and $Y = \cup_{i<\kappa} Y_i$ and X_i, Y_j are Borel separable for each i, j then X, Y are Borel separable: Just take $\cup_i \cap_j B_{ij}$ where B_{ij} Borel separates X_i, Y_j. Write X, Y as the ranges of continuous functions f, g on the Baire space (we assume that X, Y are nonempty). For each σ let X_σ, Y_σ be $f[U_\sigma], g[U_\sigma]$. Assuming that $X, Y = X_\emptyset, Y_\emptyset$ are not Borel separable we can inductively define x, y such that $X_{x|n}, Y_{y|n}$ are Borel inseparable for each n. Now $f(x) \neq g(y)$ so we can separate them with some open sets U, V. But then by the continuity of f, g we have separated $X_{x|n}, Y_{y|n}$ by U, V for large enough n, contradiction.
Remark: Note that the above proof breaks down for $\kappa > \omega$ as one cannot claim that $X_{x|n}, Y_{y|n}$ are Borel inseparable when n is infinite.
(b) The key point is that for uncountable κ, the wellfoundedness of a binary relation on κ is a closed property: R is wellfounded iff $R \cap (\alpha \times \alpha)$ is wellfounded for each $\alpha < \kappa$. Now each Borel set B is coded as $B(T)$ where T is a wellfounded tree of finite sequences whose terminal nodes are labelled with basic open sets and whose nonterminal nodes are labelled with with $\sim$ or $\cup$ (use the labels to assign a Borel set to each node of the tree; $B(T)$ is the Borel set assigned to the top node). The relation $\eta \in B(T)$ is a Δ^1_1 relation of η and T. It follows that there is a Δ^1_1 set $U(\eta, \nu)$ which is universal for Borel sets in the sense that for each η, U_η is Borel and each Borel set is of this form for some η. But then U is not Borel, else by diagonalisation, $\{\eta \mid \text{not } U(\eta, \eta)\}$ would give a contradiction. □

Theorem 6 *Let κ be regular and assume GCH. Then after forcing with $Add(\kappa, \kappa^+)$ (the forcing which adds κ^+-many κ-Cohens), every Δ^1_1 set has the Baire property.*

Proof. Let G be generic for $\text{Add}(\kappa, \kappa^+)$ and let X be Δ^1_1 in $V[G]$. Assuming that X is Δ^1_1 with parameter in V we'll show that X has the property of Baire; the general case follows from the fact that any subset of κ belongs to $G \cap \text{Add}(\kappa, \alpha)$ for some $\alpha < \kappa^+$ and $\text{Add}(\kappa, \kappa^+)$ factors as $\text{Add}(\kappa, \alpha) \times \text{Add}(\kappa, [\alpha, \kappa^+))$, the second component of this product being isomorphic to $\text{Add}(\kappa, \kappa^+)$.

We show that any basic open set U_σ contains a basic open subset U_τ on which X is either meager or comeager. Let φ, ψ be Σ^1_1 formulas (with parameter in V) defining X and the complement of X, respectively. We may assume that $G(0)$, the first κ-Cohen added by G, extends σ (if not, then change it below the length of σ so that it does). Suppose that $G(0)$ satisfies φ. Note that $V[G]$ is an extension of $V[G(0)]$ via the κ-closed forcing $\text{Add}(\kappa, [1, \kappa^+))$. We claim that $V[G(0)]$ is Σ^1_1 elementary in $V[G]$ and therefore $\varphi(G(0))$ holds in $V[G(0)]$: Indeed, suppose that φ is Σ^1_1 with parameter in $V[G(0)]$ and let T be a tree in $V[G(0)]$ on $\kappa \times \kappa$ such that cofinal branches through T correspond to pairs (x, w) where w witnesses that $\varphi(x)$ holds. Suppose that $\dot{b}$ is an $\text{Add}(\kappa, [1, \kappa^+))$-name for a branch through T; then we can build a branch through T in $V[G(0)]$ by forming a κ-sequence of conditions $p_0 \geq p_1 \geq \cdots$ deciding initial segments of $\dot{b}$. So if φ has a solution in $V[G]$ it also has one in $V[G(0)]$.

Now let τ be a κ-Cohen condition extending σ which forces $\varphi(\dot{g})$ where $\dot{g}$ denotes the κ-Cohen generic. Let M be a transitive model of ZFC^- of size κ which contains all bounded subsets of κ such that τ forces $\varphi(\dot{g})$ in M. The subsets of κ which are κ-Cohen over M form a comeager set on U_τ and if x is κ-Cohen over M extending τ then $M[x]$ and therefore $V[G]$ satisfies $\varphi(x)$. We have shown that X is comeager on U_τ. If $G(0)$ satisfies ψ, the Σ^1_1 formula that defines the complement of X, then we have shown that X is meager on U_τ. □

3.-4.Vorlesungen

Two useful facts concering Baire category do generalise nicely from classical Baire space to the generalised setting: the Kuratowski-Ulam and Mycielski theorems. As usual let κ denote an infinite cardinal such that $\kappa^{<\kappa} = \kappa$.

Theorem 7 *(Kuratowski-Ulam) Let X denote the generalised Baire space κ^κ and suppose that $A \subseteq X \times X$ has the Baire property. For each $x \in X$ let A_x denote $\{y \mid (x,y) \in A\}$. Then:*
(a) $\{x \mid A_x$ has the Baire Property$\}$ is comeager.
(b) A is meager iff $\{x \mid A_x$ is meager$\}$ is comeager (it follows that A is comeager iff $\{x \mid A_x$ is comeager$\}$ is comeager).

Proof. First suppose that A is open dense and we show that A_x is open dense for comeager-many x. Clearly A_x is open for each x, so we just have to show that A_x is dense for comeager-many x. Let $(V_i \mid i < \kappa)$ be a basis for the topology on X. Then for each i, $U_i = \{x \mid (x,y) \in A$ for some $y \in V_i\}$ is dense open since if W is nonempty and open, $A \cap (W \times V_i)$ is nonempty by the density of A. Thus for $x \in \cap_i U_i$, $A_x \cap V_i$ is nonempty for each i, i.e. A_x is dense. So we have shown that A_x is dense for all x in a comeager set.

It follows that if A is meager then A_x is meager for comeager-many x, which is the direction $\rightarrow$ of (b). To prove (a), suppose that A has the Baire property and choose an open U and meager M so that $A = U \triangle M$. Then for each x, $A_x = U_x \triangle M_x$ and M_x is meager for comeager-many x. It follows that A_x has the Baire property for comeager-many x, which is (a).

Finally we verify the direction $\leftarrow$ of (b). Suppose that A has the property of Baire and is not meager; we show that $\{x \mid A_x$ is not meager$\}$ is not meager. Write $A = U \triangle M$ where U is a nonempty open set and M is meager. U contains $V_0 \times V_1$ where V_0, V_1 are nonempty open sets. For x in V_1, if M_x is meager then $A_x = U_x \triangle M_x$ is comeager on a nonempty open set and therefore not meager. As M_x is meager for comeager-many x, it follows that the set of x such that A_x is not meager is comeager on a nonempty open set and therefore not meager. □

A subtree T of $\kappa^{<\kappa}$ is *perfect* if the limit of any increasing sequence of nodes of T of length less than κ is also a node of T (T is κ-closed) and every node of T has a splitting extension in T. T is *Sacks-perfect* if in addition the limit of any increasing sequence of splitting nodes of T of length less than κ is a splitting node of T. A subset of κ^κ is *perfect* (*Sacks-perfect*) if it consists of the κ-branches through a perfect (Sacks-perfect) subtree T of $\kappa^{<\kappa}$.

Theorem 8 *(Mycielski) Assume that κ is regular and either $\diamondsuit_\kappa$ holds, κ is inaccessible or $\kappa = \omega$. Suppose that E is a meager binary relation on* *generalised Baire space κ^κ. Then there is a Sacks-perfect set A such that $E(x, y)$ fails for all distinct x, y in A.*

Corollary 9 *Assume GCH, κ is regular and $\kappa \neq \omega_1$. Then the conclusion of the above Theorem holds.*

Proof. Write E as the union of an increasing κ-sequence $E_0 \subseteq E_1 \subseteq \cdots$ of nowhere dense sets. For each $\eta \in \kappa^{<\kappa}$ let $U(\eta)$ denote the basic open set determined by η, i.e. $\{x \in \kappa^\kappa \mid \eta \subseteq x\}$.

First suppose that κ is inaccessible or ω. We build the α-th level T_α of T by induction on α. For $\alpha = 0$, T_0 has just the single node $\emptyset$ and for limit α, T_α consists of all limits of branches through the levels T_β, $\beta < \alpha$.

Suppose that $\alpha = \beta + 1$. Then we list all pairs $(s * i, t * j)$ where s, t are on level β, i, j are 0 or 1 and $s * i \neq t * j$. As κ is inaccessible or ω there are fewer than κ such pairs. Now choose such a pair $(s * i, t * j)$ and find $(s * i)^1$ extending $s * i$ and $(t * j)^1$ extending $t * j$ so that $U((s * i)^1) \times U((t * j)^1)$ is disjoint from E_β. This is possible as E_β is nowhere dense. Then choose another pair and do the same, repeating this for all pairs and resulting in sequences $(s * i)^1 \subseteq (s * i)^2 \subseteq \cdots$ for each $s * i$. Let $(s * i)^\infty$ be the limit of this sequence and take level T_α to consist of all of these $(s * i)^\infty$'s.

The result is that if x, y are κ-branches through T and extend distinct nodes on level $\beta + 1$ of T then (x, y) does not belong to E_β and therefore (x, y) does not belong E as β can be chosen to be arbitarily large.

Now suppose that $\diamondsuit_\kappa$ holds. Fix a $\diamondsuit_\kappa$ sequence $(D_\beta \mid \beta < \kappa)$ that guesses pairs (x, y) in κ^κ, i.e., for such pair, $\{\beta \mid D_\beta = (x|\beta, y|\beta)\}$ is stationary in κ. Now repeat the above construction except at stage $\beta + 1$ only treat the four pairs $(d_0 * i, d_1 * j)$ if $D_\beta = (d_0, d_1)$ and d_0, d_1 belong to T_β, guaranteeing that if (x, y) extends (d_0, d_1) then (x, y) does not belong to E_β. Other nodes s on level β are simply extended to $s * 0$ and $s * 1$ on level $\beta + 1$. The $\diamondsuit_\kappa$ sequence guarantees that if x, y are distinct branches through the resulting Sacks-perfect tree then (x, y) does not belong to E_β for any β and therefore does not belong to E. □

Other forms of regularity

The Baire property is just one example of a regularity property for subsets of generalised Baire space. We consider now other such properties, each associated to a "κ-arboreal" forcing in the way that the Baire property is associated to κ-Cohen forcing.

A forcing $\mathcal{P}$ is *κ-arboreal* iff it is a κ-closed suborder of the set of subtrees of $\kappa^{<\kappa}$ ordered by inclusion.

Examples of κ-arboreal forcings:
κ-Cohen. These are subtrees of $2^{<\kappa}$ consisting of a stem and all nodes above it.
κ-Sacks. These are κ-closed subtrees of $2^{<\kappa}$ with the property that every node has a splitting extension and the limit of splitting nodes is a splitting node.
κ-Miller. These are κ-closed subtrees of the tree of *increasing* sequences in $\kappa^{<\kappa}$ with the property that every node can be extended to a club-splitting node and the limit of club-splitting nodes is club-splitting. We also require *continuous club-splitting*, which means that if s is a limit of club-spltting nodes then the club witnessing club-splitting for s is the intersection of the clubs witnessing club-splitting for the club-splitting proper initial segments of s (this is useful in the study of regularity properties and in the large cardinal context).
κ-Laver. These are κ-Miller trees with the property that every node beyond some fixed node (the stem) is club-splitting.

The above examples were presented assuming that κ is uncountable; for $\kappa = \omega$ take "club" to just be "infinite".

To define "$\mathcal{P}$-regularity" for the above forcing notions $\mathcal{P}$ we proceed as follows. A set A is *strictly $\mathcal{P}$-null* if every tree $T \in \mathcal{P}$ has a subtree in $\mathcal{P}$, none of whose κ-branches belongs to A. And A is *$\mathcal{P}$-null* if it is the union of κ-many strictly $\mathcal{P}$-null sets. Then A is *$\mathcal{P}$-regular* (or *$\mathcal{P}$-measurable*) if any tree $T \in \mathcal{P}$ has a subtree $S \in \mathcal{P}$ such that either all κ-branches through S, with a $\mathcal{P}$-null set of exceptions, belong to A or all κ-branches through S, with a $\mathcal{P}$-null set of exceptions, belong to the complement of A.

5.-6.Vorlesungen

Proposition 10 *A set is κ-Cohen measurable iff it has the property of Baire.*

Proof. Let $\mathcal{P}$ denote κ-Cohen forcing. First note that a set is strictly $\mathcal{P}$-null iff it is nowhere dense and therefore is $\mathcal{P}$-null iff it is meager. Now if A is $\mathcal{P}$-measurable it follows that every basic open set has a basic open subset on which A is either meager or comeager. Thus if U is the union of the basic open sets on which A is meager or comeager and U_0 is the union of the basic open sets on which A is comeager, it follows that A differs from U_0 by a meager set, as the complement of U is nowhere dense. So A has the property of Baire.

Conversely, if $A = U \triangle M$ with U open and M meager, then to verify that A is $\mathcal{P}$-measurable it suffices to show that U is $\mathcal{P}$-measurable. But this is clear, as any basic open set not disjoint from U has a basic open subset that is completely contained in U. □

Now let $\mathcal{P}$ be any of the above κ-arboreal forcings.

Proposition 11 *Any Borel set is $\mathcal{P}$-measurable.*

Proof. We may assume that $\mathcal{P}$ is not κ-Cohen, as in that case $\mathcal{P}$-measurability is the same as the property of Baire and we know that all Borel sets have the property of Baire. So assume that $\mathcal{P}$ is one of the other examples. We first treat the κ-Sacks and κ-Miller forcings.

Note that the collection of $\mathcal{P}$-measurable sets if obviously closed under complements, so it suffices to show that it is closed under κ-unions and that basic open sets are $\mathcal{P}$-measurable. For the basic open sets, note that for each of the above examples of arboreal forcings $\mathcal{P}$, if T belongs to $\mathcal{P}$ then so does $T(\eta)$ for each node η of T (where $T(\eta)$ consists of all nodes in T which are compatible with η). Now if η is an arbitrary element of $\kappa^{<\kappa}$, determining the basic open set $U(\eta)$, and T belongs to $\mathcal{P}$ then either η belongs to T, in which case $T(\eta)$ is a strengthening of T whose κ-branches are all contained in $U(\eta)$, or η does not belong to T, in which case no κ-branch of T belongs to $U(\eta)$. So $U(\eta)$ is $\mathcal{P}$-measurable.

Suppose that A is the union of A_i, $i < \kappa$ and we know that each A_i is $\mathcal{P}$-measurable. Given $T \in \mathcal{P}$ and $i < \kappa$ we can strengthen T to T_i so that either almost all κ-branches of T_i belong to A_i or almost all κ-branches of T_i do not belong to A_i, where "almost all" refers to a $\mathcal{P}$-null set of exceptions. If the former occurs for some i then almost all κ-branches of T_i also belong to

A so we are done. Otherwise we want to strengthen T to T^* so that almost no κ-branch of T^* belongs to any A_i. Of course we can do this for fewer than κ-many A_i's using the κ-closure of the forcing $\mathcal{P}$; to handle κ-many A_i's we use the fusion property. This is expressed as follows: There are partial orders $\leq_i$ on $\mathcal{P}$ which refine the standard ordering on $\mathcal{P}$ such that

(a) $i \leq j$, $T^* \leq_j T$ implies $T^* \leq_i T$.
(b) If $(T_i \mid i < \lambda)$, $\lambda \leq \kappa$, belong to $\mathcal{P}$ and for all $i \leq j < \lambda$, $T_j \leq_i T_i$ then there is $T \leq_i T_i$ for all $i < \lambda$.
(c) Suppose that T belongs to $\mathcal{P}$ and D is a set of extensions of T which is dense below T. Then for each $i < \kappa$ there are $T^* \leq_i T$ and $d \subseteq D$ of size at most κ such that each κ-branch through T^* is also a κ-branch through some element of d.

Now recall that we are given T such that for each i, the set of T^* such that almost no κ-branch through T^* belongs to A_i is open dense below T. Now use fusion to build a sequence $(T_i \mid i < \kappa)$ such that $i \leq j \rightarrow T_j \leq_i T_i$ and each κ-branch through T_{i+1} is a κ-branch through one of κ-many extensions of T_i, almost none of whose κ-branches belong to A_i. If T^* is a lower bound to the sequence of T_i's then almost no κ-branch of T^* belongs to any A_i, so we have verified the $\mathcal{P}$-measurability of A, the union of the A_i's.

We next verify the fusion property for the κ-Sacks and κ-Miller forcings.

κ-Sacks:
If T is a condition then let $f_T : 2^{<\kappa} \rightarrow T$ be the natural order-preserving bijection between the full tree $2^{<\kappa}$ and the set of splitting nodes of T. Then define $T^* \leq_i T$ iff $f_{T^*}(s) = f_T(s)$ for all $s \in 2^{<\kappa}$ of length at most i. Then property (a) is clear. Note that for limit i, this is the same as requiring this just for s of length less than i, as the limit of spltting nodes is a splitting node; this gives property (b). For (c), for each $s \in 2^{<\kappa}$ of length i and $j \in \{0,1\}$ we choose $T_{s*j} \leq T(f_T(s) * j)$ in D, let d be the set of such T_{s*j}'s and let T^* the the union of the T_{s*j}'s. As $\kappa^{<\kappa} = \kappa$, there are only κ-many such $s * j$'s.

κ-Miller:
If T is a condition then let $f_T : \kappa_{\uparrow}^{<\kappa} \rightarrow T$ be the natural order-preserving bijection between the full tree $\kappa_{\uparrow}^{<\kappa}$ and the set of splitting nodes of T. Define $T^* \leq_i T$ iff $f_{T^*}(s) = f_T(s)$ for all $s \in 2^{<\kappa}$ such that $s(\alpha) \leq i$ for all $\alpha < |s|$.

Property (a) is clear and property (b) follows using (diagonal) intersections when λ equals κ. (c) is verified as for κ-Sacks.

For κ-Laver we need to consider *pure* extensions. If $T^* \leq T$ are κ-Laver conditions then we write $T^* \leq^* T$, T^* *is a pure extension of* T, iff $T^* \leq T$ and T^* has the same stem as T. A set D is *purely dense* if any T has a pure extension in D and is *purely dense below* T if any extension of T has a pure extension in D.

A set A is *strictly purely Laver-null* if any T has a pure extension T^* such that $[T^*]$ is disjoint from A. A is *purely Laver-null* if it is the κ-union of strictly purely dense sets. And A is *purely Laver-measurable* if any T has a pure extension T^* such that $[T^*]$ is either contained in or disjoint from A modulo the ideal of purely Laver-null sets.

We will show that any Borel set is purely Laver-measurable. As the ideal of purely Laver-null sets is contained in the ideal of Laver-null sets this implies that any Borel set is Laver-measurable in the ordinary sense.

Clearly the collection of purely Laver-measurable sets is closed under complements. And basic open sets are purely Laver-measurable: For any η and T, if η is contained in the stem of T then $[T] \subseteq U(\eta)$; otherwise we can form a pure extension of T whose κ-branches are incompatible with η and hence do not belong to $U(\eta)$.

To handle κ-unions we need Pure Fusion.

Pure Fusion for κ-Laver. There are partial orders $\leq_i$ on $\mathcal{P}$ which refine the order $\leq^*$ of pure extension such that

(a) $i \leq j$, $T^* \leq_j T$ implies $T^* \leq_i T$.
(b) If $(T_i \mid i < \lambda)$, $\lambda \leq \kappa$, belong to $\mathcal{P}$ and for all $i \leq j < \lambda$, $T_j \leq_i T_i$ then there is $T \leq_i T_i$ for all $i < \lambda$.
(c) Suppose that T belongs to $\mathcal{P}$ and D is a set of extensions of T which is purely dense below T. Then for each $i < \kappa$ there are $T^* \leq_i T$ and $d \subseteq D$ of size at most κ such that each κ-branch through T^* is also a κ-branch through some element of d.

Pure Fusion implies that the κ-union of purely Laver-measurable sets is pure-ly Laver-measurable. And κ-Laver satisfies Pure Fusion: As for κ-Miller, if T

is a condition then let $f_T : \kappa_{\uparrow}^{<\kappa} \to T$ be the natural order-preserving bijection between the full tree $\kappa_{\uparrow}^{<\kappa}$ and the set of splitting nodes of T, and define $T^* \leq_i T$ iff $f_{T^*}(s) = f_T(s)$ for all $s \in 2^{<\kappa}$ such that $s(\alpha) \leq i$ for all $\alpha < |s|$. Property (a) is clear and property (b) follows using (diagonal) intersections when λ equals κ. Property (c) can be handled as in the case of κ-Miller, using the fact that D is purely dense. □

Remark 1. Using fusion the above arguments show that any $\mathcal{P}$-null set is in fact strictly $\mathcal{P}$-null for the cases of κ-Sacks and κ-Miller and any purely Laver-null set is strictly purely Laver-null in the case of κ-Laver. Therefore in the caes of κ-Sacks and κ-Miller, $\mathcal{P}$-measurability can be more simply expressed by: A is $\mathcal{P}$-measurable iff any T in $\mathcal{P}$ has a subtree T^* in $\mathcal{P}$ such that $[T^*]$ is either contained in or disjoint from A. And pure Laver-measurability just says that any κ-Laver tree T has a pure extension T^* such that $[T^*]$ is either contained in or disjoint from A.

Question. Does Laver-measurability imply Pure Laver-measurability?

Remark 2. (Yurii) Note that κ-Laver is κ^+-cc as any two conditions with the same stem are compatible. It then follows, as in the case of κ-Cohen, that A is Laver-measurable iff $A \triangle O$ is Laver-meager for some set O which is Laver-open, i.e., the union of sets of the form $[T]$, T a κ-Laver tree. Given this, the fact that the collection of Laver-measurable sets is closed under κ-unions follows easily, as in the case of κ-Cohen forcing.

7.-8.Vorlesungen

Theorem 12 *Not every Σ^1_1 set is $\mathcal{P}$-measurable.*

Proof. First we verify this for κ-Sacks. Let A consist of all $x \in 2^\kappa$ such that $\{i \mid x(i) = 0\}$ contains a club. Suppose that T is a κ-Sacks tree. Then there are κ-branches of T in A and also κ-branches of T in the complement of A: For the former simply choose a κ-branch x through T as the union of splitting nodes s_i of T of lengths α_i such that for each i, $s_{i+1}(\alpha_i) = 0$; this is possible as the limit of splitting nodes of T is also a splitting node of T. For the latter do the same, but with $s_{i+1}(\alpha_i) = 1$ for limit i.

To handle the other cases we prove the following general fact, patterned after work of Brendle-Löwe, Khomskii and Laguzzi.

Lemma 13 *Let* Γ *be a pointclass closed under continuous pre-images (like* Δ^1_n*,* Σ^1_n *or* Π^1_n*). Let* $\Gamma(\mathcal{P})$ *be the statement that every set in* Γ *is* $\mathcal{P}$*-measurable. Then:*

$\Gamma(\kappa$*-Cohen*$) \to \Gamma(\kappa$*-Miller*$)$
$\Gamma(\kappa$*-Laver*$) \to \Gamma(\kappa$*-Miller*$)$
$\Gamma(\kappa$*-Miller*$) \to \Gamma(\kappa$*-Sacks*$)$.

Proof of Lemma. For the first implication, first note the following:

Fact 1. $\Gamma(\kappa$-Cohen$)$ $(= \Gamma(2^{<\kappa}$-Cohen$))$ implies $\Gamma(\kappa_\uparrow^{<\kappa}$-Cohen$)$.

Proof of Fact 1. Note that there is $D \subseteq 2^\kappa$ which is the κ-intersection of open dense subsets of 2^κ (and therefore comeager) such that D is homeomorphic to $\kappa^\kappa_\uparrow$. We may choose D to consist of all $x \in 2^\kappa$ such that $x(i) = 1$ for cofinally many $i < \kappa$; the homeomorphism sends x to $y \in \kappa^\kappa_\uparrow$ where $x = 0^{y(0)} * 1 * 0^{y(1)} * \cdots$. If $A \subseteq \kappa^\kappa_\uparrow$ belongs to Γ then the $\kappa^{<\kappa}_\uparrow$-measurability of A follows from that of its pre-image under this homeomorphism, which in turn follows from $\Gamma(\kappa$-Cohen$)$, as D is comeager in 2^κ. $\square$ (*Fact 1*)

Now let A belong to Γ and let T be a κ-Miller tree. Under the assumption $\Gamma(\kappa$-Cohen$)$ we want to find a κ-Miller subtree of T, all of whose κ-branches belong to A or all of whose κ-branches belong to the complement of A.

Let φ be the natural order-preserving bijection between the full tree $\kappa^{<\kappa}_\uparrow$ (of increasing $< \kappa$-sequences through κ) and the splitting nodes of T. Also let φ^* denote the induced homeomorphism between $\kappa^\kappa_\uparrow$ and $[T]$, the set of κ-branches through T. Let A' be $(\varphi^*)^{-1}[A]$, which belongs to Γ as by assumption Γ is closed under continuous pre-images. Apply $\Gamma(\kappa$-Cohen$)$ to get a basic open set $U(\eta)$ such that A' is either meager or comeager on $U(\eta)$. Without loss of generality assume the latter. Now build a κ-Miller tree S' such that $[S']$ is contained in $U(\eta) \cap A'$: assume that $A' \cap U(\eta)$ contains the intersection of U_i, $i < \kappa$, where each U_i is open dense on $U(\eta)$ and ensure that any $x \in \kappa^\kappa$ extending a node on the i-th spitting level of S' belongs to U_i. We can also require that splitting nodes μ of S' are full-splitting, in the sense that if $\mu * \alpha$ belongs to S' for all $\alpha < \kappa$. Then $\varphi[S']$ consists of the splitting nodes of a κ-Miller tree S contained in T with the property that $[S]$ is contained in A.

For the second implication, note that like κ-Cohen forcing, κ-Laver forcing is κ^+-cc and we can form a topology, which we call the Laver topology, whose basic open sets are of the form $[T]$ for T a κ-Laver tree. Then in analogy to κ-Cohen forcing we have:

Fact 2. A is κ-Laver measurable iff A is of the form $O \triangle M$ where O is open in the Laver topology and M is meager in the Laver topology.

Proof of Fact 2. Note that strictly Laver-null is the same as nowhere dense in the Laver topology and Laver-null is the same as meager in the Laver topology. So if A is Laver-measurable it follows that $D =$ the set of κ-Laver trees T such that A is either meager or comeager on $[T]$ is open dense in κ-Laver forcing. Let X be a maximal antichain contained in D. Then the union O^* of the $[T]$ for T in X is open dense in the Laver topology. Let O be the union of the $[T]$ for T in X where A is comeager on $[T]$. Then as X has size at most κ, A differs from O by a meager set. Conversely, it suffices to show that open sets in the Laver topology are κ-Laver measurable. But if O is Laver-open and T is a κ-Laver tree then $[T] \cap O$ is either empty or contains $[S]$ for some κ-Laver tree S, so O is κ-Laver measurable. □ (*Fact 2*)

Now we use *Fact 2* to prove the third implication, by imitating the argument used for the second implication. Let A belong to Γ and let T be a κ-Miller tree. Under the assumption $\Gamma(\kappa\text{-Laver})$ we want to find a κ-Miller subtree of T, all of whose κ-branches belong to A or all of whose κ-branches belong to the complement of A.

We "collapse" T into a κ-Laver tree T' as follows: Define a function ψ from the splitting nodes of T to nodes of the full tree $\kappa_{\uparrow}^{<\kappa}$ by induction as follows. If η is a splitting node of T which is not the limit of splitting nodes of T then write η as $\eta_0 * \alpha * \eta_1$ where η_0 is the longest splitting node of T properly contained in η (or $\emptyset$ if η is the least splitting node of T) and set $\psi(\eta) = \psi(\eta_0) * \alpha$. If η is a limit of splitting nodes of T then set $\psi(\eta) =$ the union of the $\psi(\eta_0)$ for η_0 a splitting node of T properly contained in η. Let φ be the inverse of ψ, mapping the κ-Laver tree T' onto the splitting nodes of T, and let φ^* be the induced homeomorphism between $[T']$ and $[T]$, the sets of κ-branches of T' and T, respectively.

Now let A' be $(\varphi^*)^{-1}[A]$, which belongs to Γ as by assumption Γ is closed under continuous pre-images. Apply $\Gamma(\kappa\text{-Laver})$ to get a κ-Laver subtree of T' such that A' is either Laver-meager or Laver-comeager on $[T]$. Without loss of generality assume the latter. Now build a κ-Miller tree S' such that $[S']$ is contained in $[T] \cap A'$: assume that $A' \cap [T]$ contains the intersection of U_i, $i < \kappa$, where each U_i is Laver-open dense on $[T]$ and ensure that any $x \in \kappa^\kappa$ extending a node on the i-th spitting level of S' belongs to U_i. Then $\varphi[S']$ consists of the splitting nodes of a κ-Miller tree S contained in T with the property that $[S]$ is contained in A.

9.-10.Vorlesungen

For the third implication, let A belong to Γ and let T be a κ-Sacks tree. Under the assumption Γ(κ-Miller) we want to find a κ-Sacks subtree S of T such that $[S]$ is either contained in or disjoint from A. Define an injection φ_0 from the full tree $\kappa_\uparrow^{<\kappa}$ into $2^{<\kappa}$ as follows:

$\varphi_0(\emptyset) = \emptyset$
$\varphi_0(\eta) = (\bigcup_{\alpha<|\eta|} \varphi_0(\eta|\alpha))$, if $|\eta|$ = the length of η is a limit ordinal
$\varphi_0(\eta * \alpha) = \varphi_0(\eta) * 0^{\alpha-|\eta|} * 1$, where 0^β denotes a β-sequence of 0's.

And let φ_0^* be the injection from $\kappa_\uparrow^\kappa$ into 2^κ induced by φ_0. Also let ψ be the natural bijection between $2^{<\kappa}$ and the splitting nodes of T and ψ^* the induced bijection between 2^κ and $[T]$. Define $\varphi = \psi \circ \varphi_0$ and $\varphi^* = \psi^* \circ \varphi_0^*$.

As φ^* is continuous, $A' = (\varphi^*)^{-1}[A]$ belongs to Γ. Apply Γ(κ-Miller) to obtain a κ-Miller tree S' such that $[S']$ is either contained in or disjoint from A'. Thin S' to guarantee that if η is a splitting node of S' then the length $|\eta|$ of η is the sup of its range and $\eta * |\eta|$ belongs to S'. Then $\varphi[S'] = S$ generates a κ-Sacks subtree S of T such that $[S]$ is either contained in or disjoint from A. $\square$ (*Lemma*)

Using the Lemma, we conclude that Σ^1_1 measurability fails for κ-Miller, κ-Cohen and κ-Laver. $\square$

We have seen that Δ^1_1(κ-Cohen) is consistent, from which it follows by the above Lemma that Δ^1_1(κ-Miller) and Δ^1_1(κ-Sacks) are consistent. What about Δ^1_1(κ-Laver)? Again we can imitate the proof for the κ-Cohen case. First we need a lemma.

Lemma 14 *Let M be a transitive model of ZFC$^-$ containing κ and all bounded subsets of κ which is elementary in $H(\kappa^+)$. Then $x \in \kappa^\kappa$ is κ-Laver generic over M iff x belongs to every Borel set coded in M which is open dense in the Laver topology of M (equivalently, open dense in the Laver topology of V).*

Proof. Of course when we say that x is κ-Laver generic over M we mean that $G_x = \{T \in M \mid T$ is a κ-Laver tree of M and $x \in [T]\}$ is κ-Laver generic over M in the strict sense. If this holds and B is a Borel set coded in M which is open dense in the Laver topology of M then the set of T in M such that $M \models [T] \subseteq B$ is open dense in the κ-Laver forcing of M and therefore there is such a T in G_x; by the elementarity of M in $H(\kappa^+)$, $V \models [T] \subseteq B$ and therefore as x belongs to $[T]$ it also belongs to B. Conversely, suppose that x belongs to every Borel set coded in M which is open dense in the Laver topology of M and that $D \in M$ is open dense on the κ-Laver forcing of M. Let $X \in M$ be a maximal antichain contained in D; then X has size at most κ and $B =$ the union of the $[T]$ for T in X is a Borel set coded in M which is open dense in the κ-Laver topology of M. By hypothesis x belongs to B and therefore to some $[T]$ where T belongs to X; so G_x meets D. □

Theorem 15 *Let κ be regular and assume GCH. Then after forcing with Laver(κ, κ^+) (the iteration of κ^+-many κ-Laver forcings with support* of size *$< \kappa$), every Δ^1_1 set is κ-Laver measurable.*

Proof. Note that the forcing Laver(κ, κ^+) is κ^+-cc; this follows using a Δ-system argument and the fact that κ-Laver forcing is both κ-closed and κ-centered.

Let G be generic for Laver(κ, κ^+) and let X be Δ^1_1 in $V[G]$. We'll show that X is κ-Laver measurable in $V[G]$, i.e., any κ-Laver tree T contains a κ-Laver subtree S such that $[S]$ is either contained in or disjoint from X modulo a Laver-null set. Without loss of generality we assume that the defining parameter for X and the tree T belong to V (otherwise factor over $V[G|\alpha]$ for some large enough $\alpha < \kappa^+$). Let φ, ψ be Σ^1_1 formulas (with parameters in V) defining X and the complement of X, respectively.

Let M be a transitive elementary submodel of $H(\kappa^+)^V$ of size κ which contains all bounded subsets of κ and T. Then by the κ^+-cc of Laver(κ, κ^+), $M[G]$ is elementary in $H(\kappa^+)^V[G] = H(\kappa^+)^{V[G]}$. If α is $M \cap \kappa^+$ then $M[G] = M[G|\alpha]$; we may assume that $G(\alpha)$, the κ-Laver generic added by G at stage α, belongs to $[T]$, as it is dense to force this for some M. Note that $G(\alpha)$ is also κ-Laver generic over $M[G|\alpha]$ as this model is Σ_1 elementary in $H(\kappa^+)^V[G|\alpha]$ (and the property of being a maximal antichain is Π_1). Without loss of generality assume that $\varphi(G(\alpha))$ holds in $V[G]$ and therefore also in $V[G|\alpha][G(\alpha)]$ (as the former is a κ-closed forcing extension of the latter).

Now let S be a κ-Laver condition in $M[G|\alpha]$ extending T which forces $\varphi(\dot{g})$ where $\dot{g}$ denotes the κ-Laver generic. Now using the Lemma, the set of $x \in \kappa^\kappa_\uparrow$ which are Laver-generic over $M[G|\alpha]$ is Laver-comeager in $V[G]$ as it is the intersection of κ-many Borel sets, each of which is open dense in the Laver topology of $M[G|\alpha]$ and therefore in the Laver topology of $V[G]$. And if x is a κ-branch through S which is Laver-generic over $M[G|\alpha]$ then $M[G|\alpha][x]$ and therefore $V[G]$ satisfies $\varphi(x)$. We have shown that $[S]$ is contained in X modulo a Laver-null set and therefore X is Laver-measurable in $V[G]$. $\Box$

11.-12.Vorlesungen

Borel Reducibility

If E and F are equivalence relations on κ^κ then we say that *E is Borel reducible to F*, written $E \leq_B F$, if there is a Borel function f such that for all x, y: $E(x, y)$ iff $F(f(x), f(y))$. The relation $\leq_B$ is reflexive and transtive and we write $\equiv_B$ for the equivalence relation it induces.

We will begin by focusing on Borel reducibility between Borel equivalence relations. For such relations E, F with at most κ-many equivalence classes the notion is rather trivial: $E \equiv_B F$ iff E and F have the same number of equivalence classes. This is because if E and F have the same number of classes we may choose sets X_E and X_F of the same size selecting one element from each equivalence class of E, F respectively and then extend any bijection between X_E and X_F to a Borel reduction of E to F (and similarly obtain a Borel reduction of F to E).

So the first nontrivial question to ask is whether there is a Borel equivalence relation which is minimal with respect to Borel reducibility among those with more than κ equivalence classes.

Theorem 16 *(Silver's Dichotomy) Suppose that κ equals ω and E is a Borel equivalence relation with uncountably many classes. Then id $\leq_B E$, where id is the equivalence relation of equality on ω^ω.*

The most popular proof proof of Silver's dichotomy makes heavy use of the Gandy-Harrington topology, which I'll now introduce.

The basic open sets of the Gandy-Harrington topology are the *lightface* Σ^1_1 sets. These sets form a basis for the topology which is therefore second countable. The Gandy-Harrington topology refines the usual topology on Baire space and is Hausdorff.

But the Gandy-Harrington topology is *not* Polish (i.e. not induced by a complete metric):

Proposition 17 *The Gandy-Harrington topology is not regular, i.e., points cannot in general be separated from closed sets using open sets.*

Proof. Let X be lightface Π^1_1 but not Borel. For each basic open set disjoint from X choose, if possible, an open set U disjoint from it which contains X. If the topology were regular then the union of the basic open sets for which such a U exists would be the complement of X and therefore X would be the intersection of countably many open sets. But any open set is boldface $\mathbf{\Sigma}^1_1$ and therefore we now have that X is boldface $\mathbf{\Sigma}^1_1$, contradiction. □

The Gandy-Harrington topology does however contain an open dense subspace that is Polish. A real x is *low* if it computes only computable ordinals, i.e., $\omega_1^{ck}(x) = \omega_1^{ck}$. The set of low reals is a lightface Σ^1_1 set and therefore open in the Gandy-Harrington topology (x is low iff for all e, if $\{e\}^x$ is a wellorder then $\{e\}^x$ is isomorphic to $\{f\}$ for some f). The fact that it is dense is equivalent to:

Theorem 18 *(Gandy Basis Theorem) Suppose that A is a nonempty lightface Σ^1_1 set. Then A has a low element.*

Proof Sketch. Let CWT denote the set of indices for computable wellfounded trees. Now write A as $x \in A$ iff $T(x)$ is not wellfounded, where T is a computable tree on $\omega \times \omega$. If A is nonempty then A has an element computable in CWT.

But now consider $A^* = \{(x, y) \mid x \in A$ and y is not Hyp in $x\}$. If A is nonempty Σ^1_1 then so is A^*. Choose (x, y) in A^* which is computable in CWT. As y is not Hyp in x it follows that CWT is not Hyp in x. But then x is low, else we can choose e so that $\{e\}^x$ is a wellorder of length ω_1^{ck} and then $\{f\}$ is a wellfounded tree iff $\{f\}$ is a tree whose rank is less than $\{e\}^x$, giving a Σ^1_1 definition of CWT; this contradicts the fact that CWT is not Hyp (i.e. not Δ^1_1) in x. □

To show that X_{low}, the restriction of the Gandy-Harrington topology to the low reals, is Polish, we need:

Lemma 19 *If x is low then for any lightface Σ^1_1 set A either x belongs to A or x belongs to a lightface Δ^1_1 set disjoint from A.*

Proof. Write $y \in A$ iff $T(y)$ is not wellfounded, where T is a computable tree on $\omega \times \omega$. If x does not belong to A then $T(x)$ is wellfounded; let α be the rank of $T(x)$. As x is low, α is computable and we can take B to consist of those y such that $T(y)$ has rank α. □

Corollary 20 *X_{low} has a basis of clopen sets and is therefore regular.*

Choquet gave a topological characterisation of Polish spaces. For a topological space X the *strong Choquet game* G_X is the two-person game where Player I choose pairs (x_i, U_i) at even stages and Player II chooses sets V_i at odd stages where $x_i \in V_i \subseteq U_i$ and $U_{i+1} \subseteq V_i$ for each i. Player II wins if the intersection of the U_i's (= the intersection of the V_i's) is nonempty. The space X is *strong Choquet* iff Player II has a winning strategy in G_X.

Theorem 21 *(Choquet) A topological space is Polish iff it is second countable, regular and strong Choquet.*

We have seen that X_{low} is second countable and regular; so to see that it is Polish we just need:

Theorem 22 *X_{low} is strong Choquet.*

Proof. It suffices to show that the full Gandy-Harrington topology is strong Choquet, as X_{low} is an open subset in that topology. We describe a winning strategy for Player II in the game G_X where X is the entire space ω^ω, endowed with the Gandy-Harrington topology.

Let (x_0, U_0) be the first move by Player I. Choose a lightface Σ^1_1 subset A_0 of U_0 containing x_0 and write A_0 as the projection of a computable tree T_0. We can choose y_0 so that $(x_0, y_0) \in [T_0]$. For any pair (s, t) in a tree T on $\omega \times \omega$ we let $T_{(s,t)}$ denote the subtree of T consisting of those pairs (s', t') in T where s' is compatible with s and t' is compatible with t. Now Player II's response V_0 to Player I's first move is the projection of $(T_0)_{(s_0,t^0_0)}$ where $s_0 = x_0|1$ and $t^0_0 = y_0|1$. Then $x_0 \in V_0 \subseteq U_0$ so the rules of the game are obeyed.

Let Player I's next move be (x_1, U_1). Then as x_1 belongs to V_0 we can choose y_0' with $(x_1, y_0') \in [(T_0)_{(s_0,t_0^0)}]$. Set $s_1 = x_1|2$ and $t_1^0 = y_0'|2$. Then $s_0 \subseteq s_1$ and $t_0^0 \subseteq t_1^0$. Also choose a lightface Σ_1^1 set A_1 with $x_1 \in A_1 \subseteq U_1$, a computable tree T_1 which projects to A_1 and y_1 such that (x_1, y_1) belongs to $[T_1]$. Note that $s_0 \subseteq x_1$ so if we set $t_0^1 = y_1|1$ we have that x_1 is in the projection of $(T_1)_{(s_0,t_0^1)}$. Player II now plays $V_1 =$ the projection of $(T_0)_{(s_1,t_1^0)}$ $\cap$ the projection of $(T_1)_{(s_0,t_0^1)}$. We have $x_1 \in V_1 \subseteq A_1 \subseteq U_1$.

We continue in this way, producing a sequence of plays (x_i, U_i) and V_i together with computable trees T_n and sequences $s_0 \subseteq s_1 \subseteq \cdots$ and $t_0^n \subseteq t_1^n \subseteq \cdots$ such that:

1. $x_n \in A_n =$ the projection of T_n, $A_n \subseteq U_n$.
2. s_n has length $n+1$ and $s_n \subseteq x_n$.
3. $(s_k, t_k^n) \in T_n$.
4. V_n is the intersection of the projections of the trees $(T_0)_{(s_n,t_n^0)}$, $(T_1)_{(s_{n-1},t_{n-1}^1)} \cdots (T_n)_{(s_0,t_0^n)}$.

Now let x be the union of the s_n's and y_n the union of the t_k^n's. Then (x, y_n) belongs to $[T_n]$ for each n and therefore x belongs to $A_n =$ the projection of T_n for each n. $\square$

13.Vorlesung

We now have a clear strategy to prove Silver's Dichotomy. Suppose that E is a Borel equivalence relation with uncountably many classes. We assume that E is effectively Borel (i.e. lightface Δ_1^1), else we can relativise to a parameter. Let τ denote the Gandy-Harrington topology. We will find a nonempty Σ_1^1 set V such that E is meager on $V \times V$ in the product topology $\tau \times \tau$. Now by Gandy Basis, $U = V \cap X_{\text{low}}$ is a nonempty open set in X_{low} on which E is meager. But then applying Mycielski's theorem to the Polish space X_{low}, there is a continuous $f : 2^\omega \to X_{\text{low}}$ witnessing that id continuously reduces to E, with the standard topology on 2^ω and the Gandy-Harrington topology on X_{low}. Note that this reduction is also continuous as a function from 2^ω to ω^ω with the standard topology on both 2^ω and ω^ω, as the Gandy-Harrington topology on ω^ω refines the standard topology.

So the Silver Dichotomy reduces to:

Main Claim. Suppose that E is a lightface Δ^1_1 equivalence relation on Baire space ω^ω with uncountably many classes. Then E is meager in $V \times V$ in the topology $\tau \times \tau$, for some nonempty lightface Σ^1_1 set V.

Proof. The desired set V is:

$\{x \mid \text{There is no lightface } \Delta^1_1 \text{ set } U \text{ with } x \in U \subseteq [x]_E\}$

where $[x]_E$ denotes the E-equivalence class of x. Note that V is nonempty, else E would have only countably many equivalence classes. Also V is lightface Σ^1_1 as:

$x \notin V$ iff there exists a code c for a lightface Δ^1_1 set D_c such that $x \in D_c$ and $\forall y(y \in D_c \to xEy)$

and as the set of codes of lightface Δ^1_1 sets is a Π^1_1 set of numbers, this gives a lightface Π^1_1 definition of the complement of V. We must show that E is meager on $V \times V$ for the topology $\tau \times \tau$, where τ is the Gandy-Harrington topology.

Claim 1. If x belongs to V then there is no lightface Σ^1_1 set U such that $x \in U \subseteq [x]_E$.

Proof of Claim 1. Otherwise note that $[x]_E$ is lightface Π^1_1: yEx iff $\forall z(z \in U \to yEz)$. Now apply the Separation Theorem for Σ^1_1 sets to get a lightface Δ^1_1 set D with $U \subseteq D \subseteq [x]_E$, contradicting the fact that x belongs to V. □ (*Claim 1*)

Next note that as E is Δ^1_1, it has the Baire property in the topology $\tau \times \tau$: E is the result of applying the Suslin operation to sets which are closed in the usual product topology and therefore to sets which are closed in $\tau \times \tau$; but the collection of sets with the Baire property is closed under the Suslin operation in any topological space and therefore E has the Baire property in $\tau \times \tau$.

So by the Kuratowski-Ulam Theorem, to show that E is $\tau \times \tau$-meager on $V \times V$ it suffices to show that for $x \in V$, $[x]_E$ is τ-meager in V.

Claim 2. Let τ be the Gandy-Harrington topology on $X = \omega^\omega$ and suppose that A is τ-comeager on U, where U is τ-open. Let τ_2 denote the Gandy-Harrington topology on $X \times X$ (τ_2 is finer than $\tau \times \tau$). Then $A \times A$ is τ_2-comeager on $U \times U$.

Proof of Claim 2. It suffices to show that if $B \subseteq X$ is τ-closed and τ-nowhere dense then $B \times U$ and $U \times B$ are τ_2-closed and τ_2-nowhere dense. Without loss of generality consider $B \times U$. That it is τ_2-closed is clear, as τ_2 refines $\tau \times \tau$. If $B \times U$ contained a nonempty τ_2-open set U_2 then the projection of U_2 onto the first coordinate would be a nonempty τ-open subset of B, contradicting the τ-nowhere density of B. □ (*Claim 2*)

Now suppose that x belongs to V and $[x]_E$ is not τ-meager in V. Then for some nonempty open $U \subseteq V$, $[x]_E$ is τ-comeager on U. By Claim 2, $[x]_E \times [x]_E$ is τ_2-comeager on $U \times U$. By Claim 1, $(U \times U) \cap (\sim E)$ is a nonempty set which is τ_2-open and therefore intersects $[x]_E \times [x]_E$, which is a contradiction. □

14.-15.Vorlesungen

We have established the Silver Dichotomy for the classical Baire space: A Borel equivalence relation on ω^ω with uncountably many classes has a perfect set of classes.

However the analagous Silver Dichotomy for κ^κ when κ is uncountable fails in L:

Theorem 23 *Assume $V = L$. Then there are Borel equivalence relations E with more than κ classes which are strictly below id with respect to Borel reducibility.*

Proof. A *weak Kurepa tree on κ* is a tree T of height κ with κ^+ many branches such that the α-th splitting level of T has size at most $\text{card}(\alpha)$ for stationary-many $\alpha < \kappa$.

Lemma 24 *Suppose $V = L$ and κ is regular and uncountable. Then there exists a weak Kurepa tree on κ.*

Proof. Our tree will be a subtree of the binary tree $2^{<\kappa}$. For singular $\alpha < \kappa$ let $\beta(\alpha)$ be the least limit ordinal $\beta > \alpha$ such that α is singular in L_β.

First assume that κ is inaccessible. Then T consists of all $\sigma \in 2^{<\kappa}$ such that:

$(*)$ For singular cardinals $\alpha \leq |\sigma|$ of cofinality ω, $\sigma|\alpha$ belongs to $L_{\beta(\alpha)}$.

Any node of T can be extended to nodes in T of any greater length (jus add 0's). And any node of T of length α splits into two nodes in T of length $\alpha + 1$ so the α-th splitting level consists of nodes of length α. It follows that the α-th splitting level of T has size at most card(α) for α a singular cardinal of cofinality ω.

Main Claim. T has κ^+ many branches.

Proof of Main Claim. For a limit ordinal β between κ and κ^+ we say that β is *critical* if some subset of κ is definable over L_β but not an element of L_β. The set of critical ordinals is cofinal in κ^+ and for critical β, the Skolem hull of κ in L_β is all of L_β.

Now for each critical β define:

$(*)$ $C_\beta = \{\alpha < \kappa \mid$ The Skolem hull of α in L_β contains no ordinals between α and $\kappa\}$.

Then C_β is a club in κ for each critical β and moreover if $\beta_0 < \beta_1$ are both critical then sufficiently large elements of C_{β_1} are limit points of C_{β_0}; this is because β_0 is an element of the Skolem hull of α in L_{β_1} for a large enough α and therefore so is C_{β_0}.

In particular the C_β's for critical β are distinct. Now we claim that each C_β is a branch through T. For this we need only check that if $\alpha < \kappa$ is a singular cardinal of cofinality ω then $C_\beta \cap \alpha$ belongs to $L_{\beta(\alpha)}$. This is clear if α does not belong to C_β, for then $C_\beta \cap \alpha$ is bounded in α and therfore an element of L_α. Otherwise note that $C_\beta \cap \alpha$ is definable over $L_{\bar{\beta}+1}$ where $L_{\bar{\beta}}$ is the transitive collapse of the Skolem hull of α in L_β; as α is regular in $L_{\bar{\beta}}$ it follows that $\bar{\beta}$ is less than $\beta(\alpha)$ so $C_\beta \cap \alpha$ is an element of $L\beta(\alpha)$, as desired.

The case of a successor cardinal κ is similar, except one can now obtain a *Kurepa tree on* κ, i.e. a tree T of height κ with κ^+ many branches such that the α-th splitting level of T has size at most card(α) for *all* $\alpha < \kappa$. □ (*Lemma*)

Now note that there can be no continuous injection from 2^κ into $[T]$, the set of κ-branches through T, because this would yield a club of $\alpha < \kappa$ such that the α-th splitting level of T has 2^α many nodes. In fact there cannot be such an injection which is Borel, as any Borel function is continuous on a comeager set and any comeager set contains a copy of 2^κ.

Finally define xE_Ty iff x, y are not branches through T or $x = y$. Then E_T is a Borel equivalence relation with κ^+ classes yet id cannot Borel reduce to E_T for the reasons given above. And E_T is Borel reducible to id via the reduction that sends each branch of T to itself and the non-branches of T to some fixed non-branch of T. □

Remark. Vadim and I have improved this to get (assuming $V = L$) 2^κ Borel Reducibility Degrees below id as well as Borel equivalence relations which are incomparable with id with respect to Borel reducibility.

One might hope that if a Borel equivalence relation has not just κ^+ many classes but a large number of classes then it must have a perfect set of classes (i.e., it must Borel reduce id). But also this can consistently fail:

Theorem 25 *Let κ be regular and uncountable in L. Then in a cardinal-preserving forcing extension of L, $2^\kappa = \kappa^{+++}$ and there is a Borel equivalence relation on κ^κ with exactly κ^{++} classes. (The same holds with $\kappa^{+++}, \kappa^{++}$ replaced by any pair of cardinals $\lambda_1 \geq \lambda_0$ of cofinality greater than κ.)*

Proof. Add a (weak) Kurepa tree T on κ with κ^{++} branches. The forcing for doing this is κ-closed and κ^+-cc and therefore preserves cardinals. Then follow this by adding κ^{+++} many κ-Cohen sets (by a product with supports of size less than κ). Again cardinals are preserved. But notice that the second forcing does not add branches to T as it is κ-closed. Now (as before) take the equivalence relation E_T defined by xE_Ty iff x, y are not κ branches through T or $x = y$. □

I'll return to the Silver Dichotomy later, but now turn to:

The Harrington-Kechris-Louveau Dichotomy

In the classical case E_0 is defined by: xE_0y iff $x \triangle y$ is finite. The first question to resolve is: How shall we define E_0 on κ^κ? The next result answers this question:

Theorem 26 *For λ an infinite cardinal $\leq \kappa$ define $E_{0,\lambda}$ by $xE_{0,\lambda}y$ iff $x \triangle y$ has size less than λ. Then id $\leq_B E_{0,\lambda}$, $E_{0,\lambda}$ is Borel and:*
$(*)$ $E_{0,\kappa}$ is not Borel reducible to id but $E_{0,\lambda}$ is Borel reducible to id for $\lambda < \kappa$.

In light of this we take E_0 to be $E_{0,\kappa}$.

Proof. To prove $(*)$, first suppose that λ is less than κ. For each $\alpha < \kappa$ use the axiom of choice to choose a function $f_\alpha : 2^\alpha \to 2^\alpha$ such that for x, y in 2^α, $x \triangle y$ has size less than λ iff $f_\alpha(x) = f_\alpha(y)$. Then for x, y in 2^κ, $x \triangle y$ has size less than λ iff $f_\alpha(x|\alpha) = f_\alpha(y|\alpha)$ for all $\alpha < \kappa$ (here we use $\lambda < \kappa$). So we obtain a reduction of $E_{0,\lambda}$ to id by sending x to $(f_\alpha(x) \mid \alpha < \kappa)$.

The proof that $E_{0,\kappa}$ is *not* Borel reducible to id is just as in the classical case: Suppose that f were a reduction and let x be sufficiently κ-Cohen (i.e., κ-Cohen over a transitive model of ZFC$^-$ of size κ containing the parameter for this reduction). Define $\bar{x}(i) = 1 - x(i)$ for $i < \kappa$. As $\sim xE_0\bar{x}$ we can choose $\sigma \subseteq x$, $i < \kappa$ and $j \in \{0, 1\}$ such that for sufficiently κ-Cohen y, $f(y)(i) = j$ if y extends σ and $f(y)(i) = 1 - j$ if y extends $\bar{\sigma}$. But $y = \bar{\sigma} * (x \text{ above } \bar{\sigma})$ is E_0 equivalent to x yet $f(y) \neq f(x)$, contradiction. $\Box$

Unfortunately the Harrington-Kechris-Louveau Dichotomy is provably false for κ^κ, κ uncountable:

Theorem 27 *There is a Borel equivalence relation E_0' which is strictly above id and strictly below E_0 with respect to Borel reducibility.*

Proof. We define E_0' on 2^κ as follows:

$xE_0'y$ iff
xE_0y and $\{i < \kappa \mid x(i) \neq y(i)\}$ is a finite union of intervals.

i. id $\leq_B E_0' \leq_B E_0$.
For the first reduction use $f(x) =$ the set of codes for proper initial segments of x; then $x = y \to f(x)E_0'f(y)$ and $x \neq y \to \sim f(x)E_0f(y) \to \sim f(x)E_0'f(y)$. For the second reduction: for each $\alpha < \kappa$ choose $f_\alpha : 2^\alpha \to 2^\alpha$ such that for

$x, y \in 2^\alpha$, $\{i < \kappa \mid x(i) \neq y(i)\}$ is a finite union of intervals iff $f_\alpha(x) = f_\alpha(y)$ and for $x \in 2^\kappa$ define $f(x)$ = the set of codes for the pairs $(f_\alpha(x|\alpha), x(\alpha))$, $\alpha < \kappa$; then $xE_0'y \to f(x)E_0f(y)$ and $\sim xE_0'y \to\sim f(x)E_0f(y)$.

ii. $E_0' \not\leq_B$ id.
Otherwise let M be a transitive model of ZFC$^-$ of size κ containing all bounded subsets of κ as well as a code for the Borel reduction f. Let $x \in 2^\kappa$ be κ-Cohen generic over M and define $\bar{x}(i) = 1 - x(i)$ for each $i < \kappa$. Then as $\sim xE_0\bar{x}$ there is $\alpha < \kappa$ such that $f(x) \neq f(y)$ whenever y is κ-Cohen generic over M and extends $\bar{x}|\alpha$. But then $f(x) \neq f((\bar{x}|\alpha) * (x|[\alpha, \kappa)))$, contradicting $xE_0'((\bar{x}|\alpha) * (x|[\alpha, \kappa)))$.

iii. $E_0 \not\leq_B E_0'$.
As in the previous argument choose a reduction f, a transitive model M and $x \in 2^\kappa$ which is κ-Cohen over M. Choose α_0 so that for some ordinal $i_0 < \alpha_0$, $f(x)(i_0) \neq f(y)(i_0)$ whenever y is κ-Cohen over M and extends $\bar{x}|\alpha_0$; this is possible as $\sim xE_0\bar{x}$ and therefore $\sim f(x)E_0'f(\bar{x})$. Then choose $\alpha_1 > \alpha_0$ so that for some ordinal $i_1 \in [\alpha_0, \alpha_1)$, $f(x)(i_1) = f(y)(i_1)$ whenever y is κ-Cohen over M and extends $(\bar{x}|\alpha) * (x|[\alpha_0, \alpha_1))$; this is possible as $xE_0((\bar{x}|\alpha) * (x|[\alpha_0, \kappa)))$ and therefore $f(x)E_0'f((\bar{x}|\alpha) * (x|[\alpha_0, \kappa)))$. After ω steps we obtain $\sim f(x)E_0'f(y)$ whenever y is κ-Cohen over M and extends $(\bar{x}|\alpha_0) * (x|[\alpha_0, \alpha_1)) * (\bar{x}|[\alpha_1, \alpha_2)) * \cdots$, contradicting the fact that there is such a y which is E_0 equivalent to x. $\Box$

In summary: Even for Borel equivalence relations, the Silver Dichotomy can consistently fail and the Harrington-Kechris-Louveau Dichotomy is provably false.

But there is still some hope for the Harrington-Kechris-Louveau Dichotomy. Recall that we found a Borel equivalence relation E_0' strictly between id and E_0 with respect to Borel reducibility.

Question. Suppose that a Borel equivalence relation E is not Borel reducible to id. Then is E_0' Borel reducible to E?

This seems unlikely. But so far it has not been ruled out as a possible valid generalisation of the Harrington-Kechris-Louveau Dichotomy for κ^κ.

16..-17.Vorlesungen

Regarding the Silver Dichotomy, first consider one more negative result:

Theorem 28 *There is a Δ^1_1 equivalence relation with κ^+ classes but no perfect set of classes. So the Silver Dichotomy provably fails for Δ^1_1.*

Proof. The relation is $xE^{\text{rank}}y$ iff x, y do not code wellorders or x, y code wellorders of the same length. This has exactly κ^+ classes and is Δ^1_1. Suppose T were a perfect tree whose distinct κ-branches are E^{rank}-inequivalent. Now let x be a generic branch through T (treating T as a version of κ-Cohen forcing) and let $p \in T$ be a condition forcing that x codes a wellorder of some rank $\alpha < \kappa^+$. Then any sufficiently generic branch through T extending p codes a wellorder of rank α, which contradicts the fact that there are distinct such branches in V. $\square$

So a first step toward obtaining the consistency of Silver's Dichotomy for κ^κ is the following.

Theorem 29 *The relation E^{rank} of the previous theorem is not Borel.*

Proof. For $\alpha < \kappa$ let $\mathcal{L}_\alpha$ denote the forcing to Lévy collapse α to κ (using conditions of size less than κ). If g is $\mathcal{L}_\alpha$-generic then g^* denotes the subset of κ defined by $i \in g^*$ iff $g((i)_0) \leq g((i)_1)$ where $i \mapsto ((i)_0, (i)_1)$ is a bijection between κ and $\kappa \times \kappa$.

By induction on Borel rank we show that if B is Borel then there is a club C in κ^+ such that:

$(*)$ For $\alpha \leq \beta$ in C and (p_0, p_1) a condition in $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$, (p_0, p_1) $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$-forces that (g^*_0, g^*_1) belongs to B iff it $\mathcal{L}_\alpha \times \mathcal{L}_\beta$ forces that (g^*_0, g^*_1) belongs to B.

If B is a basic open set then we may take C to consist of all ordinals greater than κ in κ^+: If B is $U(\sigma_0) \times U(\sigma_1)$ then $(p_0, p_1) \in \mathcal{L} - \alpha \times \mathcal{L}_\beta$ forces $(g^*_0, g^*_1) \in B$ exactly if (p^*_0, p^*_1) extends $(\sigma_0, sigma_1)$ where p^*_i is the set of i such that $(i)_0, (i)_1$ are in the domain of p_0 and $p_0(((i)_0) \leq p_0((i)_1)$; this is independent of the pair α, β. Inductively, suppose that B is the intersection of Borel sets B_i, $i < \kappa$, of smaller Borel rank. By intersecting clubs obtained by applying $(*)$ to the B_i's we obtain a club C ensuring the desired conclusion for B, as (p_0, p_1) forces $(g^*_0, g^*_1) \in B$ iff for each $i < \kappa$ it forces $(g^*_0, g^*_1) \in B_i$. Finally if B is the complement of the Borel set B_0 then by induction we have a club C_0 such that for $\alpha \leq \beta$ in C_0 and $(p_0, p_1) \in \mathcal{L}_\alpha \times \mathcal{L}_\alpha$, (p_0, p_1) $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$-forces $(g^*_0, g^*_1) \in B_0$ iff it $\mathcal{L}_\alpha \times \mathcal{L}_\beta$-forces this. Now thin out the club

C_0 to a club C so that for α in C, if (p_0, p_1) in $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$ and there is some β in C_0 and some (q_0, q_1) in $\mathcal{L}_\alpha \times \mathcal{L}_\beta$ below (p_0, p_1) which $\mathcal{L}_\alpha \times \mathcal{L}_\beta$-forces (g_0^*, g_1^*) in B_0 then there is such a (q_0, q_1) in $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$. Then for $\alpha \leq \beta$ in this thinner club, (p_0, p_1) $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$-forces (g_0^*, g_1^*) in B iff none of its extensions in $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$ forces (g_0^*, g_1^*) in B_0 iff none of its extensions in $\mathcal{L}_\alpha \times \mathcal{L}_\beta$ forces (g_0^*, g_1^*) in B_0 iff (p_0, p_1) $\mathcal{L}_\alpha \times \mathcal{L}_\beta$-forces (g_0^*, g_1^*) in B, completing the induction.

It follows that E^{rank} is not Borel, as otherwise we have $g_0^* E^{\text{rank}} g_1^*$ where g_0, g_1 are sufficiently generic for $\mathcal{L}_\alpha \times \mathcal{L}_\beta$ with $\alpha < \beta$. □

Now using an analogous argument we have:

Theorem 30 *Suppose that $0^\#$ exists, κ is regular in L and λ is the κ^+ of V. Then after forcing over L with the Lévy collapse turning λ into κ^+, the Silver Dichotomy holds for κ^κ.*

Proof Sketch. Suppose that E is a Borel equivalence relation in the Lévy collapse extension with parameter in L and that p is a Lévy collapse condition forcing that the Lévy collapse names $(\sigma_\alpha \mid \alpha < \lambda)$ are pairwise E-inequivalent. We may assume that the E-equivalence class of σ_α does not depend on the choice of Lévy collapse generic containing p; otherwise the class of σ_α would be different for two Lévy collapse generics which are mutually generic and then we can build a perfect set of classes by building a perfect tree of mutual generics.

Let I be a final segment of the Silver indiscernibles between κ and λ such that p belongs to L_i for i in I. For $i < j$ in I let π_{ij} be an elementary embedding from L to L with critical point i, sending i to j. Also for each $\alpha < \lambda$ let $f(\alpha)$ denote the L-rank of the name σ_α; as f is constructible, $f(i)$ is less than the least Silver indiscernible greater than i for sufficiently large $i \in I$; we assume that this holds for all $i \in I$. For each $\alpha < \lambda$ let $\mathcal{L}_\alpha$ denote the Lévy collapse to κ just of the ordinals up to and including α.

Now in analogy to the previous proof, show by induction on the Borel rank of E that there is a club C contained in I such that for $i \leq j$ in C and $(p_0, p_1) \leq (p, p)$ in $\mathcal{L}_{f(i)} \times \mathcal{L}_{f(i)}$, (p_0, p_1) $\mathcal{L}_{f(i)} \times \mathcal{L}_{f(i)}$-forces $\sigma_i^{\dot{g}_0} E \sigma_i^{\dot{g}_1}$ iff $(p_0, \pi_{ij}(p_1))$ $\mathcal{L}_{f(i)} \times \mathcal{L}_{f(j)}$-forces $\sigma_i^{\dot{g}_0} E \sigma_j^{\dot{g}_1}$. But (p_0, p_1) does $\mathcal{L}_{f(i)} \times \mathcal{L}_{f(i)}$-force $\sigma_i^{\dot{g}_0} E \sigma_i^{\dot{g}_1}$ as the class of σ_i is independent of the choice of $\mathcal{L}_{f(i)}$-generic; it follows that for $i < j$ in C some condition below (p, p) forces $\sigma_i E \sigma_j$, contradicting our assumption that σ_i, σ_j are forced to be pairwise E-inequivalent. □

18.-19.Vorlesungen

I'll now discuss some recent work regarding the analogue of the countable Borel equivalence relations for κ^κ, i.e., those Borel equivalence relations whose classes have size at most κ.

An *orbit equivalence relation* is one induced by a Borel action of a Polish group G: xEy iff $g \cdot x = y$ for some $g \in G$. Two important facts about countable equivalence relations in the clasical setting are:

E_∞. Among orbit equivalence relations induced by a Borel action of a *countable* discrete group, there is one of maximum complexity, called E_∞.

Feldman-Moore. In fact *any* countable Borel equivalence relation is the orbit equivalence relation induced by a Borel action of a countable discrete group.

The first of these facts holds true for κ^κ, but in a surprising way:

Theorem 31 *If E is the orbit equivalence relation of a Borel action of a discrete group of size at most κ on a Borel subset of 2^κ then E is Borel reducible to E_0.*

Proof Sketch. The *shift action* of a discrete group G of size κ on its power set $\mathcal{P}(G)$ is defined by:

$g \cdot X = \{g \cdot h \mid h \in X\}$.

$\mathcal{P}(G)$ is topologised to be homeomorphic to 2^κ and the action is a continuous action. Let $E(G)$ denote the orbit equivalence relation resulting from this action. Also let $E(G)^\kappa$ denote the orbit equivalence relation induced by the action of G on $\mathcal{P}(G)^\kappa$ defined by $g \cdot (X_i \mid i < \kappa) = (g \cdot X_i \mid i < \kappa)$.

Now let F_κ denote the free group on κ generators. We show that $E(F_\kappa)$ is Borel reducible to E_0. The key observation is that F_α has cardinality less than κ for $\alpha < \kappa$ (this fails when κ equals ω). For each $\alpha < \kappa$ fix a wellorder $<_\alpha$ of $\{g \cdot X \mid g \in F_\alpha \text{ and } X \subseteq F_\alpha\}$; the latter has size at most κ.

Now to reduce $E(F_\kappa)$ to E_0, map $X \subseteq F_\kappa$ to the sequence $f(X) = (f(X)_\alpha \mid \alpha < \kappa)$ where $f(X)_\alpha =$ the $<_\alpha$-least element of $\{g_\alpha \cdot (X \cap F_\alpha) \mid g_\alpha \in F_\alpha\}$. If X, Y are equivalent under shift then $f(X)_\alpha = f(Y)_\alpha$ for α large enough so that for some $g \in F_\alpha$, $g \cdot X = Y$. Conversely, if $f(X)_\alpha = f(Y)_\alpha$ for large enough α then by Fodor we can fix some $g \in F_\kappa$ such that $g \cdot (X \cap \alpha) = g \cdot (Y \cap \alpha)$ for a cofinal set of α's and therefore $g \cdot X = Y$. This verifies that we have a continuous reduction of $E(F_\kappa)$ to E_0.

Finally, to prove the Theorem we show that any orbit equivalence relation E given by a Borel action of a discrete group G of size κ on a Borel set X is Borel reducible to $E(F_\kappa)^\kappa$. This suffices, as an argument similar to the one given in the previous paragraph shows that not only $E(F_\kappa)$, but also $E(F_\kappa)^\kappa$ is Borel reducible to E_0.

First note that E Borel reduces to $E(G)^\kappa$. To see this let $\pi : \kappa \to 2^{<\kappa}$ be a bijection and define $F(x) = (F(x)_\alpha \mid \alpha < \kappa)$ where $F(x)_\alpha = \{g \mid g \cdot x \in U(\pi(\alpha))\}$ (and where as usual $U(\sigma)$ is the basic open neighbourhood determined by $\sigma \in 2^{<\kappa}$). It is straightforward to verify that this is a reduction. Now note that G is a quotient of F_κ by one of its normal subgroups and therefore we can Borel reduce $E(G)^\kappa$ to $E(F_\kappa)^\kappa$ by sending $(X_\alpha \mid \alpha < \kappa)$ to $(Y_\alpha \mid \alpha < \kappa)$ where Y_α is the pre-image of X_α under the natural projection map of F_κ onto G. $\square$

The Feldman-Moore Theorem however consistently fails for κ^κ:

Theorem 32 *Assume $V = L$. Then there is a Borel equivalence relation with classes of size 2 which is Borel reducible to id but which is not the orbit equivalence relation of any Borel action of a group of size at most κ.*

Proof Sketch. Let X be the Borel set of (Σ_ω-) Master Codes for initial segments of L of size κ. These are the subsets of κ which code the theory of a structure $(L_\alpha, \gamma)_{\gamma<\kappa}$ where $\kappa \le \alpha < \kappa^+$. Now enumerate X in L-increasing order as $(x_\alpha \mid \alpha < \kappa^+)$ and the complement of X in L-increasing order as $(y_\alpha \mid \alpha < \kappa^+)$. The bijection $f : X \to\sim X$ defined by $f(x_\alpha) = y_\alpha$ is a Borel function whose inverse is not Borel on any non-meager Borel set (otherwise the value of its inverse on sufficiently κ-Cohen sets would code collapses of arbitrarily large ordinals less than κ^+).

Now define an equivalence relation by $E(x, y)$ iff $x = y$ or $y = f(x)$ or $x = f(y)$. E is not induced by a Borel action of a discrete group of size at most κ, else the inverse of f would be Borel on a non-meager Borel set. And E is smooth as given x we can first see if x is a Master Code; if so, send x to $f(x)$ and if not send x to itself. □

Questions. (1) Are all Borel equivalence relations with classes of size at most κ Borel reducible to E_0? (2) Is the Feldman-Moore Theorem for κ^κ consistent?

Isomorphism Relations

An important class of Σ_1^1 equivalence relations is the class of *isomorphism relations*. View the elements of κ^κ as codes for structures with universe κ (for a language of size at most κ). An *isomorphism relation* is given by specifying a sentence φ of the infinitary logic $L_{\kappa^+\kappa}$ and defining:

$xE_\varphi y$ iff x, y do not code models of φ or x, y code isomorphiic models of φ.

We can eliminate the logic using the following result:

Theorem 33 *(Vaught) X is the set of codes for models of a sentence of $L_{\kappa^+\kappa}$ iff X is Borel and invariant under isomorphism (if x belongs to X and y codes a model isomorphic to the model coded by x then y also belongs to X).*

Isomorphism relations need not be Borel and there is one of maximum complexity, the relation of isomorphism of graphs.

In the classical setting, isomorphism relations are far from complete under Borel reducibility within the class of Σ_1^1 equivalence relations as a whole:

Proposition 34 *There is a Σ_1^1 equivalence relation E on reals with an equivalence class which is not Borel.*

Proof. Let X be a Σ_1^1 set of reals which is not Borel. Define E by: $E(x, y)$ iff $x, y \in X$ or $x = y$. Then X is an equivalence class of E. □

Theorem 35 *(Scott) For any countable structure $\mathcal{A}$, the set of (codes for) countable structures which are isomorphic to $\mathcal{A}$ is Borel.*

Proof. Let φ be the Scott sentence of $\mathcal{A}$, i.e., the canonical sentence of $L_{\omega_1\omega}$ whose countable models are exactly those isomorphic to $\mathcal{A}$. This set of models is Borel, as the set of countable models of any sentence of $L_{\omega_1\omega}$ is Borel. □

It follows that isomorphism on countable structures is not complete for Σ^1_1 equivalence relations under Borel reducibility, as Borel reductions take non-Borel equivalence classes to non-Borel equivalence classes.

In the case of $\kappa = \kappa^{<\kappa}$ uncountable, Scott's theorem fails and indeed:

Theorem 36 *Assume $V = L$ and let κ be the successor of a regular cardinal. Then all Σ^1_1 equivalence relations are Borel reducible to isomorphism.*

Proof. Write $\kappa = \lambda^+$ where λ is regular, let $\mathcal{Q}$ be a λ-saturated dense linear order without endpoints and let $\mathcal{Q}_0$ be $\mathcal{Q}$ together with a least point. For any subset S of κ let $\mathcal{L}(S)$ be obtained from the natural order on κ by replacing α by $\mathcal{Q}_0$ if $0 < \alpha$ belongs to S and by $\mathcal{Q}$ if α is 0 or does not belong to S.

Lemma 37 *$\mathcal{L}(S)$ is isomorphic to $\mathcal{L}(T)$ iff $S \triangle T$ is nonstationary in κ.*

Now the key use of $V = L$ is the following.

Lemma 38 *In L, any Σ^1_1 set X is Borel reducible to the collection (ideal) of nonstationary sets in the sense that there is a Borel function f such that $x \in X$ iff $f(x)$ is nonstationary.*

Lemmas 37, 38 imply that isomorphism is *complete as a set* in the sense that if X is a Σ^1_1 set then for some Borel functions g, h:

$x \in X$ iff $g(x) \simeq h(y)$

where the values of g, h are dense linear orders. Simply choose a Borel f so that $x \in X$ iff $f(x)$ is nonstationary and then define $g(x) = \mathcal{L}(f(x))$, $h(x) = \mathcal{L}(\emptyset)$.

Proof of Lemma 37. For simplicity we'll assume that κ is ω_1, so $\mathcal{Q}$ is the rational order and $\mathcal{Q}_0$ is the rational order together with a least point. Suppose $\mathcal{L}(S)$ is isomorphic to $\mathcal{L}(T)$ via the isomorphism π. For countable α let $\mathcal{L}(S)|\alpha$ be the initial segment of $\mathcal{L}(S)$ obtained from the natural order on α by replacing $i < \alpha$ by $\mathcal{Q}_0$ if $0 < i$ belongs to S and by $\mathcal{Q}$ otherwise. Then for club-many α, the restriction of π to $\mathcal{L}(S)|\alpha$ is an isomorphism from $\mathcal{L}(S)|\alpha$ onto $\mathcal{L}(T)|\alpha$. For such α, α belongs to S iff it belongs to T, as otherwise the restrictin of π to $\mathcal{L}(S)|\alpha$ would not extend to an isomorphism from $\mathcal{L}(S)$ onto $\mathcal{L}(T)$. Thus S, T agree on a club and $S \triangle T$ is nonstationary.

Conversely, suppose that $S \triangle T$ is nonstationary and choose a club C on which S, T agree. By induction on α in C build an isomorphism between $\mathcal{L}(S)|\alpha$ and $\mathcal{L}(T)|\alpha$: The base case is easy, as there is a unique countable dense linear order without endpoints. The limit cases are trivial, as the limit of isomorphisms is an isomorphism. For the case where α is the C-successor to β, use the fact that S, T agree at β to conclude that the ordinal β is replaced by the same ordering in $\mathcal{L}(S)|\alpha$ and $\mathcal{L}(T)|\alpha$. $\square$

Proof of Lemma 38. Again for simplicity we'll assume that κ is ω_1. We can write $x \in X$ iff $\varphi(\omega_1, x)$ where φ is a Σ_1 formula with a subset of ω_1 as parameter. We'll ignore that parameter.

Claim. The following are equivalent:
(a) $\varphi(x)$ holds.
(b) The set A of those countable α for which there exists a countable limit β such that

$$L_\beta \models \alpha = \omega_1 \wedge \varphi(\alpha, x \cap \alpha))$$

contains a club in ω_1.

Proof of Claim. If $\varphi(x)$ holds then choose a continuous chain $(M_i \mid i < \omega_1)$ of elementary submodels of some large ZF^- model L_θ so that x belongs to M_0 and the intersection of each M_i with ω_1 is an ordinal α_i less than ω_1. Let C be the set of α_i's, a club in ω_1. Then any α in C belongs to A by condensation.

Conversely, if $\varphi(\omega_1, x)$ fails then let C be any club in ω_1 and let D be the club of $\alpha < \omega_1$ such that $H(\alpha)$ = the Skolem Hull in some large L_θ of α together with $\{\omega_1, C\}$ contains no ordinals in the interval $[\alpha, \omega_1)$. Let α be the least limit point of D. Then α does not belong to A: If L_β satisfies $\varphi(\alpha, x \cap \alpha)$ then β must be greater than $\bar{\beta}$ where $\overline{H(\alpha)} = L_{\bar{\beta}}$ is the transitive collapse of $H(\alpha)$, because $\varphi(\alpha, x \cap \alpha)$ fails in $\overline{H(\alpha)}$. But as $D \cap \alpha$ is an element of $L_{\bar{\beta}+2}$, it follows that α is singular in L_β. Of course α does belong to C so we have shown that A does not contain C for an arbitrary club C in ω_1. $\square$ (*Claim.*)

The Claim implies that X is Borel-reducible to the collection of subsets of ω_1 which contain a club and therefore also to its dual, the nonstationary ideal. $\square$

We finish by showing how the above argument can be extended to prove Theorem 36, again assuming for simplicity that κ is ω_1. Given a Σ^1_1 equivalence relation E on 2^{ω_1} we want to produce a Borel reduction of E to isomorphism, i.e., a Borel function f such that xEy iff $f(x)$ is isomorphic to $f(y)$. The main step is to produce a Borel reduction of E to the equivalence relation on ω^{ω_1} given by:

$\eta \sim_{NS} \xi$ iff $\{\alpha < \omega_1 \mid \eta(\alpha) = \xi(\alpha)\}$ contains a club in ω_1.

Given this, we get a Borel reduction of E to dense linear orders which are coloured with ω-many colours, using the argument of Lemma 37.

To obtain the Borel reduction of E to $(\omega^{\omega_1}, \sim_{NS})$ we refine the argument of Lemma 38 as follows. For each countable α let $\beta(\alpha)$ be the largest limit ordinal $\beta > \alpha$ such that α is the ω_1 of L_β, if such a limit ordinal β exists. The ordinal $\beta(\alpha)$ does exist for club-many α and in fact for any $x \subseteq \omega_1$, there are club-many α such that $\beta(\alpha)$ exists and $x \cap \alpha$ belongs to $L_{\beta(\alpha)}$. Also let φ be a Σ_1 formula such that xEy iff $\varphi(\omega_1, x, y)$; we assume that there is no parameter in φ (other than ω_1).

Now to each $x \subseteq \omega_1$ associate the function η'_x that for each countable α for which $\beta(\alpha)$ exists and $x \cap \alpha$ belongs to $L_{\beta(\alpha)}$ assigns the L-least $z \subseteq \alpha$ in $L_{\beta(\alpha)}$ such that "$x \cap \alpha$ and z are E-equivalent in $L_{\beta(\alpha)}$", i.e., $L_{\beta(\alpha)} \vDash \varphi(\alpha, x \cap \alpha, z)$. Then by the same argument as in the proof of Lemma 38 one has: xEy iff η'_x, η'_y agree on a club. Finally, define $\eta_x(\alpha) = \pi_\alpha \circ \eta'_x(\alpha)$ where π_α is the L-least injection of $\beta(\alpha)$ into ω, in order to obtain the desired Borel reduction of E to $(\omega^{\omega_1}, \sim_{NS})$. $\square$

Question. Is it consistent that isomorphism is *not* complete for Σ^1_1 equivalence relations under Borel reduciblity?