Ideals and Generic Elementary Embeddings

1.-2.Vorlesungen

Introduction

This course is based on Matt Foreman's article in the Handbook of Set Theory with the above title. Surely we can't cover the entire article, but I hope to unearth the highlights. Most of the following introductory comments are copied from Foreman's introduction.

Large cardinals are typically defined as critical points of elementary embeddings $j : V \to M$ which are "internal" to the universe V of all sets. The word "internal" can be taken to be "definable with parameters", or more generously, "amenable" in the sense that $j \restriction x$ belongs to V for any set x. The idea behind *generic elementary embeddings* is to allow embeddings j which are internal not necessarily to V, but to a generic extension $V[G]$ of V. Thus the universe in which j is internal can be larger than the domain of j.

The power of this idea is that by allowing j to be "external" to its domain V, we have the possibility that the critical point of j is small, perhaps even ω_1, something not possible with traditional large cardinal embeddings. As we will see, this has many interesting applications.

Important parameters for traditional large cardinal embeddings $j : V \to m$ with critical point κ are the sizes of the ordinals $j(\kappa^{+n})$, n finite, and closure properties of M. For generic elementary embeddings we have a third parameter, namely the nature of the forcing that gives rise to the model $V[G]$ to which j is internal.

Basic Facts

We review some facts about Boolean algebras. Recall that these are structures $\mathbb{B} = (B, \wedge, \vee, \neg, 0, 1)$ which are isomorphic to a field of subsets of some set Z with $\wedge, \vee, 0, 1$ corresponding to $\cap, \cup, \emptyset, Z$ and $\neg$ corresponding to complement within Z. In a Boolean algebra $\mathbb{B}$ we write $b_0 \leq b_1$ for $b_0 \wedge b_1 = b_0$.

A Boolean algebra $\mathbb{B}$ is *κ-complete* iff any subset X of B of size $< \kappa$ has a least upper bound, denoted $\sum X$. Equivalently, every such X has a greatest lower bound, denoted by $\prod X$. In forcing terms, $\sum X$ forces the generic to intersect X: $\sum X \Vdash \dot{G} \cap X \neq \emptyset$. $\mathbb{B}$ is *complete* iff it is κ-complete for all κ.

A *homomorphism* from the Boolean algebra $\mathbb{B}$ to the Boolean algebra $\mathbb{C}$ is a function that preserves $\wedge, \vee, \neg$ (but not necessarily $0, 1$). It is *κ-complete* iff it also preserves least upper bounds and greatest lower bounds of sets of size $< \kappa$.

A partial order $\mathbb{P} = (P, \leq)$ is *separative* iff whenever p, q belong to P and $p \not\leq q$, there is some $r \leq p$ which is incompatible with q. If $\mathbb{P}$ is not separative then forcing with $\mathbb{P}$ is equivalent to forcing with its *separative quotient*, the separative partial order obtained by factoring $\mathbb{P}$ by the equivalence relation: $p \sim q$ iff the elements of P compatible with p are the same as those compatible with q. And if $\mathbb{P}$ is separative then it is isomorphic to a dense subset of (the nonzero elements of) a unique complete Boolean algebra, which we denote by $\mathbb{B}(\mathbb{P})$ (even when $\mathbb{P}$ is not separative and is replaced by its separative quotient).

Let $\mathbb{B}$ be a Boolean algebra. A nonempty subset I of B is an *ideal on* $\mathbb{B}$ iff it is closed under finite joins and $\leq$. Its *dual filter* is $\breve{I} = \{\neg A \mid A \in I\}$. For any $S \subseteq B$ the ideal generated by S is denoted by $\bar{S}$. An ideal is *proper* iff it does not contain 1. We assume that all of our ideals are proper. A *prime ideal* is a maximal, proper ideal and its dual is called an *ultrafilter*.

In case $\mathbb{B}$ is a the Boolean algebra of all subsets of some set Z, instead of *ideal on* $\mathbb{B}$ we say *ideal over* Z. We assume then that the ideal is *nonprincipal*, which means that all singletons $\{z\}$ for z in Z belong to it.

An ideal I is *κ-complete* iff it is closed under joins of size $< \kappa$. For ω_1-complete we also write *countably complete* or *σ-additive*. The *completeness of I*, denoted $\mathrm{comp}(I)$, is the least κ such that I is not κ^+-complete. This is a regular cardinal.

Assume now that I is an ideal over Z.

If Y is a subset of Z, then we say that I *concentrates on* Y iff Y belongs to the dual filter $\breve{I}$. I is *uniform* iff it contains all subsets of Z of cardinality less than $\mathrm{card}(Z)$.

We are often interested in the quotient Boolean algebra $\mathcal{P}(Z)/I$, whose elements are the equivalence classes of subsets of Z under the equivalence relation $S \sim_I T$ iff $S \triangle T \in I$. The equivalence class of S is denoted $[S]_I$. If I is κ-complete then so is $\mathcal{P}(Z)/I$.

I^+ denote the *I-positive sets*, i.e., the subsets of Z not belonging to I. For S in I^*, we let $I \restriction S$ denote the ideal $I \cap \mathcal{P}(S)$. If I is κ-complete then so is $I \restriction S$. More generally, if $S \subseteq T$ are I-positive then $\mathrm{comp}(I \restriction S)$ is at least $\mathrm{comp}(I \restriction T)$: If $(A_i \mid i < \kappa)$ belong to $I \restriction S$ where κ is less than $\mathrm{comp}(I \restriction T)$ then the A_i's also belong to $I \restriction T$ and therefore have union in $I \restriction T$; of course this union also belongs to $\mathcal{P}(S)$ and therefore to $I \restriction S$.

We can think of this as an ideal over Z by identifying it with the ideal generated by it together with the set $Z \setminus S$. The quotient of $\mathcal{P}(Z)$ mod this ideal is isomorphic to $\mathcal{P}(S)/I \cap \mathcal{P}(S)$.

I is *atomless* iff $\mathcal{P}(Z)/I$ contains no atoms. This means that each set not in I can be split into two disjoint sets also not in I.

For a property φ, we say that I is *nowhere* φ iff $I \restriction S$ fails to satisfy φ for each I-positive set S.

The *saturation* of I, denoted $\mathrm{sat}(I)$, is the least κ so that $\mathcal{P}(Z)/I$ has the κ-cc, i.e., all antichains are of size $< \kappa$. I is *λ-saturated* iff $\mathrm{sat}(I) \leq \lambda$. This is always a regular cardinal.

Generic Ultrapowers

Suppose that I is an ideal over Z. Then forcing over V with the positive elements of $\mathcal{P}(Z)/I$ produces an ultrafilter G on $\mathcal{P}(Z)/I$ with the following property: If $([S_j]_I \mid j \in J)$ is a maximal antichain in V below some element $[S]_I$ of G, then $[S_j]_I$ belongs to G for some $j \in J$. If $\mathcal{P}(Z)/I$ is complete then this is equivalent to saying that if $[T_j]_I$ belongs to G for each $j \in J$ then $\prod_j [T_j]_I$ also belongs to G. Instead of working with equivalence classes mod I we could also work directly with $\mathcal{P}(Z) \setminus I$, producing an ultrafilter on $\mathcal{P}(Z)^V$, which we identify with G.

Now form the "generic ultrapower" by taking functions $f : Z \to V$ which belong to V and say that two such functions f, g are equivalent iff $\{z \in$

$Z \mid f(z) = g(z)\}$ belongs to G. We write $[f]$ for the equivalence class of f and introduce the "membership relation E of the ultrapower" by $[f]E[g]$ iff $\{z \in Z \mid f(z) \in g(z)\}$ belongs to G. In this way we obtain a structure $(V^Z/G, E)$ with an elementary embedding

$$j : V \to V^Z/G$$

that is definable in $V[G]$ via $j(x) = [c_x]$, where c_x denotes the constant function on Z with value x.

Lemma 1 *Suppose that I is an ideal over Z, let $G \subseteq \mathcal{P}(Z)/I$ be generic and $j : V \to V^Z/G$ the associated generic elementary embedding. Let id denote the identity function on Z. Then:*

1. For $A \subseteq Z$, A belongs to G iff $[id]Ej(A)$, where E is the membership relation of the ultrapower.
2. For all $g : Z \to V$ in V, $[g] = j(g)([id])$.

This follows from the Loś theorem, because $[\mathrm{id}]Ej(A)$ means that $\{z \in Z \mid z \in A\}$ belongs to G, i.e., A belongs to G. Also as for any $z \in Z$, $g(z) = g(\mathrm{id}(z))$, it follows that $[g]$ equals $j(g)([\mathrm{id}])$.

Now we turn to the question of well-foundedness for V^Z/G. We say that I is *precipitous* iff V^Z/G is well-founded for all generic G.

If V^Z/G is well-founded then we can replace it by its transitive collapse M, which is a submodel of $V[G]$. We also think of $j : V \to V^Z/G$ as an embedding from V to M via this identification. If $g : Z \to V$ belongs to V then we denote the unique element of M corresponding to $[g]$ by $[g]^M$. The embedding $j : V \to M$, if not the identity, must move some ordinal, as it must move the least rank of a set which is moved. As usual, the least ordinal moved is called the *critical point*, denoted $\mathrm{crit}(j)$.

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Proposition 2 *Let I be precipitous, $G \subseteq \mathcal{P}(Z)/I$ generic and $j : V \to M \subseteq V[G]$ the associated embedding. Then j is not the identity and crit(j) is the largest κ such that there is an $S \in G$ with comp($I \restriction S$) $= \kappa$.*

Proof. Recall that if $S \subseteq T$ are I-positive then $\mathrm{comp}(I \restriction S) \geq \mathrm{comp}(I \restriction T)$. Also note that if $\kappa = \mathrm{comp}(I \restriction T)$ and T is the union of κ-many sets $(A_\alpha \mid \alpha < \kappa)$ from I, then in fact $\mathrm{comp}(I \restriction S) = \mathrm{comp}(I \restriction T)$ for all I-positive $S \subseteq T$: Define $B_\alpha = A_\alpha \cap S$ and note that the union of the B_α's is all of S. Let us say that an I-positive T is *I-exact* iff T is the union of $\mathrm{comp}(I \restriction T)$-many sets in I.

Now observe that any I-positive T has an I-exact subset S: Let $(A_\alpha \mid \alpha < \kappa)$ be sets in $I \restriction T$ with union not in $I \restriction T$, where $\kappa = \mathrm{comp}(I \restriction T)$. Then the union S of the A_α's is an I-positive subset of T and $\mathrm{comp}(I \restriction S)$ is at least κ, as S is a subset of T, and is at most κ, as S is the union of κ-many sets in $I \restriction S$. If follows that S is I-exact.

So by genericity, there is an I-exact element T of G. Also note that $\kappa = \mathrm{comp}(I \restriction T)$ is the largest possible value of $\mathrm{comp}(I \restriction S)$ for S in G, as G is a filter. Let $(A_\alpha \mid \alpha < \kappa)$ be sets in I with union T.

Define $F : Z \to \kappa$ by sending z in T to the least α such that z belongs to A_α and z not in T to 0. Then $j(\alpha) < [F]$ for each $\alpha < \kappa$ because $\alpha < F(z)$ for I-almost all z for each such α, and $[F] < j(\kappa)$ because $F(z) < \kappa$ for all z. So j has a critical point and $\mathrm{crit}(j) = \gamma$ is at most κ. But now choose $S \subseteq T$ in G and F' in V so that S forces $[F'] = \gamma = \mathrm{crit}(j)$. Then $F'(z) < \gamma$ for I-almost all $z \in S$ so we may assume that this is the case for all $z \in S$. For $\alpha < \gamma$ let B_α be $\{z \in S \mid F'(z) < \alpha\}$; then each B_α belongs to $I \restriction S$ but the union of the B_α's is all of S. It follows that $\mathrm{comp}(I \restriction S) = \kappa$ is at most $\gamma = \mathrm{crit}(j)$, as desired. □

Remarks. (a) Note that the above proof shows that the critical point of j is $\mathrm{comp}(I \restriction S)$ for any I-exact S in G. (b) It follows from the Proposition that any precipitous ideal I is countably complete: Otherwise let S be the union of countably many sets in I which does not belong to I and let G be generic containing S. Then by the previous, the critical point of j would be ω, which is impossible.

We next present a combinatorial equivalent of precipitousness. For a partial order P, a *tree of maximal antichains* is a sequence $(A_n \mid n \in \omega)$ of maximal antichains of P such that A_{n+1} refines A_n, i.e., each element of A_{n+1} extends an element of A_n. A *branch* through such a tree is a descending sequence $(p_n \mid n \in \omega)$ such that p_n belongs to A_n for each n.

In the following we identify elements a of $\mathcal{P}(Z)$ with their equivalence classes $[a]_I$ in $\mathcal{P}(Z)/I$.

Proposition 3 *I is precipitous iff for any I-positive set S and any tree of maximal antichains $(A_n \mid n \in \omega)$ below S there is a branch $(a_n \mid n \in \omega)$ through this tree such that $\bigcap_n a_n \neq \emptyset$.*

Proof. Suppose that I is precipitous and $(A_n \mid n \in \omega)$ is a tree of maximal antichains below the I-positive set S. Let G be generic below S and $j : V \to M$ the associated embedding. Since the A_n's are maximal antichains, for each n there is a_n in A_n belonging to G, i.e., $[\mathrm{id}]_I$ belongs to $j(a_n)$. Thus the tree

$$\{(a_0^*, \ldots, a_k^*) \mid a_i^* \in j(A_i), [\mathrm{id}]_I \in a_i^*, [a_{i+1}^*]_{j(I)} \leq [a_i^*]_{j(I)} \text{ for all } i\}$$

has an infinite branch. Therefore it has an infinite branch in M. So M satisfies that there is a sequence $(a_n^* \mid n \in \omega)$ with $a_n^* \in j(A_n)$ so that $\bigcap_n a_n^* \neq \emptyset$; the statement of the Proposition then follows by elementarity.

Now suppose that I is not precipitous. Choose an I-positive S and names $\dot{F}_n$ so that S forces $\dot{F}_n : Z \to V$ belongs to V and $[\dot{F}_{n+1}]E[\dot{F}_n]$ for all n, where as usual E denotes the membership relation of the ultrapower. Now build a tree of antichains A_n so that each a in A_n forces $\dot{F}_n \restriction a$ to be a particular function f_n^a in V and if $a_{n+1} \in A_{n+1}$ is a subset of $a_n \in A_n$ then $f_{n+1}^{a_{n+1}}(z) \in f_n^{a_n}(z)$ for all z in a_{n+1}. Then this tree has no infinite branch with nonempty intersection, as if z belonged to the intersection we would get an infinite descending $\in$-chain $(f_n^{a_n}(z) \mid n \in \omega)$. □

Precipitousness also has a game-theoretic characterisation: Players I and II alternate moves, resulting in an ω-sequence $S_0 \supseteq A_1 \supseteq \cdots$ of I-positive sets. Player II wins if the intersection of the S_n's is nonempty. Then I is precipitous iff Player I does *not* have a winning strategy. This is not hard to see, as strategies for Player I correspond to trees of antichains.

The Disjointing Property

An ideal I over Z has the *disjointing property* iff every antichain in $\mathcal{P}(Z)/I$ has a pairwise disjoint set of representativies.

Proposition 4 *If I is κ^+-saturated and κ-complete then I has the disjointing property.*

Proof. If $(S_\alpha \mid \alpha < \lambda)$ forms an antichain, then replace S_α by $T_\alpha = S_\alpha \setminus \bigcup_{\beta<\alpha} S_\beta$. By κ^+-saturation, λ is at most κ and therefore by κ-completeness, T_α differs from S_α only by a set in I. □

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An easy consequence of the disjointing property is the following:

Proposition 5 *Suppose that I has the disjointing property and suppose that P = the positive elements of $\mathcal{P}(Z)/I$ forces that $\dot{f}$ is a function in V with domain Z. Then there is $g : Z \to V$ in V such that P forces $[\dot{f}] = [g]$.*

Proof. Let $(S_\alpha \mid \alpha < \lambda)$ be an antichain in P such that for each α, S_α forces $\dot{f} = g_\alpha$ for some particular $g_\alpha : Z \to V$ in V. By the disjointing property we can assume that the S_α's are disjoint subsets of Z. Now define $g(z) = g_\alpha(z)$ if z belongs to S_α, $g(z) = 0$ if z belongs to no S_α. Then each S_α forces $\dot{f}$ to equal g on a set in the generic, so we are done. □

The disjoint property is important for the following reason:

Proposition 6 *Suppose that I is countably complete and has the disjointing property. Then I is precipitous and if $j : V \to M \subseteq V[G]$ is the generic ultrapower given by the $\mathcal{P}(Z)/I$-generic G, then $M^\kappa \cap V[G] \subseteq M$, where $\kappa = crit(j)$.*

Proof. For the precipitousness it suffices to show that for any I-positive set S and any tree of maximal antichains $(A_n \mid n \in \omega)$ below S there is a branch $(S_n \mid n \in \omega)$ through this tree such that $\bigcap_n S_n \neq \emptyset$.

By the disjointing property, we can assume that the elements of A_n are pairwise disjoint. We can also assume (without using the disjointing property again) that A_{n+1} strongly refines A_n in the sense that if S_{n+1} belongs to A_{n+1} then for all S_n in A_n, either S_{n+1} is a subset of S_n or is disjoint from S_n.

As each A_n is a maximal antichain, the complement of $\cup A_n$ belongs to I. By the countable completeness of I, there is some $z \in Z$ which belongs to $\cup A_n$ for each n. Then we get the desired branch $(S_n \mid n \in \omega)$ by choosing S_n to be the unique element of A_n such that z belongs to S_n.

To prove the second conclusion, let $(\dot{x}_\alpha \mid \alpha < \kappa)$ be a sequence of names for elements of M and let S be an element of G which forces $\mathrm{crit}(j) = \kappa$. Using the disjointing property and Proposition 5, choose a fixed $k : Z \to V$ in V so that S forces $[k]^M = \kappa$. Again by the disjointing property we can choose g_α's in V so that S forces $[g_\alpha]^M = \dot{x}_\alpha$ for each $\alpha < \kappa$. Now define $g : Z \to V$ by $g(z) = (g_\alpha(z) \mid \alpha < k(z))$; then S forces $[g]^M = (\dot{x}_\alpha \mid \alpha < \kappa)$: To see this, write $j((g_\alpha \mid \alpha < \kappa))$ as $(g^*_\alpha \mid \alpha < j(\kappa))$ and note that $g^*_\alpha = j(g_\alpha)$ for $\alpha < \kappa$. So S forces the following:

$[g]^M = j(g)([\mathrm{id}]) =$
$(g^*_\alpha([\mathrm{id}]) \mid \alpha < j(k)([\mathrm{id}])) =$
$(g^*_\alpha([\mathrm{id}]) \mid \alpha < [k]^M) =$
$(j(g_\alpha)([\mathrm{id}]) \mid \alpha < \kappa) =$
$([g_\alpha]^M \mid \alpha < \kappa) = (\dot{x}_\alpha \mid \alpha < \kappa)$, as desired. □

Another consequence of the disjointing property is the following.

Theorem 7 *If I has the disjointing property then $\mathcal{P}(Z)/I$ is a complete Boolean algebra.*

Proof. If not let κ be least so that some subset B of $\mathcal{P}(Z)/I$ has no least upper bound. By the leastness of κ we may assume that B can be enumerated as $(b_\alpha \mid \alpha < \kappa)$ where the b_α's are increasing. We can also assume that $b_0 = 0$ and b_λ is the least upper bound of the b_α, $\alpha < \lambda$, for limit $\lambda < \kappa$. Set $a_\alpha = b_{\alpha+1} - b_\alpha$ for each $\alpha < \kappa$; then the set $\mathcal{A}$ of the a_α's form an antichain without a least upper bound, because any least upper bound for it would also be a least upper bound for B.

Enlarge $\mathcal{A}$ to a maximal antichain $\mathcal{A} \cup \mathcal{C}$ and choose disjoint representatives $\{A_\alpha \mid \alpha < \kappa\} \cup \{C_\beta \mid \beta < \gamma\}$ of the elements of this maximal antichain. Then we claim that the class of the union A of the A_α's is the least upper bound of the classes of the A_α's: Otherwise, there is some B almost containing each A_α but not almost containing A. As $A \setminus B$ is I-positive it must have I-positive intersection with some A_α or some C_β. But as B almost contains each A_α, $A \setminus B$ is almost disjoint from each A_α, and as A is disjoint from each C_β so is $A \setminus B$. Contradiction! □

Normal Ideals

Normality is a notion that applies to ideals I over Z where Z is of the form $\mathcal{P}(X)$ (or a subset of this) for some set X. A function $f : A \to X$, where A is a subset of $\mathcal{P}(X)$, is *regressive* iff $f(a) \in a$ for each $a \in A$. Then I is *normal* iff whenever $f : A \to X$ is regressive and A is I-positive, there is an I-positive $B \subseteq A$ on which f is constant.

Normality can also be phrased in terms of diagonal intersections or unions. If $(A_x \mid x \in X)$ is a collection of subsets of Z then the *diagonal union* $\triangledown(A_x \mid x \in X)$ is the set of z such that z belongs to A_x for some $x \in z$; the *diagonal intersection* $\triangle(A_x \mid x \in X)$ is the set of z such that z belongs to A_x for all $x \in z$. Then I is normal iff it is closed under diagonal unions iff its dual filter $\breve{I}$ is closed under diagonal intersections.

If I is a normal ideal over a subset of $\mathcal{P}(X)$ and f is a function from X to X then I-almost all $A \subseteq X$ are closed under f, provided I is also *fine*, i.e., for each $x \in X$, I-almost all A contain x as an element. For, if A were not closed under f for an I-positive set of A's, we could choose a regressive g so that $f(g(A)) \notin A$ for all such A and then by normality get a single value z so that $f(z) \notin A$ for an I-positive set of A's. But this contradicts fineness.

If I is countably complete we can generalise the previous to countably many functions of any positive arity. Thus:

Proposition 8 *I is a countably complete, normal and fine ideal over a subset of $\mathcal{P}(X)$ and for each $i \in \omega$, f_i is a function from X^{n_i} to X for some finite n_i, then A is closed under each f_i for I-almost all A.*

Sums in $\mathcal{P}(Z)/I$ when I is a normal, fine ideal on Z (and Z is a subset of $\mathcal{P}(X)$ for some X) can be described in terms of diagonal unions.

Proposition 9 *Suppose that I is a normal, fine ideal over $Z \subseteq X$ and $\mathcal{A}_x$ is subset of Z for each x in X. Then the least upper bound in $\mathcal{P}(Z)/I$ of the classes of the $\mathcal{A}_x$'s is the class of $\triangledown(\mathcal{A}_x \mid x \in X)$.*

Proof. For $x \in X$ we know by fineness that x belongs to A for I-almost all A; so for I-almost all A, if A belongs to to $\mathcal{A}_x$ then A also belongs to $\triangledown(\mathcal{A}_x \mid x \in X)$, by the definition of diagonal union. In other words, $\mathcal{A}_x$ is I-almost contained in $\triangledown(A_x \mid x \in X)$ for each x. For the converse it suffices to show that if $\mathcal{B}$ is I-positive and contained in $\triangledown(\mathcal{A}_x \mid x \in X)$ then $\mathcal{B} \cap \mathcal{A}_x$

is I-positive for some x. For each B in $\mathcal{B}$ choose $x(B) \in B$ so that B belongs to $\mathcal{A}_{x(B)}$. Apply normality to fix $x(B) = x$ for an I-positive set of B's in $\mathcal{B}$. Thus $\mathcal{B} \cap \mathcal{A}_x$ is I-positive, as desired. □

Recall that κ^+-saturated and κ-complete ideals have the disjointing property; in the case of normal ideals there is a sufficient condition which does not assume any completeness.

Proposition 10 *Suppose that I is a normal, fine ideal over $Z \subseteq \mathcal{P}(X)$ and I is card$(X)^+$-saturated. Then I has the disjointing property.*

Proof. If $\mathcal{A}$ is an antichain then by the hypothesis we can assume that $\mathcal{A}$ is of the form $\{[A_x] \mid x \in X_0\}$ where X_0 is a subset of X. By fineness we can also assume that x belongs to each element of A_x for each $x \in X_0$. For each distinct pair x, y in X_0 choose a set $C_{x,y}$ in $\breve{I}$ = the filter dual to I so that $A_x \cap A_y \cap C_{x,y}$ is empty. Then $C = \{z \mid z \in C_{x,y}$ for all $x, y \in z\}$ also belongs to $\breve{I}$, by normality. Now we can disjointify the A_x's by replacing A_x with $A_x \cap C$ for each $x \in X_0$. □

Note that the disjointing property for an ideal over $Z \subseteq \mathcal{P}(X)$ immediately gives card$(Z)^+$-saturation and therefore $(2^{\text{card}(X)})^+$-saturation.

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Recall that the generic ultrapower via a countably complete ideal with the disjointing property is closed under κ-sequences, where κ is the critical point of the associated generic embedding. If we also assume normality then we get even more closure.

Theorem 11 *Suppose that I is a normal, fine, precipitous ideal over $Z \subseteq \mathcal{P}(X)$, let λ be the cardinality of X and suppose that $G \subseteq \mathcal{P}(Z)/I$ is generic. Let $j : V \to M$ be the associated embedding. Then $\mathcal{P}(\lambda) \cap V \subseteq M$. If in addition I has the disjointing property then $M^\lambda \cap V[G] \subseteq M$.*

Proof. We can assume that $X = \lambda$.

Claim. Let I be a normal fine ideal on $Z \subseteq \mathcal{P}(X)$ and G, $j : V \to M$ as above. Then $[\text{id}] = j[\lambda]$ (id "represents" $j[\lambda]$).

Proof of Claim. By fineness, for each $\alpha < \lambda$, $\{z \mid \alpha \in z\}$ belongs to the filter dual to I. So for each $\alpha < \lambda$, $[\mathrm{id}]Ej(\{z \mid \alpha \in z\})$ and of course $j(\{z \mid \alpha \in z\}) = \{z \mid j(\alpha) \in z\}$; i.e., $j(\alpha)E[\mathrm{id}]$ for each $\alpha < \lambda$. Conversely, if $[f]E[\mathrm{id}]$ then $A = \{z \mid f(z) \in z\}$ belongs to G; by normality it is dense below A to force that for some $\alpha < \lambda$, $\{z \mid f(z) = \alpha\}$ belongs to the generic. So by the genericity of G, $[f] = [c_\alpha] = j(\alpha)$ for some $\alpha < \lambda$ (where c_α is the constant function with value α). □ (Claim)

To prove the Theorem, suppose that A is a subset of λ in V. Consider the function $f_A(z) = A \cap z$. Then $[f_A] = j(f_A)([\mathrm{id}]) = j(f_A)(j[\lambda]) = j(A) \cap j[\lambda]$. As A can be easily recovered from $j(A) \cap j[\lambda]$ and $j[\lambda]$ and these both belong to M, it follows that A belongs to M.

Assume now that I has the disjointing property. Let $(\dot{a}_\alpha \mid \alpha < \lambda)$ be a sequence of names of elements of M. Use the disjointing property to obtain functions $\mathcal{G} = (g_\alpha \mid \alpha < \lambda)$ such that $[g_\alpha] = \dot{a}_\alpha$ is forced. Write $j(\mathcal{G})$ as $(j(g)_\alpha \mid \alpha < j(\lambda))$. Now define $g : Z \to V$ by $g(z) = (g_\alpha(z) \mid \alpha \in z)$. Then $[g] = j(g)(j[\lambda]) = (j(g)_\beta(j[\lambda]) \mid \beta \in j[\lambda])$. So the function that sends $\alpha < \lambda$ to $j(g)_{j(\alpha)}(j[\lambda]) = j(g_\alpha)(j[\lambda]) = [g_\alpha]$ belongs to M and is interpreted by G to be G's interpretation of $(\dot{a}_\alpha \mid \alpha < \lambda)$. □

More general facts

We look at a limitation on the closure of the generic ultrapower M in $V[G]$ as well as continuity points of the embedding $j : V \to M$ associated to the generic ultrapower.

Proposition 12 *Suppose that I is an ideal over Z and $j : V \to M$ is the embedding associated with a well-founded generic ultrapower V^Z/G. Then $j[\mathit{card}(Z)^+]$ does not belong to M.*

Proof. If not, let $a \in \mathcal{P}(Z)$ force $[f] = j[\mathrm{card}(Z)^+]$. Let $a_0 = \{z \in a \mid \mathrm{card}(f(z)) > \mathrm{card}(Z)\}$ and $a_1 = \{z \in a \mid \mathrm{card}(f(z)) \leq \mathrm{card}(Z)\}$. Choose an ordinal $\alpha < \mathrm{card}(Z)^+$ which does not belong to any $f(z)$, $z \in a_1$. Now choose a function $g : a \to \mathrm{card}(Z)^+$ so that $g(z) = \alpha$ for $z \in a_1$, g is injective on a_0 and $g(z) \in f(z)$ for all $z \in a_0$. Now there are two cases: If a_1 belongs to G then $j(\alpha)$ is not E-below $[f]$. If a_0 belongs to G then $[g]$ is E-below $[f]$ but as g is injective, $[g] \neq j(\beta)$ for $\beta < \mathrm{card}(Z)^+$. Both cases contradict our hypothesis about f. □

Proposition 13 *Suppose that $j : V \to M$ is a generic elementary embedding. Let κ be the critical point of j and suppose that $\eta < \kappa$ is regular, λ is an ordinal and $V, V[G]$ agree on which ordinals $< \lambda$ have cofinality η. Then $j[\lambda]$ is η-closed in $V[G]$ (i.e., if $\alpha < \sup j[\lambda]$ is a limit point of $j[\lambda]$ and has cofinality η in $V[G]$, then α belongs to $j[\lambda]$).*

Proof. If not, then choose an ordinal $\alpha < \sup j[\lambda]$ which is a limit point of $j[\lambda]$, does not belong to $j[\lambda]$ and has cofinality η in $V[G]$. Let $\delta < \lambda$ be least so that $j(\delta)$ is greater than α. As α has cofinality η in $V[G]$ it follows that δ also has cofinality η in $V[G]$ and therefore cofinality η in V. But η is less than the critical point of j so it follows that j is continuous at δ, contradiction. □

There is a partial converse to the previous.

Proposition 14 *Suppose that $j : V \to M$ is a generic elementary embedding, $j[\lambda]$ is η-closed in $V[G]$ and M is closed under η-sequences in $V[G]$. Then $V, V[G]$ agree on which ordinals $< \lambda$ have cofinality η.*

Proof. If not, let μ be the least counterexample. Then μ is V-regular, else $\mathrm{cof}^V(\mu)$ would be a smaller counterexample. So $j(\mu)$ is regular in M. As $j[\lambda]$ is η-closed, it follows that j is continuous at μ and therefore the cofinality of $j(\mu)$ in $V[G]$ is η. As M is closed under η-sequences in $V[G]$ it follows that $j(\mu)$ has cofinality η in M and therefore as $j(\mu)$ is regular in M, we have $j(\mu) = \eta$. But this contradicts the fact that η is less than μ. □

Canonical functions

Suppose that I is a normal, fine countably complete ideal over $Z \subseteq \mathcal{P}(\lambda)$. We define functions $(f_\alpha \mid \alpha < \lambda^+)$ such that f_α is forced to represent α in the generic ultrapower for each $\alpha < \lambda^+$.

First define f_α for $\alpha < \lambda$ by

$$f_\alpha(z) = \mathrm{ot}\ (z \cap \alpha).$$

Then to define f_α for α at least λ choose a bijection $g : \lambda \to \alpha$ and inductively define

$$f_\alpha(z) = \sup\{f_{g(\eta)}(z) + 1 \mid \eta \in z\}.$$

Note that if g_1, g_2 are any two bijections from λ onto α then $\{g_1(\eta) \mid \eta \in z\} = \{g_2(\eta) \mid \eta \in z\}$ for a set of z in the filter dual to I, by the normality of I; it follows that f_α is well-defined modulo I. The f_α's are called the *canonical functions for normal ideals on $Z \subseteq \mathcal{P}(\lambda)$.*

Proposition 15 *Let G be generic for $\mathcal{P}(Z)/I$ where $Z \subseteq \mathcal{P}(X)$ and X has cardinality λ. Suppose that I is normal, fine and countably complete. Then $(\lambda^+)^V$ is included in the well-founded part of V^Z/G and for each $\alpha < (\lambda^+)^V$, $[f_\alpha]^G = \alpha$.*

Proof. We may assume that X equals λ. We have seen that if $j : V \to V^Z/G$ is the canonical embedding, then $[\mathrm{id}]^G = j[\lambda]$. So V^Z/G is a model of ZFC which contains a well-founded set of ordertype λ. It follows that the ordinals of V^Z/G have an initial segment of ordertype λ.

We show that in fact that the ordinals of V^Z/G are well-founded up to λ^+: Let $(f_\alpha \mid \alpha < \lambda^+)$ be the canonical functions. Then:

1. For $\alpha < \beta < \lambda^+$, $\{z \in \mathcal{P}(\lambda) \mid f_\alpha(z) < f_\beta(z)\}$ belongs to the filter dual to I.
2. For any normal ideal I and $\alpha < \lambda^+$, if $h(z) < f_\alpha(z)$ for an I-positive set A of z, then there is an I-positive subset B of A and $\beta < \alpha$ such that $h(z) = f_\beta(z)$ for z in B.

So for generic $G \subseteq \mathcal{P}(Z)/I$, the $[f_\alpha]^G$, $\alpha < \lambda^+$ form a well-ordered initial segment of the ordinals of V^Z/G or ordertype λ^+. □

Example. Let M be a well-founded model of $V = L$ and suppose that $G \subseteq (\mathcal{P}(\omega_1)/NS_{\omega_1})^M$ is generic over M and let N be the generic ultrapower. By the above, N is well-founded up to ω_2^M. Moreover ω_1^M is countable in N. So a bijection between ω and ω_1^M appears at some ordinal level of the L-hierarchy of N; this level must be beyond the first ω_2^M ordinals of N. If we define $f : \omega_1 \to \omega_1$ in M by $f(\alpha) =$ the least β such that α is countable in L_β then $[f]^G = j(f)([\mathrm{id}]) = j(f)(\omega_1^M)$ is greater than the first ω_2^M ordinals of N and therefore for any $\delta < \omega_2^M$, $f(\alpha) > f_\delta(\alpha)$ for a set of α in G. As G is arbitrary, it follows that f dominates each f_δ on a closed unbounded set of α's. (This last fact can however be shown quite easily without generic ultrapowers.)

9.-10.Vorlesungen

Ideals and Changing Cofinalities

Large cardinals can be used to change cofinalities in interesting ways, using Prikry-style forcings. Generic ultrapowers provide another way of doing this, with dramatic effects.

Lemma 16 *Suppose that I is a normal, fine, precipitous ideal over $P_\kappa\lambda$ (= $[\lambda]^{<\kappa}$, the set of subsets of λ of size less than κ). Let $G \subseteq \mathcal{P}(P_\kappa\lambda)/I$ be generic and $j : V \to M$ the resulting embedding. Then:*

1. Suppose that μ, ν are less than κ and let A be the set of $z \in P_\kappa\lambda$ such that:
$z \cap \kappa \in \kappa$,
$card(z) = card(z \cap \kappa)$,
$cof\,(z \cap \kappa) = \mu$ and
$cof\,(\sup(z)) = \nu$.
Then if A belongs to G we have the following in M: μ, ν are regular, $card(\lambda) = card(\kappa)$, $cof\,(\kappa) = \mu$ and $cof\,(\lambda) = \nu$.

2. Suppose that $\kappa = \rho^+$, ρ a cardinal, and let B be the collection of $z \in P_\kappa\lambda$ such that:
$z \cap \kappa \in \kappa$,
$card(z) = card(z \cap \kappa)$ and
$cof\,(z \cap \kappa) = cof\,(\sup(z)) \neq cof\,(\rho)$.
Then if B belongs to G, both ρ and $cof\,(\rho)$ remain cardinals in M and the following hold in M: $card(\lambda) = card(\kappa) = \rho$, $cof\,(\lambda) = cof\,(\kappa)$ and $cof\,(\lambda) \neq cof\,(\rho)$.

Moreover, if I is κ^+-saturated then the conclusions of 1 and 2 above hold with M replaced by $V[G]$.

Proof. We first show that j is the identity below κ. By fineness, $\{z \mid \beta \in z\}$ belongs to G for each $\beta < \kappa$. And by hypothesis $\{z \mid z \cap \kappa \in \kappa\}$ also belongs to G, so $\{z \mid \beta \subseteq z\}$ belongs to G for each $\beta < \kappa$. Recall that $j[\lambda]$ is represented by id so we can apply Łoś to conclude that $j(\beta) \subseteq [\mathrm{id}] = j[\lambda]$ for $\beta < \kappa$. So the critical point of j is at least β for each $\beta < \kappa$.

In fact the critical point of j equals κ: Otherwise $\kappa = j(\kappa) \subseteq j[\lambda] = [\mathrm{id}]$ so by Łoś $\{z \mid \kappa \subseteq z\}$ belongs to G; but this contradicts the hypothesis that $\{z \mid z \cap \kappa \in \kappa\}$ belongs to G.

To prove 1: As κ is the critical point of j, then μ, ν, which are regular in V and less than κ, must also be regular in M. As id represents $j[\lambda]$ and $j[\lambda] \cap j(\kappa) = \kappa$, it follows from the hypothesis $\mathrm{card}(z) = \mathrm{card}(z \cap \kappa)$ for $z \in A$ that $\mathrm{card}(\lambda) = \mathrm{card}(j[\lambda]) = \mathrm{card}([\mathrm{id}]) = \mathrm{card}([\mathrm{id}] \cap j(\kappa)) = \mathrm{card}(j[\lambda] \cap j(\kappa)) = \mathrm{card}(\kappa)$ in M. Also in M, $\mathrm{cof}\,(\kappa) = \mathrm{cof}\,(j[\lambda] \cap j(\kappa)) = j(\mu)$, as by hypothesis $\{z \mid \text{cof } (z\cap\kappa) = \mu\}$ belongs to G. As $j(\mu) = \mu$ we get cof $(\kappa) = \mu$ in M. Finally, in M we have cof $(\lambda) =$ cof $(\sup(j[\lambda]) = j(\nu)$, as by hypothesis $\{z \mid \text{cof } (\sup(z)) = \nu\}$ belongs to G; as $j(\nu) = \nu$ we get cof $(\lambda) = \nu$ in M.

To prove 2: Again $\rho, \text{cof } (\rho)$ are cardinals in M because they are less than the critical point of j. The hypothesis that $\{z \mid \text{card}(z) = \text{card}(z \cap \kappa)\}$ belongs to G gives that card$(\lambda) =$ card(κ) in M. But as κ is the critical point, it follows that $\kappa = \rho^+ < j(\rho^+) = (\rho^+)^M$ and therefore κ has cardinality ρ in M. Similarly, the hypothesis that $\{z \mid \text{cof } (z \cap \kappa) = \text{cof } (\sup(z)) \neq \text{cof } (\rho)\}$ belongs to G gives that in M, cof $(j[\lambda] \cap \kappa) =$ cof $(\sup(j[\lambda]) \neq$ cof $(j(\rho))$; but this says cof $(\kappa) =$ cof $(\lambda) \neq$ cof (ρ).

Finally, if I is λ^+-saturated then M and $V[G]$ have the same λ-sequences, so we can replace M by $V[G]$ in the above conclusions. □

Examples of Ideals

Natural Examples

Some interesting examples of ideals can be defined explicitly. The ideal of bounded subsets of a regular cardinal is an example of such an ideal. This ideal is never precipitous: Suppose that $G \subseteq \mathcal{P}(\kappa)/I$ is generic where I denotes this ideal. By a density argument, there exists a sequence $(Y_n \mid n \in \omega)$ of sets in G such that for all n and all $\alpha < \kappa$, the α-th element of Y_{n+1} is greater than the α-the element of Y_n. Then define $f_n : Y_n \to \kappa$ by $f_n(\beta) = \alpha$ where β is the α-th element of Y_n; we have that $f_{n+1}(\beta) < f_n(\beta)$ for all β, n. So $([f_n]^G \mid n \in \omega)$ is a descending sequence of ordinals in the generic ultrapower.

More generally, consider $I_\kappa\lambda =$ the smallest κ-complete, fine ideal on $P_\kappa\lambda$. This is the ideal dual to the filter $F_\kappa\lambda = \{X \subseteq P_\kappa\lambda \mid$ For some $a \in P_\kappa\lambda$, X contains all supersets of $a\}$. We'll show that neither is this ideal precipitous.

Definition. Let I be a countably complete, non-atomic ideal over a set Z. Then $\pi(I)$ is the least cardinality of an I-positive set and $\gamma(I)$ is the least cardinality of a set that generates I by taking subsets.

For example, suppose that $I = I_\kappa\lambda = \{X \subseteq P_\kappa\lambda \mid$ For some $a \in P_\kappa\lambda$, X does not contain any superset of $a\}$. Then $X \subseteq P_\kappa\lambda$ is I-positive iff every

$a \in P_\kappa\lambda$ can be covered by (i.e., is a subset of) an element of X. It follows that $\pi(I)$ is the least cardinality of a subset of $P_\kappa\lambda$ which covers every element of $P_\kappa\lambda$. Also, if every $a \in P_\kappa\lambda$ can be covered by an element of X then I is generated from $\{X_a \mid a \in X\}$ where $X_a = \{b \mid b \text{ does not cover } a\}$, by taking subsets, so $\gamma(I) \leq \pi(I)$. The other direction $\pi(I) \leq \gamma(I)$ is true in general: Suppose that $\mathcal{J} \subseteq I$ generates I by taking subsets. For A in $\mathcal{J}$ choose z_A in $Z \setminus A$. Then the set T of the z_A's is I-positive as otherwise there would be an $A \in \mathcal{J}$ such that $T \subseteq A$.

So we conclude that $\pi(I_\kappa\lambda) = \gamma(I_\kappa\lambda)$.

Theorem 17 *If $\pi(I) = \gamma(I)$ then I is not precipitous.*

Proof. First we prove:

Claim. Let $\kappa = \pi(I)$. For all I-positive sets X and injective $f : X \to \kappa$ there is an I-positive $Y \subseteq X$ and injective $g : Y \to \kappa$ such that $g(y) < f(y)$ for all $y \in Y$.

Proof of Claim. Suppose that $\{X_\alpha \mid \alpha < \kappa\}$ generates I by taking subsets and inductively choose $y_\alpha \in X \setminus (X_\alpha \cup f^{-1}[\alpha + 1] \cup \{y_\beta \mid \beta < \alpha\})$. This is possible as X is I-positive but neither the X_α's nor sets of size $< \kappa$ are I-positive. Let Y be $\{y_\alpha \mid \alpha < \kappa\}$ and $g(y_\alpha) = \alpha$. Then Y is I-positive as it is not a subset of any X_α and $g(y_\alpha) < f(y_\alpha)$ as y_α does not belong to $f^{-1}[\alpha + 1]$. □ (Claim)

Note that by a similar argument to the above, any I-positive set contains an I-positive subset of size κ. Now using the Claim, build maximal antichains $A_n \subseteq \mathcal{P}(Z)/I$ such that:

1. A_{n+1} refines A_n.
2. For each $X \in A_n$ we have an injective $f_X : X \to \kappa$.
3. If $X_{n+1} \in A_{n+1}$, $X_n \in A_n$ and $X_{n+1} \subseteq X_n$ then for all $y \in X_{n+1}$, $f_{X_{n+1}}(y) < f_{X_n}(y)$.

Clearly there is no branch $(X_n \mid n \in \omega)$ through the A_n's with $\cap_n X_n$ nonempty, so I is not precipitous. □

11.-12.Vorlesungen

The nonstationary ideal

We take an *algebra* on a set X to be a structure of the form $\mathcal{A} = (X, f_n)_{n\in\omega}$ where the f_n is an n-ary function from X to X. For such an algebra $\mathcal{A}$ we define $C_\mathcal{A}$ to be the set of $z \subseteq X$ which are closed under the f_n's. The resulting sets $C_\mathcal{A}$ are the *strongly closed unbounded sets* and the filter they generate is the *strongly closed unbounded filter* over $Z = \mathcal{P}(X)$. The ideal dual to this filter is the *strongly nonstationary ideal* and a set is *weakly stationary* if it is positive for this ideal.

Lemma 18 *The filter of strongly closed unbounded sets is fine and normal.*

Proof. Fineness is clear, as for any $x \in X$ we can consider the algebra $\mathcal{A} = (X, f)$ where f is unary with constant value x; then $C_\mathcal{A}$ consists only of subsets of X which have x as an element, and therefore $\{z \mid x \in z\}$ belongs to the filter generated by the $C_\mathcal{A}$'s.

Suppose that $F : A \to X$ is regressive, where A is positive for the strongly nonstationary ideal I, i.e., A is weakly stationary. If F is not constant on an I-positive subset of A then for each $x \in X$ we can choose an algebra $\mathcal{A}(x) = (X, f_n^x)_{n\in\omega}$ so that $F(z) \neq x$ for all z in $A \cap C_{\mathcal{A}(x)}$. But now define the algebra $\mathcal{A} = (X, f_n)_{n\in\omega}$ by: $f_{n+1}(x, x_1, \ldots, x_n) = f_n^x(x_1, \ldots, x_n)$. As A is weakly stationary we can choose $z \in A$ closed under the f_n's. It follows that z is closed under the f_n^x's for each x in z and therefore z belongs to $\mathcal{A}(x)$ for all x in z. But this gives a contradiction to the choice of $\mathcal{A}(x)$ when $x = F(z)$. $\square$

Note that any weakly stationary subset of $\mathcal{P}(X)$ contains elements which are countable. Now there are other notions of closed unbounded filter and stationary set for which this is not the case; we show that they can be obtained from the strongly closed unbounded filter and weakly stationary sets by restricting to an appropriate weakly stationary set.

For example, suppose $X = \kappa$, a regular uncountable cardinal. Consider the set $A = \{x \subseteq \kappa \mid x \cap \kappa \in \kappa\}$; A is weakly stationary. And the standard closed unbounded filter on κ is obtained as the filter of strongly closed unbounded sets restricted to the set A.

More generally, consider $P_\kappa\lambda$, where $\kappa \leq \lambda$ are cardinals and κ is regular and uncountable. We say that $D \subseteq P_\kappa\lambda$ is *closed* iff it is closed under directed unions of size less than κ and is *unbounded* iff it is cofinal in $P_\kappa\lambda$ under inclusion. Then we have:

Theorem 19 *The filter generated by the closed unbounded subsets of $P_\kappa\lambda$ is the same as that generated by the strongly closed unbounded filter on $\mathcal{P}(\lambda)$ together with the set $\{z \mid z \cap \kappa \in \kappa\}$.*

Proof. First note that any strongly closed unbounded subset of $P_\kappa\lambda$ is both closed and unbounded in the above sense. Also the set $\{z \in P_\kappa\lambda \mid z \cap \kappa \in \kappa\}$ is closed and unbounded. So it only remains to show that if $D \subseteq P_\kappa\lambda$ is closed and unbounded then there exists a function $F : \lambda^{<\omega} \to \lambda$ such that any $z \in P_\kappa\lambda$ closed under F and satisfying $z \cap \kappa \in \kappa$ belongs to D.

First note the following: If $\mathcal{B}$ is a structure for a countable language whose universe contains λ as a subset, then there is a function $F : \lambda^{<\omega} \to \lambda$ with the property that if $z \in P_\kappa\lambda$ is closed under F, then z is the intersection with λ of an elementary submodel of $\mathcal{B}$ of size less than κ. This is because we can assume that $\mathcal{B}$ has Skolem functions g_n, $n \in \omega$, where g_n is n-ary and then we set $F(\alpha_1, \ldots, \alpha_n) = g_n(\alpha_1, \ldots, \alpha_n)$ if the latter belongs to λ, 0 otherwise.

Now to prove the theorem, choose $\mathcal{B}$ to be a structure of the form $(H(\theta), \in, D, \ldots)$ for some large θ, where $\ldots$ denotes a sequence of Skolem functions and define F from $\mathcal{B}$ as above. Suppose that $z \in P_\kappa\lambda$ is closed under F and $z \cap \kappa \in \kappa$. Then $z = N \cap \lambda$ for some elementary submodel N of $\mathcal{B}$ of size less than κ and as $N \cap \kappa \in \kappa$ it follows that $x \subseteq N$ for any $x \in N \cap P_\kappa\lambda$. It follows that $z = N \cap \lambda$ is precisely the union of the elements of $N \cap D$ and therefore z belongs to D. □

Proposition 20 *The ideal of strongly nonstationary subsets of $Z = \mathcal{P}(X)$ is the smallest fine, normal ideal on Z.*

Proof. Let I be a fine, normal ideal on Z. If B is strongly nonstationary but does not belong to I then choose an algebra $(X, f_n)_{n\in\omega}$ under which the elements of B are not closed and define a regressive function F on B by sending $z \in B$ to an n-tuple from z whose value under f_n does not belong to z. By the normality of I we get an I-positive set of z not containing

$f_n(x_1, \ldots, x_n)$ for some fixed n and $(x_1, \ldots, x_n)$, contradicting the fineness of I. $\square$

Other exaamples of natural ideals include the meager and null ideals, the approachability ideals, the club guessing ideals, the diamond ideals, the uniformisation ideals and the weakly compact ideals. For these ideals there are interesting questions of properness, normality, saturation and precipitousness, not all of which have been thoroughly investigated.

CHANGE OF TOPIC!!!

I couldn't make sense out of Foreman's article past the above point. So I decided to switch topics:

Large cardinals and Combinatorial Principles

I look at the extent to which some well-known combinatorial principles can tolerate large cardinal hypotheses. *We assume GCH throughout.* The combinatorial principles that I have in mind are:

(a) $\square_\alpha$ and its variants.
(b) Various forms of Stationary Reflection at α.
(c) $\diamondsuit_\alpha$ and its variants.

The large cardinals I have in mind are:

Large cardinals of "Strong" type: κ is the critical point of $j : V \to M$
where j is
α-strong: $H(\alpha) \subseteq M$
Superstrong: $H(j(\kappa)) \subseteq M$
n-Superstrong: $H(j^n(\kappa)) \subseteq M$
ω-Superstrong: $H(j^\omega(\kappa)) \subseteq M$
($H(j^\omega(\kappa)^+) \subseteq M$ is inconsistent!)

"Subcompact" Large cardinals:
κ is *α-Subcompact* iff for each $A \subseteq H(\alpha)$ there are $\bar\kappa < \kappa$, $\bar\alpha \leq \alpha$, $\bar A \subseteq H(\bar\alpha)$ and an elementary $\pi : (H(\bar\alpha), \bar A) \to (H(\alpha), A)$ with critical point $\bar\kappa$, sending $\bar\kappa$ to κ.

About $\square$

As far as forcing $\square$ to hold we have the following limitation:

Theorem 21 *(Jensen) Suppose that κ is subcompact (i.e., κ^+ subcompact). Then $\Box_\kappa$ fails.*

More generally:

Theorem 22 *Suppose that κ is α^+ subcompact, $\kappa \leq \alpha$. Then $\Box_\alpha$ fails.*

Proof. Suppose that $\vec{C} = (C_\beta \mid \beta < \alpha^+)$ were a $\Box_\alpha$ sequence and apply α^+ subcompactness to the predicate $\vec{C}$: The result is

$$\pi : (H(\bar{\alpha}^+), \bar{\vec{C}}) \to (H(\alpha^+), \vec{C})$$

with critical point $\bar{\kappa}$, where π sends $\bar{\kappa}$ to κ.

Claim. There is π as above where $\bar{\alpha}$ is less than α.

Given the Claim, let λ be the supremum of $\pi[\bar{\alpha}^+]$. Then λ has cofinality $\bar{\alpha}^+ < \alpha^+$. Let D be the intersection Lim $(C_\lambda) \cap$ Range (π). Note that Lim (C_λ) is club in λ, Range (π) is ω-club in λ and therefore D is cofinal in λ. By coherence, if $\beta_0 < \beta_1$ belong to Lim (C_λ) then $ot(C_{\beta_0}) < ot(C_{\beta_1})$ and therefore ot is an injective function from D into α. But the range of ot on D is contained in Range (π) and therefore Range $(\pi) \cap \alpha$ has cardinality at least $\bar{\alpha}^+$, the cofinality of λ. This is impossible, because Range $(\pi) \cap \alpha = \pi[\bar{\alpha}]$ has ordertype $\bar{\alpha}$.

Proof of Claim. Let $f : [\alpha]^\omega \to \alpha$ be ω-Jónsson for α, that is, for any subset X of α of cardinality α, the range of f on $[X]^\omega$ is all of α. Such functions were shown to exist for all α by Erdős and Hajnal. Note that f is a subset of H_{α^+}, so there will be an elementary embedding $\pi : (H_{\bar{\alpha}^+}, \bar{A}, \bar{f}) \to (H_{\alpha^+}, A, f)$ witnessing the α^+-subcompactness of κ for A and f. We claim that this π, when considered as a function from $(H_{\bar{\alpha}^+}, \bar{A})$ to (H_{α^+}, A), satisfies the requirements of the Claim, namely that $\bar{\alpha} < \alpha$. The proof is exactly as in Kunen's proof that there can be no nontrivial elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$: Suppose $\bar{\alpha}$ were to equal α. Then since card$(j``\bar{\alpha}) = \bar{\alpha}$ we would have $f``[j``\bar{\alpha}]^\omega = \alpha$, and so there would be some $s \in [j``\bar{\alpha}]^\omega$ such that $f(s) = \bar{\kappa}$. But now since $\omega < \bar{\kappa}$, s is of the form $j(t)$ for some $t \in [\bar{\alpha}]^\omega$. By elementarity, $j(\bar{f}(t)) = f(j(t))$, so $\bar{\kappa}$ is in the range of j, contradiction. □

13.-14.Vorlesungen

The proof of the above Claim can be used to show:

Proposition 23 *Suppose that κ is α^+-subcompact. Then for each $A \subseteq H(\alpha^+)$ there is $\pi : (H(\bar{\alpha}^+), \bar{A}) \to (H(\alpha^+), A)$ with critical point $\bar{\kappa}$, sending $\bar{\kappa}$ to κ, where $\bar{\alpha}$ is less than κ.*

Proof. We may assume that $A \subseteq H(\alpha^+)$ codes an ω-Jónsson function for α in the sense that such a function is definable over $(H(\alpha^+), A)$. Let $\bar{\alpha}$ be least so that there exists $\pi : (H(\bar{\alpha}^+), \bar{A}) \to (H(\alpha^+), A)$ with critical point $\bar{\kappa}$, sending $\bar{\kappa}$ to κ. The claim is that $\bar{\alpha}$ is less than κ. If not, then apply the $\bar{\alpha}^+$-subcompactness of κ to get $\tau : (H(\bar{\bar{\alpha}}^+), \bar{\bar{A}}) \to (H(\bar{\alpha}^+), \bar{A})$ with critical point $\bar{\bar{\kappa}}$ strictly between $\bar{\kappa}$ and κ. Then the composition $\pi \circ \tau$ from $(H(\bar{\bar{\alpha}}^+), \bar{\bar{A}})$ to $(H(\alpha^+), A)$ is elementary and has critical point $\bar{\kappa}$, sending $\bar{\kappa}$ to κ. So $\pi \circ \tau$ witnesses the α^+-subcompactness of κ for A and therefore $\bar{\bar{\alpha}}$ equals $\bar{\alpha}$, by the choice of $\bar{\alpha}$. But by the proof of the Claim of the previous proof, this is impossible. $\square$

The result about $\square_\alpha$ failing due to subcompactness is optimal in the following sense:

Theorem 24 *Let $I = \{\alpha \mid \kappa$ is α^+-subcompact for some $\kappa \leq \alpha\}$. Then there is a cofinality-preserving $\mathbb{P}$ so that for P-generic G:*
1. $I = I^{V[G]}$.
2. ω-superstrongs are preserved.
3. $\square_\alpha$ holds in $V[G]$ for all $\alpha \notin I$.

Proof. The partial order $\mathbb{P}$ will be a reverse Easton forcing iteration. At stage α for α a cardinal not in I, we force with the usual size α^+ (thanks to the GCH), $< \alpha^+$-strategically closed partial order $\mathbb{S}_\alpha$ due to Jensen to obtain $\square_\alpha$, which uses initial segments of the generic $\square_\alpha$ sequence as conditions. At all other stages we take the trivial forcing. Thus, the iteration preserves cofinalities, and $\square_\alpha$ holds in $V[G]$ for all $\alpha \notin I^V$. It therefore only remains to show that forcing with $\mathbb{P}$ preserves the α-subcompactness of any κ that is α-subcompact in V, and that ω-superstrongs are preserved.

So suppose κ is α-subcompact in V. By the definition of I, the forcing is trivial on the interval $[\kappa, \alpha)$. Also, the tail of the iteration starting at stage α is $< \alpha^+$-strategically closed since each iterand is. Hence, no new subsets of α are added by this part of the forcing. We wish to consider arbitrary subsets of $H(\alpha)$ in the generic extension. By the GCH it suffices to consider names for subsets of α. Moreover, we may take these names σ to be $\mathbb{P}_\kappa$-names, where $\mathbb{P}_\kappa$ denotes the κ-stage iteration that is the initial part of $\mathbb{P}$ up to κ. In particular, σ can be taken to be a subset of $H(\alpha)$.

Applying the α-subcompactness of κ in V, let

$$\pi : (H(\bar{\alpha}), \bar{\sigma}) \to (H(\alpha), \sigma)$$

be elementary, with critical point $\bar{\kappa}$ taken by π to κ. We wish to lift π to an elementary embedding $\pi' : (H(\bar{\alpha})^{V[G]}, \bar{\sigma}_G) \to (H(\alpha)^{V[G]}, \sigma_G)$. Now $\bar{\kappa}$ is $< \bar{\alpha}$-subcompact if α is a limit cardinal and $\bar{\beta}$-subcompact if $\bar{\alpha} = \bar{\beta}^+$, so in either case $\mathbb{P}$ is trivial on the interval $[\bar{\kappa}, \bar{\alpha})$. Furthermore, even if the forcing iterand at stage $\bar{\alpha}$ is non-trivial, it will be $< \bar{\alpha}^+$-strategically closed, and hence adds no new sets to $H(\bar{\alpha})$. Indeed the tail of the forcing from stage $\bar{\alpha}$ on is $< \bar{\alpha}^+$-strategically closed. Therefore $H(\bar{\alpha})^{V[G]} = H(\bar{\alpha})^{V[G_{\bar{\kappa}}]}$, so combining this with $H(\alpha)^{V[G]} = H(\alpha)^{V[G_\kappa]}$ our goal becomes to lift π to

$$\pi' : (H(\bar{\alpha})^{V[G_{\bar{\kappa}}]}, \bar{\sigma}_{G_{\bar{\kappa}}}) \to (H(\alpha)^{V[G_\kappa]}, \sigma_{G_\kappa}),$$

for which it suffices by the usual (Silver) argument to show that $\pi[G_{\bar{\kappa}}] \subseteq G_\kappa$. But π is the identity below $\bar{\kappa}$, so this is immediate.

To show that ω-superstrongs are preserved we again use Silver's method of lifting embeddings. Let κ be ω-superstrong, let $j : V \to M$ witness this, let λ be the supremum of the $j^n(\kappa)$'s and suppose we have chosen j in such a way that every element of M is of the form $j(f)(a)$ for some a in $H(\lambda)$ and f with domain $H(\lambda)$. It follows from ω-superstrength that κ is α-subcompact for every $\alpha < \lambda$, that is, $< \lambda$-subcompact. Thus, our forcing $\mathbb{P}$ is trivial between κ and λ. Also, since the definition of $I \cap H(\lambda)$ is absolute for models containing $H(\lambda)$, $j(\mathbb{P}^V_\lambda) = \mathbb{P}^M_\lambda = \mathbb{P}^V_\lambda$ (hence the "non-trivial support" of $\mathbb{P}$ will also be bounded below κ). Below λ, therefore, we may just take the generic for M to be the generic for V, G_λ, and we get a lift j' of j from $V[G_\lambda]$ to $M[G_\lambda]$.

We claim that for the tail of the forcing, the pointwise image of the tail of the generic for V, $j'[G^\lambda]$, generates a generic filter for M, by the λ^+-distributivity of this tail forcing. Indeed this is standard for preservation results about ω-superstrongs. To be explicit: every element of $M[G_\lambda]$ is of the form σ_{G_λ} for some $\sigma = j(f)(a)$ with $a \in H(\lambda)$. Suppose D is a a dense class in the tail of forcing iteration, defined in $M[G_\lambda]$ as $\{p \mid \psi(p, d)\}$ for some parameter $d = j(f)(a)_{G_\lambda}$ with $a \in H(\lambda)$. Since the tail $\mathbb{P}^\lambda$ of the forcing is $< \lambda^+$-strategically closed and $|H(\lambda)| = \lambda$, it is dense for $q \in \mathbb{P}^\lambda$ to extend an element of $D_x = \{p \mid \psi(p, f(x)_{G_\lambda})\}$ whenever $x \in H(\lambda)$ and D_x is dense in $\mathbb{P}^\lambda$. We may therefore take such a q lying in G^λ, and by elementarity have that $j(q)$ extends D. That is, $j'[G^\lambda]$ indeed generates a generic filter over M for $(\mathbb{P}^\lambda)^M$. □

Stationary reflection

For regular $\kappa > \lambda$, $\mathrm{SR}(\kappa, \lambda)$ is the statement that for every stationary subset S of $\kappa \cap \mathrm{Cof}(\lambda)$ there is a $\gamma < \kappa$ such that $S \cap \gamma$ is stationary in γ. Note that $\square_\alpha$ refutes $\mathrm{SR}(\alpha^+, \lambda)$ for every $\lambda \leq \alpha$: the function $\xi \mapsto \mathrm{ot}\,(C_\xi)$ from $(\alpha^+ \setminus \alpha + 1) \cap \mathrm{Cof}(\lambda)$ to $\alpha + 1$ is regressive, and so is constant on a stationary set S. But now if $S \cap \gamma$ is stationary in γ, then a pair of distinct elements of $S \cap \lim(C_\gamma)$ can be found, violating coherence.

For any cardinal α, we say that a cardinal $\kappa \leq \alpha$ is α^+*-stationary subcompact* iff for each $A \subseteq H(\alpha^+)$ and stationary $S \subseteq \alpha^+$ there are $\bar\kappa < \kappa$, $\bar\alpha \leq \alpha$, $\bar A$, $\bar S$ and embedding $\pi : (H(\bar\alpha^+), \bar A, \bar S) \to (H(\alpha^+), A, S)$ sending its critical point to κ such that $\bar S$ is stationary in $\bar\alpha^+$.

Note that in the previous definition the set A can be coded into the stationary set S and therefore can be omitted from the definition. We will use this fact in the proofs below.

Since $H(\beta^+)$ is correct for stationarity of subsets of β, we have that if $\kappa < \beta^+ < \alpha$ and κ is α-subcompact, then κ is β^+-stationary subcompact, and moreover if $\pi : (H(\bar\alpha), \bar A) \to (H(\alpha), A)$ is an embedding with critical point $\bar\kappa$ witnessing the α-subcompactness of κ for some $A \subseteq H(\alpha)$, then for all $\bar\beta^+ < \bar\alpha$, $\bar\kappa$ is $\bar\beta^+$-stationary subcompact.

This strengthened subcompactness notion is sufficient to obtain stationary reflection.

Proposition 25 *If there exists some $\kappa \leq \alpha$ such that κ is α^+-stationary subcompact, then $SR(\alpha^+, \omega)$ holds.*

Proof. Suppose $\kappa \leq \alpha$ is α^+-stationary subcompact, let S be a stationary subset of $\alpha^+ \cap \mathrm{Cof}(\omega)$, and let $\pi : (H(\bar\alpha^+), \bar S) \to (H(\alpha^+), S)$ with critical point $\bar{\kappa}$ and $\bar{S}$ stationary in $\bar{\alpha}^+$ witness α^+-subcompactness of κ for S. We may assume that $\bar{\alpha}$ is less than α. Let $\lambda = \sup(\pi[\bar{\alpha}^+])$; we claim that $S \cap \lambda$ is stationary in λ. The pointwise image of $\bar{\alpha}^+$ in α^+ is countably closed and unbounded in λ, so for any club $C \subseteq \lambda$, $C \cap \pi[\bar{\alpha}^+]$ is also countably closed and unbounded in λ. Therefore, $\pi^{-1}C$ is countably closed and unbounded in $\bar{\alpha}^+$, and hence has nonempty intersection with $\bar{S}$. But now taking $\xi \in \bar{S} \cap \pi^{-1}C$, we have $\pi(\xi) \in S \cap C$, and so $S \cap \lambda$ is stationary. □

Again, we have a complementary result under the GCH.

Theorem 26 *Suppose the GCH holds. Let* $I = \{\alpha \mid \exists\kappa \leq \alpha(\kappa \textit{ is } \alpha^+\textit{-subcompact})\}$ *be as before, and similarly let*

$$J = \{\alpha \mid \exists\kappa \leq \alpha(\kappa \textit{ is } \alpha^+\textit{-stationary subcompact})\} \subseteq I.$$

Then there is a cofinality-preserving partial order $\mathbb{P}$ *such that for any* $\mathbb{P}$*-generic* G *the following hold.*

1. $I^{V[G]} = I$ *and* $J^{V[G]} = J$.
2. *SR*(α^+, ω) *fails in* $V[G]$ *for all* $\alpha \notin J$.
3. $\Box_\alpha$ *holds in* $V[G]$ *for all* $\alpha \notin I$.
4. ω*-superstrongs are preserved.*

15.-16.Vorlesungen

Theorem 27 *Suppose the GCH holds. Let* $I = \{\alpha \mid \exists\kappa \leq \alpha(\kappa \textit{ is } \alpha^+\textit{-subcompact})\}$ *be as before, and similarly let*

$$J = \{\alpha \mid \exists\kappa \leq \alpha(\kappa \textit{ is } \alpha^+\textit{-stationary subcompact})\} \subseteq I.$$

Then there is a cofinality-preserving partial order $\mathbb{P}$ *such that for any* $\mathbb{P}$*-generic* G *the following hold.*

1. $I^{V[G]} = I$ *and* $J^{V[G]} = J$.
2. *SR*(α^+, ω) *fails in* $V[G]$ *for all* $\alpha \notin J$.
3. $\Box_\alpha$ *holds in* $V[G]$ *for all* $\alpha \notin I$.

4. ω-superstrongs are preserved.

Proof. Again $\mathbb{P}$ will be a reverse Easton iteration. At stage α for $\alpha \in J$, we take the trivial forcing. For $\alpha \in I \setminus J$, we take the forcing $\mathbb{R}_\alpha$ that adds a non-reflecting stationary set to $\alpha^+ \cap \mathrm{Cof}(\omega)$ by initial segments; this forcing is α^+-strategically closed and (by the GCH) of size α^+. For $\alpha \notin I$, we take a three stage iteration, first forcing with $\mathbb{R}_\alpha$. Next, we force with the partial order $\mathbb{C}^{\mathbb{R}}_\alpha$ that makes the generic stationary set from $\mathbb{R}_\alpha$ non-stationary by shooting a club through its complement. Third, we force to make $\square_\alpha$ hold with $\mathbb{S}_\alpha$. The two stage iteration $\mathbb{R}_\alpha * \dot{\mathbb{C}}^{\mathbb{R}}_\alpha$ is $< \alpha^+$-strategically closed, as is $\mathbb{S}_\alpha$, so $\mathbb{R}_\alpha * \dot{\mathbb{C}}^{\mathbb{R}}_\alpha * \dot{\mathbb{S}}_\alpha$ is $< \alpha^+$-strategically closed. It also has a dense subset of size α^+. Thus, our reverse Easton iteration will indeed preserve cofinalities. We will denote by $\mathbb{P}_\beta$ the iteration below stage β and by G_β the corresponding generic; for κ inaccessible, $\mathbb{P}_\kappa$ is a direct limit, so we can and will identify $\mathbb{P}_\kappa$ with $\bigcup_{\gamma<\kappa} \mathbb{P}_\gamma$.

It only remains to show that the classes I and J are preserved by the forcing. We show that if β belongs to J and $\kappa \leq \beta$ is any β^+-stationary subcompact in V then κ remains β^+-stationary subcompact in the generic extension $V[G]$. Also, if β belongs to $I \setminus J$ and $\kappa \leq \beta$ is the least β^+-subcompact in V then κ remains β^+-subcompact in $V[G]$. These facts suffice to show that I and J are preserved, as $\square_\beta$ will hold in $V[G]$ for $\beta \notin I$ and $\mathrm{SR}(\beta^+, \omega)$ will fail in $V[G]$ for $\beta \notin J$.

If β belongs to J and κ is β^+-stationary subcompact, then the forcing is trivial at stages from κ up to and including β, and is $< \beta^{++}$-strategically closed from stage β^+ onward, so no new subsets of $H(\beta^+)$ are added after stage κ. Thus, to show that β^+-stationary subcompactness is preserved, it suffices to show that for any condition p and any $\mathbb{P}_\kappa$-name σ forced by $p \restriction \kappa$ to be a stationary subset of β^+, there is an extension q of p forcing the existence of an embedding from $(H(\bar{\beta}^+)^{V[G]}, \bar{\sigma}^G)$ to $(H(\beta^+)^{V[G]}, \sigma^G)$ witnessing the β^+-stationary subcompactness of κ for σ^G in $V[G]$. As σ is forced by $p \restriction \kappa$ to be stationary in β^+ and $\mathbb{P}_\kappa$ is only of cardinality κ, there is some q extending p and some $S \in V$ stationary in β^+ such that $q \restriction \kappa \Vdash S \subseteq \sigma$. In V let $\pi : (H(\bar{\beta}^+), \bar{q}, \bar{S}, \bar{\sigma}) \to (H(\beta^+), q \restriction \kappa, S, \sigma)$ with critical point $\bar{\kappa}$ and $\bar{S}$ stationary in $\bar{\beta}^+$ witness the β^+-stationary subcompactness of κ for $(q \restriction \kappa, S, \sigma)$. Then $\bar{q} = q \restriction \kappa$ since the latter is bounded below κ, and $q \restriction \kappa$ forces $\bar{S}$ to be a stationary subset of $\bar{\sigma}$. Now by Silver's lifting of embeddings method, q forces π to lift to an elementary embedding π^* : $(H(\bar\beta^+)^{V[G_{\bar\kappa}]}, \bar\sigma_{G_{\bar\kappa}}) \to (H(\beta^+)^{V[G_\kappa]}, \sigma_{G_\kappa})$, since $\pi[G_{\bar\kappa}] = G_{\bar\kappa} \subseteq G_\kappa$. That is, we have $\pi^* : (H(\bar\beta^+)^{V[G]}, \bar\sigma^G) \to (H(\beta^+)^{V[G]}, \sigma^G)$ with $\bar\sigma^G$ stationary, as required.

If β belongs to $I \setminus J$ and κ is β^+-subcompact then the forcing is trivial on $[\kappa, \beta)$, is $\mathbb{R}_\beta$ at stage β, and is $< \beta^{++}$-strategically closed thereafter. Let κ be the least β^+-subcompact and for $\bar\kappa < \kappa$ choose $A(\bar\kappa) \subseteq \beta^+$ so that $\bar\kappa$ is not β^+-subcompact with respect to the predicate $A(\bar\kappa)$; this is possible as κ is the least β^+-subcompact and any subset of $H(\beta^+)$ can be coded by a subset of β^+. Let A be the join of the $A(\bar\kappa)$'s: $A = \{(\bar\kappa, \gamma) \mid \gamma \in A(\bar\kappa)\}$.

Now suppose that p forces the $\mathbb{P}_{\beta+1}$-name σ to be a subset of β^+. Note that $p \restriction \beta + 1$ is comprised of $p \restriction \kappa$, a $\mathbb{P}_\kappa$-condition that is thus bounded below κ, and a name $\dot p(\beta)$ for an $\mathbb{R}_\beta$-condition. Take $\pi : (H(\bar\beta^+), \bar p, \bar\sigma, \bar A) \to (H(\beta^+), p \restriction \beta + 1, \sigma, A)$ witnessing the β^+-subcompactness of κ for $(p \restriction \beta + 1, \sigma, A)$ with $\bar\kappa =$ (the critical point of π) least, and $\bar\beta$ least for this choice of $\bar\kappa$.

Claim. $\bar\beta$ does not belong to I.

We must show that no $\bar{\bar\kappa} \le \bar\beta$ is $\bar\beta^+$-subcompact. If $\bar{\bar\kappa}$ is less than $\bar\kappa$ then $\bar{\bar\kappa}$ cannot be $\bar\beta^+$-subcompact for the predicate $\bar A(\bar{\bar\kappa})$, else by composing with the map π we would get the β^+-subcompactness of $\bar{\bar\kappa}$ for the predicate $A(\bar{\bar\kappa})$ ($\bar{\bar\kappa}$ is not moved by π). If $\bar{\bar\kappa}$ equals $\bar\kappa$ then it cannot be $\bar\beta^+$-subcompact for the predicate (p, σ, A), else by composing with π we would contradict the leastness of $\bar\kappa$. Finally, if $\bar{\bar\kappa}$ lies in the interval $(\bar\kappa, \bar\beta]$ then by composing with π we contradict the leastness of $\bar\beta$. This proves the Claim.

Now $\bar\kappa$ is $\bar\gamma^+$-stationary subcompact for $\bar\gamma < \bar\beta$, so the forcing is trivial on $[\bar\kappa, \bar\beta)$, is $\mathbb{R}_{\bar\beta} * \dot{\mathbb{C}}^{\mathbb{R}}_{\bar\beta} * \dot{\mathbb{S}}_{\bar\beta}$ at stage $\bar\beta$, and is $< \bar\beta^{++}$-strategically closed thereafter. In particular, $H(\bar\beta^+)$ receives no new elements from stage $\bar\kappa$ of the forcing onward. Note that $\bar p \restriction \bar\kappa = p \restriction \kappa$, $\bar p(\bar\beta)$ is a name for an $\mathbb{R}_{\bar\beta}$ condition, and $\bar\sigma$ is a $\mathbb{P}_{\bar\beta} * \dot{\mathbb{R}}_{\bar\beta}$-name for a subset of $\bar\beta^+$. Extend p to q_0 extending $\bar p(\bar\beta)$; this is possible as $p \restriction \beta = p \restriction \bar\kappa$ so p is trivial at $\bar\beta < \beta$. Now, the key point is that q_0 can be extended to a condition q forcing the generic $G_{\beta, \mathbb{R}_\beta}$ for $\mathbb{R}_\beta$ to extend $\pi[G_{\bar\beta, \mathbb{R}_{\bar\beta}}]$, since the pointwise image of the $\mathbb{C}^{\mathbb{R}}_{\bar\beta}$-generic verifies that the pointwise image of the $\mathbb{R}_{\bar\beta}$-generic has union non-stationary in $\sup(\pi[\bar\beta^+])$.

Hence, q forces that π lifts to an embedding

$$\pi^* : (H(\bar\beta^+)^{V[G_{\bar\beta,\mathbb{R}_{\bar\beta}}]}, \bar\sigma^{G_{\bar\beta,\mathbb{R}_{\bar\beta}}}) \to (H(\beta^+)^{V[G_{\beta+1}]}, \sigma^{G_{\beta+1}}),$$

But this is the same as

$$\pi^* : (H(\bar\beta^+)^{V[G]}, \bar\sigma^{G}) \to (H(\beta^+)^{V[G]}, \sigma^{G}),$$

and we are done for this case.

The preservation of ω-superstrongs is as in the previous proof. $\Box$

17.-18.Vorlesungen

For any cardinal α, a $\Box_{\alpha,<\mu}$*-sequence* is a sequence $\langle \mathcal{C}_\beta \mid \beta \in \alpha^+ \cap \text{Lim} \rangle$ such that for every $\beta \in \alpha^+ \cap \text{Lim}$:

$\mathcal{C}_\beta$ is a set of closed unbounded subsets of β
$1 \le |\mathcal{C}_\beta| < \mu$
ot $(C) \le \alpha$ for every $C \in \mathcal{C}_\beta$
For any $C \in \mathcal{C}_\beta$ and $\gamma \in \lim(C)$, $C \cap \gamma \in \mathcal{C}_\gamma$.

We say $\Box_{\alpha,<\mu}$ holds if there exists a $\Box_{\alpha,<\mu}$-sequence, and we write $\Box_{\alpha,\nu}$ for $\Box_{\alpha,<\nu^+}$.

Of course, $\Box_{\alpha,1}$ is simply $\Box_\alpha$, and the strength of the statement $\Box_{\alpha,<\mu}$ can only decrease as μ increases. (In fact, Jensen shows that the strength can strictly decrease as μ increases.) *Weak square*, denoted $\Box^*_\alpha$, is $\Box_{\alpha,\alpha}$, and $\Box_{\alpha,\alpha^+}$ is provable in ZFC for all α.

Theorem 28 *Suppose κ is α^+-subcompact for some $\kappa \le \alpha$. Then $\Box_{\alpha,<cof(\alpha)}$ fails.*

Proof. Suppose for contradiction that $\mathcal{C} = \langle \mathcal{C}_\beta \mid \beta \in \alpha^+ \cap \text{Lim} \rangle$ is a $\Box_{\alpha,<cof(\alpha)}$-sequence. We can take an α^+-subcompactness embedding

$$\pi : (H_{\bar\alpha^+}, \in, \bar{\mathcal{C}}) \to (H_{\alpha^+}, \in, \mathcal{C})$$

with critical point some $\bar\kappa < \bar\alpha^+$ such that $\pi(\bar\kappa) = \kappa$, and $\bar\alpha < \alpha$. Let λ be the supremum of $\pi[\bar\alpha^+]$, let C be an arbitrary member of $\mathcal{C}_\lambda$, and consider the inverse image $\bar D$ of $\lim(C)$ under π. Then $\bar D$ is $(< \bar\kappa)$-closed and unbounded in $\bar{\alpha}^+$, so we may take some $\bar{\beta} \in \bar{D}$ of cofinality different from cof $(\bar{\alpha})$ such that $|\bar{D} \cap \bar{\beta}| = \bar{\alpha}$. Let β denote $\pi(\bar{\beta})$.

Now, for any $\bar{\gamma} < \bar{\beta}$ in $\bar{D}$, $\pi(\bar{\gamma}) \in C \cap \pi(\bar{\beta}) \in \mathcal{C}_\beta$, so by elementarity there is some $\bar{C} \in \mathcal{C}_{\bar{\beta}}$ with $\bar{\gamma} \in \bar{C}$. But there are fewer than cof $(\bar{\alpha})$ elements of $\mathcal{C}_{\bar{\beta}}$, and as cof $(\bar{\beta}) \neq$ cof $(\bar{\alpha})$, each of them has ordertype strictly less that $\bar{\alpha}$; so $|\bigcup \mathcal{C}_{\bar{\beta}}| < \bar{\alpha}$, and not all $\gamma \in \bar{D} \cap \bar{\beta}$ can be covered in this way. □

If $\square^*_\alpha$ holds then there is a $\square^*_\alpha$ sequence (an *improved* square sequence, $\square^{imp}_{\alpha,\alpha}$) with the added property that for all limit $\beta < \alpha^+$, there is a $C \in \mathcal{C}_\beta$ with ot $(C) =$ cof (β). Indeed, if we choose an arbitrary sequence $\langle D_\gamma \mid \gamma < \alpha\rangle$ such that D_γ is a club in γ of order type cof (γ), then for any $\square^*_\alpha$-sequence $\mathcal{C}$, we may obtain a $\square^{imp}_{\alpha,\alpha}$-sequence by adding $\{\delta \in C \mid \text{ot } (C \cap \delta) \in D_\gamma\}$ to $\mathcal{C}_\beta$ for every $C \in \mathcal{C}_\beta$ and γ such that ot $(C) \in$ Lim $(D_\gamma) \cup \{\gamma\}$.

Theorem 29 *(GCH) Suppose κ is α^+-subcompact for some $\kappa \leq \alpha$ with cof $(\alpha) < \kappa$. Then $\square_{\alpha,\alpha}$ fails.*

Proof. Suppose for contradiction that $\mathcal{C}$ is a $\square^{imp}_{\alpha,\alpha}$ sequence, and let

$$\pi : (H_{\bar{\alpha}^+}, \in, \bar{\mathcal{C}}) \to (H_{\alpha^+}, \in, \mathcal{C})$$

be an embedding witnessing the α^+-subcompactness of κ for $\mathcal{C}$. The hypothesis cof $(\alpha) < \kappa$ implies that in fact cof $(\alpha) =$ cof $(\bar{\alpha}) < \bar{\kappa}$. Let $\lambda = \sup(\pi[\bar{\alpha}^+])$, and take $C \in C_\lambda$ with ot $(C) = \bar{\alpha}^+ =$ cof (λ). Let $\bar{D}$ be the preimage of Lim C under π; it is an ω-closed unbounded subset of $\bar{\alpha}^+$. Let ζ be the $\bar{\alpha}$-th element of $\bar{D}$. As $\pi(\zeta)$ is a limit point of C, it follows by coherence that $C \cap \pi(\zeta) \in \mathcal{C}_{\pi(\zeta)}$. Now for every subset X of $\bar{D} \cap \zeta$ of size less than $\bar{\kappa}$, $\pi(X) = \pi[X] \subseteq C \cap \pi(\zeta) \in \mathcal{C}_{\pi(\zeta)}$, so by elementarity, there is an element $\bar{C}_X$ of $\bar{\mathcal{C}}_\zeta$ such that $X \subseteq \bar{C}_X$. But there are $\bar{\alpha}^{<\bar{\kappa}} > \bar{\alpha}$ such subsets X of $\bar{D} \cap \zeta$ and only $\bar{\alpha}$ elements of $\bar{\mathcal{C}}_\zeta$, so some single element of $\bar{\mathcal{C}}_\zeta$ must cover more than $\bar{\alpha}$ many such X. This is impossible as each element of $\bar{\mathcal{C}}_\zeta$ has ordertype less than $\bar{\alpha}$. □

We now sketch the proof of a result establishing the optimality of Theorems 28 and 29.

Theorem 30 *Let K denote $\{\alpha \mid$ There is an α^+-subcompact κ with cof $(\alpha) < \kappa\}$. Then there is a cofinality and ZFC-preserving definable class forcing P such that for P-generic G:*

(a) If κ is α^+-subcompact in V then it remains so in $V[G]$.
(b) $\Box_{\alpha,cof(\alpha)}$ holds in $V[G]$ for $\alpha \notin K$.
(c) ω-superstrongs are preserved.

Proof sketch. P is the reverse Easton iteration which at stages α in K does nothing and otherwise adds a $\Box_{\alpha,cof(\alpha)}$-sequence by the following forcing:

Let cof $(\alpha) = \mu$ and fix $(\alpha_i \mid i < \mu)$ an increasing sequence of regular cardinals such that $\mu < \alpha_0$ and $\sup_i \alpha_i = \alpha$. Conditions are of the form

$$p = (C_{\beta,i} \mid \beta \text{ limit},\ \beta \leq |p|,\ i^p(\beta) \leq i < \mu),$$

where:

1. $|p|$ is a limit ordinal $< \alpha^+$.
2. $i^p(\beta) < \mu$ for limit $\beta \leq |p|$.
3. If $i^p(\beta) \leq i < \mu$ then $C_{\beta,i}$ is club in β of ordertype $< \alpha_i$.
4. If $i^p(\beta) \leq i < j < \mu$ then $C_{\beta,i} \subseteq C_{\beta,j}$.
5. If $i^p(\gamma) \leq i < \mu$ and β is a limit point of $C_{\gamma,i}$ then $i^p(\beta) \leq i$ and $C_{\beta,i} = C_{\gamma,i} \cap \beta$.
6. If β and γ are limit ordinals with $\beta < \gamma \leq |p|$ then β is a limit point of $C_{\gamma,i}$ for sufficiently large $i < \mu$.

Extension is defined by: $q \leq p$ iff $|p| \leq |q|$ and for limit $\beta \leq |p|$, $i^p(\beta) = i^q(\beta)$ and $C^p_{\beta,i} = C^q_{\beta,i}$ for all i with $i^p(\beta) \leq i < \mu$.

It can be verified that P is $< \mu$-directed closed, is $< \alpha$-strategically closed and adds a $\Box_{\alpha,cof(\alpha)}$-sequence. So P preserves cofinalities and ZFC, and we get conclusion (b) of the Theorem. (The verification of (c) is as in earlier proofs.) It remains to show that instances of subcompactness are preserved.

Suppose that κ is α^+-subcompact in V and p is a condition forcing that the name σ denotes a subset of α^+. We can assume that σ is in fact a $P_{\alpha+1}$-name as the forcing after stage α does not add subsets of α^+. Applying subcompactness to the name σ and the condition $p_{\alpha+1}$, which can be viewed as subsets of α^+, we get:

$$\pi : (H(\bar{\alpha}^+), \bar{\sigma}, \bar{p}) \to (H(\alpha^+), \sigma, p_{\alpha+1})$$

with critical point $\bar{\kappa}$ and $\bar{\alpha} < \kappa$, $\pi(\bar{\kappa}) = \kappa$. We want a condition q extending p which forces that π can be lifted to:

$$\pi : (H(\bar{\alpha}^+)^{V[G]}, \bar{\sigma}^G, \bar{p}) \to (H(\alpha^+)^{V[G]}, \sigma^G, p_{\alpha+1}).$$

For this it suffices that q forces $\pi[G_{\bar{\alpha}+1}] \subseteq G_{\alpha+1}$.

Now the forcing $P_{\alpha+1}$ factors as $P_\kappa * P[\kappa, \alpha]$ where P_κ forces $P[\kappa, \alpha]$ to be $< \kappa$-directed closed: This is because the forcing $P[\kappa, \alpha]$ is nontrivial at stage $\beta \in [\kappa, \alpha]$ only if $\text{cof}\,(\beta) \geq \kappa$ (else β belongs to K), in which case it is $< \text{cof}\,(\beta)$ and therefore $< \kappa$-directed closed. It now follows that we can extend p to q forcing $\pi[G_{\bar{\alpha}}] \subseteq G_{\alpha+1}$ as $\pi[G_{\bar{\kappa}}] = G_{\bar{\kappa}} \subseteq G_\kappa$ and by directed closure we can form a master condition extending $\pi[G[\bar{\kappa}, \bar{\alpha}]]$ to ensure that it is contained in $G[\kappa, \alpha]$. So the α^+-subcompactness of κ is preserved, as desired. $\square$