Isomorphism Relations, Sommersemester 2008

1.-2.Vorlesungen

Analytic equivalence relations

This course will focus on definable equivalence relations which arise naturally in model theory. They form a special case of *analytic equivalence relations on a Polish space*, which I now describe briefly.

A *Polish space* is a complete, separable, metrizable space. The best example is *Baire space* ω^ω, consisting of all ω-sequences of natural numbers with topology generated by the clopen sets. Other examples are the reals, the Cantor space 2^ω. The *Borel sets* of a Polish space are the elements of the least σ-algebra generated by the open sets of the space. A function between Polish spaces is *Borel* iff the preimage under the function of each Borel set is Borel. The *analytic sets* are the continuous images of Borel sets, i.e., sets of the form $f[A]$ where A is a Borel subset of some Polish space and f is a continous map between Polish spaces. Equivalently, a subset A of a Polish space X is *analytic* iff for some Polish space Y and Borel subset B of $X \times Y$, A is the projection of B, i.e., $\{x \mid \exists y (x, y) \in B\}$.

If X is a Polish space then so is X^n (with the product topology) for each n; thus we may talk about *analytic* n-ary relations on Polish spaces. In particular, an equivalence relation R on a Polish space X is *analytic* iff it is an analytic subset of $X \times X$.

There are a number of interesting results about general analytic equivalence relations (on Polish spaces), some of which I will discuss in this course. But my main focus is on a smaller class:

Orbit equivalence relations

If G is a group and X a set then an *action* of G on X is a map $a : G \times X \to X$, usually written $a(g, x) = g \cdot x$, such that $e \cdot x = x$ and $g \cdot (h \cdot x) = gh \cdot x$.

A *topological group* is a group $(G, \cdot, e)$ together with a topology on G such that $(x, y) \mapsto xy^{-1}$ is continuous from G^2 to G. A *Polish group* is a topological group whose topology is Polish. An action of G on a topological space X is *continuous* iff it is continuous as a map from $G \times X$ to X. A *Polish action* is a continuous action of a Polish group on a Polish space.

Examples of Polish groups
Countable groups with the discrete topology
$(R^n, +)$
The torus $(R/Z, +)$
$(Z_2^N, +)$ (the Cantor group)
Lie groups, like $GL(n, C)$, $U(n)$
The unitary group $U(H)$ where H is an infinite-dimensional Hilbert space (i.e., the infinite-dimensional analog of $U(n)$
The group of homeomorphisms of a compact metrizable space X
The group of isometries of a complete, separable metric space
$(X, +)$ where X is a separable Banach space
S_∞, the permutation group of N

Examples of Polish actions
Polish groups act on themselves: $g \cdot x = gx$ (left action), $g \cdot x = xg^{-1}$ (right action), $g \cdot x = gxg^{-1}$ (conjugation).
If G is a group of permutations of X then G acts on X in the obvious way: $g \cdot x = g(x)$.
$U(n)$ acts on $n \times n$ matrices by conjugation. The infinitary analog is the action of $U(H)$ on $L(H)$ where the latter is the set of bounded linear operators on the Hilbert space H. (But $L(H)$ is not obviously Polish.)
Measure-preserving transformations of a Polish space, equipped with a Borel measure (ergodic theory).

A group action $a : G \times X \to X$ induces an *orbit equivalence relation*:

$$xE_a y \text{ iff } \exists g \in G(g \cdot x = y).$$

The equivalence classes of E_a are called the *orbits.*

A (proper) subclass of the analytic equivalence relations are the orbit equivalence relations arising from Polish actions.

Logic equivalence relations (LER's)

Of central interest in this course are the orbit equivalence relations induced by a continuous action of the permutation group of N, called S_∞. These actions are called the *logic actions*, and the resulting equivalence relations we call the *logic equivalence relations* for reasons we now describe.

Let $\mathcal{L}$ denote a relational first-order language with relation symbols R_i, $i \in I$, where I is nonempty and countable and R_i is n_i-ary. Denote by $X_{\mathcal{L}}$ the space $\prod_i 2^{N^{n_i}}$, which is homeomorphic to Cantor space. We can view X_L as the space of countably infinite structures for $\mathcal{L}$ by identifying $x = (x_i)_{i \in I}$ with the structure $\mathcal{A}_x = (N, R_i^{\mathcal{A}_x})_{i \in I}$ where for $x \in N^{n_i}$, $R_i^{\mathcal{A}_x}(s)$ holds iff $x_i(x) = 1$. The *logic action* of S_∞ on $X_{\mathcal{L}}$ is defined by setting $J_{\mathcal{L}}(g, x) = y$ iff g is an isomorphism of $\mathcal{A}_x$ onto $\mathcal{A}_y$. The associated orbit equivalence relation is the isomorphism relation $\simeq_{\mathcal{L}}$ on $\mathcal{L}$-structures with universe N.

More generally, consider a sentence φ of $\mathcal{L}_{\infty\omega}$ (for example, the conjunction of a first-order theory in the language $\mathcal{L}$). Let $\mathrm{Mod}(\varphi)$ denote the set of $\mathcal{L}$-structures with universe N which satisfy φ. Then $\mathrm{Mod}(\varphi)$ is a Borel subset of $X_{\mathcal{L}}$ and $\simeq_{\mathcal{L}}$ restricted to $\mathrm{Mod}(\varphi)$ is an analytic equivalence relation on the Borel set $\mathrm{Mod}(\varphi)$. A *logic equivalence relation (LER)* is an equivalence relation of this form, i.e., the isomorphism relation restricted to $\mathrm{Mod}(\varphi)$ for a sentence φ in $\mathcal{L}_{\infty\omega}$ for some countable relational language $\mathcal{L}$. Thus an LER is the restriction to an invariant Borel set of a continuous action of S_∞ It can be shown that any Polish action restricted to an invariant Borel set B is in fact a continuous action on B for a topology refining the topology that B inherits as a subspace. Thus any LER is in fact the orbit equivalence relation of a continuous action of S_∞ on a Polish space. Conversely, it can be shown that any continuous action of S_∞ on a Polish space is "equivalent" (in a sense made precise below) to an LER.

Borel reducibility

How do we compare equivalence relations? If E, F are equivalence relations on Polish spaces X, Y then we write

$$E \leq_B F$$

iff there is a Borel reduction from E to F, i.e., a Borel function $f : X \to Y$ so that for any x_0, x_1 in X:

$$x_0 E x_1 \text{ iff } f(x_0) F f(x_1).$$

We write $\sim_B$ for ($\leq_B$ and $\geq_B$) and $<_B$ for ($\leq_B$ and $\not\geq_B$).

There are $\leq_B$-complete LER's, to which all LER's are Borel-reducible (examples below). A $\leq_B$-complete LER is necessarily analytic and not Borel. I next describe a $\leq_B$-cofinal hierarchy of Borel LER's, together with some examples that occur at particular levels of this hierarchy. The hierarchy looks like this:

$0 < 1 < \cdots < \omega < R < E_0 < E_\infty < F_2 < F_3 < \cdots < F_\alpha < \cdots (\alpha < \omega_1)$

1. Borel-equivalent to ω.

Finite linear orderings

2. Borel-equivalent to R

Orderings of type ω with a unary predicate

Smooth = Borel-reducible to R.

3. E_0 = equality mod finite on subsets of ω

Subgroups of $(Q, +)$

4. $E_\infty = \leq_B$-largest *countable* Borel equivalence relation

Countable = has countably many equivalence classes.

Fact: Any countable Borel equivalence relation is an LER.

Locally-finite, connected graphs
Finitely-generated groups
Fields of finite transcendence degree over Q

5. F_α

xF_2y iff $\{x_n \mid n \in \omega\} = \{y_n \mid n \in \omega\}$

xF_3y iff $\{\{(x_m)_n \mid n \in \omega\} \mid m \in \omega\}$ = same for y

etc.

Each Borel LER is Borel-reducible to some F_α

Equivalent to F_2:

locally-finite graphs
Archimedean totally-ordered Abelian groups with a distinguished positive element

6. Beyond Borel

$\leq_B$-complete LER's: graphs, trees, fields, groups, linear orderings, Boolean algebras

Abelian groups: Invariants are elements of $2^{<\omega_1}$ (Ulm invariants). Not known to be complete.

Abelian p-groups: Not complete.

7. Beyond LER's

Hjorth: An orbit equivalence relation is reducible to an LER iff it is not "turbulent".

Non-turbulent = invariants are given by countable structures

Examples of turbulent actions:

Conjugacy on the homeomorphism group of the unit square
Conjugacy of ergodic, measure-preserving transformations
Unitary equivalence of unitary operators
Biholomorphic equivalence of 2-dimensional complex manifolds

8. Beyond orbit equivalence relations

$E_1 : xE_1y$ iff $\{x_n \mid n \in \omega\}$ almost equals $\{y_n \mid n \in \omega\}$

Composant equivalence relation for certain indecomposable continua

3.-4.Vorlesungen

More about LER's

We now justify some of our earlier claims about LER's, and prove some additional ones. Recall that the LE's are those analytic equivalence relations (on a Borel subset of a Polish space) which arise as the isomorphism relation restricted to models of some sentence of $\mathcal{L}_{\omega_1\omega}$. We describe how these arise from group actions of S_∞, the Polish group of permutations of N. Suppose that $\mathcal{L} = \{R_i\}_{i\in I}$ is a countable relational language where R_i is m_i-ary. We write $\mathrm{Mod}_\mathcal{L}$ for $\prod_{i\in I}\mathcal{P}(N^{m_i})$, the space of $\mathcal{L}$-structures with universe N. The logic action $j_\mathcal{L}$ of S_∞ on $\mathrm{Mod}_\mathcal{L}$ is defined as follows: If $x = \{x_i\}_{i\in I} \in \mathrm{Mod}_\mathcal{L}$ and $g \in S_\infty$ then $j_\mathcal{L}(g, x) = \{y_i\}_{i\in I} \in \mathrm{Mod}_\mathcal{L}$, where

$$(k_1, \dots, k_{m_i}) \in x_i \leftrightarrow (g(k_1), \dots, g(k_{m_i})) \in y_i$$

for all $i \in I$ and $(k_1, \dots, k_{m_i}) \in N^{m_i}$. Then $(\mathrm{Mod}_\mathcal{L}, j_\mathcal{L})$ is a Polish S_∞-action and we denote the resulting orbit equivalence relation by $\simeq_\mathcal{L}$. If G is a subgroup of S_∞ then the orbit equivalence relation of $j_\mathcal{L}$ restricted to G is denoted by $\simeq^G_\mathcal{L}$.

A set $M \subseteq \mathrm{Mod}_\mathcal{L}$ is *invariant* iff it is closed under isomorphism. Let $\mathcal{L}_{\omega_1\omega}$ denote the infinitary language which results by closing the first-order language $\mathcal{L}$ under countable conjunctions and disjunctions (of sets of formulas whose free variables are contained within some fixed finite set). If x belongs to $\mathrm{Mod}_\mathcal{L}$ then $x \vDash \varphi(i_1, \dots, i_n)$ has the obvious meaning, where φ is a formula of $\mathcal{L}_{\omega_1\omega}$ and $i_1, \dots, i_n$ belong to N.

Theorem 1 *A set $M \subseteq Mod_\mathcal{L}$ is invariant and Borel iff for some sentence φ of $\mathcal{L}_{\omega_1\omega}$, $M = \{x \in Mod_\mathcal{L} \mid x \vDash \varphi\}$.*

Corollary 2 *The LER's are exactly the restrictions to invariant Borel sets of the orbit equivalence relation induced by the logic action $j_\mathcal{L}$, for some countable relational language $\mathcal{L}$.*

Proof of Theorem 1. Suppose that M is the set of models of some sentence. Then M is obviously invariant. To see that M is Borel, we show that for any formula $\varphi(v_0, \dots, v_{n-1})$ and $(i_0, \dots, i_{n-1}) \in N^n$, the set of x such that $x \vDash \varphi(i_0, \dots, i_{n-1})$ is Borel. If φ is atomic then this set is clopen. The connective cases are immediate, as the collection of Borel sets is closed under complement and countable unions and intersections. Finally, suppose that $\varphi(v_0, \dots, v_{n-1})$ is $\forall v_n \psi(v_0, \dots, v_n)$; by induction, the set B_i of x such that $x \vDash \psi(i_0, \dots, i_{n-1}, i)$ is Borel for each $i \in N$. It follows that the set of x such that $x \vDash \varphi(i_0, \dots, i_{n-1})$ is Borel, as this set is the intersection of the Borel sets B_i.

For the other direction, let M be invariant and Borel and set $B_s = \{g \in S_\infty \mid s \subseteq g\}$ for any injective $s \in N^{<\omega}$. Each B_s is clopen. For $A \subseteq S_\infty$ write $s \Vdash A(\dot{g})$ iff the set $B_s \cap A$ is co-meager in B_s.

Claim 1. $M = \{x \in \mathrm{Mod}_\mathcal{L} \mid \emptyset \Vdash \dot{g}^{-1} \cdot x \in M\}$, i.e., M consists of those x such that the set of $g \in S_\infty$ with $g^{-1} \cdot x \in M$ is co-meager in S_∞.

Claim 2. For *any* Borel $M \subseteq \mathrm{Mod}_\mathcal{L}$ and any n there is a formula $\varphi^n_M(v_0, \ldots, v_{n-1})$ of $\mathcal{L}_{\omega_1\omega}$ such that for every $x \in \mathrm{Mod}_\mathcal{L}$ and every injective $s \in N^n$, we have: $x \vDash \varphi^n_M(s_0, \ldots, s_{n-1})$ iff $s \Vdash \dot{g}^{-1} \cdot x \in M$.

The Theorem clearly follows from these two claims. Claim 1 is clear, as if x belongs to M then $g^{-1} \cdot x$ belongs to x for every $g \in S_\infty$, and conversely, if x does not belong to M, then $g^{-1} \cdot x$ belongs to M for no $g \in S_\infty$.

Claim 2 is proved by induction on the Borel rank of M. For simplicity assume that $\mathcal{L}$ has just one binary relation R. If M is of the form $\{x \in N^2 \mid (k, l) \in x\}$ then if k, l are less than n take $\varphi^n_M(v_0, \ldots, v_{n-1})$ to be $R(v_k, v_l)$ and otherwise take $\varphi^n_M(v_0, \ldots, v_{n-1})$ to be any contradiction. If $\varphi^n_M(v_0, \ldots, v_{n-1})$ is defined then take $\varphi^n_{\sim M}(v_0, \ldots, v_{n-1})$ to be:

$\bigwedge_{k \geq n}$ (If $(u_0, \ldots, u_{k-1})$ is an injective sequence extending $(v_0, \ldots, v_{n-1})$ then $\sim \varphi^k_M(u_0, \ldots, u_{k-1})$).

(This works because $\sim M$ is co-meager below s iff the complement of M is not co-meager below any extension of s.) Finally, if M is the intersection of M_j, $j \in N$ then take $\varphi^n_M(v_0, \ldots, v_{n-1})$ to be $\bigwedge_j \varphi^n_{M_j}(v_0, \ldots, v_{n-1})$. $\square$

Definition 3 *An equivalence relation (on a Borel subset of a Polish space) is* classifiable by countable structures *iff it is Borel-reducible to the isomorphism relation $\simeq_\mathcal{L}$ on the countable structures for a countable language $\mathcal{L}$.*

LER's are obtained by restricting the logic action $j_\mathcal{L}$ to an invariant Borel set B. Such a relation is trivially Borel-reducible to $\simeq_\mathcal{L}$, via the identity restricted to B.

We can also restrict $j_\mathcal{L}$ by replacing S_∞ by one of its closed subgroups G. The resulting equivalence relation $\simeq^G_\mathcal{L}$ is expressed by: $x \simeq^G_\mathcal{L} y$ iff x and y are isomorphic via a permutation in G.

Theorem 4 *The equivalence relation $\simeq^G_{\mathcal{L}}$ induced by a closed subgroup G of S_∞ is classifiable by countable structures.*

Proof. For any $x \in \mathrm{Mod}_{\mathcal{L}}$ define $\mathrm{Aut}_x = \{g \in S_\infty \mid g \cdot x = x\}$, the group of automorphisms of x.

Lemma 5 *The closed subgroups of S_∞ are exactly the automorphism groups of countable structures for a countable language.*

Proof of Lemma 5. Clearly each automorphism group is a closed subgroup. For the other direction, let G be a closed subgroup of S_∞. Let I_n be the set of G-orbits on N^n, i.e., the set of equivalence classes of elements of N^n under $s \sim t$ iff $t = g \circ s$ for some $g \in G$. I_n is a countable subset of $\mathcal{P}(N^n)$. Let I be the union of the I_n's, for any $i \in I_n$ let R_i be an n-ary relation symbol and let $\mathcal{L}$ be the resulting language. Now define $x \in \mathrm{Mod}_{\mathcal{L}}$ as follows: For $i \in I_n$, $x \models R_i(k_0, \ldots, k_{n-1})$ iff $(k_0, \ldots, k_{n-1})$ belongs to i. Then $G = \mathrm{Aut}_x$, using the fact that G is closed. □ (Lemma 5)

Now to show that $\simeq^G_{\mathcal{L}}$ is classifiable by countable structures for a closed subgroup G of S_∞, write G as Aut_y for some countable structure y for a language $\mathcal{L}'$ disjoint from $\mathcal{L}$. The map $x \mapsto (x, y)$ is then a Borel reduction of $\simeq^G_{\mathcal{L}}$ to $\simeq_{\mathcal{L}\cup\mathcal{L}'}$. □

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Theorem 6 *The equivalence classes of any LER are Borel.*

Proof. We use "Scott analysis". Let $\mathcal{A}$, $\mathcal{B}$ be structures for a countable relational language $\mathcal{L}$ and let $\bar{a}$, $\bar{b}$ be finite sequences from the universes of $\mathcal{A}$, $\mathcal{B}$ of the same length. We define the relation $(\mathcal{A}, \bar{a}) \sim^\alpha (\mathcal{B}, \bar{b})$ by induction on α as follows:

$(\mathcal{A}, \bar{a}) \sim^0 (\mathcal{B}, \bar{b})$ iff $\bar{a}$, $\bar{b}$ satisfy the same atomic formulas in $\mathcal{A}$, $\mathcal{B}$, respectively.
$(\mathcal{A}, \bar{a}) \sim^{\alpha+1} (\mathcal{B}, \bar{b})$ iff for any c there is a d such that $(\mathcal{A}, \bar{a}, c) \sim^\alpha (\mathcal{B}, \bar{b}, d)$ and conversely, for any d there is a c such that $(\mathcal{A}, \bar{a}, c) \sim^\alpha (\mathcal{B}, \bar{b}, d)$.
For limit λ, $(\mathcal{A}, \bar{a}) \sim^\lambda (\mathcal{B}, \bar{b})$ iff $(\mathcal{A}, \bar{a}) \sim^\alpha (\mathcal{B}, \bar{b})$ for all $\alpha < \lambda$.

For a countable $\mathcal{A}$, the least ordinal α such that $(\mathcal{A}, \bar{a}) \sim^\alpha (\mathcal{A}, \bar{b})$ implies $(\mathcal{A}, \bar{a}) \sim^{\alpha+1} (\mathcal{A}, \bar{b})$ for all $\bar{a}$, $\bar{b}$ is called the *Scott rank of* $\mathcal{A}$, denoted $\mathrm{sr}(\mathcal{A})$.

Then $\mathrm{sr}(\mathcal{A})$ is a countable ordinal and by a back-and-forth argument, if $\mathcal{A}$ and $\mathcal{B}$ are countable and have the same Scott rank, then for any $\bar{a}$, $\bar{b}$, if $(\mathcal{A}, \bar{a}) \sim^{sr(\mathcal{A})} (\mathcal{B}, \bar{b})$, then $(\mathcal{A}, \bar{a})$ is isomorphic to $(\mathcal{B}, \bar{b})$. In particular, for any countable ordinal α, the isomorphism relation restricted to models of an $\mathcal{L}_{\omega_1\omega}$ sentence σ which have Scott rank at most α is Borel.

The equivalence relations $\sim^\alpha$ are closely related to the infinitary logic $\mathcal{L}_{\omega_1\omega}$. For any formula φ of that logic, define the *quantifier-rank* of φ, $\mathrm{qr}(\varphi)$, as follows:

$\mathrm{qr}(\varphi) = 0$ for atomic φ.
$\mathrm{qr}(\sim \varphi) = \mathrm{qr}(\varphi)$.
$\mathrm{qr}(\forall v \varphi) = \mathrm{qr}(\exists v \varphi) = \mathrm{qr}(\varphi) + 1$.
$\mathrm{qr}(\bigwedge_n \varphi_n) = \mathrm{qr}(\bigvee_n \varphi_n) = \sup_n \mathrm{qr}(\varphi_n)$.

Then $(\mathcal{A}, \bar{a}) \sim^\alpha (\mathcal{B}, \bar{b})$ iff $(\mathcal{A}, \bar{a})$ and $(\mathcal{B}, \bar{b})$ satisfy the same sentences of quantifier-rank $\leq \alpha$.

In fact, we can assign to any $(\mathcal{A}, \bar{a})$ and ordinal α a formula $\varphi_{\bar{a}}^{\mathcal{A},\alpha}(x_1, \ldots, x_n)$ (where $\bar{a}$ has length n) such that $(\mathcal{A}, \bar{a}) \sim^\alpha (\mathcal{B}, \bar{b})$ iff $\mathcal{B} \models \varphi_{\bar{a}}^{\mathcal{A},\alpha}(\bar{b})$:

$\varphi_{\bar{a}}^{\mathcal{A},0}$ is the conjunction of all atomic and negatomic formulas satisfied by $\bar{a}$ in $\mathcal{A}$.
$\varphi_{\bar{a}}^{\mathcal{A},\alpha+1} = \varphi_{\bar{a}}^{\mathcal{A},\alpha} \wedge \bigwedge_{a_{n+1}} (\exists x_{n+1}) \varphi_{\bar{a},a_{n+1}}^{\mathcal{A},\alpha} \wedge (\forall x_{n+1}) \bigvee_{a_{n+1}} \varphi_{\bar{a},a_{n+1}}^{\mathcal{A},\alpha}$.
$\varphi_{\bar{a}}^{\mathcal{A},\lambda}$ is the conjunction of the $\varphi_{\bar{a}}^{\mathcal{A},\alpha}$ for $\alpha < \lambda$.

For any $\mathcal{A}$ the following sentence expresses the fact that the Scott rank of $\mathcal{A}$ is at most α:

$$\psi^{\mathcal{A},\alpha} \sim \bigwedge_{a_1,\ldots,a_n} (\forall x_1, \ldots, x_n) \varphi_{a_1,\ldots,a_n}^{\mathcal{A},\alpha}(x_1, \ldots, x_n) \rightarrow \varphi_{a_1,\ldots,a_n}^{\mathcal{A},\alpha+1}(x_1, \ldots, x_n).$$

Then for any countable $\mathcal{A}$, a countable $\mathcal{B}$ is isomorphic to $\mathcal{A}$ iff $\mathcal{B}$ satisfies the sentence $\varphi_\emptyset^{sr(\mathcal{A})} \wedge \psi^{\mathcal{A},sr(\mathcal{A})}$. I.e., the isomorphism type of $\mathcal{A}$ is described by a single sentence of $\mathcal{L}_{\omega_1\omega}$, called the *Scott sentence* of $\mathcal{A}$ and therefore the isomorphism type of $\mathcal{A}$ is Borel. □

If there is a fixed countable bound α on the Scott ranks of the countable models of the $\mathcal{L}_{\omega_1\omega}$ sentence σ, then the entire isomorphism relation on countable models of σ is Borel, as we have seen that isomorphism restricted to models of Scott rank at most α is Borel. By the same argument, the relation "$(\mathcal{A}, \bar{a})$ is isomorphic to $(\mathcal{B}, \bar{b})$" where $\mathcal{A}$, $\mathcal{B}$ are countable models of σ (and $\bar{a}$, $\bar{b}$ are finite sequences from $\mathcal{A}$, $\mathcal{B}$, respectively) is Borel.

Conversely, we have:

Theorem 7 *Suppose that σ is a sentence of $\mathcal{L}_{\omega_1\omega}$, where $\mathcal{L}$ is a countable relational language. Suppose that the relation "$(\mathcal{A}, \bar{a})$ is isomorphic to $(\mathcal{B}, \bar{b})$" where $\mathcal{A}$, $\mathcal{B}$ are countable models of σ, is Borel. Then there is a countable bound on the Scott ranks of the models of σ.*

Proof. We use some descriptive set theory. For any real x let $\mathcal{A}_x$ denote the countable structure coded by x. For simplicity, assume that the language $\mathcal{L}$ can be coded recursively. Then for any x and any $\bar{a}$, $\bar{b}$ of the same length from $\mathcal{A}_x$, we have:

If $(\mathcal{A}_x, \bar{a}) \not\cong (\mathcal{A}_x, \bar{b})$ then for some ordinal α less than ω_1^x such that $(\mathcal{A}_x, \bar{a}) \nsim^\alpha (\mathcal{A}_x, \bar{b})$,

where ω_1^x denotes the least ordinal not recursive in x. By hypothesis, the set $B = \{(x, \bar{a}, \bar{b}) \mid \mathcal{A} \models \sigma \text{ and } (\mathcal{A}_x, \bar{a}) \not\cong (\mathcal{A}_x, \bar{b})\}$ is Borel, and we have:

$\forall (x, \bar{a}, \bar{b}) \in B\ \exists n(n$ codes an x-recursive wellordering $<_n^x$ and $(\mathcal{A}_x, \bar{a}) \nsim^\alpha (\mathcal{A}_x, \bar{b})$ where α is the length of $<_n^x)$.

Thus we have a total, $\mathbf{\Pi}_1^1$ relation $R(x, n)$ on $B \times \omega$. By "easy uniformisation" for $\mathbf{\Pi}_1^1$ relations, there is a Borel function f defined on B such that $R(x, f(x))$ for all x in B. But then the set of $<_{f(x)}^x$, $x \in B$, is a $\mathbf{\Sigma}_1^1$ set of wellorderings. The Boundedness Theorem says that there is a countable bound α on the lengths of any $\mathbf{\Sigma}_1^1$ set of wellorderings, and therefore we have:

For models A_x of σ, if $(\mathcal{A}_x, \bar{a}) \not\cong (\mathcal{A}_x, \bar{b})$ then $(\mathcal{A}_x, \bar{a}) \nsim^\alpha (\mathcal{A}_x, \bar{b})$,

which implies that the Scott rank of each model $\mathcal{A}_x$ of σ is at most α. □

Remark. The above proof only used the weaker hypothesis that the relation "$(\mathcal{A}, \bar{a}) \simeq (\mathcal{A}, \bar{b})$" is Borel for models $\mathcal{A}$ of σ. It can be shown that it is also sufficient to assume that the relation "$\mathcal{A} \simeq \mathcal{B}$" is Borel.

A related result of Sami is the following:

Theorem 8 *An LER is Borel iff there is a countable bound on the Borel ranks of its equivalence classes.*

Proof. If E is an LER of Borel rank α, then α is also a bound on the Borel ranks of its equivalence classes. For the converse, we consider the relations:

$xE^\alpha y$ iff for all $\mathbf{\Pi}^0_\alpha$ E-invariant A, x belongs to A iff y belongs to A.

If α is a bound on the Borel ranks of E's equivalence classes, then $E = E^\alpha$. So it suffices to show that each E^α is $\mathbf{\Pi}^1_1$, for then E is both $\mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^1_1$, and therefore Borel.

Let $U \subseteq \omega^\omega \times \mathrm{Mod}(\sigma)$ be a universal $\mathbf{\Pi}^0_\alpha$ relation (where E is the isomorphism relation on models of σ). Define $T \subseteq \omega^\omega \times \mathrm{Mod}(\sigma)$ by: $(a, x) \in T$ iff $\{g \mid (a, g \cdot x) \in U\}$ is comeager (where $g \cdot x$ is the action of S_∞ on $\mathrm{Mod}(\sigma)$). It can be shown that T is Borel and the $T_a = \{x \mid (a, x) \in T\}$, $a \in \omega^\omega$, are precisely the E-invariant $\mathbf{\Pi}^0_\alpha$ sets. Thus:

$$xE^\alpha y \text{ iff for all } z, \ (x \in T_z \text{ iff } y \in T_z),$$

and so E^α is $\mathbf{\Pi}^1_1$. □

Remark. It is not in general true that if an analytic equivalence relation has Borel equivalence classes of bounded rank then it must be Borel. Here is a simple example due to Su Gao: Let X be Polish and consider $Y = X \times \{0, 1\}$. Fix an analytic non-Borel $A \subseteq X$. Consider then the equivalence relation on Y defined by:

$$(x, m)E(y, n) \text{ iff } ((x = y \in A \text{ or } (x, m) = (y, n)).$$

Its equivalence classes are $\{x\} \times \{0, 1\}$, if $x \in A$, and $\{(x, n)\}$, if $x \notin A$. It follows that $x \in A$ iff $(x, 0)E(x, 1)$, and so E is not Borel. Every E-class has at most two elements!

7.-8.Vorlesungen

Theorem 9 *There is a complete LER, i.e., an LER which is largest among LER's under Borel-reducibility. An example is the isomorphism relation for binary relations on N.*

Proof. If $\mathcal{L}$ is the relational language with infinitely-many n-place relations for each positive n, then the isomorphism relation on $\mathcal{L}$-structures is clearly complete, as any LER reduces to it via the identity.

To prove the second statement, let $\mathrm{HF}(N)$ denote the set of all hereditarily finite sets over the set N, where N is viewed as a set of atoms (i.e., elements of N have no elements), and let ϵ denote the membership relation for $\mathrm{HF}(N)$. Also let $\simeq_{\mathrm{HF}(N)}$ be the isomorphism relation for binary relations on the set $\mathrm{HF}(N)$. It suffices to show that any $\simeq_{\mathcal{L}}$ is Borel-reducible to $\simeq_{\mathrm{HF}(N)}$.

Define an action of S_∞ on $\mathrm{HF}(N)$ by: $g \cdot n = g(n)$, $g \cdot \{a_1, \ldots, a_n\} = \{g \cdot a_1, \ldots, g \cdot a_n\}$. Then for any fixed $g \in S_\infty$, the map $a \mapsto g \cdot a$ is an ϵ-isomorphism of $\mathrm{HF}(N)$.

Lemma 10 *Suppose that X, Y are ϵ-transitive subsets of HF(N), the sets $N \setminus X$, $N \setminus Y$ are infinite and $\epsilon \restriction X \simeq_{HF(N)} \epsilon \restriction Y$. Then there is $f \in S_\infty$ such that $Y = f \cdot X = \{f \cdot s \mid s \in X\}$.*

Proof. It follows from the hypothesis $\epsilon \restriction X \simeq_{\mathrm{HF}(N)} \epsilon \restriction Y$ that there is an ϵ-isomorphism π from X onto Y. Then π restricted to $X \cap N$ is a bijection from $X_0 = X \cap N$ onto $Y_0 = Y \cap N$ and therefore there is $f \in S_\infty$ such that f agrees with π on X_0. Then we have $f \cdot s = \pi(s)$ for any $s \in X$. □ (Lemma 10)

Now to prove Theorem 9, we first show that $\simeq_{G(m)}$ is Borel-reducible to $\simeq_{\mathrm{HF}(N)}$ for any m, where $G(m)$ is the language with a single m-ary relation symbol. For each x in $\mathrm{Mod}_{G(m)} = \mathcal{P}(N^m)$ set $\Theta(x) = \{\theta(s) \mid s \in x\}$, where $\theta(\langle i_1, \ldots, i_m\rangle) =$ the transitive closure under ϵ of $\{\langle 2i_1, \ldots, 2i_m\rangle\}$. It follows from the lemma that x is isomorphic to y iff $\epsilon \restriction \Theta(x)$ is isomorphic to $\epsilon \restriction \Theta(y)$. Thus $\simeq_{G(m)}$ is Borel-reducible to $\simeq_{\mathrm{HF}(N)}$.

Finally we must show that $\simeq_{\mathcal{L}'}$ is Borel-reducible to $\simeq_{\mathrm{HF}(N)}$ where $\mathcal{L}'$ is the language with infinitely-many binary relation symbols. In this case $\mathrm{Mod}_{\mathcal{L}'} = \mathcal{P}(N^2)^N$ and we can assume that every $x \in \mathrm{Mod}_{\mathcal{L}'}$ has the form $x = \{x_n\}_{n \geq 1}$ with $x_n \subseteq (N \setminus \{0\})^2$ for all n. Let $\Theta(x)$ be $\{s_n(k,l) \mid n \geq 1$ and $(k,l) \in x_n\}$ for any such x, where

$$s_n(k,l) = \mathrm{TC}_\epsilon(\{\{\cdots\{\langle k,l\rangle\}\cdots\}, 0\}),$$

where there are $n+2$ pairs of braces and TC_ϵ denotes ϵ-transitive closure. Then Θ is a continuous reduction of $\simeq_{\mathcal{L}'}$ to $\simeq_{\mathrm{HF}(N)}$. □

Theorem 11 *$(R,=)$ is the $\leq_B$-smallest Borel LER with uncountably-many equivalence classes.*

Vaught's Conjecture. $(R,=)$ is the $\leq_B$-smallest LER with uncountably-many equivalence classes.

To prove Theorem 11, we first study the topology generated by the nonempty lightface analytic (Σ^1_1) sets.

The Gandy-Harrington Topology

Definition 12 *Suppose that $\mathcal{F}$ is a family of sets in a topological space. Then a subfamily $\mathcal{D}$ of $\mathcal{F}$ is* dense *in $\mathcal{F}$ iff every element of $\mathcal{F}$ contains an element of $\mathcal{D}$ as a subset. We say that $\mathcal{F}$ is* Polish-like *iff there exists a collection $\{\mathcal{D}_n \mid n \in \omega\}$ of dense subfamilies of $\mathcal{F}$ such that $\bigcap_n F_n$ is nonempty whenever $F_0 \supseteq F_1 \supseteq \ldots$ is a decreasing sequence of sets in $\mathcal{F}$ such that $\{F_n \mid n \in \omega\}$ intersects each $\mathcal{D}_n$.*

Lemma 13 *The collection $\mathcal{F}$ of nonempty Σ^1_1 subets of Baire space is Polish-like.*

Proof. Let $\mathcal{X}$ denote Baire space. For any $P \subseteq \mathcal{X} \times \mathcal{X}$ define proj $P = \{x \mid \exists y P(x,y)\}$. If $P \subseteq \mathcal{X} \times \mathcal{X}$ and $s,t \in N^{<\omega}$ then set $P_{st} = \{(x,y) \in P \mid s \subseteq x$ and $t \subseteq y\}$. Let $\mathcal{D}(P,s,t)$ be the collection fo all nonempty Σ^1_1 sets X such that either $X \cap$proj P_{st} is empty or $X \subseteq P_{s*i,t*j}$ for some i,j. Let $(\mathcal{D}_n \mid n \in \omega)$ be an enumeration of all $\mathcal{D}(P,s,t)$ where $P \subseteq \mathcal{X} \times \mathcal{X}$ is Π^0_1. Then each $\mathcal{D}_n$ is a dense subfamily of $\mathcal{F}$.

Now consider a decreasing sequence $X_0 \supseteq X_1 \supseteq \cdots$ of nonempty Σ^1_1 sets such that $\{X_n \mid n \in \omega\}$ intersects each $\mathcal{D}_n$; we show that $\bigcap_n X_n$ is nonempty. Say that X is *positive* iff $X_n \subseteq X$ for some n. For each n fix a Π^0_1 set $P^n \subseteq \mathcal{X} \times \mathcal{X}$ such that $X_n =$ proj P^n. For any $s,t \in N^{<\omega}$, if proj P^n_{st} is positive, then there is a unique i and some j such that proj $P^n s * i, t * j$ is also positive. It follows that there is a unique $x = x_n \in \mathcal{X}$ and some $y = y_n \in \mathcal{X}$ such that proj $P^n_{x\restriction k, y\restriction k}$ is positive for each k. As P^n is closed, we have $P^n(x,y)$ and therefore $x_n = x \in X_n$. But $x_m = x_n$ for each m,n, as if both P_{st} and $Q_{s't'}$ are positive, then either $s \subseteq s'$ or $s' \subseteq s$. □

The topology generated by the (lightface) Σ^1_1 sets is called the *Gandy-Harrington topology.* It is not metrizable, but does satisfy the Baire category theorem.

Proof of Theorem 11. In fact we show that $(R, =)$ is Borel-reducible to any $\mathbf{\Pi}^1_1$ equivalence relation with uncountably many equivalence classes. Let E be a $\mathbf{\Pi}^1_1$ equivalence relation (on Baire space); for simplicity, we assume that E is in fact lightface Π^1_1, as the proof below easily relativises to a parameter. A set X is *pairwise E-equivalent* iff any two elements of X are E-equivalent to each other.

If every x belongs to a Δ^1_1 pairwise E-equivalent set, then of course E has at most countably many equivalence classes. So we assume that this is not the case and let H be the nonempty set of x which do *not* belong to a Δ^1_1 pairwise E-equivalent set.

Lemma 14 *H is Σ^1_1 and has no nonempty Σ^1_1, pairwise E-equivalent subset.*

Proof. Let $\langle W_e \mid e \in I \rangle$ be a nice Π^1_1 enumeration of the Δ^1_1 sets; i.e., the W_e's are the Δ^1_1 sets and the relation $\{(x, e) \mid e \in I \text{ and } x \in W_e\}$ is the restriction to $\mathcal{X} \times I$ of a Σ^1_1 relation, where I is Π^1_1. Then x belongs to H iff for all e, if x belongs to W_e then there are $y, z \in W_e$ which are E-inequivalent. As E is Π^1_1, it follows that H is Σ^1_1.

If X is a nonempty pairwise E-equivalent Σ^1_1 set then $B = \bigcap_{x \in X} [x]_E$ is a Π^1_1 E-equivalence class and X is a subset of B. By the Separation Theorem for Σ^1_1 sets, there is a Δ^1_1 set C such that $X \subseteq C \subseteq B$. Then by the definition of H, C is disjoint from H, and therefore X is not a subset of H. □

Now let $\mathcal{M}$ be a countable elementary submodel of $H(\omega_1)$, the collection of hereditarily countable sets. Then $\mathcal{M}$ satisfies ZFC minus the powerset axiom. Let $P \in M$ be the forcing whose conditions are (codes for) nonempty Σ^1_1 sets under inclusion, and let $G \subseteq P$ be P-generic over $\mathcal{M}$. Then by Lemma 13, the intersection of the Σ^1_1 sets (coded) in G is nonempty. Let x_G be the unique real in this intersection. Such a real is said to be *P-generic* over $\mathcal{M}$.

Now consider the forcing $P^2 = P \times P$. If G is P^2-generic over $\mathcal{M}$ then the intersection of the $X \times Y$ for $(X, Y) \in G$ is a pair of reals x_0^G, x_1^G. Let $\dot{x}_0$, $\dot{x}_1$ be names for these reals.

Lemma 15 *The condition $H \times H$ in P^2 forces that $\dot{x}_0$ is E-inequivalent to $\dot{x}_1$.*

Proof. If not, then choose a condition $X \times Y \subseteq H \times H$ which forces $\dot{x}_0 E \dot{x}_1$. Now if x_0, x_1 are any P-generic reals in X, there is a P-generic real y in Y such that both pairs (x_0, y) and (x_1, y) are P^2-generic; it follows that x_0 and x_1 are E-equivalent.

Now consider the forcing P_2, consisting of all nonempty Σ^1_1 subsets of $X \times X$. If G is P_2-generic then the intersection of the sets in G is a pair (x_0, x_1) of reals in X such that both x_0 and x_1 are P-generic and therefore by the above are E-equivalent. But if G contains the condition $(X \times X) \setminus E$ (which by Lemma 33 is nonempty), then the resulting x_0, x_1 cannot be E-equivalent as the pair (x_0, x_1) belongs to the complement of E, contradiction. $\square$

Now to finish the proof of Theorem 11, build a perfect set A of reals in H such that any distinct pair from A is P^2-generic. Then by Lemma 15, distinct reals in A are E-inequivalent, and therefore we have a Borel reduction of $(R, =)$ to E. $\square$

9.-10.Vorlesungen

We next take a closer look at the the Borel LER's.

Definition 16 *If E is an equivalence relation on a Polish space X then E^+ is the equivalence relation on the Polish space X^N defined by: xE^+y iff $\{[x(n)]_E \mid n \in N\} = \{[y(n)]_E \mid n \in N\}$. If E_i is an equivalence relation on X_i for each $i \in I$ then $\bigvee_{i \in I} E_i$ is the disjoint union of the E_i's, i.e., the equivalence relation on $\bigcup_{i \in I} \{i\} \times X_i$ defined by $(i, x) \sim (j, y)$ iff $i = j$ and xE_iy. Define the relations F_α, $\alpha < \omega_1$ as follows: F_1 is equality on Baire space, $F_{\alpha+1} = F_\alpha^+$, and $F_\lambda = \bigvee_{\alpha < \lambda} F_\alpha$ for countable limit ordinals λ.*

Theorem 17 *An LER is Borel iff it is Borel-reducible to some F_α, $\alpha < \omega_1$.*

Proof. Clearly each F_α is a Borel equivalence relation, so it follows that any LER Borel-reducible to some F_α must also be Borel.

To prove the converse, we again use Scott analysis. By induction on $\alpha < \omega_1$ we define equivalence relations $\equiv^\alpha_{st}$ on $\mathcal{P}(N^2)$ (= binary relations on N) for $s, t \in N^{<\omega}$ of the same length as follows:

$A \equiv^0_{st} B$ iff $A(s_i, s_j) \leftrightarrow B(t_i, t_j)$ for $i, j <$ length $s =$ length t.
$A \equiv^{\alpha+1}_{st} B$ iff $\forall k \exists l (A \equiv^\alpha_{s*k,t*l} B)$ and $\forall l \exists k (A \equiv^\alpha_{s*k,t*l} B)$.
$A \equiv^\lambda_{st} B$ iff $A \equiv^\alpha_{st} B$ for all $\alpha < \lambda$, λ limit.

By a back-and-forth argument, $A \simeq B$ (more precisely, $(N, A) \simeq (N, B)$) iff $A \equiv^\alpha_{\emptyset\emptyset} B$ for all $\alpha < \omega_1$.

Lemma 18 *Suppose that P is a Σ^1_1 set of pairs (A, B) where A, B are binary relations on N and suppose that P is* unbounded *in the sense that for each $\alpha < \omega_1$, P contains a pair (A, B) such that $A \equiv^\alpha_{\emptyset\emptyset} B$. Then P contains a pair (A, B) such that $A \simeq B$.*

Proof of Lemma. Let F be a continuous map from N^N onto P. For $u \in N^{<\omega}$ let P_u be $\{F(a) \mid u \subseteq a\}$. There is a smallest n_0 such that $P_{\langle n_0 \rangle}$ is still unbounded. Let $k_0 = 0$. Then there is l_0 such that $P_{\langle n_0 \rangle}$ is still unbounded "over $\langle k_0 \rangle, \langle l_0 \rangle$" in the sense that for each $\alpha < \omega_1$, $P_{\langle n_0 \rangle}$ contains a pair (A, B) such that $A \equiv^\alpha_{\langle k_0 \rangle, \langle l_0 \rangle} B$. Iterating this, we can produce numbers n_m, k_m, l_m such that both $\langle k_m \mid m \in \omega \rangle$ and $\langle l_m \mid m \in \omega \rangle$ are permutations of N and for any m, the set $P_{\langle n_0, \ldots, n_m \rangle}$ is unbounded over $\langle k_0, \ldots, k_m \rangle, \langle l_0, \ldots, l_m \rangle$. Let a be $\langle n_m \mid m \in N \rangle$ and $F(a) = (A, B) \in P$.

Then the permutation $f(k_m) = l_m$ witnesses $A \simeq B$. □ (Lemma 18)

Now suppose that E is a Borel LER. Then E is Borel-reducible to $\simeq$ on binary relations; let Θ be a Borel reduction. Then $\{(\Theta(x), \Theta(y)) \mid \sim xEy\}$ is a Σ^1_1 set and does not intersect $\simeq$. It follows from the lemma that for some $\alpha < \omega_1$, Θ reduces E to $\equiv^\alpha_{\emptyset\emptyset}$. Thus to finish the proof we only need the following:

Lemma 19 *Let $\equiv^\alpha$ denote $\equiv^\alpha_{\emptyset\emptyset}$ for any $\alpha < \omega_1$. Then $\equiv^\alpha$ is Borel-reducible to some F_β.*

Proof of Lemma. We show that each $\equiv^\alpha_{st}$ is Borel-reducible to some F_β. The relations $\equiv^0_{st}$ have countably-many clopen equivalence classes, and hence are Borel-reducible to F_0, which we can take to be equality on N. For each α, the function $(s, A) \mapsto \{(s * k, A) \mid k \in N\}$ is a Borel reduction of $\equiv^{\alpha+1}$ to $(\equiv^\alpha)^+$. For limit $\lambda < \omega_1$, write λ as the limit of an increasing sequence $\langle \lambda_n \mid n \in \omega \rangle$; then $(s, A) \mapsto \{(m, s, A) \mid m \in N\}$ is a Borel reduction of $\equiv^\lambda$ to $(\bigvee_{n \in N} \equiv^{\lambda_n})^+$. □

Remark. It is also not difficult to show that each F_α is an LER. This is because the collection of LER's is closed under the operations $^+$ and $\bigvee_{n\in N}$.

Note that Vaught's conjecture would follow from Silver's theorem if we could show that each LER is either Borel or complete, as any complete LER has continuum-many equivalence classes. Unfortunately:

Theorem 20 *There is an LER which is neither Borel nor complete.*

To prove this we take an excursion into the theory of Abelian torsion groups, i.e., Abelian groups $(G, +)$ with the property that for every $g \in G$ there is $n \in N$ such that $g + g + \cdots + g$ (n times) equals 0. The following lemmas show that $\simeq_{Atg}$, the isomorphism relation on these groups, serves as a witness to our theorem.

Lemma 21 *(Ulm invariants) There is an "absolute reduction" of $\simeq_{Atg}$ to equality on $N^{<\omega_1}$.*

We can take "absolutely reducible" to mean: some formula defines a reduction which works in any transitive model of (enough) set theory.

Lemma 22 *Not every LER is absolutely reducible to equality on $N^{<\omega_1}$.*

Lemma 23 *$\simeq_{Atg}$ is complete analytic as a binary relation.*

Lemmas 21 and 24 imply that $\simeq_{Atg}$ is not complete as an LER and Lemma 23 implies that it is not Borel.

Proof of Lemma 21. Any Abelian torsion group G can be written as a direct sum $\sum_p G_p$, where p varies over primes and G_p is a *p-group* (i.e., for each x there is an n such that $p^n \cdot x = x + x + \cdots + x$ (p^n times) equals 0. If G is a p-group then G in turn can be written as $G_r \oplus G_d$ where G_d is *divisible* (i.e., for each x there is y such that $p \cdot y = x$) and G_r is *reduced* (i.e., G_r has no divisible subgroups other than $\{0\}$). Now G_d is just a direct sum of copies of $Z(p^\infty)$, the group of p^n-th complex roots of unity, $n > 0$. So to classify Abelian torsion groups, it suffices to classify reduced p-groups, for the various primes p.

Fix a prime p. For any countable P-group G, pG denotes $\{p \cdot x \mid x \in G\}$. Then define:

$p^0G = G$
$p^{\alpha+1}G = p(p^\alpha G)$
$p^\lambda G = \bigcap_{\alpha<\lambda} p^\alpha G$, λ limit.

Then $p^\infty G$ ($= p^\alpha G$ for sufficiently large α) is the divisible part G_d of G; of course if G is reduced then $p^\infty G = \{0\}$. The least α such that $p^\alpha G = p^{\alpha+1}G$ is called the *length of* G, written $l(G)$.

For $\alpha < l(G)$ we associate a cardinal number $U_G(\alpha)$, called the *α-th Ulm invariant of* G. To define this, note that every element of $p^\alpha G/p^{\alpha+1}G$ has order p, and therefore this quotient can be regarded as a vector space over Z_p, the field with p elements, with some dimension. We let $U_G(\alpha)$ be this dimension, which is an element of $\{1, 2, \ldots, \infty\}$. For $\alpha \geq l(G)$, we set $U_G(\alpha) = 0$.

Ulm's Theorem. If G and H are reduced p-groups then G and H are isomorphic iff they have the same Ulm invariants, i.e., iff $U_G(\alpha) = U_H(\alpha)$ for every α.

Lemma 21 now follows: Associate to each countable Abelian torsion group G its sequence of Ulm invariants, together with a 0 or 1, indicating whether or not G is reduced. This association is an absolute reduction of $\simeq_{Atg}$ to equality on $N^{<\omega_1}$. $\Box$

Proof of Lemma 23. It is a result of hyperarithmetic theory that there is a Borel map $L \mapsto L^*$ from linear orders of ω to itself, such that L is a wellordering iff L^* is a wellordering iff $L^* + L^*$ is not isomorphic to L^*. To each linear ordering L associate the tree T_L, consisting of finite descending L-sequences.

Now via a construction of Feferman, further associate to each tree T of finite sequences from N an Abelian 2-group G_T. The generators of G are g_s, $s \in T$, subject to the following relations:

For s of length greater than 1, $g_s + g_s = g_{s^-}$ (where s^- is the immediate predecessor of s in T).
For s of length 1, $g_s + g_s = 0$.

Then it is not hard to see that if T is a well-founded tree of rank α, then the Ulm rank of G_T is approximately α. In particular, if T' is another well-founded tree whose rank is much bigger than α, then G_T and $G_{T'}$ are not isomorphic. So we have that for any linear ordering L:

L is not a wellordering iff
L^* is not a wellordering iff
$L^* + L^*$ is isomorphic to L^* iff
T_{L^*} is isomorphic to $T_{L^*+L^*}$ iff
$G_{T_{L^*}}$ is isomorphic to $G_{T_{L^*+L^*}}$.

The last equivalence holds because if L^* is a wellordering of length α, then the rank of $T_{L^*+L^*}$ is twice that of T_{L^*}. So we have Borel-reduced the set of wellorderings of N, a complete analytic set, to the isomorphism relation for Abelian 2-groups. □

11.-12.Vorlesungen

Lemma 24 *Not every LER is absolutely reducible to equality on $N^{<\omega_1}$.*

Proof. An example is E_0, equality mod finite for subsets of ω. Suppose that F were an absolute reduction from E_0 to $N^{<\omega_1}$, or equivalently, to countable sets of countable ordinals. Let M be a countable transitive model of enough set theory. Add a Cohen real g over M, and consider the E_0-equivalence class $[g]$ of g. Then $F([g])$ is a countable set of countable ordinals in $M[g]$, by the absoluteness of the reduction F. But by the homogeneity of Cohen forcing, the statement "α belongs to $F([g])$" is decided by the empty condition of Cohen forcing for each countable ordinal α of $M[g]$ and therefore $F([g])$ is in fact independent of the choice of the Cohen real g. But there are distinct Cohen reals g_0, g_1 with $[g_0]$ unequal to $[g_1]$, contradicting the fact that F is a reduction. □

Theorem 25 *There is a size continuum antichain in the ordering of countable Borel equivalence relations (and therefore in the ordering of LER's) under Borel-reducibility.*

Embeddability relations: Logic Quasi-Orders (LQO's)

A *quasi-order* is a reflexive, transitive relation. If R is a quasi-order then the *equivalence relation derived from R* is given by xEy iff (xRy and yRx).

A *logic quasi-order*, or *LQO*, is the restriction of the quasi-order of embeddability to the countable models of a sentence of $\mathcal{L}_{\omega_1\omega}$ (for some countable language $\mathcal{L}$).

First observe the following.

Theorem 26 *There is a $\le_B$-complete analytic quasi-order.*

Proof. Let W_0 be an analytic subset of $(2^\omega)^3$ which is universal for analytic subsets of $(2^\omega)^2$, i.e., any analytic subset of $(2^\omega)^2$ is of the form $\{(y,z) \mid (x,y,z) \in W_0\}$ for some x. Define W_1 by: $(x_1,x_2)W_1(y_1,y_2)$ iff $(x_1 = y_1 \wedge (x_1,x_2,y_2) \in W_0)$, and let W_2 be the least reflexive and transitive relation containing W_1. Then W_2 is a complete analytic quasi-order: Clearly it is an analytic quasi-order, as analytic relations are closed under existential quantification over reals. If R is any analytic quasi-order on 2^ω with W_0-code x, then

$$y_1 R y_2 \leftrightarrow (x,y_1)W_2(x,y_2),$$

and therefore the map $y \mapsto (x,y)$ reduces R to W_2. □

Theorem 27 *Let E be an analytic equivalence relation on a Polish space X. Then E is $\le_B$-complete as an analytic equivalence relation iff E is the equivalence relation $\equiv_R$ derived from a $\le_B$-complete analytic quasi-order R on X.*

Proof. Suppose that R is a complete analytic quasi-order. If F is an analytic equivalence relation on a Polish space, then F is in particular a quasi-order and therefore is Borel-reducible to R. But then the same reduction shows that F also Borel-reduces to $\equiv_R$, which is therefore a complete analytic equivalence relation.

Conversely, suppose that E is a complete analytic equivalence relation on X and let R_0 be a complete analytic quasi-order on 2^ω. Let $f : 2^\omega \to X$ be a Borel reduction of $\equiv_{R_0}$, the equivalence relation derived from R_0, to E. Define:

$$xRy \leftrightarrow xEy \vee \exists a \exists b(xEf(a) \wedge yEf(b) \wedge aR_0b).$$

Then R is analytic and contains E. Let X_0 be $\{x \mid \exists a(xEf(a))\}$ and $X_1 = X \setminus X_0$. Note that for $x,y \in X_0$ with $x(R \setminus E)y$ and any a,b with $xEf(a)$ and $yEf(b)$, one must have aR_0b, since f reduces $\equiv_{R_0}$ to E. Also points in

X_0, X_1 are R-unrelated and R and E agree on X_1. It follows that R is indeed a quasi-order, $\equiv_R$ equals E and f is a reduction of R_0 to R, so that R is complete. □

Recall that no LER is complete as an analytic equivalence relation. However for LQO's we have:

Theorem 28 *There is an LQO which is complete as an analytic quasi-order (and therefore there is a bi-embeddability relation which is complete as an analytic equivalence relation).*

Proof. First we introduce a particular complete analytic quasi-order $\leq_{max}$, and then use it to show that a certain LQO is also complete.

If s, t are finite sequences from ω of the same length, then we write $s \leq t$ iff $s(i) \leq t(i)$ for all $i < |s|$ and $s + t$ for the sequence of length $|s|$ whose value at i is $s(i) + t(i)$. For any set X, a *tree* on X is a subset of $X^{<\omega}$ closed under restriction. If T is a tree on $X \times \omega$ then we say that T is *normal* iff whenever (u, s) belongs to T and $s \leq t$ then (u, t) belongs to T. For $s \in \omega^{<\omega}$ set $T(s) = \{u \in X^{<\omega} \mid |u| = |s| \wedge (u, s) \in T\}$. Thus for normal T we have $s \leq t \rightarrow T(s) \subseteq T(t)$.

A function $f : \omega^{<\omega} \rightarrow \omega^{<\omega}$ is *Lipschitz* iff f preserves both length and extension.

Definition 29 *Let $\mathcal{T}$ be the space of normal trees on $2 \times \omega$, with its natural Polish topology. Define $\leq_{max}$ on $\mathcal{T}$ by:*

$$S \leq_{max} T \leftrightarrow \exists \text{ Lipschitz } f : \omega^{<\omega} \rightarrow \omega^{<\omega} \ \forall s \in \omega^{<\omega} \ S(s) \subseteq T(f(s)).$$

This is a strong way of saying that the projection of S is included in the projection of T.

13.Vorlesung

$\leq_{max}$ is an analytic quasi-order on $\mathcal{T}$. To prove that it is complete, we use the following "normal form" result for analytic quasi-orders on 2^ω.

Lemma 30 *Let R be an analytic quasi-order on 2^ω. Then there is a tree S on $2 \times 2 \times \omega$ such that:*
(i) R is the projection of S, i.e., xRy iff for some z, $(x|n, y|n, z|n) \in S$ for all n.
(ii) S is normal, i.e., if (u, v, s) belongs to S and $s \leq t$ then (u, v, t) belongs to S.
(iii) If $u \in 2^{<\omega}$ and $s \in \omega^{<\omega}$ have the same length, then (u, u, s) belongs to S.
(iv) If (u, v, s) and (v, w, t) belong to S then so does $(u, w, s + t)$.

Proof. Start with any tree T_0 on $2 \times 2 \times \omega$ with R the projection of T_0. If we set $T_1 = \{(u, v, t) \mid \exists s \leq t(u, v, s) \in T_0\}$ then T_1 is normal and we have that R is also the projection of T_1. Also, if we let T_2 be $T_1 \cup \{(u, u, s) \mid |u| = |s|\}$ then T_2 satisfies (i),(ii) and (iii).

Finally we define S by: $(\emptyset, \emptyset, \emptyset) \in S$ and for $k, n \in \omega$, $u, v \in 2^k$, $s \in \omega^k$, $i, j \in 2$: $(u * i, v * j, n * s) \in S$ iff $\exists u_0, u_1, \ldots, u_n \in 2^k(u_0 = u \wedge u_n = v \wedge \forall l < n(u_l, u_{l+1}, s) \in T_2)$. (So if $n = 0$, $(u * i, v * j, 0 * s) \in S$ iff $u = v$.)

Then S works: Clearly it is a tree. To check (i), note first that if (x, y, a) is a branch through T_2 then $(x, y, 1*a)$ is a branch through S. So R is a subset of the projection of S. Conversely, suppose that $(x, y, n*a)$ is a branch through S. If $n = 0$, $x = y$ and $(x, y) \in R$. If $n > 0$ we get for each k sequences $(u_i^k)_{i \leq n}$ in 2^k with $u_0^k = x|k$, $u_n^k = y|k$ and for $i < n$, $(u_i^k, u_{i+1}^k, a|k) \in T_2$. By the compactness of 2^ω, we can find a subsequence (k_l) and for $i \leq n$ elements z_i of 2^ω such that $u_i^{k_l} \to z_i$, as $l \to \infty$. But then for $i < n$, (z_i, z_{i+1}, a) is a branch through T_2, hence $z_i R z_{i+1}$. As $z_0 = x$ and $z_n = y$, by transitivity we get xRy, as desired.

To check (ii), let $(u, v, s) \in S$ and $t \geq s$. The case of $(\emptyset, \emptyset, \emptyset)$ is trivial. So suppose $u = u' * i$, $v = v' * j$, $s = n * s'$ and $t = m * t'$, with $n \leq m$ and $s' \leq t'$. As T_2 is normal we also have $(u, v, n * t') \in S$, with the same witnesses $(u_i)_{i \leq n}$. Also, using property (iii) of T_2 we can repeat the witness u_0 $(m - n)$ times to get witneses for $(u, v, m * t') \in S$, as desired.

(iii) follows from (ii) and the fact that if $|u| = |s|$ and $s(0) = 0$ then $(u, u, s) \in S$.

Finally, to check (iv), let $u = u' * i$, $v = v' * j$, $w = w' * k$, $s = n * s'$ and $t = m * t'$ satisfy $(u, v, s) \in S$ and $(v, w, t) \in S$. By (ii) we also have $(u, v, n*(s'+t')) \in S$ and $(v, w, m*(s'+t')) \in S$, as witnessed by say $(u_i)_{i \leq n}$,

$(v_j)_{j\leq m}$. But then $(u_i)_{i<n}*(v_j)_{j\leq m}$ is a witness that $(u, w, (n+m)*(s'+t')) \in S$, as desired. □ (Lemma 30)

Now we show that any analytic quasi-order R on 2^ω is Borel-reducible to $\leq_{max}$. Let S be the tree associated to R by the lemma and define $f : 2^\omega \to \mathcal{T}$ by

$$f(x) = S^x = \{(u, s) \in (2 \times \omega)^{<\omega} \mid (u, x \restriction |u|, s) \in S\}.$$

The tree S^x is normal as S is. We check that f is the desired Borel reduction. Suppose first that $S^x \leq_{max} S^y$, witnessed by the Lipschitz map $\varphi : \omega^{<\omega} \to \omega^{<\omega}$. Then the sequences $\varphi(0^k)$, $k \in \omega$, extend each other and hence build some $a \in \omega^\omega$. By property (iii), for all k, $(x|k, 0^k) \in S^x$, hence $(x|k, \varphi(0^k)) \in S^y$. So (x, y, a) is a branch through S and by (i), xRy.

Conversely, suppose xRy and let a be such that (x, y, a) is a branch through S. Define $\varphi : \omega^{<\omega} \to \omega^{<\omega}$ by $\varphi(s) = s + (a \restriction |s|)$. The map φ is clearly Lipschitz. Also, if $s \in 2^k$ and u is such that $(u, s) \in S^x$ we get $(u, x|k, s) \in S$ and $(x|k, y|k, a|k) \in S$. Hence by property (iv) of S, $(u, y|k, \varphi(s)) \in S$ and $(u, \varphi(s)) \in S^y$. So φ witnesses $S^x \leq_{max} S^y$, as desired.

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Finally we Borel-reduce $\leq_{max}$ to a particular LQO, namely, the embeddability relation on (countable) *combinatorial trees*, i.e., symmetric, irreflexive, connected, acyclic binary relations. Fix some injection θ of $2^{<\omega}$ into ω such that $|s| \leq |t|$ implies $\theta(s) \leq \theta(t)$. For each $T \in \mathcal{T}$ we describe the combinatorial tree G_T.

First we add, for each $s \in \omega^{<\omega} \setminus \{\emptyset\}$ another vertex s^* and put edges between s^* and s and between s^* and the predecessor s^- of s. This defines a combinatorial tree G_0. Then for each pair $(u, s) \in T$ we add vertices (u, s, x) where x is either 0^k or $0^{2\theta(u)+2} * 1 * 0^k$, for $k \in \omega$: Also, we link each (u, s, x) to (u, s, x') where x' is the predecessor of x (as a sequence) and link $(u, s, \emptyset)$ to s. This completely describes the combinatorial tree G_T.

We make some simple observations about G_T. First, one can compute the valence v_T (number of neighbours) of vertices in G_T: elements in $\omega^{<\omega}$ have valence ω, elements $(u, s, 0^{2\theta(u)+2})$, for $(u, s) \in T$, have valence 3, and all other vertices have valence 2. Next consider the distance d_T between vertices. The distance between vertices in $\omega^{<\omega}$ is even, and the distance between a vertex

$(u, s, 0^{2\theta(u)+2})$ and points in $\omega^{<\omega}$ is odd and at least $2\theta(u) + 3$ (obtained at s).

Suppose that $S \leq_{max} T$. Then there is in fact a 1-1 Lipschitz map $f : \omega^{<\omega} \to \omega^{<\omega}$ with $S(s) \subseteq T(f(s))$ for $s \in \omega^{<\omega}$. Define an embedding of G_S into G_T as follows: Send $s \in \omega^{<\omega}$ to $f(s)$ and s^* to $f(s)^*$. This defines an embedding of G_0 into itself. Next if $(u, s) \in S$ we have $(u, f(s)) \in T$ so we can send (u, s, x) to $(u, f(s), x)$. Thus G_S embeds into G_T.

Conversely, suppose that g is an embedding of G_S into G_T. Then we have $v_T(g(y)) \geq v_S(y)$ and $d_T(g(y), g(z)) = d_S(y, z)$ for all vertices y, z in the domain of G_S. Thus g must send elements in $\omega^{<\omega}$ to elements of $\omega^{<\omega}$, i.e., defines a map $f : \omega^{<\omega} \to \omega^{<\omega}$. We claim that f witnesses $S \leq_{max} T$. First we show $f(\emptyset) = \emptyset$: Consider $x = (\emptyset, \emptyset, 0^2)$. It is a vertex in G_S of valence 3 and d_S-distance 3 from $\emptyset$. So it must be sent to some vertex of valence at least 3 in G_T, witih d_T-distance 3 from $f(\emptyset)$. But there is only one possible such vertex, namely, $(\emptyset, \emptyset, 0^2)$, as points in $\omega^{<\omega}$ are at even distance from $f(\emptyset)$ and the other vertices of valence 3 are at a larger distance. This implies that $f(\emptyset) = \emptyset$. Second we show that f is Lipschitz by induction on the length of s. The first step was done above. As $s * n$ is within distance 2 from s in G_S, $f(s * n)$ must be within distance 2 of $f(s)$ in G_T; it cannot be $f(s)^-$, which is $f(s^-)$ by induction. So it is $f(s) * k$ for some k, so f is Lipschitz. Finally, we show that if $(u, s) \in S$ then $(u, f(s)) \in T$. Consider the vertex $x = (u, s, 0^{2\theta(u)+2})$ in G_S. It must be sent by g to some vertex y in G_T of valence at least 3 and at distance $2\theta(u) + 3$ from $f(s)$. Again points in $\omega^{<\omega}$ are forbidden by parity, so $y = (v, t, 0^{2\theta(v)+2})$ for some $(v, t) \in T$. But as the path in G_S joining s to x does not contain s^-, the path in G_T joining $f(s)$ to y does not contain $f(s^-) = f(s)^-$, and t must extend $f(s)$. But if it extends it strictly, we get $|v| > |u|$ and $\theta(v) > \theta(u)$ so that the distance is too big. So $t = f(s)$ and $\theta(v) = \theta(u)$, hence $v = u$ and finally $(u, f(s)) \in T$, as desired. □

Question. Is every analytic quasi-order Borel bi-reducible to an LQO?

A remark about countable Borel equivalence relations

We show that any countable Borel equivalence relation is an LER, and there is a $\leq_B$-largest countable Borel equivalence relation, E_∞.

E_∞ is defined as follows: Given a group G and a set X, G acts on X^G by the *shift action*: For any $g \in G$ and $F \in X^G$, $(g \cdot F)(h) = F(g^{-1}h)$. If X is Polish, G is countable and X^G is given the product topology, then this is a Polish group action, written as $E(G, X)$. Then E_∞ is the orbit equivalence relation of the group action $E(F_2, 2)$, where F_2 is the free group on two generators.

Theorem 31 *Suppose that E is a countable Borel equivalence relation on a Polish space X.*
(a) E is induced by a Polish action of a countable group on X.
(b) E is Borel-reducible to E_∞.

Proof. (a) We assume that X is Cantor space 2^N. A basic result of descriptive set theory is that there is a sequence $(f_n \mid n \in \omega)$ of Borel maps such that for each $a \in 2^N$, $[a]_E = \{f_n(a) \mid n \in \omega\}$. For each n let Γ'_n be the graph of f_n and put $\Gamma_n = \Gamma'_n \setminus \bigcup_{k<n} \Gamma'_k$. The sets $P_{nk} = \Gamma_n \cap \Gamma_k^{-1}$ form a partition of E into countably many Borel relations, each of which is injective. Let $(D_m \mid m \in \omega)$ be an enumeration of all nonempty sets of the form $P_{nk} \setminus \Delta$, where $\Delta = \{(a, a) \mid a \in 2^N\}$ denotes the diagonal. Intersecting the sets D_m with the rectangles of the form R_s and R_s^{-1}, where

$$R_s = \{(a, b) \mid s * 0 \subseteq a \wedge s * 1 \subseteq b\},$$

we reduce the general case to the case where $\mathrm{dom} D_m \cap \mathrm{Range}\ D_m = \emptyset$ for all m.

Now for any m define $h_m(a) = b$ whenever either $(a, b) \in D_m$, $(a, b) \in D_m^{-1}$ or $a = b \notin \mathrm{dom} D_m \cup \mathrm{Range}\ D_m$. Then h_m is a Borel injection. Thus $(h_m \mid m \in \omega)$ is a family of Borel automorphisms of 2^N such that $[a]_E = \{h_m(a) \mid m \in \omega\}$. This system can be expanded to a Borel action of F_ω, the free group on countably-many generators, on 2^N, whose induced equivalence relation is E.

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(b) By the above, E is Borel-reducible to R, where R is induced by a Borel action $\cdot$ of F_ω on 2^N. The map $I(a) = (g^{-1} \cdot a)_{g \in F_\omega}$ is a Borel reduction of R to $E(F_\omega, 2^N)$. As F_ω admits an injective homomorphism into F_2 (send any a_n to $a^n b^n$, where a, b are the generators of F_2), it follows E is Borel-reducible $E(F_2, 2^N)$. So it remains to show that $E(F_2, 2^N)$ is Borel-reducible to $E(F_2, 2) = E_\infty$.

First reduce $E(F_2, 2^N)$ to $E(F_2, 2^{Z\setminus\{0\}})$. Then reduce the latter to $E(F_2 \times Z, 3)$ using the map sending $(a_g)_{g\in F_2}$ $(a_g \in 2^{Z\setminus\{0\}})$ to $(b_{gj})_{g\in F_2, j\in Z}$, where $b_{gj} = a_g(j)$ for $j \neq 0$ and $b_{g0} = 2$. Now for any group G, $E(G, 3)$ is Borel-reducible to $E(G \times Z_2, 2)$ via the map sending $(a_g)_{g\in G}$ $(a_g \in \{0, 1, 2\})$ to $(b_{gi})_{g\in G, i\in Z_2}$ where:

$b_{gi} = 0$ if $a_g = 0$ or ($a_g = 1$ and $i = 0$)
$b_{gi} = 1$ if $a_g = 2$ or ($a_g = 1$ and $i = 1$).

Thus $E(F_2, 2^N)$ is Borel-reducible to $E(F_2 \times Z \times Z_2, 2)$. However there is an injective homomorphism of $F_2 \times Z \times Z_2$ onto F_ω and then from F_ω into F_2, so $E(F_2, 2^N)$ is Borel-reducible to $E(F_2, 2)$, as required. □

As we have seen that any LER with countable equivalence classes (indeed with equivalence classes of bounded Borel rank) is Borel, we have that the countable Borel equivalence relations are exactly the countable LER's.

Adams-Kechris constructed an antichain of size continuum in the partial order of countable Borel equivalence relations under Borel reducibility; it follows that there is also such an antichain within the LER's.

The Silver dichotomy revisited

The Silver dichotomy says that if E is a Borel equivalence relation then either E is Borel-reducible to $(\omega, =)$ or $(R, =)$ is Borel-reducible to E. In particular, E has either countably many or $2^{\aleph_0}$ equivalence classes. Whether or not this holds for arbitrary LER's which are not Borel remains an open problem. We prove here the weaker claim that if an LER has more than ω_1 equivalence classes then it has $2^{\aleph_0}$ many, a theorem of Morley.

A *fragment* of $\mathcal{L}_{\omega_1,\omega}$ is a set of formulas of $\mathcal{L}_{\omega_1,\omega}$ containing all first-order formulas and closed under subformulas, finitary connectives and quantifiers. For a fragment F, two structures $\mathcal{M}$, $\mathcal{N}$ for $\mathcal{L}$ are *elementarily equivalent over* F, written $\mathcal{M} \equiv_F \mathcal{N}$, iff they satisfy the same F-sentences. A *complete F-type* is a set of the form $p = \{\varphi(v_1, \ldots, v_n) \mid \mathcal{M} \models \varphi(a_1, \ldots, a_n)\}$, for some $\mathcal{L}$-structure $\mathcal{M}$ and $a_1, \ldots, a_n$ in M. We say that p is *realised* in $\mathcal{M}$ by

$a_1, \ldots, a_n$. For an $\mathcal{L}_{\omega_1\omega}$ sentence ψ let $S_n(F, \psi)$ denote the set of F-types in n variables realised in some model of ψ. It is not difficult to show that if F is a countable fragment then any F-type realised in some model of ψ can be realised in a countable model.

Theorem 32 *For a countable fragment F, $S_n(F, \psi)$ is either countable or has cardinality $2^{\aleph_0}$.*

Proof. List the formulas of F with free variables $v_1, \ldots, v_n$ as $\varphi_1, \varphi_2, \ldots$. For any p in $S_n(F, \psi)$ let p^* denote $\{k \mid \varphi_k \in p\}$ and let $S_n^*(F, \psi)$ denote $\{p^* \mid p \in S_n(F, \psi)\}$. Then p^* belongs to $S_n^*(F, \psi)$ iff there is a model $\mathcal{M}$ of ψ with universe ω such that $p^* = \{k \mid \mathcal{M} \models \varphi_k(1, \ldots, n)\}$. This gives a Σ_1^1 definition of $S_n^*(F, \psi)$, i.e., a definition of the form:

$p^* \in S_n^*(F, \psi)$ iff $\exists m \subseteq \omega((p^*, m) \in B)$

where B is a Borel subset of $\mathcal{P}(\omega) \times \mathcal{P}(\omega)$. As a Σ_1^1 set is either countable or has $2^{\aleph_0}$ elements; it follows that this is the case for $S_n^*(F, \psi)$ and hence also for $S_n(F, \psi)$. □ (Theorem 32)

A sentence ψ of $\mathcal{L}_{\omega_1\omega}$ is *scattered* iff for each n and countable fragment F of $\mathcal{L}$, the set $S_n(F, \psi)$ of F-types in n variables realised in a model of ψ is countable. If ψ is not scattered, then ψ must have $2^{\aleph_0}$ models, as by Theorem 32, there are $2^{\aleph_0}$ F-types realised in some model of ψ for some countable F, each such F-type can be realised in some countable model of ψ and each countable model of ψ can realise at most countably many F-types.

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Assume now that ψ is scattered. Build a sequence $\langle F_\alpha \mid \alpha < \omega_1 \rangle$ of countable fragments as follows. F_0 is the least fragment containing the sentence ψ. For α limit, F_α is the union of F_β, $\beta < \alpha$. Given F_α, define $F_{\alpha+1}$ as follows. For each F_α-type p realised in a model of ψ let Φ_p be the formula $\bigwedge_{\varphi \in p} \varphi$. As F_α is countable, Φ_p is a countable formula, and as ψ is scattered, there are only countably many such formulas. Let $F_{\alpha+1}$ be the smallest fragment containing F_α and all of the Φ_p, p an F_α-type.

For a countable model $\mathcal{M}$ of ψ and $\bar{a} = (a_1, \ldots, a_n)$ in M, let $\mathrm{tp}_\alpha^{\mathcal{M}}(\bar{a})$ denote the F_α-type realised by $\bar{a}$ in $\mathcal{M}$. As $\mathcal{M}$ is countable, there will be a countable ordinal γ such that whenever $\bar{a}$, $\bar{b}$ in M have the same length and realise the same F_γ-type in $\mathcal{M}$, they also realise the same F_α-type in $\mathcal{M}$ for all α. We refer to the least such γ as the *height* of $\mathcal{M}$.

Lemma 33 *Suppose that $\mathcal{M}$, $\mathcal{N}$ are countable models of ψ with the same height γ and that $\mathcal{M}$, $\mathcal{N}$ are $F_{\gamma+1}$-elementarily equivalent (i.e., $tp^{\mathcal{M}}_{\gamma+1}(\emptyset) = tp^{\mathcal{N}}_{\gamma+1}(\emptyset)$). Then $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.*

Proof. We show that if $\bar{a}$ in M, $\bar{b}$ in N and $\mathrm{tp}^{\mathcal{M}}_\gamma(\bar{a}) = \mathrm{tp}^{\mathcal{N}}_\gamma(\bar{b})$ then for any $a \in M$ there exists $b \in N$ such that $\mathrm{tp}^{\mathcal{M}}_\gamma(\bar{a}, a) = \mathrm{tp}^{\mathcal{N}}_\gamma(\bar{b}, b)$ and for any $b \in N$ there exists $a \in M$ such that $\mathrm{tp}^{\mathcal{M}}_\gamma(\bar{a}, a) = \mathrm{tp}^{\mathcal{N}}_\gamma(\bar{b}, b)$. Then by a back-and-forth construction using the countability of M, N, it follows that $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.

Given $a \in M$ let p be the F_γ-type realised by $(\bar{a}, a)$ in $\mathcal{M}$. The sentence $\exists\bar{v}\exists w \bigwedge_{\varphi\in p} \varphi(\bar{v}, w)$ belongs to the fragment $F_{\gamma+1}$. By hypothesis this sentence is true in $\mathcal{N}$ and therfore there exists $(\bar{c}, d)$ in N which realises p in $\mathcal{N}$. Now $\bar{c}$ realises the same F_γ-type in $\mathcal{N}$ as does $\bar{a}$ in $\mathcal{M}$ and therefore the same F_γ-type that $\bar{b}$ realises in $\mathcal{N}$. By hypothesis, $\bar{b}$, $\bar{c}$ realise the same $F_{\gamma+1}$-type in $\mathcal{N}$. Therefore since $\exists w \bigwedge_{\varphi\in p} \varphi(\bar{v}, w)$ is a formula of $F_{\gamma+1}$ satisfied in $\mathcal{N}$ by $\bar{c}$, this formula is also satisfied in $\mathcal{N}$ by $\bar{b}$, and therefore there is $b \in N$ such that $(\bar{b}, b)$ realises p in $\mathcal{N}$, as desired. A symmetric argument produces the desired $a \in M$, given $b \in N$. □ (Lemma 33)

The theorem now follows: Suppose that ψ is scattered. Then by Lemma 33, each countable model $\mathcal{M}$ of ψ is uniquely determined up to isomorphism by thet pair $(\gamma, \mathrm{tp}^{\mathcal{M}}_{\gamma+1}(\emptyset))$, where γ is the height of $\mathcal{M}$. As there are only countably many possibilities for $\mathrm{tp}^{\mathcal{M}}_{\gamma+1}(\emptyset)$ for each γ and only ω_1 possibilities for γ, it follows that there are at most ω_1 possible countable models of ψ.

The Glimm-Effros Dichotomy

Recall that in the case of the LER determined by Abelian torsion groups, the equivalence classes are classified by countable sets of countable ordinals. It turns out that this type of classification is quite general for LER's to which E_0 is not Borel-reducible. We say that an equivalence relation E on a Polish space X is *absolutely reducible to equality on* $2^{<\omega_1}$ iff there is a function $F : X \to 2^{<\omega_1}$ such that xEy iff $F(x) = F(y)$ and for some formular φ with a real parameter, $F \restriction M$ is defined by φ over M for any transitive model

M of ZF^- containing that real parameter ($\mathrm{ZF}^- = \mathrm{ZF}$ minus the power set axiom).

Theorem 34 *Let E be an LER (or any orbit equivalence relation induced by a Polish group action on a Polish space). Then exactly one of the following holds:*

(1) E is absolutely-reducible to equality on $2^{<\omega_1}$.
(2) E_0 is Borel-reducible to E.

The proof will make use of the following earlier result of Harrington-Kechris-Louveau:

Theorem 35 *Suppose that E is a Borel equivalence relation. Then exactly one of the following holds:*

(1) E is Borel-reducible to equality on R.
(2) E_0 is Borel-reducible to E.

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We turn now to a proof of this latter theorem. We assume that E is Δ^1_1 without parameters. Consider the relation $\hat{E}$ defined by $x\hat{E}y$ iff x, y belong to the same E-invariant Δ^1_1 sets. (X is *E-invariant* iff X is the union of E-equivalence classes.) Then $E \subseteq \hat{E}$.

Lemma 36 *$\hat{E}$ is Σ^1_1.*

Proof. There is a Π^1_1 set of numbers C whose elements serve as codes for Δ^1_1 sets. An element c of C codes an E-invariant Δ^1_1 set W_c iff

$$\forall x, y((x \in W_c \wedge xEy) \to y \in W_c).$$

So the set of codes for E-invariant Δ^1_1 sets forms a Π^1_1 set. And then:

$x\hat{E}y$ iff whenever c codes an E-invariant Δ^1_1 set W_c, $x \in W_c$ iff $y \in W_c$.

So $\hat{E}$ is Σ^1_1. □

Now we have:

Case 1. E equals $\hat{E}$.

In this case we show that E is smooth, i.e., reducible to equality on the reals. Consider the relation $R = \{(\langle x, y\rangle, c) \mid c$ codes an E-invariant Δ^1_1 set W_c, $x \in W_c$ and $y \notin W_c\}$. Then the domain of R is the complement of E and is therefore Δ^1_1, so R is a Π^1_1 relation on a Δ^1_1 domain with countable sections. It follows that R can be uniformised by a Δ^1_1 function f, whose range is a Σ^1_1 subset of C^*, the set of codes for E-invariant Δ^1_1 sets. It follows from Σ^1_1 separation that for some Δ^1_1 set A, Range $(f) \subseteq A \subseteq C^*$. It follows that $g(x) = \{c \in A \mid x \in W_c\}$ defines a Δ^1_1 reduction of E to equality on subsets of ω.

Case 2. E is properly contained in $\hat{E}$. In this case we show that E_0 is reducible to E.

Define $H = \{x \mid [x]_E \neq [x]_{\hat{E}}\}$. H is Σ^1_1 and $\hat{E}$-invariant.

Lemma 37 *If X is a nonempty Σ^1_1 subset of H then E and $\hat{E}$ disagree on X.*

Proof. We may assume that X is E-invariant, as if E and $\hat{E}$ agree on X then they must also agree on $[X]_E$. As X is a nonempty subset of H, X is a proper subset of $[X]_{\hat{E}}$. Now $Y = [X]_{\hat{E}} \setminus X = \{x \mid$ For some y, $x\hat{E}y$ and $\sim xEy\}$ is Σ^1_1, as E is Δ^1_1. Therefore X and Y are disjoint, E-invariant Σ^1_1 sets. Now we use the following fact:

Fact. If F is a Σ^1_1 equivalence relation and X, Y are disjoint F-invariant Σ^1_1 sets, then there is an F-invariant Δ^1_1 set D which contains X and is disjoint from Y.

Proof of Fact. We only prove the non-effective version, which asserts that there is a Borel set D satisfying the conclusion. An effective version of the proof we give yields the desired result.

There is a Δ^1_1 set A_0 which contains X and is disjoint from Y, by the usual form of Σ^1_1 separation. Now $A_0' = [A_0]_F$ is also disjoint from Y, as Y is F-invariant. Applying separation to the Σ^1_1 sets A_0' and Y, we get a Δ^1_1 set A_1 containing A_0' and disjoint from Y. Then the union of the A_n's is a Borel set containing X and disjoint from Y. □ (Fact)

But as each element of X belongs to D, it follows by the definition of $\hat{E}$ that $[X]_{\hat{E}} \subseteq D$, contradiction. □ (Lemma 37)

Lemma 38 *If A, B are nonempty Σ^1_1 subsets of H with $[A]_E = [B]_E$ then there exists nonempty disjoint Σ^1_1 $A' \subseteq A$, $B' \subseteq B$ satisfying $[A']_E = [B']_E$.*

Proof. We first show that there are $a \neq b$, $a \in A$, $b \in B$ with aEb. If not, then E agrees with equality on $X = A \cup B$. Now take any $x \neq y$ in X and let U be a clopen set containing x but not y. Then $[U \cap X]_E$ and $[X \setminus U]_E$ are disjoint E-invariant sets containing x, y, respectively. But then $\sim x\hat{E}y$ by the Fact above. So we have shown that $\hat{E}$ agrees with E on X, which contradicts Lemma 37.

Now let U be a clopen set containing a but not b and put $A' = A \cap U \cap [B \cap \sim U]_E$, $B' = B \cap \sim U \cap [A \cap U]_E$. □

(Lemma 40)

Now we use Gandy-Harrington forcing. Fix a countable, transitive model M of ZF$^-$. Let $P^2|E$ consist of all $X \times Y$ where $X, Y \subseteq N^N$ are nonempty Σ^1_1 sets and $[X]_E = [Y]_E$. A $P^2|E$-generic G yields a pair (x^G_l, x^G_r), where x^G_l, x^G_r are generic for $P =$ Gandy-Harrington forcing (whose conditions are nonempty Σ^1_1 sets). Let $\dot{x}_l, \dot{x}_r$ be canonical names for x^G_l, x^G_r, respectively.

Lemma 39 *Any condition $X \times Z$ in $P^2|E$ forces $\dot{x}_l\hat{E}\dot{x}_r$, and $H \times H$ forces $\sim \dot{x}_l E \dot{x}_r$.*

Proof. To see that $\dot{x}_l\hat{E}\dot{x}_r$ is forced, suppose otherwise. Then by the definition of $\hat{E}$, there is $X \times Z \in P^2|E$ and an E-invariant Δ^1_1 set B such that $X \times Z$ forces $\dot{x}_l \in B$ and $\dot{x}_r \notin B$. Then X must be a subset of B and Z must be disjoint from B, contradicting $[X]_E = [Z]_E$.

To see that $H \times H$ forces $\sim \dot{x}_l E \dot{x}_r$, suppose that $X \times Z \in P^2|E$ with $X \cup Z \subseteq H$ forces $\dot{x}_l E \dot{x}_r$; thus xEz holds for every $P^2|E$-generic pair $(x, z) \in X \times Z$.

Claim. If $x, y \in X$ are P-generic over M and $x\hat{E}y$ then xEy.

Given the Claim, it follows that $\hat{E}$ agrees with E on X, as otherwise $S = \{(x, y) \in X^2 \mid x\hat{E}y \wedge \sim xEy\}$ is a nonempty Σ^1_1 set, and any generic pair (x, y) for the Gandy-Harrington forcing on $(N^N)^2$ below S yields a contradiction to the Claim. But by the definition of H, X is disjoint from H, contrary to hypothesis.

So it only remains to prove the Claim. We first assert that x and y belong to the same E-invariant Σ^1_1 sets. If not, then choose an E-invariant Σ^1_1 set A such that $x \in A$, $y \notin A$; then by the genericity of y there is a Σ^1_1 set C with $y \in C$ and A disjoint from C. As A is E-invariant, the Fact from the proof of Lemma 37 yields an E-invariant Δ^1_1 set B containing C and disjoint form A. Then $x \notin B$ but $y \in B$, contradicting $x\hat{E}y$.

Now let $\{D_n\}_{n\in N}$ enumerate the dense subsets of $P^2|E$ which are coded in M. We define sequence $P_0 \supseteq P_1 \supseteq \cdots$ and $Q_0 \supseteq Q_1 \supseteq \cdots$ of conditions $P_n = X_n \times Z_n$ and $Q_n = Y_n \times Z_n$ in $P^2|E$ so that $P_0 = Q_0 = X \times Z$, $x \in X_n$ and $y \in Y_n$ for each n, and $P_n, Q_n \in D_{n-1}$ for $n > 0$. Once this is done, we have a real z (the only element of $\bigcap_n Z_n$) such that both (x, z) and (y, z) are $P^2|E$-generic, and hence both xEz and yEz by our hypothesis, implying xEy.

Suppose that P_n, Q_n are defined. As x is generic, there is a condition $P' = A \times C \in D_n$, $P' \subseteq P_n$ such that $x \in A$. Let B be $Y_n \cap [A]_E$. Then $y \in B$ by our first assertion and $[B]_E = [C]_E = [A]_E$ (as $[X_n]_E = [Z_n]_E = [Y_n]_E$); thus $B \times C$ belongs to $P^2|E$. So there is a condition $Q' = V \times W \in D_n$ contained in $B \times C \subseteq Q_n$, with $y \in V$. Finally, put $Y_{n+1} = V$, $Z_{n+1} = W$ and $X_{n+1} = A \cap [W]_E$. $\square$ (Lemma 39)

To prove that E_0 is Borel-reducible to E we use a *splitting system*, defined as follows.

Fix enumerations $\{D(n)\}_{n\in N}$, $\{D_2(n)\}_{n\in N}$ and $\{D^2(n)\}_{n\in N}$ of all elements of M which are dense on P, P_2 and $P^2|E$, respectively (where P is Gandy-Harrington forcing on N^N, P_2 is Gandy-Harrington forcing on $(N^N)^2$ and $P^2|E$ is defined above). We assume that these sequences are descending under inclusion. If $u, v \in 2^m$ have the form $u = 0^k * 0 * w$, $v = 0^k * 1 * w$ for some k, w then we call (u, v) a *crucial pair*. For each m, 2^m under the crucial pair relation is a connected graph.

We define a splitting system to be sequences of sets X_u, $u \in 2^{<\omega}$, and R_{uv}, (u, v) a crucial pair, which obey the following requirements:

(1) $X_u \in P$ and $X_\emptyset \subseteq H$.
(2) $X_u \in D(n)$ for any $u \in 2^n$.
(3) $X_{u*i} \subseteq X_u$ for all u, i.
(4) $R_{uv} \in P_2$ and $R_{uv} \in D_2(n)$ for any crucial pair (u, v) in 2^n.
(5) $R_{uv} \subseteq E$ and $X_u R_{uv} X_v$ for any crucial pair (u, v) in 2^n.

(6) $R_{u*i,v*i} \subseteq R_{uv}$.
(7) If $u, v \in 2^n$ and $u(n-1) \neq v(n-1)$ then $X_u \times X_v \in D^2(n)$ and X_u is disjoint from X_v.

(5) implies that $X_u E X_v$ for any crucial pair (u, v), hence also for any pair (u, v) from 2^n. It follows that $X_u \times X_v$ belongs to $P^2|E$ for any pair $u, v \in 2^n$, for any n.

Now assume that such a splitting system has been defined. Then for any $a \in 2^N$ the sequence $\{X_{a|n}\}_{n \in N}$ is P-generic over M by (2) and therefore the intersection of the $X_{a|n}$'s is $\{x_a\}$ where x_a is P-generic. Moreover the map $a \mapsto x_a$ is continuous since the diameters of the X_u converge to 0 as the length of u goes to infinity, and this map is 1-1 by the last condition of (7).

Let a, b be elements of 2^N. If $\sim aE_0b$ then by (7), (x_a, x_b) is $P^2|E$-generic and hence $\sim x_a E x_b$ by Lemma 39. Now suppose that aE_0b and we wish to show that $x_a E x_b$. We can assume that $a = w*0*c$, $b = w*1*c$ for some w, c. Then (x_a, x_b) is P_2-generic as it is the unique member of the intersection of the $R_{w*0*c|n, w*1*c|n}$'s, by (4) and (5). In particular, $x_a E x_b$ because $R_{uv} \subseteq E$ for all u, v. So E_0 is in fact continuously, 1-1 reducible to E.

To complete the proof of the Glimm-Effros Dichomtomy in the Borel case, it remains only to construct a splitting system.

20.Vorlesung

To complete the proof of the Glimm-Effros Dichomtomy in the Borel case, it remains only to construct a splitting system. For this we will need the following earlier lemma:

Lemma 40 *If A, B are nonempty Σ^1_1 subsets of H with $[A]_E = [B]_E$ then there exists nonempty disjoint Σ^1_1 $A' \subseteq A$, $B' \subseteq B$ satisfying $[A']_E = [B']_E$.*

Let $X_\emptyset$ be any set in $D(0)$ contained in H.

Now suppose that X_s and R_{st} have been defined for all s of length n and all crucial pairs (s, t) of length n. Temporarily define $X_{s*i} = X_s$ and $R_{s*i,t*i} = R_{st}$; this leaves $R_{0^n*0,0^n*1}$ undefined, which we take to be $E \cap (X_{0^n} \times X_{0^n})$. With these definitions we have a "splitting pre-system", i.e., we have satisfied all the requirements for strings of length $n+1$ with the exception of membership in the dense sets of (2), (4) and (7) and the disjointness requirement of (7). We show how to produce a splitting system consisting of sets and relations contained in those of this pre-system.

Step 1. We arrange $X_u \in D(n+1)$ for $u \in 2^{n+1}$. Choose some $u_0 \in 2^{n+1}$. Choose $X'_{u_0} \subseteq X_{u_0}$ in $D(n+1)$. Suppose that (u_0, v) is a crucial pair. Put $R'_{u_0,v} = \{(x,y) \in R_{u_0 v} \mid x \in X'_{u_0}\}$ and $X'_v = \text{Ran } R'_{u_0 v}$. In this way we obtain a splitting pre-system satisfying $X'_{u_0} \in D(n+1)$. Do this consecutively for all $u_0 \in 2^{n+1}$. The result is a splitting pre-system with $X_u \in D(n+1)$ for all u. Note that we still have $X_{0^n*0} = X_{0^n*1}$ and $R_{0^n*0,0^n*1} = E \cap (X_{0^n*0} \times X_{0^n*1})$.

Step 2. We arrange $X_{s*0} \times X_{t*1} \in D^2(n+1)$ for all s, t of length $n+1$. Consider a pair $u_0 = s_0 * 0$ and $v_0 = t_0 * 1$ of length $n+1$. Choose $X'_{u_0} \times X'_{v_0} \in D^2(n+1)$ contained in $X_{u_0} \times X_{v_0}$. We have $[X'_{u_0}]_E = [X'_{v_0}]_E$ and thanks to Lemma 40, we can ensure that X'_{u_0}, X'_{v_0} are disjoint. Now define $R'_{0^n*0,0^n*1} = E \cap (X'_{0^n*0} \times X'_{0^n*1})$. The result is a splitting pre-system which satisfies the $D(n+1), D^2(n+1)$ requirements of (2) and (7).

Step 3. We arrange $R_{uv} \in D_2(n+1)$ for any crucial pair (u,v) of length $n+1$, together with the disjointness of X'_{0^n*0}, X'_{0^n*1}. Consider any such crucial pair (u_0, v_0). If this pair is not $(0^n * 0, 0^n * 1)$ then let $R'_{u_0 v_0}$ be any subset of $R_{u_0 v_0}$ in $D_2(n+1)$. Otherwise choose disjoint nonempty Σ^1_1 sets $U \subseteq X_{0^n*0}$, $V \subseteq X_{0^n*1}$ with $[U]_E = [V]_E$ (which is possible by Lemma 40), and then $R'_{u_0 v_0} \subseteq E \cap (U \times V)$ in $D_2(n+1)$. In all cases put $X'_{u_0} = \text{Dom } R'_{u_0 v_0}$, $X'_{v_0} = \text{Ran } R'_{u_0 v_0}$. Then handle other crucial pairs as in Step 1. Performing such a reduction for all crucial pairs (u_0, v_0) of length $n+1$ one at a time, we end up with the desired splitting system. □

21.Vorlesung

Now using the Glimm-Effros dichotomy for the Borel case, we prove:

Theorem 41 *Let E be an LER (or any orbit equivalence relation induced by a Polish group action on a Polish space). Then exactly one of the following holds:*

(1) E is absolutely-reducible to equality on $2^{<\omega_1}$.
(2) E_0 is Borel-reducible to E.

Proof. We use the following.

Lemma 42 *There is a sequence A_ξ, $\xi < \omega_1$, of E-invariant sets such that:*
(1) The union of the A_ξ's is the entire Polish space.
(2) The $E|A_\xi$'s are uniformly Borel*: For some parameter z, A_ξ and $E|A_\xi$ are uniformly $\Delta^1_1(w, z)$ in any code w for ξ.*
(3) Relative to some parameter, there is an absolute function c such that for each x, $f(x)$ is a code for an ordinal ξ such that x belongs to A_ξ.

We now prove the theorem, given this lemma. For simplicity we assume that the parameters in (2) and (3) are just 0.

Suppose that E_0 is not Borel-reducible to E. Then E_0 is Borel-reducible to none of the $E|A_\xi$'s, and therefore each $E|A_\xi$ is smooth (i.e., Borel-reducible to $=$ on R). The smoothness of an equivalence relation F is equivalent to the existence of a *separating family*, i.e., a sequence of Borel sets B_n, $n \in \omega$, such that xFy iff $\forall n(x \in B_n \leftrightarrow y \in B_n)$. In the present case we have in fact *uniform* separating families for the $E|A_\xi$'s, i.e., for each code w for ξ, there is a separating family $(S^w_n)_{n\in\omega}$ for $E|A_\xi$ such that the S^w's are uniformly Δ^1_1 for $w \in \mathrm{WO}$ = the set of codes for wellorderings.

We now define a new separating family $(S^{\xi,\xi_0,\ldots,\xi_{k-1}}_n)_{n,\xi_0,\ldots,\xi_{k-1}}$ for $E|A_\xi$, for each ξ. For each injection $f \in (\xi)^\omega$, the Polish space of injections from ω into ξ (assume that ξ is infinite), let $w_f \in \mathrm{WO}$ be the associated code for ξ, given by $w_f(m,n) = 1 \leftrightarrow f(m) < f(n)$. Then $S^{\xi,\xi_0,\ldots,\xi_{k-1}}_n$ consists of all x such that the set of $f \in (\xi)^\omega$ with $x \in S^{w_f}_n$ is comeager on $N_{\xi_0,\ldots,\xi_{k-1}}$ (the basic open set consisting of those injections f satisfying $f(i) = \xi_i$ for each $i < k$). We claim that $(S^{\xi,\xi_0,\ldots,\xi_{k-1}}_n)_{n,\xi_0,\ldots,\xi_{k-1}}$ is indeed a separating family for $E|A_\xi$.

Suppose that xEy. If x belongs to $S^{\xi,\xi_0,\ldots,\xi_{k-1}}_n$ then so does y, as each $S^{w_f}_n$ is E-invariant. Conversely, suppose that $\sim xEy$. Then for any code w for an ordinal ξ, there is some n such that $x \in S^w_n$ but $y \notin S^w_n$. It follows by a category argument that for some distinct $\xi_0, \ldots, \xi_{k-1} < \xi$ and $n \in \omega$, $(x \in S^{w_f}_n$ and $y \notin S^{w_f}_n)$ holds for comeager many f in $N_{\xi_0,\ldots,\xi_{k-1}}$. Thus x belongs to $S^{\xi,\xi_0,\ldots,\xi_{k-1}}_n$ and y does not. So we do have a separating family.

Finally put $V(x) = \{(\xi, \xi_0, \ldots, \xi_{k-1}, n) \mid$ The ξ_i's are distinct, $\xi_i < \xi$ for each $i < k$ and $x \in S^{\xi,\xi_0,\ldots,\xi_{k-1}}_n\}$, where ξ is chosen so that x belongs to A_ξ. Then xEy iff $V(x) = V(y)$, which shows that E is absolutely-reducible to equality on $2^{<\omega_1}$.

Proof sketch of Lemma 42. Let G denote the group S_∞ and $\cdot$ the logic action on some invariant Borel set X, whose orbit equivalence relation is E. We consider the product action

$$g \cdot (x, F) = (g \cdot x, gFg^{-1})$$

of G on $X \times \mathcal{S}(G)$, where $\mathcal{S}(G)$ is the Borel space of closed subgroups of G. Then $P = \{(x, G_x) \mid x \in X\}$ is invariant under this action, where G_x denotes the stabilizer of x in the logic action. Moreoever, P is Π^1_1. It follows that there is a Π^1_1-rank $\varphi : P \to \omega_1$ which is also invariant under this action, and in fact:

(1) There is a Π^1_1 function g such that for each $(x, G_x) \in P$, $g(x, G_x) \in \mathrm{WO}$ codes the ordinal $\varphi(x, G_x)$.
(2) For any ξ, the set $P^\xi = \{(x, G_x) \in P \mid \varphi(x, G_x) \leq \xi\}$ is uniformly Δ^1_1 in any code for ξ.

Now for each ξ let P_ξ be $\{(x, G_x) \mid \varphi(x, G_x) = \xi\}$. Then P_ξ is uniformly Δ^1_1 in any code for ξ. Put $A_\xi = \{x \mid (x, G_x) \in P_\xi\}$. Then both A_ξ and $E|A_\xi$ are uniformly Δ^1_1 in any code for ξ and witness the lemma. □