$\mathbb{P}_{max}$

1.Vorlesung

Introduction

The beginning of the $\mathbb{P}_{max}$ story is the following result of Woodin:

Theorem 1 *If $NS_{\aleph_1}$ is saturated and there is a measurable cardinal then δ^1_2 equals $\aleph_2$.*

Here NS_κ denotes the ideal of nonstationary subsets of κ. The word "saturated" here means "$\aleph_2$-saturated", i.e., there is no antichain of size $\aleph_2$ in the quotient $\mathcal{P}(\omega_1)/\mathrm{NS}_{\aleph_1}$. The ordinal δ^1_2 is the supremum of the ranks of Δ^1_2 definable prewellorderings of the reals. It is not known if $\mathrm{NS}_{\aleph_1}$ can be saturated in the presence of CH; by this result it cannot be if a measurable cardinal exists.

A key step in the proof of the above result is that every element of $H(\aleph_2)$ belongs to a "generic iterate" of a countable model of ZFC. Woodin used this to define a forcing in $L(\mathbb{R})$ called $\mathbb{P}_{max}$ which when applied for $L(\mathbb{R})$ yields a version of $H(\aleph_2)$ which satisfies AC and has some restricted[1] but attractive absoluteness properties.

In this course we'll follow Paul Larson's article in the *Handbook of Set Theory*, which presents the basics of the $\mathbb{P}_{max}$ theory.

Iterations

Suppose that I is a normal ideal on ω_1 containing all countable subsets of ω_1. "Normal" means that I is not all of $\mathcal{P}(\omega_1)$ and whenever A is an I-positive set (i.e. a subset of ω_1 not belonging to I), $f : A \to \omega_1$ is regressive then f is constant on an I-positive set. An example is the ideal of nonstationary subsets of ω_1.

If we force with the quotient $\mathcal{P}(\omega_1)/I$ then the result is a V-ultrafilter on ω_1 (i.e., a filter on ω_1 which for every A in V contains either A or $\sim A$) and this ultrafilter U is V-normal (i.e., normal for functions in V).

[1] One cannot hope for too much absoluteness. Indeed absoluteness for class forcing extensions is not possible, nor is absoluteness for set forcing extensions with regard to arbitrary sentences about $H(\aleph_2)$.

If we form $\mathrm{Ult}(V, U)$, the ultrapower of V by U, we don't necessarily have a wellfounded model, but the canonical elementary embedding $j : V \to \mathrm{Ult}(V, U)$ has critical point ω_1^V and (identifying the wellfounded part of $\mathrm{Ult}(V, U)$ with its transitive collapse) ω_2^V is an initial segment of the ordinals of $\mathrm{Ult}(V, U)$ (as using j any subset of ω_1^V in V also belongs to $\mathrm{Ult}(V, U)$). Since I is normal:

$$A \in U \text{ iff } \omega_1^V \in j(A)$$

for subsets A of ω_1^V in V.

Sometimes we will need a weakening of ZFC, denoted by ZFC°. For now we omit the details of the definition of ZFC°. The existence of transitive models of ZFC° is provable in ZFC.

Now we turn to iterated generic ultrapowers. Suppose that M is a model of ZFC°, $I \in M$ is a normal ideal on ω_1^M and $\mathcal{P}(\mathcal{P}(\omega_1))^M$ is countable. Then there exist generics for $(\mathcal{P}(\omega_1)/I)^M$. Moreover, if $j : M \to N$ is a resulting generic ultrapower embedding then $\mathcal{P}(\mathcal{P}(\omega_1))^N$ is also countable and so there also exist generics for $(\mathcal{P}(\omega_1)/j(I))^N$. We can continue this process for ω_1 stages, as in the following definition.

Definition 2 *Let M be a model of ZFC°, I a normal ideal on ω_1^M and $\gamma \leq \omega_1$. An* iteration of (M, I) of length γ *consists of models $(M_\alpha \mid \alpha \leq \gamma)$, sets $(G_\alpha \mid \alpha < \gamma)$ and a commuting family of elementary embeddings $(j_{\alpha\beta} : M_\alpha \to M_\beta \mid \alpha \leq \beta \leq \gamma)$ such that*

1. $M_0 = M$
2. G_α is generic for $(\mathcal{P}(\omega_1)/j_{0\alpha}(I))^{M_\alpha}$
3. $j_{\alpha\alpha}$ is the identity
4. $j_{\alpha(\alpha+1)}$ is the ultrapower embedding induced by G_α
5. For limit $\beta \leq \gamma$, M_β is the direct limit of the system $(M_\alpha, j_{\alpha\delta} \mid \alpha \leq \delta < \beta)$ and for $\alpha < \beta$, $j_{\alpha\beta}$ is the induced embedding into this direct limit.

If in the above iteration γ equals ω_1 and each $\omega_1^{M_\alpha}$ is wellfounded then the set of these ordinals forms a club in ω_1. Also note that each of the embeddings $j_{\alpha\beta}$ is cofinal into the ordinals of M_β.

The models that appear in an iteration of (M, I) are called *iterates* of (M, I). In case I equals $\mathrm{NS}^M_{\aleph_1}$ then we talk about an iteration and iterates of

M. When we say that $j : (M, I) \to (M^*, I^*)$ is an *iteration* we mean that j is $j_{0\gamma}$ for an iteration of (M, I) as above with $M_\gamma = M^*$ and $I^* = j(I)$.

(M, I) is *iterable* if every iterate of (M, I) is wellfounded. This is equivalent to saying that every iterate which arises through a countable iteration of (M, I) is wellfounded.

Conditions for iterability

The basic lemma which yields iterability is the following. An ideal I is *precipitous* if each of its generic ultrapower is wellfounded.

Lemma 3 *Suppose that M is a transitive model of enough of ZFC, I is a normal precipitous ideal on ω_1^M in M. Suppose that $j : (M, I) \to (M^*, I^*)$ is an iteration of (M, I) of length $\gamma \leq \omega_1$ and γ belongs to M. Then M^* is wellfounded.*

2.-3.Vorlesungen

The following provides a sufficient condition for (generic) iterability.

Lemma 4 *Suppose that M is a transitive model of enough of ZFC, I is a normal precipitous ideal on ω_1^M in M. Suppose that $j : (M, I) \to (M^*, I^*)$ is an iteration of (M, I) of length $\gamma \leq \omega_1$ and γ belongs to M. Then M^* is wellfounded.*

Proof. It suffices to show that iterations of length γ of $(H(\kappa)^M, I)$ are wellfounded for each regular κ of M greater than the M-cardinality of $\mathcal{P}(\mathcal{P}(\omega_1))^M$, for M^* is the union of the $j(H(\kappa)^M)$ for such κ and $j \restriction H(\kappa)^M$ results from an iteration of $(H(\kappa)^M, I)$.

If not, let (γ, κ, η) be the lexicographically least triple such that for some iteration $(N_\alpha, G_\beta, j_{\alpha\delta} \mid \beta < \gamma,\ \alpha \leq \delta \leq \gamma)$ of $(H(\kappa)^M, I)$, $j_{0\gamma}(\eta)$ is illfounded. γ is a limit ordinal because I is precipitous. The triple (γ, κ, η) is definable in M as it is the least triple (γ, κ, η) for which the existence of such an iteration is forced by the Lévy collapse to ω of sufficiently large ordinals of M. Fix such an iteration $(N_\alpha, G_\beta, j_{\alpha\delta} \mid \beta < \gamma,\ \alpha \leq \delta \leq \gamma)$ and choose $\gamma^* < \gamma$, $\eta^* < j_{0\gamma^*}(\eta)$ so that $j_{\gamma^*\gamma}(\eta^*)$ is illfounded. This is possible as both γ and η are limit ordinals. Also note that the above iteration lifts to an iteration $(M_\alpha, G_\beta, j^*_{\alpha\delta} \mid \beta < \gamma,\ \alpha \leq \delta \leq \gamma)$ of (M, I).

Now by elementarity, $(j^*_{0\gamma^*}(\gamma), j^*_{0\gamma^*}(\kappa), j^*_{0\gamma^*}(\eta))$ is the lexicographically least triple such that for some iteration of $(H(j_{0\gamma^*}(\kappa))^{M_{\gamma^*}}, j^*_{0\gamma^*}(I))$, the ordinal $j^*_{0\gamma^*}(\eta)$ is sent by the iteration into the illfounded part. But there is a lexicographically smaller such triple: Consider the tail of the iteration $(N_\alpha, G_\beta, j_{\alpha\delta} \mid \beta < \gamma,\ \alpha \le \delta \le \gamma)$ starting at N_{γ^*}. This gives rise to a triple $(\gamma', \kappa', \eta^*)$ whose first component γ' is $\gamma - \gamma^*$, surely at most $j^*_{0\gamma^*}(\gamma)$, whose second component κ' equals $j^*_{0\gamma^*}(\kappa)$ and whose third component η^* is stricly less than $j^*_{0\gamma^*}(\eta)$. This is a contradiction. □

We'll also need the following two little facts.

Lemma 5 *Suppose that M is a countable transitive model of enough of ZFC and (M, I) is iterable, where $I \in M$ is a normal ideal on th ω_1 of M. Let x be a real coding the pair (M, I). Then whenever $L_\gamma[x]$ models ZFC, iterations of (M, I) of length less than γ yields models of height less than γ.*

Proof. The point is that the set of heights of models which result from a generic iteration of length δ is Σ^1_1 in x together with a code for δ and therefore bounded by an ordinal admissible in x together with this code. If δ is less than γ and $L_\gamma[x]$ models ZFC then there is a code for δ in $L_\gamma[x][g]$ where g is generic for the Lévy collapse to ω of δ and therefore the heights of models which arise from a generic iteration of length δ will be less than γ. □

Lemma 6 *Suppose that (M, I) is iterable where M satisfies enough of ZFC. Then M is closed under #'s for subsets of ω_1^M.*

Proof. If $j : (M, I) \to (M_1, I_1)$ is a generic ultrapower of (M, I) via the $(\mathcal{P}(\omega_1)/I)^M$-generic ultrafilter G_1 then in $M[G_1]$ we see that there is an elementary embedding of the $L[x]$ of M to itself for each real $x \in M$. So $M[G_1]$ thinks that every real of M has a # and therefore so does M (as set-forcing does not create new #'s). Moreover, if $A \in M$ is a subset of ω_1^M then $A = j(A) \cap \omega_1^M$ is countable in M_1 and therefore has a # in $M_1 \subseteq M[G_1]$, again implying that A also has a # in M. The fact that (M, I) is iterable implies that M is elementarily embeddable into a model containing all countable ordinals and therefore M's version of $A^\#$ for $A \subseteq \omega_1^M$ is the correct $A^\#$. □

$\mathbb{P}_{max}$

We now define the $\mathbb{P}_{max}$ forcing. When we write MA we are referring to Martin's Axiom for ccc partial orders and collections of dense sets of size ω_1. A condition in $\mathbb{P}_{max}$ is a pair $((M, I), a)$ such that:

1. M is a countable transitive model of enough of ZFC + MA.
2. I is a normal ideal in M.
3. (M, I) is iterable.
4. a belongs to $\mathcal{P}(\omega_1)^M$.
5. For some real x in M, ω_1^M equals $\omega_1^{L[a,x]}$.

$((M, I), a) \leq ((N, J), b)$ iff N belongs to $H(\omega_1)^M$ and there exists an iteration $j : (N, J) \to (N^*, J^*)$ such that:

i. $j(b) = a$.
ii. j, N^* belong to M.
iii. $I \cap N^* = J$.

4.-5.Vorlesungen

We now define the $\mathbb{P}_{max}$ forcing. When we write MA we are referring to Martin's Axiom for ccc partial orders and collections of dense sets of size ω_1. A condition in $\mathbb{P}_{max}$ is a pair $((M, I), a)$ such that:

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$((M, I), a) \leq ((N, J), b)$ iff N belongs to $H(\omega_1)^M$ and there exists an iteration $j : (N, J) \to (N^*, J^*)$ such that:

i. $j(b) = a$.
ii. j, N^* belong to M.
iii. $I \cap N^* = J^*$.

We make some remarks. (1) Suppose that $((M, I), a)$ is a condition. As M is closed under #'s for reals, a cannot be coded by a real and is therefore unbounded in ω_1^M. It follows that if $((M, I), a)$ extends $((N, J), b)$ then the iteration which shows this has length ω_1^M. (2) The requirement (ii) in the definition of extension implies that the ordering on conditions in transitive: If j_0 witnesses $((M_1, I_1), a_1) \leq ((M_0, I_0), a_0)$ and j_1 witnesses $((M_2, I_2), a_2) \leq ((M_1, I_1), a_1)$ then $j_1(j_0)$ witnesses $((M_2, I_2), a_2) \leq ((M_0, I_0), a_0)$. (3) The requirement of MA will be used to show that any iteration of (M, I) is uniquely determined by the image of a and therefore there is a unique iteration which witnesses that one condition extends another. The argument will be via almost disjoint coding.

Lemma 7 *Let $((M, I), a)$ be a $\mathbb{P}_{max}$ condition and A a subset of ω_1. Then there is at most one iteration of (M, I) for which A is the image of a, and if this iteration exists then it belongs to $L[((M, I), a), A]$.*

Proof. Choose a real x such that $\omega_1^M = \omega_1^{L[a,x]}$. By induction on $\alpha < \omega_1^M$ choose z_α^* to be the $L[a, x]$-least real distinct from the z_β^*, $\beta < \alpha$, and make the z_α^*'s almost disjoint by replacing z_α^* by $z_\alpha =$ the set of codes for finite initial segments of z_α^*. Suppose that

$$\mathcal{I} = (M_\alpha, G_\beta, j_{\delta\mu} \mid \alpha \leq \omega_1, \beta < \omega_1, \delta \leq \mu \leq \omega_1)$$

and

$$\mathcal{I}' = (M'_\alpha, G'_\beta, j'_{\delta\mu} \mid \alpha \leq \omega_1, \beta < \omega_1, \delta \leq \mu \leq \omega_1)$$

are two iterations of (M, I) such that $j_{0\omega_1}(a) = A = j'_{0\omega_1}(a)$. Then $j_{0\omega_1}(Z) = j'_{0\omega_1}(Z)$ as well, where Z is the sequence of z_α's. Write the latter as $(z_\alpha \mid \alpha < \omega_1)$.

We show by induction on $\alpha < \omega_1$ that $G_\alpha = G'_\alpha$, which implies that the two iterations are the same. Suppose that $G_\beta = G'_\beta$ for $\beta < \alpha$ and we want to show $G_\alpha = G'_\alpha$. If B is a subset of $\omega_1^{M_\alpha}$ in $M_\alpha = M'_\alpha$ then B belongs to G_α iff $\omega_1^{M_\alpha}$ belongs to $j_{\alpha\alpha+1}(B)$. Since M_α satisfies MA, there is a real y in M_α such that for $\eta < \omega_1^{M_\alpha}$, η belongs to B iff y, z_η are almost disjoint. By elementarity, $\omega_1^{M_\alpha}$ belongs to $j_{\alpha\alpha+1}(B)$ iff $y, z_{\omega_1^{M_\alpha}}$ are almost disjoint. As the latter holds also for $j'_{\alpha\alpha+1}$ we have that G_α and G'_α are the same. Moreover this gives a definition of the sequence of G_α's in terms of $(M, I), a$ and Z and hence this sequence belongs to $L[((M, I), a), A]$. $\square$

A consequence of this lemma is that if G is $\mathbb{P}_{max}$ generic over $L(\mathbb{R})$ then $L(\mathbb{R})[G] = L(\mathbb{R})[A]$ where A is the union of the a such that $((M, I), a)$ belongs to G for some (M, I).

We say that (M, I) is a *precondition* iff for some a, $((M, I), a)$ is a condition.

Lemma 8 *If (M, I) is a precondition and J is a normal ideal on ω_1 then there exists an iteration $j : (M, I) \to (M^*, I^*)$ such that $j(\omega_1^M) = \omega_1$ and $I^* = J \cap M^*$.*

Proof. First note that if $j : (M, I) \to (M^*, I^*)$ is any iteration of (M, I) of length ω_1 then I^* is contained in J. To see this, write the iteration as $(M_\alpha, G_\beta, j_{\delta\mu} \mid \alpha \leq \omega_1, \beta < \omega_1, \delta \leq \mu \leq \omega_1)$ and note that if E belongs to $I^* = j_{0\omega_1}(I)$ then $E = j_{\alpha\omega_1}(E')$ for some countable α and $E' \in j_{0\alpha}(I)$. Then for all countable $\beta \geq \alpha$, $j_{\alpha\beta}(E') \notin G_\beta$ so $\omega_1^{M_\beta} \notin E$. As the set of such $\omega_1^{M_\beta}$'s forms a club, it follows that E is nonstationary and therefore belongs to J by the normality of J.

Choose a family $(A_{i\alpha} \mid i < \omega, \alpha < \omega_1)$ of pairwise disjoint members of $\mathcal{P}(\omega_1) \setminus J$. (This is possible as there is no countably additive ideal on ω_1 containing all finite sets which is ω_1-saturated; the proof of this fact uses Ulam matrices.) We describe an iteration $(M_\alpha, G_\beta, j_{\delta\mu} \mid \alpha \leq \omega_1, \beta < \omega_1, \delta \leq \mu \leq \omega_1)$ of (M, I) by inductively choosing the G_β's. We simultaneously choose enumerations $(B_i^\alpha \mid i < \omega)$ of $\mathcal{P}(\omega_1)^{M_\alpha} \setminus j_{0\alpha}(I)$.

Given $(M_\alpha, G_\beta, j_{\delta\mu} \mid \alpha \leq \gamma,\ \beta < \gamma,\ \delta \leq \mu \leq \gamma)$, if $\omega_1^{M_\gamma}$ belongs to $A_{i\alpha}$ for some $i < \omega$ and $\alpha \leq \gamma$ then we let G_γ be any $(\mathcal{P}(\omega_1)/j_{0\gamma}(I))^{M_\gamma}$-generic over M_γ which contains $j_{\alpha\gamma}(B_i^\alpha)$. If $\omega_1^{M_\gamma}$ does not belong to any $A_{i\alpha}$ for $i < \omega$, $\alpha \leq \gamma$ then we let G_γ be any $(\mathcal{P}(\omega_1)/j_{0\gamma}(I))^{M_\gamma}$-generic over M_γ.

Now suppose that E belongs to $\mathcal{P}(\omega_1)^{M_{\omega_1}} \setminus j_{0\omega_1}(I)$. We want to show that E does not belong to J. Fix $i < \omega$ and $\alpha < \omega_1$ such that $E = j_{\alpha\omega_1}(B_i^\alpha)$. Then $\omega_1^{M_\beta}$ belongs to $j_{\alpha,\beta+1}(B_i^\alpha)$ (and therefore to E) whenever it belongs to $A_{i\alpha}$. It follows that E contains the intersection of a club with a set not in J and therefore does not belong to J. □

We next show that $\mathbb{P}_{max}$ is homogeneous in the following sense: Any two $\mathbb{P}_{max}$ conditions p_0, p_1 have extensions q_0, q_1 such that the suborders of $\mathbb{P}_{max}$ below q_0 and q_1 are isomorphic.

Lemma 9 *Assume that for every real x there is an inner model V_0 containing x and a measurable cardinal κ in V_0 whose power set in V_0 is countable in V. (This follows from the existence of "daggers", less than the existence of two measurable cardinals.) Then $\mathbb{P}_{max}$ is homogeneous.*

Proof. The hypothesis of the lemma implies that any real x belongs to the model M of some precondition (M, I): Let V_0 be an inner model containing x with a measurable cardinal κ whose power set in V_0 is countable in V. Then in V there is a generic for the forcing that over V_0 that Lévy collapses κ to become ω_1 and then forces MA. In this generic extension there is a normal precipitous ideal on κ and therefore a precondition (M, I) with M containing x.

Now suppose that $p_0 = ((M_0, I_0), a_0)$ and $p_1 = ((M_1, I_1), a_1)$ are $\mathbb{P}_{max}$ conditions. Fix a precondition (N, J) such that p_0, p_1 belong to $H(\omega_1)^N$. Applying the previous lemma in N, choose iterations $j_0 : (M_0, I_0) \to (M_0^*, I_0^*)$ and $j_1 : (M_1, I_1) \to (M_1^*, I_1^*)$ such that $I_0^* = J \cap M_0^*$ and $I_1^* = J \cap M_1^*$. Let $a_0^* = j_0(a_0)$, $a_1^* = j_1(a_1)$ and consider the conditions $q_0 = ((N, J), a_0^*)$, $q_1 = ((N, J), a_1^*)$. Then j_0, j_1 witness that q_0, q_1 are extensions in $\mathbb{P}_{max}$ of p_0, p_1.

We claim that the suborders of $\mathbb{P}_{max}$ below q_0 and q_1 are isomorphic. Indeed, suppose that $q_0' = ((N', J'), a')$ is a condition below q_0 and the iteration $j' : (N, J) \to (N', J')$ witnesses this. Then $a' = j'(a_0^*)$ and $q_1' = ((N', J'), j'(a_1^*))$ is a condition below q_1. Let π be the map defined on $\mathbb{P}_{max}$ below q_0 that sends $((N', J'), a')$ to $((N', J'), j'(a_1^*))$ as above. Then π is an isomorphism onto $\mathbb{P}_{max}$ below q_1 using the fact that iterations are uniquely determined by where they send the last component of a $\mathbb{P}_{max}$ condition. □

6.-7.Vorlesungen

$\mathbb{P}_{max}$ is countably closed

Assume that every real belongs to some $\mathbb{P}_{max}$ precondition and suppose that for each finite i, $p_i = ((M_i, I_i), a_i)$ is a $\mathbb{P}_{max}$ condition and $j_{i,i+1} : (M_i, I_i) \to (M_i^*, I_i^*)$ is an iteration witnessing $p_{i+1} < p_i$. We want to find a $\mathbb{P}_{max}$ condition below all of the p_i's. Let $(j_{ik} \mid i \leq k < \omega)$ be the commuting family of embeddings generated by the $j_{i,i+1}$'s and $a = \cup_i a_i$. Then for each i there is a unique iteration $j_{i\omega} : (M_i, I_i) \to (N_i, J_i)$ sending a_i to

a and $\omega_1^{N_i} = \omega_1^{N_0}$ for all i. We would like to put the (N_i, J_i, a)'s together to get a $\mathbb{P}_{max}$ condition below each of the p_i's. To do this we need to discuss "iterations" of the structure $((N_i, J_i) \mid i < \omega)$ and prove its "iterability". Then we can easily generalise our earlier lemma about iterating to the restriction of an arbitrary normal ideal as follows.

Lemma 10 *Suppose that I is a normal ideal on ω_1. Then there is an iteration $j^* : ((N_i, J_i) \mid i < \omega) \to ((N_i^*, J_i^*) \mid i < \omega)$ such that $j^*(\omega_1^{N_0}) = \omega_1$ and $J_i^* = I \cap N_i^*$ for each i.*

Now to complete the proof of ω-closure, choose a $\mathbb{P}_{max}$ precondition (M, I) such that $H(\omega_1)^M$ contains $(p_i \mid i < \omega)$ and apply the above lemma in M to obtain j^*. Then for each i the embedding $j^*(j_{i\omega})$ witnesses that $((M, I), j^*(a))$ is a $\mathbb{P}_{max}$ condition below p_i for each i.

An *iteration* of $((N_i, J_i) \mid i < \omega)$ of length $\gamma \leq \omega_1$ consists of sequences $((N_i^\alpha, J_i^\alpha) \mid i < \omega)$ $(\alpha \leq \gamma)$ together with normal ultrafilters G_α on $\cup_i N_i^\alpha$ $(\alpha < \gamma)$ and a commuting family of embeddings $j_{\alpha\beta} : ((N_i^\alpha, J_i^\alpha) \mid i < \omega) \to ((N_i^\beta, J_i^\beta) \mid i < \omega)$ such that

$((N_i^0, J_i^0) \mid i < \omega) = ((N_i, J_i) \mid i < \omega)$.
$j_{\alpha,\alpha+1}$ is the embedding resulting by taking the ultrapower of the $((N_i^\alpha, J_i^\alpha) \mid i < \omega)$ using G_α.
For limit β, $((N_i^\beta, J_i^\beta) \mid i < \omega)$ is the direct limit of the $((N_i^\alpha, J_i^\alpha) \mid i < \omega)$ for $\alpha < \beta$ with induced embeddings $j_{\alpha\delta}$ $(\alpha \leq \delta < \beta)$.

We claim that any iterate of $((N_i, J_i) \mid i < \omega)$ is wellfounded. It suffices to show that the ω_1 of each iterate of $((N_i, J_i) \mid i < \omega)$ is wellfounded, as for each iterate $((N_i^\alpha, J_i^\alpha) \mid i < \omega)$ of $((N_i, J_i) \mid i < \omega)$, the ordinal height of N_i^α is less than the least x_i-indiscernible greater than $\omega_1^{N_0^\alpha}$ where x_i is some real in N_{i+1}^α and hence must be wellfounded assuming that $\omega_1^{N_0^\alpha}$ is. Now we prove that the ω_1 of each iterate is wellfounded by induction on the length of the iteration. As the limit case is immediate and the general successor case follows from the case of a single ultrapower we just consider the latter. We want to see that if G is a normal ultrafilter on $\cup_i N_i$ and j the induced ultrapower embedding then $j(\omega_1^{N_0}) = \sup_i \mathrm{Ord}(N_i)$ and is therefore wellfounded. Note that by choosing reals x_i in N_{i+1} with $\mathrm{Ord}(N_i)$ less than the least x_i-indiscernible greater than $\omega_1^{N_0}$, if we let $f_i(\alpha)$ be the least x_i-indiscernible above α then $j(\omega_1^{N_0}) \geq \sup_i j(f_i)(\omega_1^{N_0}) = \sup_i \mathrm{Ord}(N_i)$. For the other direction, let $h : \omega_1^{N_0} \to \omega_1^{N_0}$ be a function in some N_i. Then the closure points of h contain a final segment of the x_i-indiscernibles and therefore $f_i > h$ on a final segment of $\omega_1^{N_0}$; it follows that $[f_i]_G > [h]_G$ so we get $j(\omega_1^{N_0}) = \sup_i \mathrm{Ord}(N_i)$.

Generalised Iterability

Let A be a set of reals. We say that a precondition (M, I) is *A-iterable* iff it is iterable, $A \cap M$ is an element of M and for any iteration $j : (M, I) \to (M^*, I^*)$ we have $j(A \cap M) = A \cap M^*$.

We show that if AD holds in $L(\mathbb{R})$ and A is a set of reals in $L(\mathbb{R})$ then there is a $\mathbb{P}_{max}$ precondition (M, I) such that $(H(\omega_1)^M, A \cap M)$ is elementary in $(H(\omega_1), A)$ and (M, I) is A-iterable. For this we need the following.

Lemma 11 *Assume AD. Then every set of ordinals belongs to an inner model in which some V-countable ordinal is measurable.*

Proof. Fix a set of ordinals Z. For each increasing $f : \omega \to \omega_1$ let $s(f)$ be the sup of the range of f and let $F(f)$ be the filter on $s(f)$ consisting of all subsets of $s(f)$ which contain all but finitely many members of Range f. Also let $N(f)$ be the inner model $L[Z, F(f)]$, a model of choice. We claim that for some f, $F(f)$ restricted to $N(f)$ is countably complete in $N(f)$, i.e., every function from $s(f)$ to ω in $N(f)$ is constant on a set in $F(f)$. It then follows that some ordinal at most $s(f)$ is measurable in $N(f)$, which proves the lemma.

Suppose that $F(f)$ is not countably complete in $N(f)$ for each f. Notice that if the ranges of f_0 and f_1 are equal modulo a finite set then $F(f_0)$ equals $F(f_1)$ so the models $N(f_0)$ and $N(f_1)$, as well as their canonical wellorders, are the same. Also note that using the canonical wellorder of $N(f)$ we can choose a function G such that $G(f) : s(f) \to \omega$ is a counterexample to the countable completeness in $N(f)$ of $F(f)$ for each f.

We use the following consequence of AD: For every function from the set of increasing ω-sequences through ω_1 to the reals there is an uncountable $E \subseteq \omega_1$ such that this function is constant on the increasing ω-sequences through E.

Now for each increasing $f : \omega \to \omega_1$ let $P(f) : \omega \to \omega$ be defined by $P(f)(n) = G(f)(f(n))$. Let E be an uncountable subset of ω_1 such that P is constant on all increasing $f : \omega \to E$. Choose $i : \omega \to \omega$ such that for all increasing $f : \omega \to E$, $G(f)(f(n)) = i(n)$ for all n. But then i must be a constant function, as if $i(n) \neq i(0)$ and we choose increasing $f, g : \omega \to E$ so that $g(m) = f(m+n)$ then $G(g)(g(0)) = i(0) \neq i(n) = G(f)(f(n)) = G(f)(g(0))$, contradicting $F(f) = F(g)$. As i is a constant function we get that $G(f)$ is constant on a set in $F(f)$ for each increasing $f : \omega \to E$, contradicting the choice of $G(f)$. $\square$

Now we show:

Theorem 12 *Assume $AD^{L(\mathbb{R})}$ and let A be a set of reals in $L(\mathbb{R})$. Then there is a $\mathbb{P}_{max}$ condition $((M, I), a)$ such that*

1. $A \cap M \in M$
2. $(H(\omega_1)^M, A \cap M)$ is elementary in $(H(\omega_1), A)$
3. (M, I) is A-iterable
4. If M^+ is a forcing extension of M and J is a normal precipitous ideal on $\omega_1^{M^+}$ in M^+ then $A \cap M^+$ is an element of A^+ and (M^+, J) is A-iterable. Moreover if $j : (M^+, J) \to (M^*, J^*)$ is an iteration of (M^+, J) then $(H(\omega_1)^{M^*}, A \cap M^*)$ is elementary in $(H(\omega_1), A)$.

Proof. Assume that there is a counterexample A. By choosing A to be definable over $L_\alpha(\mathbb{R})$ for the least possible α, we can assume that A is Δ^2_1 definable in $L(\mathbb{R})$ (relative to a real parameter). In $L(\mathbb{R})$ every Δ^2_1 set is the projection of a tree on $\omega \times \mu$ for some ordinal μ and this implies that there are trees T_0, T_1 such that any transitive model N with T_0, T_1 as members satisfies $A \cap N \in N$ and $(H(\omega_1)^N, A \cap N)$ is elementary in $(H(\omega_1), A)$. Moreover if $j : N \to N^*$ is elementary then the same holds for N^* using the trees $j(T_0), j(T_1)$.

By the lemma choose an inner model N of ZFC and a countable ordinal γ such that N contains the trees T_0, T_1 and γ is measurable in N. Let δ be a strongly inaccessible cardinal of N between γ and ω_1^V. If G is generic over N_δ for the Lévy collapse of γ to ω_1 followed by the ccc iteration to make MA true, then we obtain an iterable precondition $(N_\delta[G], I)$. It suffices to show that if M^+ is a forcing extension of $N_\delta[G]$ in which there is a normal precipitous ideal J on $\omega_1^{M^+}$ then M^+ and J satisfy conclusion 4 of the theorem.

Let N^+ be the corresponding forcing extension of $N[G]$. Then $A \cap N^+$ belongs to N^+ and $(H(\omega_1)^{N^+}, A \cap N^+)$ is elementary in $(H(\omega_1), A)$ since T_0, T_1 belong to N. Fix an iteration $j : (M^+, J) \to (M^*, J^*)$. This lifts to an iteration $j^* : (N^+, J) \to (N^*, J^*)$. Then $A \cap M^* = j(A \cap M^+) \in M^*$ and $(H(\omega_1)^{M^*}, A \cap M^*)$ is elementary in $(H(\omega_1), A)$ as N^* contains $j^*(T_0), j^*(T_1)$. $\square$

8.-9.Vorlesungen

We now prove one of Woodin's main theorems about the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$.

Lemma 13 *Suppose that $V = L(\mathbb{R})$ and AD holds. Let G be $\mathbb{P}_{max}$-generic over V, $A_G = \cup\{a \mid ((M, I), a)$ belongs to G for some $(M, I)\}$. Then in $V[G]$, if E is a subset of ω_1 then there are $((M, I), a) \in G$ and $e \in \mathcal{P}(\omega_1)^M$ such that $j(e) = E$ where j is the unique iteration of (M, I) sending a to A_G. Moreover E is nonstationary iff we can take e to belong to I.*

Proof. For the proof we need two facts. If p is a $\mathbb{P}_{max}$ condition, J is a normal ideal on ω_1 and B is a subset of ω_1 then let $\mathcal{G}_{\omega_1}(p, J, B)$ be the game where Players I and II build a descending ω_1-sequence of $\mathbb{P}_{max}$ conditions $p_\alpha = ((M_\alpha, I_\alpha), a_\alpha)$ below p where it is I's turn to choose p_α if $\alpha \notin B$ and it is II's turn to choose p_α if $\alpha \in B$; II wins iff, letting A be the union of the a_α's and $j_\alpha : (M_\alpha, I_\alpha) \to (M^*_\alpha, I^*_\alpha)$ the iteration of (M_α, I_α) sending a_α to A, $j_\alpha(I_\alpha) = J \cap M^*_\alpha$ for all α.

The following is a straightforward generalisation of an earlier lemma.

Fact 1. Player II has a winning strategy in $\mathcal{G}_{\omega_1}(p, J, B)$ iff $B \notin J$.

We also need:

Fact 2. Let $p_0 = ((M, I), a)$ be a $\mathbb{P}_{max}$ condition in G and $P \in M$ a set of $\mathbb{P}_{max}$ conditions extended by p_0. Let j be the iteration of (M, I) sending a to A_G. Then every condition in $j(P)$ belongs to G.

Proof of Fact 2. Let $(M_\alpha, G_\beta, j^*_{\alpha\delta} \mid \alpha \leq \omega_1, \beta < \omega_1, \alpha \leq \delta \leq \omega_1)$ be the iteration given by j and fix $q = ((N_0, J_0), b_0)$ in $j(P)$. Fix α_0 sich that $q \in j^*_{0\alpha_0}(P)$ and as q is extended by $j^*_{0\alpha_0}(p_0)$ we can choose $j_q \in M_{\alpha_0}$ to be the iteration of (N_0, J_0) sending b_0 to $j^*_{0\alpha_0}(a)$. Choose $p_1 = ((N_1, J_1), b_1) \in G$ such that $p_1 \leq p_0$ and $\alpha_0 < \omega_1^{N_1}$. Then $(M_\alpha, G_\beta, j^*_{\alpha\delta} \mid \alpha \leq \omega_1^{N_1}, \beta < \omega_1^{N_1}, \alpha \leq \delta \leq \omega_1^{N_1})$ is in $M_{\omega_1^{N_1}}$ and is the unique iteration of (M, I) sending a to b_1. Since $j_q(J_0) = j^*_{0\alpha_0}(I) \cap j_q(N_0)$ and $j^*_{0\omega_1^{N_1}}(I) = J_1 \cap M_{\omega_1^{N_1}}$ it follows that $j^*_{\alpha_0\omega_1^{N_1}}(j_q)$ witnesses $q \geq p_1$. □

Using these Facts we prove the lemma. Let τ be a $\mathbb{P}_{max}$-name for a subset of ω_1 and let A be a set of reals coding the set of triples (p, α, i) such that $p \in \mathbb{P}_{max}$, $\alpha < \omega_1$, $i \in 2$ and $p \Vdash \alpha \in \tau$ if $i = 1$, $p \Vdash \alpha \notin \tau$ if $i = 0$. Let $p = ((N, J), d)$ be any condition and let (M, I) be an A-iterable precondition such that p belongs to $H(\omega_1)^M$ and $(H(\omega_1)^M, A \cap M) \prec (H(\omega_1), A)$.

Applying *Fact 1* in M (where B is the set of countable limit ordinals) we obtain a descending ω_1^M-sequence of conditions $p_\alpha = ((N_\alpha, J_\alpha), d_\alpha)$ such that:

(1) $p_0 = p$
(2) $p_{\alpha+1}$ decides "$\alpha \in \tau$".
(3) Letting D be the union of the d_α's, $j_\alpha(J_\alpha) = I \cap j_\alpha(N_\alpha)$, where j_α is the iteration of (N_α, J_α) sending d_α to D.

It follows that $((M, I), D)$ is a condition below each p_α. Let e be a subset of ω_1^M in M such that for each α, $\alpha \in e$ iff $p_{\alpha+1} \Vdash \alpha \in \tau$. Suppose that $((M, I), D)$ belongs to a generic G' and let $(M_\alpha, G'_\beta, j'_{\alpha\delta} \mid \alpha \leq \omega_1, \beta < \omega_1, \alpha \leq \delta \leq \omega_1)$ be the iteration of (M, I) sending D to $A_{G'}$. We show that $j'_{0\omega_1}$ sends e to $\tau_{G'}$. Write $j'_{0\omega_1}((p_\alpha \mid \alpha < \omega_1^M))$ as $(q_\alpha \mid \alpha < \omega_1)$. Then for each $\gamma < \omega_1$, $q_{\gamma+1} \Vdash \gamma \in \tau$ iff $\gamma \in j'_{0\omega_1}(e)$ and $q_{\gamma+1} \Vdash \gamma \notin \tau$ iff $\gamma \notin j'_{0\omega_1}(e)$. By *Fact 2*, each q_γ belongs to G' so $j'_{0\omega_1}(e) = \tau_{G'}$.

Finally note that if $E = j(e)$ where $((M, I), a)$ belongs to G, e belongs to I and j is the iteration sending a to A_G, then E is disjoint from the critical sequence of the iteration j and is therefore nonstationary. Conversely, if E is nonstationary then choose a club C disjoint from E and $((M, I), a) \in G$, $e, c \in \mathcal{P}(\omega_1)^M$ such that $j(e) = E, j(c) = C$ where j is the iteration of (M, I) sending a to A_G; then c must be a club in M so e must belong to I. □

Theorem 14 *Suppose that $V = L(\mathbb{R})$ and AD holds. Let G be $\mathbb{P}_{max}$-generic over V. Then in $V[G]$, $\delta^1_2 = \omega_2$.*

Proof. (a) It suffices to show that for every $\gamma < \omega_2$ there is a real x such that the least x-indiscernible above ω_1 is greater than γ. Working in the $\mathbb{P}_{max}$ extension $V[G]$, fix a wellorder π of ω_1 of length γ. By the previous lemma we may choose a condition $((M, I), a) \in G$ and $e \in \mathcal{P}(\omega_1)^M$ such that $j(e) = \pi$, where j is the iteration of (M, I) sending a to A_G. Then γ is in $j(M)$ and so for any real c coding (M, I) is less than the least c-indiscernible above ω_1. □

10.-11.Vorlesungen

Theorem 15 *Suppose that $V = L(\mathbb{R})$ and AD holds. Let G be $\mathbb{P}_{max}$-generic over V. Then in $V[G]$, NS_{ω_1} is ω_2-saturated.*

Proof. We show that if D is dense in $\mathcal{P}(\omega_1) \setminus \mathrm{NS}$ then D contains a subset D' of size ω_1 whose diagonal union contains a club.

Let τ be a name for D and let A be the set of reals coding pairs (p, e) where $p = ((M, I), a)$ is a $\mathbb{P}_{max}$ condition, $e \in \mathcal{P}(\omega_1)^M \setminus I$ and p forces that $j(e) \in \tau$, where j is the iteration of (M, I) sending a to A_G.

Let $p = ((N, J), b)$ be any $\mathbb{P}_{max}$ condition and let (M, I) be an A-iterable precondition such that $p \in H(\omega_1)^M$ and $(H(\omega_1)^M, A \cap M)$ is elementary in $(H(\omega_1), A)$. Fix a partition $(B_i^\alpha \mid \alpha < \omega_1, i < \omega)$ in M of ω_1^M into I-positive sets and an injection $g : \omega_1^M \times \omega \to \omega_1^M$ in M such that $g(\alpha, i) \geq \alpha$ for all (α, i).

Working in M our aim is to build a descending ω_1^M-sequence of conditions $p_\alpha = ((N_\alpha, J_\alpha), b_\alpha)$ (with embeddings $j_{\alpha\beta}$ witnessing $p_\alpha \geq p_\beta$) together with enumerations $(e_i^\alpha \mid i \in \omega)$ of $\mathcal{P}(\omega_1)^{N_\alpha} \setminus J_\alpha$ and sets d_α such that $p_0 = p$ and:

(1) $d_\alpha \in \mathcal{P}(\omega_1)^{N_{\alpha+1}} \setminus J_{\alpha+1}$, $p_{\alpha+1}$ forces that $j(d_\alpha) \in \tau$, where j is the iteration of $(N_{\alpha+1}, J_{\alpha+1})$ sending $b_{\alpha+1}$ to A_G, and if $\alpha = g(\beta, i)$ for some $\beta \leq \alpha$ and $i \in \omega$, then $d_\alpha \subseteq j_{\beta,\alpha+1}(e_i^\beta)$.
(2) Each $B_i^{g(\beta,i)} \setminus j_{g(\beta,i)+1,\ \omega_1^M}(d_{g(\beta,i)})$ is nonstationary.

These conditions imply that if B is the union of the b_α's then for each α, $j_\alpha(J_\alpha) = I \cap j_\alpha(N_\alpha)$ where j_α is the iteration of (N_α, J_α) sending b_α to B, and $((M, I), B)$ extends each p_α.

Assume that we can construct the sequence as above. For each (α, i) let $d'_{\alpha i}$ be $j_{g(\alpha,i)+1,\ \omega_1^M}(d_{g(\alpha,i)})$. Then the diagonal union of $\mathcal{A}$ = the set of $d'_{\alpha i}$'s is I-large. Thus if $((M, I), B)$ belongs to the generic G and j is the iteration of (M, I) sending B to A_G then the diagonal union of $j(\mathcal{A})$ contains the critical sequence and therefore a club. We claim that $j(\mathcal{A})$ is a subset of τ_G: Write $j((p_\alpha \mid \alpha < \omega_1^M))$ as $(q_\alpha \mid \alpha < \omega_1)$. By *Fact 2*, each q_α belongs to G and since (M, I) is A-iterable, each member of $j(\mathcal{A})$ is forced to be in τ_G by some q_α, so $j(\mathcal{A})$ is contained in τ_G.

It remains to construct the above sequence satisfying conditions (1) and (2). Condition (1) is easily achieved: As τ names a dense subset of $\mathcal{P}(\omega_1)\backslash\mathrm{NS}$, for each $\alpha < \omega_1^M$ there is a pair (p^*, d^*) such that $p^* \leq p_\alpha$ and (1) holds with (p^*, d^*) in the role of $(p_{\alpha+1}, d_\alpha)$.

To achieve condition (2), fix a ladder system $(h_\alpha \mid \alpha < \omega_1,\ \alpha$ limit$)$ in M (i.e., h_α maps ω increasingly and cofinally into α for each limit α). Assuming that the p_α's have been constructed below a limit β, let $(((N_i^\beta, J_i^\beta) \mid i < \omega), b_\beta^*)$ be the limit sequence corresponding to the sequence $(p_{h_\beta(i)} \mid i < \omega)$ and for each i let $j'_{i\beta}$ be the unique iteration of $(N_{h_\beta(i)}, J_{h_\beta(i)})$ sending $b_{h_\beta(i)}$ to b_β^*. Fix a precondition (N_β, J_β) in M with $((N_i^\beta \mid i < \omega), b_\beta^*) \in H(\omega_1)^{N_\beta}$. By an earlier lemma we can choose an iteration j'_β of $((N_i^\beta, J_i^\beta) \mid i < \omega)$ in N_β such that $j'_\beta(J_i^\beta) = J_\beta \cap j'_\beta(N_i^\beta)$ for each i; also require that $\omega_1^{N_0^\beta} \in B_k^\gamma$ for some $\gamma < \beta$ and $k < \omega$ with $g(\gamma, k) < \beta$. Then, letting i' be the least i such that $h_\beta(i) \geq g(\gamma, k)$,

$$j'_{i'\beta}(j_{g(\gamma,k)+1,\ h_\beta(i')}(d_{g(\gamma,k)}))$$

is in the filter corresponding to the first step of the iteration of the sequence $((N_i^\beta, J_i^\beta) \mid i < \omega)$, ensuring (provided we let b_β be $j'_\beta(b_\beta^*)$) that $\omega_1^{N_0^\beta} \in j_{g(\gamma,k)+1,\ \beta}(d_{g(\gamma,k)})$. As the set of $\omega_1^{N_0^\beta}$'s for limit β is a club, condition (2) is thereby satisfied. □

The $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$ is a model of choice:

Theorem 16 *Assume AD in $V = L(\mathbb{R})$ and let G be $\mathbb{P}_{max}$-generic. Then AC holds in $V[G]$.*

Proof. It suffices to show that in $V[G]$, the subsets of ω_1 can be wellordered in ordertype ω_2.

First note that we at least have some choice in $V[G]$: By absoluteness, any $\mathbb{P}_{max}$ condition can be α-iterated for any countable α. It follows that if $((M, I), a)$ belongs to G then (M, I) can be ω_1-iterated, by a density argument. Thus there is an embedding j sending ω_1^M to ω_1; as M satisfies choice, it contains an injection of ω_1^M into the reals of M and by applying j we get an injection of ω_1 into the reals. This is enough to partition ω_1 into ω_1-many stationary sets.

For any $\gamma < \omega_2$ if $f_\gamma : \omega_1 \to \gamma$ is a surjection then define the "canonical function " $g_\gamma : \omega_1 \to \omega_1$ by $g_\gamma(\beta) =$ ordertype $(f_\gamma[\beta])$. Without choice we cannot choose the f_γ's, but the g_γ's are unique modulo the nonstationary ideal and so we can choose for each γ the equivalence class of g_γ modulo NS.

Claim. In $V[G]$, if A, B are stationary, costationary subsets of ω_1 then $A = g_\gamma^{-1}[B]$ mod NS for some γ.

Proof of Claim. We first use choice to show:

$(*)$ If (M, I) is a precondition and $A, B \in M$ are I-positive, co-I-positive subsets of ω_1^M in M and J is a normal ideal on ω_1 then there are an iteration $j : (M, I) \to (M^*, I^*)$ of (M, I) of length ω_1 and an ordinal $\gamma < \omega_2$ such that $I^* = J \cap M^*$ and $j(A) = g_\gamma^{-1}[j(B)]$ mod NS.

Then given any $\mathbb{P}_{max}$ condition $p_0 = ((M_0, I_0), a_0)$ forcing τ_0, τ_1 to be stationary, costationary subsets of ω_1 we can choose an A-iterable (M, I) (for an appropriate A) with $p_0 \in H(\omega_1)^M$, and apply $(*)$ in M with the ideal I to obtain an extension $((M, I), a)$ of p_0 forcing the conclusion of the Claim for τ_0^G, τ_1^G.

To prove $(*)$ let x be a real coding (M, I) and form an iteration of (M, I) so that at stage $\alpha^* =$ the least x-indiscernible greater than α, $j_{0\alpha^*}(B)$ belongs to G_{α^*} iff $j_{0\alpha}(A)$ belongs to G_α. This is possible as A, B are both I-positive and co-I-positive. The result is that for a club of x-indiscernible α, $\alpha \in j(A)$ iff $\alpha^* \in j(B)$ and therefore $A = g_\gamma^{-1}[B]$ mod NS, where γ is the least x-indiscernible greater than ω_1. $\square$ (*Claim*)

Obviously if $A_0 = g_{\gamma_0}^{-1}[B]$ mod NS, $A_1 = g_{\gamma_1}^{-1}[B]$ mod NS and $A_0 \triangle A_1$ is stationary, then $\gamma_0 \neq \gamma_1$. Now fix a stationary, costationary B and let $(A_\alpha \mid \alpha < \omega_1)$ be a partition of ω_1 into stationary sets. For X a subset of ω_1 (other than $\emptyset$ or all of ω_1) choose γ_X such that $A_X = g_{\gamma_X}^{-1}[B]$ where A_X is the union of the A_α's for α in X. Then the function $X \mapsto \gamma_X$ is an injection of a set of size $\mathcal{P}(\omega_1)$ into ω_2. $\Box$

12.Vorlesung

$\Pi_2(H(\omega_2))$ *Invariance of the* $\mathbb{P}_{max}$ *extension*

We show, assuming large cardinals, that any $\Pi_2(H(\omega_2))$ sentence that holds in a set-forcing extension of the universe also holds in the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$. By "sentence" we of course mean "sentence without parameters" as there are even $\Sigma_1(H(\omega_2))$ sentences with parameters from $H(\omega_2)$ which can be forced over $L(\mathbb{R})[G]$ but do not hold there (just take a stationary, co-stationary subset of ω_1 and add a club subset of it). However the parameter ω_1 is allowed because any $\Pi_2(H(\omega_2))$ sentence using it is equivalent to one without it.

We will use (but not prove) the following

Lemma 17 *If δ is a Woodin cardinal then the Lévy collapse $Coll(\omega_1, < \delta)$ forces that NS is precipitous.*

Theorem 18 *Suppose that there is a proper class of Woodin cardinals and P is a set partial order which forces that the Π_2 sentence φ holds in $H(\omega_2)$. Then φ holds in the $H(\omega_2)$ of the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$.*

Proof. Write φ as $\exists X \forall Y \psi(X, Y)$ where ψ is Δ_0. It suffices to show that for every $\mathbb{P}_{max}$ condition $p = ((M, I), a)$ and every $x \in H(\omega_2)^M$ there is a $\mathbb{P}_{max}$ condition $q = ((N, J), b)$ extending p so that if $j : (M, I) \to (M^*, I^*)$ is the unique iteration sending a to b then

$$H(\omega_2)^N \models \exists y \psi(j(x), y).$$

Given this, for any X in the $H(\omega_2)$ of the $\mathbb{P}_{max}$ extension we can write X as $j(x)$ where $((M, I), a)$ belongs to the $\mathbb{P}_{max}$-generic, $j : (M, I) \to (M^*, I^*)$ is the iteration of (M, I) taking a to A_G and $\psi(X, Y)$ holds in $H(\omega_2)^{M^*}$ for some Y; but then $\psi(X, Y)$ also holds in the $H(\omega_2)$ of the $\mathbb{P}_{max}$ extension because ψ is Δ_0.

Let Z be a countable elementary submodel of a large $H(\theta)$ with $((M, I), a)$, P and δ as members where δ is a Woodin cardinal such that P belongs to $H(\delta)$. Let N be the transitive collapse of Z. We know that any forcing extension of M in which NS is precipitous is iterable with respect to its NS. Let $N[g_0]$ be a $\bar{P}$-generic extension of N where $\bar{P}$ is the image of P under the transitive collapse of Z to N and let $j : (M, I) \to (M^*, I^*)$ be an iteration in $N[g_0]$ such that $I^* = \mathrm{NS}^{N[g_0]} \cap M^*$. As φ holds in $H(\omega_2)^{N[g_0]}$ there is $y \in H(\omega_2)^{N[g_0]}$ such that $\psi(j(x), y)$ holds in $H(\omega_2)^{N[g_0]}$. In $N[g_0]$ the image $\bar{\delta}$ of δ under the transitive collapse of Z is Woodin; let $N[g_0][g_1]$ be a $\mathrm{Coll}(\omega_1, <\bar{\delta})^{N[g_0]}$-generic extension of $N[g_0]$ and let $N^* = N[g_0][g_1][g_2]$ be a ccc forcing extension of $N[g_0][g_1]$ in which MA holds. Then $((N^*, \mathrm{NS}^{N^*}), j(a))$ is the desired $\mathbb{P}_{max}$ condition extending p. □

Remarks. (a) The previous result also holds if we replace $H(\omega_2)$ by $(H(\omega_2), A)$ for any set of reals A in $L(\mathbb{R})$. (b) Viale has pointed out the following variant of the previous theorem (perhaps also due to Woodin): Let $(*)$ be the axiom that AD holds in $L(\mathbb{R})$ and $L(\mathcal{P}(\omega_1))$ is a $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$. If $(*)$ holds and there is a proper class of Woodin cardinals then set-forcings which preserve $(*)$ cannot affect the truth of arbitrary first-order properties of $H(\omega_2)$. This can be viewed as an analogue to the fact that if there is a proper class of Woodin cardinals then no set-forcing can affect the truth of first-order properties of $H(\omega_1)$. These results are part of a general programme of showing that the truth of certain statements about some $H(\lambda)$ is not affected by certain set-forcings which preserve the truth of certain axioms. (c) One should not hope for too much with these "truth-invariance" results. Indeed, they appear to fall apart when replacing set-forcing by class-forcing. And no large cardinal axiom is able to ensure invariance of even $\Sigma_2(H(\omega_1))$ truth with respect to arbitrary (non-generic) extensions which satisfy it.

13.Vorlesung

Theorem 19 *Suppose that NS is saturated and there is a measurable cardinal. Then $\delta^1_2 = \omega_2$ and therefore CH fails.*

Proof. Recall that δ^1_2 is the supremum of the $(\omega_1^V)^+$ of $L[R]$ for reals R.

Suppose $\alpha < \omega_2$. Form the structure $\mathcal{A} = (H(\mu), <, \{\alpha\})$ where $<$ is a wellorder of $H(\mu)$. Then by virtue of the measurability of μ, there is an ω-sequence $(i_n \mid n < \omega)$ of ordinals less than μ such that:

1. The i_n's are indiscernibles for $\mathcal{A}$.
2. Let N be the Skolem hull of the i_n's in $\mathcal{A}$ and for any limit ordinal γ let N_γ be the "stretch" of N to γ indiscernibles, i.e., the structure generated from γ-many indiscernibles in the same way tha N is generated from the i_n's. Then N_γ is wellfounded and for $\gamma_0 < \gamma_1$, N_{γ_0} is isomorphic to an initial segment of N_{γ_1}.

As NS is saturated it is precipitous and therefore $N \vDash \text{NS}$ is precipitous. It follows that generic iterations of (N, NS) of length less than ordertype $(N \cap \text{Ord})$ are wellfounded. But for any limit ordinal γ, generic iterations of (N, NS) lift to generic iterations of (N_γ, NS) and therefore generic iterations of (N, NS) of any length are wellfounded.

Claim. Let $\bar{N}$ be the transitive collapse of N and $\bar{I} = \text{NS}^{\bar{N}}$. Then there is a generic iteration $j : (\bar{N}, \bar{I}) \to (N^*, I^*)$ of length ω_1 such that $\text{Ord}(N^*) > \alpha$.

The Theorem follows from the Claim as if we let R be a real coding the countable model $\bar{N}$ we see that α is less than $(\omega_1^V)^+$ of $L[R]$.

We prove the Claim by inductively defining iterates $\bar{N}_\gamma$ of $\bar{N}$ together with embeddings $j_\gamma : \bar{N}_\gamma \to \mathcal{A}$. Suppose that $\bar{N}_\gamma$, j_γ are defined.

Let δ_γ be the ω_1 of $\bar{N}_\gamma$ and U_γ the ultrafilter on δ_γ derived from j_γ, i.e., $X \subseteq \delta_\gamma$ belongs to U_γ iff $\delta_\gamma \in j_\gamma(X)$.

Then as $\text{NS}^{\bar{N}_\gamma}$ is saturated, U_γ is generic for $(\mathcal{P}(\omega_1)/\text{NS})^{\bar{N}_\gamma}$: Indeed, if $\bar{A} \in \bar{N}_\gamma$ is a maximal antichain in this forcing then $\bar{A}$ is a collection $(\bar{X}_i \mid i < \delta_\gamma)$ of stationary sets whose diagonal union contains a club in δ_γ, and therefore the diagonal union of $j_\gamma(\bar{A}) = (X_i \mid i < \omega_1)$ contains a club in ω_1. It follows that δ_γ belongs to this diagonal union and therefore for some $i < \delta_\gamma$, δ_γ belongs to X_i. It follows that $\bar{X}_i$ belongs to U_γ.

Now let $\bar{N}_{\gamma+1}$ be the ultrapower of N_γ by U_γ and define $j_{\gamma+1} : \bar{N}_{\gamma+1} \to \mathcal{A}$ by $j_{\gamma+1}([f]) = j_\gamma(f)(\delta_\gamma)$. At limit stages we take a direct limit and embed it into $\mathcal{A}$ in the natural way. Note that if M_γ denotes the range of j_γ, then for each γ, $M_{\gamma+1}$ is the Skolem hull in $\mathcal{A}$ of $M_\gamma \cup \{\delta_\gamma\}$. As the union M^* of the M_γ's contains α as an element and ω_1 as a subset, it follows that M^* also contains $\alpha+1$ as a subset and therefore its transitive collapse N^*, the direct limit of the $\bar{N}_\gamma$'s, has ordinal height greater than α. □

The Stationary Tower

14.-15.Vorlesungen

We now switch topics from $\mathbb{P}_{max}$ to stationary tower forcing, based on Paul Larson' book on this topic. This forcing can be used to collapse the successor of a singular cardinal using less than a measurable, to show that if there is a proper class of Woodin cardinals then truth in $L(\mathbb{R})$ is invariant under set forcing and, using Martin-Steel's work on projective determinacy, show that AD in $L(\mathbb{R})$ follows from the existence of infinitely many Woodin cardinals with a measurable above.

Generalised Stationarity

If X is any nonempty set, then a subset C of $\mathcal{P}(X)$ is *CUB* iff it is of the form $\{a \subseteq X \mid F[a^{<\omega}] \subseteq a\}$ for some $F : [X]^{<\omega} \to X$. And $S \subseteq \mathcal{P}(X)$ is *stationary* iff it intersects all CUB sets, i.e., iff for any $F : [X]^{<\omega} \to X$ there exists $a \in S$ such that $F[a^{<\omega}] \subseteq a$.

For any infinite cardinal $\kappa \leq \text{Card}\ (X)$, the set of subsets of X of cardinality κ is a stationary subset of $\mathcal{P}(X)$. If $X = \alpha$ is an ordinal of uncountable cofinality then a subset of α is also a subset of $\mathcal{P}(\alpha)$ and it is stationary in the above sense iff it is stationary in the usual sense.

Another way of expressing stationarity is in terms of structures for a countable language: $S \subseteq \mathcal{P}(X)$ is stationary iff every structure $\mathcal{A}$ with universe X has an elementary substructure with universe in S.

The following are left as exercises.

Lemma 20 *(Projection and Lifting) Suppose $X \subseteq Y$.*
(a) If S is a stationary subset of $\mathcal{P}(Y)$ then $S_X = \{a \cap X \mid a \in S\}$ is stationary in $\mathcal{P}(X)$.
(b) If S is a stationary subset of $\mathcal{P}(X)$ then $S^Y = \{a \subseteq Y \mid a \cap X \in S\}$ is a stationary subset of $\mathcal{P}(Y)$.

Lemma 21 *(Fodor) Suppose that $S \subseteq \mathcal{P}(X)$ is stationary and $F : S \to X$ is regressive, i.e., $F(a) \in a$ for each $a \in S$. Then there is $x \in X$ such that $F(a) = x$ for stationary-many a in S.*

Now we force with the associated ideals of nonstationary sets. For any X let $\mathbb{P}_X$ be the partial order of stationary subsets of $\mathcal{P}(X)$, ordered by inclusion. If G is $\mathbb{P}_X$ generic then G defined an ultrafilter U on $\mathcal{P}(X)^V$ and we can form the ultrapower $j : V \to \mathrm{Ult}(V, U) = (M, E)$. Of course the elements of M are the equivalence classes $[f]_U$ of functions $f : \mathcal{P}(X) \in V$ in V. Let id denote the identity function on $\mathcal{P}(X)$. Then id "represents" $j[X]$ in M, i.e.

Lemma 22 *$j[X]$ equals $\{m \in M \mid mE[id]_U\}$.*

Proof of lemma. Suppose that x belongs to X. Then by the definition of j, $j(x)$ is $[c_x]_U$ where c_x is the constant function on $\mathcal{P}(X)$ with value x. Now $c_x(a) = x \in a = \mathrm{id}(a)$ for CUB-many $a \in \mathcal{P}(X)$ so it follows by Łoś that $j(x) = [c_x]_U E[\mathrm{id}]_U$. Conversely, suppose that $mE[\mathrm{id}]_U$ and write $m = [f]_U$. Then $f(a) \in \mathrm{id}(a) = a$ for a set of a in U. By Fodor and genericity, there is $x \in X$ such that $f(a) = x$ for a set of a in U. but then $m = [f]_U = [c_x]_U = j(x)$. $\Box$

This lemma implies that $j[X] \cap \mathrm{Ord}^M = j[X \cap \mathrm{Ord}]$ is represented in M and therefore so are all of its initial segments. It follows that the ordertype of $X \cap \mathrm{Ord}$ is represented in M and therefore belongs to the wellfounded part of M (if we identify the wellfounded part of M with its transitive collapse).

Stationary Tower Embeddings

Note that if S is a stationary subset of $\mathcal{P}(X)$ then $X = \cup S$. So we just say that S is *stationary* iff $\cup S$ is nonempty and S is stationary in $\mathcal{P}(\cup S)$.

Definition 23 *(The Stationary Tower) Let κ be strongly inaccessible. The* full stationary tower up to *κ, denoted $\mathbb{P}_{<\kappa}$, consists of stationary $a \in H(\kappa)$, ordered as follows:*

$b \leq a$ iff
$\cup a \subseteq \cup b$ and $b_{\cup a} \subseteq a$, i.e., $z \cap (\cup a) \in a$ for each $z \in b$.

We associate a generic elementary embedding $j : V \to (M, E)$ to a $\mathbb{P}_{<\kappa}$-generic G as follows. For each nonempty $X \in H(\kappa)$ define $U_X = \{b_X \mid b \in G$ and $X \subseteq \cup b\}$, where as before b_X is the projection of b to X, i.e. the set of all $z \cap X$ for z in b.

Claim. U_X is an ultrafilter on $\mathcal{P}(X)^V$ extending the CUB filter on $\mathcal{P}(X)^V$. And for $X \subseteq Y$, S belongs to U_X iff $S^Y = \{Z \subseteq Y \mid Z \cap X \in S\}$ belongs to U_Y.

Proof. Any CUB subset of $\mathcal{P}(X)$ is compatible with each stationary set and therefore belongs to G and hence to U_X. We must show that U_X is an ultrafilter on $\mathcal{P}(X)^V$. It suffices to show that if $S \subseteq \mathcal{P}(X)$ then any b can be extended to a c such that c_X is contained in or disjoint from S. We may assume that X is a subset of $\cup b$. Let b^+ be set set of $z \in b$ such that $z \cap X$ belongs to S and b^- the set of $z \in b$ such that $z \cap X$ does not belong to S. Then b^+_X is contained in S and b^-_X is disjoint from S. Let c be b^+ if this is stationary and otherwise b^-. The last claim follows easily from the definitions. □

Now for each X form the ultrapower by U_X to get $j_X : V \to (M_X, E_X)$. And for $X \subseteq Y$ define $j_{XY} : M_X \to M_Y$ by $j_{XY}([f]_{U_X}) = [f_Y]_{U_Y}$ where $f_Y : \mathcal{P}(Y) \to V$ is defined by $f_Y(Z) = f(Z \cap X)$. This defines a direct system of models (M_X, E_X) with embeddings. Let (M, E) denote the direct limit of this directed system and j the corresponding embedding of V into this direct limit. For each $a \in G$ and $f : \cup a \to V$ in V we let $[f]_G$ denote the member of M represented by f. The following is a straightforward adaptation of Lemma 22.

Fact. The identity function id_X on $\mathcal{P}(X)$ represents $j[X]$ in M, i.e., $j[X] = \{b \in M \mid bE[\mathrm{id}_X]_G\}$.

Identify the wellfounded part of M with its transitive collapse. Then by this Fact, X and $j \restriction X$ belong to M for each $X \in H(\kappa)$ and therefore $H(\kappa)$ is a subset of M. Also, as $j[X]$ is an element of M we obtain the usual description of an ultrafilter in terms of its associated ultrapower embedding: $U_X = \{a \subseteq \mathcal{P}(X) \mid j[X] \in j(a)\}$. Thus $a \in G$ iff $j[\cup a] \in j(a)$ and $[f]_G = j(f)(j[\cup a])$ when f has domain $\mathcal{P}(\cup a)$.

As $j \restriction H(\alpha)$ belongs to M for each cardinal $\alpha < \kappa$, it follows that $G \cap H(\alpha) = \{a \in H(\alpha) \mid j[\cup a] \in j(a)\}$ also belongs to M for each cardinal $\alpha < \kappa$.

Fact. For $\alpha < \kappa$, α is represented in M by the function $f : \mathcal{P}(\alpha) \to \alpha$ given by $f(Z) = \mathrm{ot}(Z)$.

It follows that for $\beta \subseteq \cup a$:

$a \Vdash j(\alpha) \leq \beta$ iff
$\mathrm{ot}(Z \cap \beta) \geq \alpha$ for "almost all" Z in a (i.e., for some CUB C in $\mathcal{P}(\cup a)$, $\mathrm{ot}(Z \cap \beta) \geq \alpha$ for all Z in $a \cap C$).

Thus a forces $j(\alpha) = \alpha$ iff $\mathrm{ot}(Z \cap \alpha) = \alpha$ for almost all Z in a.

Completely Jónsson Cardinals

κ is *completely Jónsson* iff it is strongly inaccessible and for each stationary $a \in H(\kappa)$, the set of $X \subseteq H(\kappa)$ such that $X \cap (\cup a) \in a$ and X has cardinality κ is stationary in $\mathcal{P}(H(\kappa))$.

Ramsey cardinals are completely Jónsson and measurable cardinals are Ramsey, so these cardinals are not very big. Also, as complete Jónsson-ness is a Π^1_1 property, it follows that measurable cardinals are also limits of completely Jónsson cardinals.

Completely Jónsson cardinals are relevant for the following reason. Suppose that κ is a strongly inaccessible limit of completely Jónsson cardinals. Then any $a \in \mathbb{P}_{<\kappa}$ has an extension b forcing $j(\alpha) = \alpha$ for some $\alpha < \kappa$: Choose $\alpha < \kappa$ to be completely Jónsson and such that a belongs to $H(\alpha)$ and let b be $\{Z \subseteq H(\alpha) \mid Z \cap \alpha$ has cardinality β and $Z \cap (\cup a) \in a\}$; then b extends a and forces $j(\alpha) = \alpha$. Thus if κ is a strongly inaccessible limit of completely Jónsson cardinals, it follows that j has unboundedly many fixed points below κ. In fact κ is also a fixed point of j as it is not hard to show that j is continuous at strongly inaccessibles.

Also note that if κ is a strongly inaccessible limit of completely Jónsson cardinals then any set in the $H(\kappa)$ of $V[G]$ belongs to the wellfounded part of M, κ belongs to the wellfounded part of M (as $j(\kappa) = \kappa$) and so the $H(\kappa)$ of M equals the $H(\kappa)$ of $V[G]$. Thus $j(H(\kappa)) = H(\kappa)$ of $V[G]$.

Forcing Applications

Example 1. (Universality for set forcing) Suppose that there is a proper class of completely Jónsson cardinals. Let $\mathbb{P}_\infty$ denote the stationary tower class forcing, using arbitrary stationary sets a as conditions. Then $\mathbb{P}_\infty$ is universal for set-forcing in the sense that any set forcing is a regular subforcing of $\mathbb{P}_\infty$. To see this, suppose that $Q \in V$ is a set forcing and choose a cardinal α such that the power set of Q in V belongs to $H(\alpha)$. Then consider the stationary set $a =$ the set of countable subsets of $H(\alpha)$. If G is $\mathbb{P}_\infty$ generic over V below the condition a then $H(\alpha)^V$ is countable in $V[G]$. It follows that the V-power set of Q is countable in $V[G]$ and therefore there are Q-generics in $V[G]$.

Example 2. (Stretching a "core model") Again suppose that there is a proper class of completely Jónsson cardinals and let $\mathbb{P}_\infty$ be as in Example 1. If G is $\mathbb{P}_\infty$ generic over V then we get an elementary embedding from V into $V[G]$. In particular unlike L, any formula "$V = K$" satisfied by an inner model K with a proper class of competely Jónsson cardinals is also satisfied by one of its nontrivial class generic extensions: if φ were such a formula then φ would also be true in $K[G]$ when G is $\mathbb{P}_\infty$ generic over K. However it must be said that $K[G]$ may fail to obey replacement when K is adjoined as an additional predicate.

16.Vorlesung

Example 3. (Generalised Namba forcing) Again suppose that there is a proper class of completely Jónsson cardinals and let $\gamma < \lambda$ be regular. Let a be $\{\alpha < \lambda \mid \text{cof }(\alpha) = \gamma\}$, a stationary subset of $\mathcal{P}(\lambda)$. Suppose that a belongs to a $\mathbb{P}_\infty$ generic G with associated $j : V \to V[G]$. Since a belongs to G, $j[\lambda] \in j(a)$ and since a consists of ordinals, so does $j(a)$. Thus $j[\lambda]$ is an ordinal and therefore j is the identity on λ. Moreover by elementarity, $j(a)$ consists of those ordinals less than $j(\lambda)$ which have cofinality $j(\gamma) = \gamma$ in $V[G]$; so in fact $j[\lambda] = \lambda$ is an ordinal less than $j(\lambda)$ of cofinality γ in $V[G]$. As j is the identity on λ, cardinals below λ are preserved and if 2^δ is less than λ then no new subsets of δ are added.

For example, we could have GCH in V and with $\mathbb{P}_\infty$ add no new bounded subsets of $\aleph_\omega$ but change the cofinality of $\aleph_{\omega+1}$ to $\aleph_7$. By core model theory, such a weird effect cannot be achieved if ZFC is preserved by adding V as an additional predicate, without using more than a Woodin cardinal and probably this would need a supercompact cardinal.

Wellfoundedness

Suppose that G is $\mathbb{P}_{<\delta}$ generic with resulting embedding $j : V \to M$. We'll show that if δ is a Woodin cardinal then M is wellfounded and closed in $V[G]$ under sequences of length less than δ.

Suppose that D is a subset of $\mathbb{P}_{<\delta}$. Then $Y \prec V_{\delta+1}$ *captures* D iff there is $d \in D \cap Y$ such that $Y \cap (\cup d) \in d$. If D is an antichain then the choice of d is unique: if $d' \in Y \cap D$ is distinct from d there is a function h such that no Z closed under h satisfies both $Z \cap (\cup d) \in d$ and $Z \cap (\cup d') \in d'$; as h may be chosen in Y and Y is closed under such an h one cannot have $Y \cap (\cup d') \in d'$. Also note that if $A \subseteq V_{\delta+2}$ and stationary-many $Y \in A$ capture the antichain D then in the forcing $\mathbb{P}_\infty$, A is compatible with some $d \in D$: By Fodor we can thin A to A' consisting of Y which capture D with the same choice of $d \in D \cap Y$; then A' extends both A and d.

We also define $\mathrm{sp}(D)$ as follows. For sets $X \subseteq Y$, we say that Y *end extends* X iff $X = Y \cap V_\alpha$ where α is the rank of X (i.e. the least α such that X is a subset of V_α). Then $\mathrm{sp}(D)$ consists of all $X \prec V_{\delta+1}$ of size $< \delta$ such that $D \in X$ and there exists $Y \prec V_{\delta+1}$ such that:

(1) X is a subset of Y.
(2) Y end extends $X \cap V_\delta$.
(3) Y captures D.

D is *semiproper* iff $\mathrm{sp}(D)$ contains a club in $\mathcal{P}_\delta(V_{\delta+1})$.

Lemma 24 *Let η be an infinite cardinal less than δ. Suppose that for each sequence $(D_\alpha \mid \alpha < \eta)$ of predense subsets of $\mathbb{P}_{<\delta}$ there are arbitrarily large strongly inaccessible $\gamma < \delta$ such that $D_\alpha \cap \mathbb{P}_{<\gamma}$ is semiproper in $\mathbb{P}_{<\gamma}$ for all $\alpha < \eta$. Then the ultrapower (M, E) arising from a $\mathbb{P}_{<\delta}$ generic G is closed under sequences of length η in $V[G]$. In particular, this ultrapower is wellfounded.*

Lemma 25 *Suppose that δ is a Woodin cardinal. Then for each sequence $(D_\alpha \mid \alpha < \delta)$ of predense subsets of $\mathbb{P}_{<\delta}$ there are arbitrarily large strongly inaccessible $\gamma < \delta$ such that $D_\alpha \cap \mathbb{P}_{<\gamma}$ is semiproper in $\mathbb{P}_{<\gamma}$ for each $\alpha < \gamma$.*

17.Vorlesung

Lemma 26 *Let η be an infinite cardinal less than δ. Suppose that for each sequence $(D_\alpha \mid \alpha < \eta)$ of predense subsets of $\mathbb{P}_{<\delta}$ there are arbitrarily large strongly inaccessible $\gamma < \delta$ such that $D_\alpha \cap \mathbb{P}_{<\gamma}$ is semiproper in $\mathbb{P}_{<\gamma}$ for all $\alpha < \eta$. Then the ultrapower (M, E) arising from a $\mathbb{P}_{<\delta}$ generic G is closed under sequences of length η in $V[G]$. In particular, this ultrapower is wellfounded.*

Lemma 27 *Suppose that δ is a Woodin cardinal. Then for each sequence $(D_\alpha \mid \alpha < \delta)$ of predense subsets of $\mathbb{P}_{<\delta}$ there are arbitrarily large strongly inaccessible $\gamma < \delta$ such that $D_\alpha \cap \mathbb{P}_{<\gamma}$ is semiproper in $\mathbb{P}_{<\gamma}$ for each $\alpha < \gamma$.*

Proof of Lemma 26. Fix $a_0 \in \mathbb{P}_{<\delta}$ and a term τ for an η-sequence of ordinals in (M, E). For $\alpha < \eta$ let A_α be a maximal antichain of conditions a such that $a \Vdash \tau(\alpha) = [f]_G$ for some $f : a \to \text{Ord}$. By the hypothesis of the lemma there is a strongly inaccessible $\gamma < \delta$ such that:

(1) $a_0 \in V_\gamma$, $\eta < \gamma$.
(2) $A_\alpha \cap P_{<\gamma}$ is semiproper for each $\alpha < \eta$.

Let a be the set of $X \prec V_{\gamma+1}$ such that:

X has size less than γ.
$X \cap (\cup a_0) \in a_0$.
X captures A_α for each $\alpha \in X \cap \eta$ (i.e., for $\alpha \in X \cap \eta$ there is $b \in X \cap A_\alpha$ such that $X \cap (\cup b) \in b$).

Claim. a is stationary in $\mathcal{P}_\gamma(V_{\gamma+1})$.

Proof of Claim. Fix $H : [V_{\gamma+1}]^{<\omega} \to V_{\gamma+1}$. Since a_0 is stationary we may choose $X_0 \prec V_\delta$ of size less than γ containing all relevant parameters (including H) such that $X_0 \cap (\cup a_0) \in a_0$. Define an elementary chain $(X_\alpha \mid \alpha \in X_0 \cap \eta)$ as follows: If $\alpha \in X_0 \cap \eta$ is a limit ordinal then let X_α be the union of the X_β, $\beta \in X_0 \cap \alpha$. At successor stages, since $A_\alpha \cap \mathbb{P}_{<\gamma}$ is semiproper we can choose $Y \prec V_\delta$ of size less than γ such that:

(1) X_α is a subset of Y.
(2) $Y \cap V_\gamma$ end-extends $X_\alpha \cap V_\gamma$.
(3) Y captures A_α (i.e., $Y \cap (\cup b) \in b$ for some $b \in A_\alpha \cap Y$).

(Formally speaking, we only get $Y \prec V_{\gamma+1}$ but this can be easily improved to $Y \prec V_\delta$.) Choose $X_{\alpha+1}$ to be such a Y. Let X be the union of the X_α, $\alpha \in X_0 \cap \eta$. Then X has size less than γ and as X_0 contains H, $X \cap V_{\gamma+1}$ is closed under H. And, for each $\alpha \in X_0 \cap \eta$, as $X_{\alpha+1}$ captures A_α and $X \cap V_\gamma$ end-extends $X_{\alpha+1} \cap V_\gamma$ it follows that X captures A_α. Thus a is stationary in $\mathcal{P}_\gamma(V_{\gamma+1})$ and the Claim is proved.

Now we define a function $f : a \to V$ that is forced to represent τ. Recall that if we define $g_\eta : a \to V$ by $g_\eta(X) = X \cap \eta$ then g_η represents $j[\eta]$. We define f so that for each $X \in a$, $f(X)$ is a function with domain $X \cap \eta$, so that $[f]_G$ will be a function in the ultrapower with domain $j[\eta]$. What we want to have is: $[f]_G(j(\alpha)) = \tau_G(\alpha)$ for each $\alpha < \eta$. For then f represents the function from $j[\eta]$ to M given by $j(\alpha) \mapsto \tau_G(\alpha)$ and therefore τ_G belongs to M.

Fix $X \in a$ and $\alpha \in X \cap \eta$. As X captures A_α we can choose $b \in X \cap A_\alpha$ such that $X \cap (\cup b) \in b$. The choice of b is unique as A_α is an antichain. Now as b belongs to A_α we can choose f_α such that $b \Vdash [f_\alpha]_{\dot G} = \tau(\alpha)$ and we define:

$$f(X)(\alpha) = f_\alpha(X \cap (\cup b)).$$

We claim that f works, i.e., for each $\alpha < \eta$, $a \Vdash [f]_{\dot G}(j(\alpha)) = \tau(\alpha)$.

Fix $\alpha < \eta$ and G generic for $\mathbb{P}_{<\delta}$, a an element of G. Let $\bar a \in G$ consist of those $X \in a$ such that $\alpha \in X$. Now each $X \in \bar a$ captures A_α with a unique $b \in A_\alpha \cap X$ such that $X \cap (\cup b) \in b$. By normality and the genericity of G we may fix $a_1 \le \bar a$ and $b_1 \in A_\alpha \cap \mathbb{P}_{<\gamma}$ such that $a_1 \in G$ and for all $Y \in a_1$, $b_1 \in Y \cap A_\alpha$ and $Y \cap (\cup b_1) \in b_1$. As a_1 extends b_1 it follows that b_1 belongs to G. So since $b_1 \Vdash [f_\alpha]_{\dot G} = \tau(\alpha)$ and $f(X)(\alpha) = f_\alpha(X \cap (\cup b_1))$ for each $X \in a_1$, it follows that $[f]_G(j(\alpha)) = \tau_G(\alpha)$, as desired. □

18.Vorlesung

Lemma 28 *Suppose that δ is a Woodin cardinal. Then for each sequence $(D_\alpha \mid \alpha < \delta)$ of predense subsets of $\mathbb{P}_{<\delta}$ there are arbitrarily large strongly inaccessible $\gamma < \delta$ such that $D_\alpha \cap \mathbb{P}_{<\gamma}$ is semiproper in $\mathbb{P}_{<\gamma}$ for each $\alpha < \gamma$.*

Proof of Lemma 28. Recall that δ is Woodin iff for each $f : \delta \to \delta$ there is an elementary embedding $j : V \to M$ with critical point $\gamma < \delta$ such that γ is closed under f and $V_{j(f)(\gamma)}$ is contained in M.

Now fix an $f : \delta \to \delta$ with limit ordinal values such that $\gamma < f(\gamma)$ for all $\gamma < \delta$ and for all strongly inaccessible $\gamma < \delta$ closed under f:

(a) For all $\alpha < \gamma$, $D_\alpha \cap \mathbb{P}_{<\gamma}$ is predense in $\mathbb{P}_{<\gamma}$.
(b) If $\alpha < \gamma$ is such that $D_\alpha \cap \mathbb{P}_{<\gamma}$ is not semiproper in $\mathbb{P}_{<\gamma}$, there exists a condition in $D_\alpha \cap V_{f(\gamma)}$ compatible with

$$a = \{X \prec V_{\gamma+1} \mid \operatorname{card}(X) < \gamma \text{ and } X \notin \operatorname{sp}(D_\alpha \cap \mathbb{P}_{<\gamma})\}.$$

Note that (b) is possible as since $D_\alpha \cap \mathbb{P}_{<\gamma}$ is not semiproper in $\mathbb{P}_{<\gamma}$ the set a above is stationary and therefore compatible with an element of D_α as D_α is predense.

Now apply Woodinness to get $j : V \to M$ with critical point $\gamma < \delta$ closed under f such that $V_{j(f)(\gamma)}$ is contained in M. We claim that γ works, i.e., $D_\alpha \cap \mathbb{P}_{<\gamma}$ is semiproper for all $\alpha < \gamma$. Fix such an α and suppose that $D_\alpha \cap \mathbb{P}_{<\gamma}$ is not semiproper. Let a be as in (b) above; thus a is stationary. Then $\mathbb{P}^M_{<\gamma} = \mathbb{P}_{<\gamma}$ and $j(D_\alpha) \cap \mathbb{P}_{<\gamma} = D_\alpha \cap \mathbb{P}_{<\gamma}$ is not semiproper in M so by the elementarity of j there exists $b \in j(D_\alpha) \cap V^M_{j(f)(\gamma)}$ which is compatible with $a^M = a$ in $j(\mathbb{P}_{<\delta})$. Note that b is stationary in V since $V_{j(f)(\gamma)}$ is contained in M. Let c be the greatest lower bound of a, b.

We may assume that $j(\delta) = \delta$. Choose $X \prec V_\delta$ such that $X \cap (\cup c) \in c$ and $b, j \restriction V_{\gamma+1}$ and $<$ belong to X where $<$ is a wellorder of $j(V_{\gamma+1})$ which belongs to M. As $\cup a = V_{\gamma+1}$, a consists of sets of size less than γ and c extends a, it follows that $X \cap V_{\gamma+1}$ has size less than γ. So $j(X \cap V_{\gamma+1}) = j[X \cap V_{\gamma+1}]$ and the latter belongs to $j(a)$ and hence not to $j(\operatorname{sp}(D_\alpha \cap \mathbb{P}_{<\gamma}))$. We obtain a contradiction by obtaining a witness Y to the fact that $j[X \cap V_{\gamma+1}]$ does in fact belong to $j(\operatorname{sp}(D_\alpha \cap \mathbb{P}_{<\gamma}))$.

We take Y to be the Skolem hull in $j(V_{\gamma+1})$ of $\{b\} \cup j[X \cap V_{\gamma+1}] \cup (X \cap (\cup b))$, using the wellorder $<$ in $X \cap M$. Note that as all of these sets belong to M, so does Y. And as all of these sets are subsets of X and $j(V_{\gamma+1})$ is an element of X, it follows that Y is a subset of X. Now Y contains $j[X \cap V_{\gamma+1}]$ and since $j[X \cap V_{\gamma+1}] \cap V_{j(\gamma)} = j[X \cap V_{\gamma+1}] \cap V_\gamma$, it follows that Y end-extends $j[X \cap V_{\gamma+1}]$ below $j(\gamma)$. Finally, b witnesses that Y captures $j(D_\alpha \cap \mathbb{P}_{<\gamma})$ since b belongs to Y and $Y \cap (\cup b) = X \cap (\cup b) \in b$. □