Topics in Set Theory, Wintersemester 2006

1.Vorlesung

Stationary reflection

If S is a set of ordinals and α is an ordinal of uncountable cofinality, we say that S is *stationary in* α iff S intersects every closed unbounded subset of α. We say that *stationary reflection holds at* α, abbreviated $\mathrm{SR}(\alpha)$ iff every S which is stationary in α is also stationary in some smaller $\bar{\alpha}$ of uncountable cofinality.

Note that $\mathrm{SR}(\alpha)$ is equivalent to $\mathrm{SR}(\mathrm{cof}\ \alpha)$, so we will just study $\mathrm{SR}(\kappa)$ for regular cardinals κ.

Theorem 1 *κ weakly compact $\to$ SR(κ).*

Proof. Recall that κ is weakly compact iff κ is Π^1_1 *reflecting*, i.e., for any $S \subseteq \kappa$, if φ is a Π_1 formula true in $(H_{\kappa^+}, \in, S)$ then φ is also true in $(H_{\alpha^+}, \in, S \cap \alpha)$ for some $\alpha < \kappa$. As the property "S is stationary in κ" is a Π_1 property of $(H_{\kappa^+}, \in, S \cap \kappa)$, stationary reflection follows. □

Theorem 2 *In L, SR(κ) $\to$ κ weakly compact.*

Proof. Assume $V = L$. First assume that κ is inaccessible. Let $\langle C_\alpha \mid \alpha$ a singular cardinal$\rangle$ be a square sequence on the singular cardinals, i.e., for each singular cardinal α, C_α is a closed unbounded subset of α of ordertype less than α and if $\bar{\alpha}$ is a limit point of C_α then $\bar{\alpha}$ is a singular cardinal and $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$.

Assume that κ is not weakly compact and choose $A \subseteq \kappa$ and a Π_1 formula φ so that φ holds in $(H_{\kappa^+}, \in, A) = (L_{\kappa^+}, \in, A)$ but not in $(H_{\alpha^+}, \in, A \cap \alpha) = (L_{\alpha^+}, \in, A \cap \alpha)$ for any $\alpha < \kappa$. Let S_0 consist of all singular cardinals $\alpha < \kappa$ such that φ holds in $(L_\beta, \in, A \cap \alpha)$ provided $\beta < \alpha^+$ is a limit ordinal and α is regular in L_β.

Claim 1. S_0 is stationary in κ.

Proof. Suppose that C is closed unbounded in κ and choose a limit $\beta < \kappa^+$ so that A and C belong to L_β. As φ is Π_1, it holds in $\mathcal{S} = (L_\beta, \in, A)$. For each cardinal $\alpha < \kappa$ let M_α be the least Σ_1 elementary submodel of $\mathcal{S}$ containing $\alpha \cup \{A, C\}$ as a subset. Then $C_0 = \{\alpha < \kappa \mid \alpha = M_\alpha \cap \kappa\}$ is a closed unbounded subset of C which is definable over $\mathcal{S}$. If α is the ω-th element of C_0, then α belongs to S_0, as α is singular definably over the transitive collapse $(L_{\bar{\beta}}, \in, A \cap L_\alpha)$ of M_α and φ holds in this structure. $\square(Claim 1)$

Claim 2. S_0 is not stationary in α for any regular $\alpha < \kappa$.

Proof. Suppose that $\alpha < \kappa$ is regular and choose a limit ordinal $\beta < \alpha^+$ large enough so that $A \cap \alpha$ belongs to L_β and φ does not hold in $\mathcal{S} = (L_\beta, \in, A \cap \alpha)$. Much as in the previous proof, for each cardinal $\bar{\alpha} < \alpha$ let $M_{\bar{\alpha}}$ be the least Σ_1 elementary submodel of $\mathcal{S}$ containing $\bar{\alpha} \cup \{A \cap L_\alpha\}$ as a subset and $C_0 = \{\bar{\alpha} < \alpha \mid \bar{\alpha} = M_{\bar{\alpha}} \cap \alpha\}$, a closed unbounded subset of α. Then no $\bar{\alpha}$ in C_0 belongs to S_0. $\square(Claim 2)$

Now we thin out S_0 to a stationary subset that is not stationary in any $\alpha < \kappa$. For each α in S_0 let $f(\alpha)$ be the ordertype of C_α, the closed unbounded subset of α assigned by our square sequence on the singular cardinals. Let S be a stationary subset of S_0 on which f is constant.

Claim 3. S is not stationary in any $\alpha < \kappa$.

Proof. Suppose that $S \cap \alpha$ were stationary in α; then α must be a singular cardinal of uncountable cofinality, and $S \cap \text{Lim } C_\alpha$ is unbounded in α. But f is constant on S and 1-1 on $\text{Lim } C_\alpha$, by the coherence property of the square sequence. Contradiction! $\square(Claim 3)$

Thus S is a stationary subset of κ which is not stationary in any $\alpha < \kappa$, so $\text{SR}(\kappa)$ fails.

If $\kappa = \lambda^+$ is a successor cardinal, then we use a $\square_\lambda$ sequence, i.e., a sequence $\langle C_\alpha \mid \lambda < \alpha < \lambda^+, \alpha \text{ limit}\rangle$ such that C_α is closed unbounded in α of ordertype $\leq \lambda$ and $\bar{\alpha} \in \text{Lim } C_\alpha \to C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$. As above choose $S \subseteq (\lambda, \kappa)$ to be a stationary set of limit ordinals on which the function $\alpha \mapsto$ ordertype C_α is constant. Then S is not stationary in any $\alpha < \kappa$. $\square$

2.Vorlesung

Theorem 3 *Relative to a weakly compact, it is consistent that $SR(\kappa)$ does not imply that κ is weakly compact.*

Proof. Suppose that κ is weakly compact. Then κ is weakly compact in L. Let P_κ be the reverse Easton iteration of length κ which at inaccessible $\alpha < \kappa$ adds an α-Cohen set. Let G_κ be P_κ-generic over L.

Now over $L[G_\kappa]$, consider the following forcing Q, due to Kunen, for adding a κ-Suslin tree:

For an ordinal α, an *α-tree* is a subset T of $2^{<\alpha}$ closed under initial segment such that for each $\beta < \alpha$, some element of T has length β. We refer to α as the *height* of T. For limit α, we say that an α-tree T is *homogeneous* iff for any s in T, $T_s = \{t \mid s * t \in T\}$ equals T and an $\alpha + 1$-tree T is *homogeneous* iff for $s \in T$ of length less than α, T_s equals T. For limit α, a homogeneous α-tree exists iff α is indecomposable, i.e., $\beta + \gamma$ is less than α whenever β and γ are less than α. If T is an $\alpha + 1$-tree then not only does T have a path of length α, but every node of T of length less than α can be extended to such a path.

The forcing Q consists of the one-point tree $\{\emptyset\}$, together with homogeneous trees T of successor height less than κ such that both $\langle 0\rangle$ and $\langle 1\rangle$ belong to T. Q is ordered by end-extension.

If T is a homogeneous α-tree, α limit, and s is any path through T of length α then there is a minimal extension $m(T, s)$ of T to a condition of height $\alpha + 1$ which contains s, namely $T \cup \{s_0 * (s \setminus \beta) \mid s_0 \in T \text{ and } \beta < \alpha\}$, where for each $\beta < \alpha$, $s \setminus \beta$ is such that $(s \restriction \beta) * (s \setminus \beta) = s$.

Claim 1. Q is κ-distributive and adds a κ-Suslin tree.

Proof. Q may fail to be κ-closed, as if $T_0 \geq T_1 \geq \cdots$ is a descending sequence through Q of limit length $\lambda < \kappa$, then although the union T_λ^- of the T_i's is homogeneous, it may have no path of length $\text{Height}(T_\lambda^-)$ and therefore not be extendible to a condition. However if in addition to the T_i's we have paths $s_i \in T_i$ of length $\text{Height}(T_i) - 1$ such that $i < j \to s_i \subseteq s_j$, then the union s_λ of the s_i's forms a path through T_λ^- of length $\text{Height}(T_\lambda^-)$, and we can extend T_λ^- to a condition $T_\lambda = m(T_\lambda^-, s_\lambda)$ below each of the T_i's which contains s_λ. This implies that we can inductively extend any condition to meet a sequence of fewer than κ open dense sets, i.e., the forcing Q is κ-distributive. It follows from this and induction on $\alpha < \kappa$, that any condition can be extended to one of height at least α, and therefore the union of a Q-generic is indeed a κ-tree which we denote as T_Q.

We now check that T_Q is κ-Suslin. Suppose that $T \Vdash \dot{A}$ is a maximal antichain in T_Q. Define a descending sequence of conditions T_ξ, $\xi < \kappa$, of height $\gamma_\xi + 1$ together with elements s_ξ of T_ξ of length γ_ξ which end-extend each other so that for limit ξ, $T_\xi = m(\bigcup_{\xi' < \xi} T_{\xi'}, s_\xi)$, and

1. For any ξ, if $s \in T_\xi$ then $T_{\xi+1}$ decides $s \in \dot{A}$.
2. For any $s \in \bigcup_{\xi<\kappa} T_\xi$ and $\alpha < \kappa$ there is an η such that $\gamma_\eta > \alpha$ and T_η forces that some proper initial segment of $s * (s_\eta \setminus \alpha)$ belongs to $\dot{A}$.

To achieve 2, consider how to handle a particular s and α. Choose a limit ξ such that γ_ξ is greater than both α and the length of s. If T_ξ forces that some proper initial segment of $s * (s_\xi \setminus \alpha)$ belongs to $\dot{A}$ then take η to be ξ. Otherwise there is a T' extending T_ξ and an s_1 such that $s * (s_\xi \setminus \alpha) * s_1$ belongs to T' and T' forces $s * (s_\xi \setminus \alpha) * s_1$ to belong to $\dot{A}$. Let $T_{\xi+1}$ extend T' and satisfy 1. Choose $s_{\xi+1}$ to be a path through $T_{\xi+1}$ extending s_ξ so that $s_{\xi+1} \setminus \alpha$ extends $(s_\xi \setminus \alpha) * s_1$. Then 2 is satisfied with η equal to $\xi + 1$.

There must be a limit ordinal ξ such that for $\alpha < \gamma_\xi$ and $s \in \bigcup_{\xi'<\xi} T_{\xi'}$ there is an initial segment of $s * (s_\xi \setminus \alpha)$ that is forced to belong to $\dot{A}$. It follows that every point in $T_\xi = m(\bigcup_{\xi'<\xi} T_{\xi'}, s_\xi)$ of length γ_ξ is forced to lie above some point in $\dot{A}$, so T_ξ forces that $\dot{A} \subseteq T_\xi$ has size less than κ. This proves that the generic tree is κ-Suslin. $\Box(Claim 1)$

Claim 2. Let T_Q denote the κ-Suslin tree added by Q. Then the 2-step iteration $Q * T_Q$ is equivalent to κ-Cohen.

Proof. The forcing $Q * T_Q$ has $R = \{(T, s) \mid T \text{ has height Dom } (s) + 1 \text{ and } s \text{ belongs to } T\}$ as a dense subforcing. But then both R and κ-Cohen are κ-closed forcings of cardinality κ and therefore generate isomorphic complete Boolean algebras. It follows that $Q * T_Q$ is equivalent to κ-Cohen. $\Box(Claim 2)$

It follows that $P_\kappa * Q * T_Q$ is equivalent to $P_\kappa * \kappa$-Cohen.

Claim 3. $P_\kappa * \kappa$-Cohen preserves the weak compactness of κ.

Proof. We must show that κ satisfies Π^1_1 reflection in $L[G((\leq \kappa)] = L[G_\kappa][G(\kappa)]$, where G_κ is generic over L for P_κ and $G(\kappa)$ is generic over $L[G_\kappa]$ for κ-Cohen.

Suppose that $(p, \dot{q})$ is a condition in $P_\kappa * \kappa$-Cohen which forces $\dot{A}$ to be a subset of κ and the Π_1 sentence φ to hold in the structure $(L_{\kappa^+}[G(\leq \kappa)], \in, \dot{A})$. As P_κ is κ-cc, we may assume that $\dot{q}$ belongs to L_κ. And we may assume that the name $\dot{A}$ is a subset of L_κ. Now the statement

$(p, \dot{q}) \Vdash \varphi$ holds in $(L_{\kappa^+}[\dot{G}(\leq \kappa)], \in, \dot{A})$

is a Π_1 statement about the structure $(L_{\kappa^+}, \in, \dot{A}, p, \dot{q})$ and therefore by Π^1_1 reflection in L there exists a cardinal $\alpha < \kappa$ such that $(p, \dot{q})$ belongs to L_α and

$(p, \dot{q}) \Vdash_\alpha \varphi$ holds in $(L_{\alpha^+}[\dot{G}(\leq \alpha)], \in, \dot{A} \cap L_\alpha)$,

where $\Vdash_\alpha$ refers to the forcing $P_\alpha * \alpha$-Cohen and $\dot{G}(\leq \alpha)$ refers to the generic for that forcing. Now choose a condition extending $(p, \dot{q})$ which forces (in $P_\kappa * \kappa$-Cohen) that $\dot{G}(\kappa) \restriction \alpha = \dot{G}(\alpha)$, and therefore that $\dot{A} \cap \alpha$ equals $(\dot{A} \cap L_\alpha)^{\dot{G}(\leq\alpha)}$. Then this condition forces (in $P_\kappa * \kappa$-Cohen) that $H(\alpha^+)$ of $L[\dot{G}(\leq \kappa)] = L_{\alpha^+}[\dot{G}(\leq \alpha)]$ and φ holds in $(L_{\alpha^+}[\dot{G}(\leq \alpha)], \in, \dot{A} \cap \alpha)$, as desired. $\Box(Claim 3)$

Now let H be Q-generic over $L[G_\kappa]$. Then in $L[G_\kappa][H]$, κ is not weakly compact as there is a κ-Suslin tree. However, if S is a stationary subset of κ in this model, then since the forcing T_Q is κ-cc, S is also stationary in the larger model $L[G_\kappa][H][B]$, where B is T_Q-generic over $L[G_\kappa][H]$. As κ is weakly compact in $L[G_\kappa][H][B]$, it follows that S is stationary in some $\alpha < \kappa$. Thus $L[G_\kappa][H]$ is the desired model where κ is not weakly compact but where SR(κ) holds. $\Box$

3.Vorlesung

Can SR(κ) hold for a successor cardinal κ?

Proposition 4 *SR(κ) fails if κ is the successor of a regular cardinal.*

Proof. Suppose that $\kappa = \gamma^+$, γ regular. Then $S = \{\alpha < \kappa \mid \text{cof}\, \alpha = \gamma\}$ is stationary in κ but not in any $\bar{\kappa} < \kappa$. $\Box$

Theorem 5 *If λ is a singular limit of λ^+-supercompact cardinals then SR(λ^+) holds.*

Proof. Recall that κ is *μ-supercompact* iff there is an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \mu$ and $M^\mu \subseteq M$.

Now suppose that S is stationary in λ^+. Then for some λ^+-supercompact $\kappa < \lambda$, $T = S \cap \text{Cof}(< \kappa)$ is stationary. Let $j : V \to M$ witness the λ^+-supercompactness of κ. We show that $T \cap \alpha$ is stationary for some $\alpha < \lambda^+$. Let γ be the supremum of $j[\lambda^+]$; as $j \restriction \lambda^+$ belongs to M, $\text{cof}^{\,M}(\gamma) = \lambda^+$, and therefore γ is less than $j(\lambda^+)$, which is regular in M. It suffices to show that $M \vDash j(T) \cap \gamma$ is stationary, for then by elementarity, $V \vDash T \cap \alpha$ is stationary for some $\alpha < \lambda^+$.

Suppose that C is closed unbounded in γ. As j is continuous at ordinals of cofinality $< \kappa$, $j[\lambda^+]$ is $< \kappa$-closed, i.e., contains all of its limit points of cofinality less than κ. It follows that Range $(j) \cap C$ is unbounded in γ and therefore $D = j^{-1}[C] \subseteq \lambda^+$ is unbounded in λ^+. And again since j is continuous at ordinals of cofinality $< \kappa$, D is $< \kappa$-closed. Since T is a stationary subset of $\text{Cof}(< \kappa) \cap \lambda^+$, it follows that $T \cap D$ is nonempty and therefore $j[T \cap D] \subseteq j(T) \cap C$ is nonempty, as desired. □

Theorem 6 *Assume GCH and suppose that $\kappa_0 < \kappa_1 < \cdots$ is an ω sequence of supercompact cardinals. Define $P_1 = Coll\ (\omega, < \kappa_0)$, $P_{n+1} = P_n * Coll\ (\kappa_{n-1}, < \kappa_n)$ for finite $n > 0$ and $P_\omega =$ Inverse limit of the P_n's. Then P_ω forces SR($\aleph_{\omega+1}$).*

Proof. Let λ be the supremum of the κ_n's and let G_ω be P_ω-generic, $G_n = G_\omega \restriction P_n$.

Claim 1. In $V[G_\omega]$, $\kappa_n = \aleph_{n+1}$, $\lambda = \aleph_\omega$ and $\lambda^+ = \aleph_{\omega+1}$.

Proof. The forcing Coll $(\omega, < \kappa_0)$ makes everything less than κ_0 countable and is κ_0-cc. So κ_0 is $\aleph_1$ in $V[G_1]$. The rest of the iteration is κ_0-closed, so κ_0 is also $\aleph_1$ in $V[G_\omega]$. A similar argument shows that each κ_n is $\aleph_{n+1}$, and therefore that λ is $\aleph_\omega$. If λ^+ were collapsed then it would be given a cofinality less than some κ_n; but for large enough m, the iteration P_ω factors as $P_m * P_{m,\omega}$ where P_m has size less than λ and $P_{m,\omega}$ is κ_n-closed; it follows that λ^+ cannot have cofinality less than κ_n in $V[G_\omega]$. □(Claim 1)

4.Vorlesung

Claim 2. For each n there is a generic extension $V[G_\omega][H_n]$ of $V[G_\omega]$ in which there is a definable elementary embedding $k_n : V[G_\omega] \to M_n \subseteq V[G_\omega][H_n]$ with critical point κ_n such that $k_n \restriction \lambda^+$ belongs to M_n and $k_n(\kappa_n) > \lambda^+$. Moreover the forcing to add H_n is $\aleph_n$-closed.

Proof. Let $j : V \to M$ witness that κ_n is λ^+ supercompact. We wish to extend j to the k_n of the Claim. To do so, we need to find, in an $\aleph_n$-closed generic extension of $V[G_\omega]$, a $j(P_\omega) = P_\omega^M$-generic G_ω^M over M which contains $j[G_\omega]$ as a subset.

The forcing P_ω is the ω-iteration Coll $(\omega, < \kappa_0) *$ Coll $(\kappa_0, < \kappa_1) * \cdots$ and therefore $j(P_\omega) = P_\omega^M$ is the ω-iteration in M given by Coll $^M(\omega, < \kappa_0) *$ Coll $^M(\kappa_0, < \kappa_1) * \cdots *$ Coll $^M(\kappa_{n-2}, < \kappa_{n-1}) *$ Coll $^M(\kappa_{n-1}, < j(\kappa_n)) *$ Coll $^M(j(\kappa_n), < j(\kappa_{n+1})) * \cdots$. The first n factors of these two iterations are the same and so we choose G_n^M to be G_n, yielding a lifting of j to an elementary embedding $j^* : V[G_n] \to M[G_n^M]$. The next factor Coll $(\kappa_{n-1}, < \kappa_n)$ of the V iteration is included as a subforcing of the next factor Coll $^M(\kappa_{n-1}, < j(\kappa_n)) =$ Coll $(\kappa_{n-1}, < j(\kappa_n))$ of the M-iteration and indeed the latter factors as Coll $(\kappa_{n-1}, < \kappa_n) \times$ Coll $(\kappa_{n-1}, [\kappa_n, j(\kappa_n))$. Note that the forcing Coll $(\kappa_{n-1}, [\kappa_n, j(\kappa_n))$ is $\kappa_{n-1} = \aleph_n$-closed. So we choose a generic for this product whose first factor equals the generic specified by G_{n+1}, thereby lifting j^* to $j^{**} : V[G_{n+1}] \to M[G_{n+1}^M]$.

Now the remainder P^{n+1} of the P_ω iteration (where $P_\omega = P_{n+1} * P^{n+1}$) has size $(\kappa_\omega^+)^V$ and $j(\kappa_n)$ is greater than $(\kappa_\omega^{++})^V$; therefore in $M[G_{n+1}^M]$, P^{n+1} is an $\aleph_n$-closed forcing with only $\aleph_n$ maximal antichains in $V[G_{n+1}]$. It follows that in $M[G_{n+1}^M]$ there is a generic for P^{n+1} over $V[G_{n+1}]$, which we may assume equals G^{n+1}. As the remainder $P^{M,n+1}$ of the iteration P_ω^M (where $P_\omega^M = P_{n+1}^M * P^{M,n+1}$) is $j(\kappa_n)$-closed and therefore $(\lambda^{++})^V$-closed, there is a single condition in $P^{M,n+1}$ which is below each condition in $j^{**}[G^{n+1}]$; so we force below that condition. The result is that in a κ_n-closed forcing extension we have lifted j to $k_n : V[G_\omega] \to M[G_\omega^M]$, as desired. □(Claim 2)

5.Vorlesung

Claim 3. Suppose that $n > 0$ is finite and $V[G_\omega] \models S \cap$ Cof$(< \aleph_n)$ is stationary. Then S remains stationary in all $\aleph_n$-closed forcing extensions of $V[G_\omega]$.

Given this last Claim, we finish the proof of the Theorem as follows. Suppose that $V[G_\omega] \models S \subseteq \aleph_{\omega+1}$ is stationary. Then for some finite $n >$

0, $V[G_\omega] \models S \cap \text{ Cof}(< \aleph_n)$ is stationary. By Claim 2, in some $\aleph_n$-closed forcing extension $V[G_\omega][H_n]$ of $V[G_\omega]$ there is an embedding $k_n : V[G_\omega] \to M_n \subseteq V[G_\omega][H_n]$ with critical point $\kappa_n = \aleph_{n+1}^{V[G_\omega]}$, $k_n \restriction \aleph_{\omega+1}^{V[G_\omega]} \in M_n$ and $k_n(\aleph_{n+1}^{V[G_\omega]}) > \aleph_{\omega+1}^{V[G_\omega]}$. By Claim 3, S is still stationary in $V[G_\omega][H_n]$. Let γ be the supremum of $k_n[\aleph_{\omega+1}^{V[G_\omega]}]$. Then γ has cofinality $\aleph_n$ in M_n and therefore γ is less than $k_n(\aleph_{\omega+1}^{V[G_\omega]})$, which is regular in M_n.

We claim that $k_n(S) \cap \gamma$ is stationary in M_n. Suppose that $C \subseteq \gamma$ is closed unbounded, $C \in M_n$. As k_n is continuous at ordinals of $V[G_\omega]$-cofinality $< \aleph_n$ and $V[G_\omega]$, $V[G_\omega][H_n]$ have the same $< \aleph_n$ sequences of ordinals, it follows that Range $(k_n \restriction \aleph_{\omega+1}^{V[G_\omega]})$ is $< \aleph_n$-closed in M_n. Therefore Range $(k_n) \cap C$ is unbounded in γ. Let D be $k_n^{-1}[C]$. Then D is unbounded in $\aleph_{\omega+1}^{V[G_\omega]}$ and moreover is $< \aleph_n$-closed. As S is a subset of $\text{Cof}(< \aleph_n)$ which is stationary in $V[G_\omega][H_n]$, it follows that $S \cap D$ is nonempty, and therefore that $k_n(S) \cap C$ is nonempty, as desired.

As $M_n \models k_n(S) \cap \gamma$ is stationary, it follows that $V[G_\omega] \models S \cap \alpha$ is stationary for some $\alpha < \aleph_{\omega+1}$, thereby proving $\text{SR}(\aleph_{\omega+1})$ in $V[G_\omega]$.

Proof of Claim 3. We use the following Lemma of Shelah:

Lemma 7 *In $V[G_\omega]$ there is a sequence $\langle x_\alpha \mid \alpha < \aleph_{\omega+1} \rangle$ of bounded subsets of $\aleph_{\omega+1}$ such that for all α in a closed unbounded subset C of $\aleph_{\omega+1}$ there is a closed unbounded $c \subseteq \alpha$ of ordertype cof (α) such that all proper initial segments of c are of the form x_β for some $\beta < \alpha$.*

Now suppose that in $V[G_\omega]$, $S \subseteq \text{ Cof}(< \aleph_n)$ is stationary and P is an $\aleph_n$-closed forcing. Let $\langle x_\alpha \mid \alpha < \aleph_{\omega+1} \rangle$ and C be as in the Lemma. Given $p \in P$ which forces $\dot{D}$ to be closed unbounded in $\aleph_{\omega+1}$, we must find an extension q of p which forces that some α is in $S \cap \dot{D}$.

In $V[G_\omega]$ let $(N, \in, <_N)$ be an elementary submodel of some large $(H_\theta, \in, <_\theta)$ (where $<_\theta$ is a well-ordering of H_θ) which contains $P, p, \dot{D}, \langle x_\alpha \mid \alpha < \aleph_{\omega+1} \rangle, D$ and such that $N \cap \aleph_{\omega+1}$ is an ordinal $\alpha \in C \cap S$. This is possible as S is stationary in $V[G_\omega]$. Let $c \subseteq \alpha$ be of ordertype cof $(\alpha) < \aleph_n$ with all of its proper initial segments of the form x_γ for some $\gamma < \alpha$. It follows that all of the proper initial segments of c belong to N.

Now build a descending chain of conditions $\langle p_i \mid i < \text{cof}\,(\alpha)\rangle$ such that $p_0 = p$ and p_j is the $<_N$-least extension of p_i, $i < j$, which forces some ordinal greater than the j-th element of c into $\dot{D}$. Then for each $j < \text{cof}\,(\alpha)$, the sequence $\langle p_i \mid i < j\rangle$ belongs to N and by the $< \aleph_n$-closure of P there is a condition q below each of the p_i, $i < \text{cof}\,(\alpha)$. Then $q \leq p$ forces that α belongs to $S \cap \dot{D}$, as desired. $\Box$(Claim 3).

This completes the proof of the theorem.

6.Vorlesung

Saturated Ideals

Let κ be an uncountable regular cardinal and I a nonprincipal κ-complete ideal on κ, i.e., a collection of subsets of κ, including all bounded subsets of κ, with the following properties:

1. $A \subseteq B \in I \to A \in I$.
2. $\alpha < \kappa$, $A_i \in I$ for each $i < \alpha \to \bigcup_{i<\alpha} A_i \in I$.
3. $\kappa \notin I$.

For a cardinal λ, I is λ-*saturated* iff the Boolean algebra $\mathcal{P}(\kappa)/I$ has the λ-cc. Equivalently: If A_i, $i < \lambda$ are subsets of κ not in I, then $A_i \cap A_j$ belongs to I for some distinct pair $i, j < \lambda$. We say that κ *carries a* λ*-saturated ideal* iff there exists a λ-saturated, κ-complete ideal on κ.

I is 2-saturated iff I is a maximal ideal, and therefore κ carries a 2-saturated ideal iff κ is measurable. However even $\aleph_1$-saturation does not imply measurability, as the next result shows.

Theorem 8 *If κ is measurable then in some cofinality-preserving forcing extension, $2^{\aleph_0} = \kappa$ and κ carries an $\aleph_1$-saturated ideal.*

Proof. Let P be the forcing that adds κ Cohen reals, by a finite support product. As P is ccc, cofinalities are preserved. In the extension $2^{\aleph_0} = \kappa$. Let I be a κ-complete maximal ideal on κ, whose existence is guaranteed by the measurability of κ. We claim that in $V[G]$, where G is P-generic, the ideal $J = \{X \subseteq \kappa \mid X \subseteq Y \text{ for some } Y \in I\}$ is a κ-complete, $\aleph_1$-saturated ideal.

First we prove that J is κ-complete. Suppose that $p \Vdash \dot{X}_\alpha \in J$ for each $\alpha < \lambda$, where λ is less than κ. For each $\alpha < \lambda$ let A_α be a maximal antichain of conditions q below p which force $\dot{X}_\alpha$ to be a subset of some $Y^\alpha_q \in I$. p forces $\dot{X}_\alpha$ to be a subset of the union of the Y^α_q's. It follows that p forces $\bigcup_{\alpha<\lambda} \dot{X}_\alpha$ to belong to J, as it forces it to be a subset of $\bigcup_{\alpha<\lambda, q\in A_\alpha} Y^\alpha_q$, which belongs to I as I is κ-complete and each A_α has size less than κ (in fact, each A_α is countable).

To prove $\aleph_1$-saturation, suppose that $\dot{X}_\alpha$, $\alpha < \omega_1$, is forced by a condition p to be a sequence of subsets of κ not in J whose pairwise intersections are in J. By the $\aleph_1$-completeness of J, we may in fact assume that p forces $\dot{X}_\alpha \cap \dot{X}_\beta$ to be empty for distinct $\alpha, \beta < \omega_1$. For each $\alpha < \omega_1$, let Y_α be the set of ordinals which are forced into $\dot{X}_\alpha$ by some condition below p. As $\dot{X}_\alpha$ is not in J, it follows that Y_α is not in I and therefore as I is an $\aleph_2$-complete maximal ideal, the intersection Y of the Y_α, $\alpha < \omega_1$, belongs to I. Let γ belong to Y. Then for each $\alpha < \omega_1$ there is an extension q_α of p which forces $\gamma \in \dot{X}_\alpha$. By the ccc, there exist distinct $\alpha, \beta < \omega_1$ such that q_α, q_β are compatible; but then a common extension of q_α, q_β forces that $\dot{X}_\alpha \cap \dot{X}_\beta$ is nonempty, contradiction. □

7.Vorlesung

Thus κ can carry an $\aleph_1$-saturated ideal without being strongly inaccessible. However:

Theorem 9 *If κ carries a κ-saturated ideal then κ is weakly inaccessible.*

Proof. We must show that κ is a limit cardinal. Suppose not and let $\kappa = \lambda^+$, λ an infinite cardinal. For $\xi < \lambda^+$ let f_ξ be a surjection of λ onto ξ. For $\alpha < \lambda^+$ and $\eta < \lambda$ define $A_{\alpha,\eta} = \{\xi \mid f_\xi(\eta) = \alpha\}$. Then for each $\eta < \lambda$, $A_{\alpha,\eta}$ and $A_{\beta,\eta}$ are disjoint for distinct $\alpha, \beta < \lambda^+$. And for each $\alpha < \lambda^+$, the union of the $A_{\alpha,\eta}$, $\eta < \lambda$, contains all sufficiently large ordinals $< \lambda^+$.

Now suppose that I were a λ^+-saturated ideal on λ^+. It follows from the λ^+-completeness of I that for each $\alpha < \lambda^+$, A_{α,η_α} does not belong to I for some $\eta_\alpha < \lambda$. . Therefore for some fixed $\eta < \lambda$, $A_{\alpha,\eta}$ does not belong to I for λ^+-many $\alpha < \lambda^+$. But as $A_{\alpha,\eta}$ and $A_{\beta,\eta}$ are disjoint for distinct $\alpha, \beta < \lambda^+$, this contradicts the λ^+-saturation of I. □

Can a successor cardinal κ carry a κ^+-saturated ideal? We give a positive answer using forcing axioms.

Definition. Let P be a forcing and $p \in P$. The *proper game for P below p* is defined as follows: Player I plays P-names $\dot{\alpha}_n$ for ordinals and II plays ordinals β_n. II wins iff there is some $q \leq p$ which forces that for each n, $\dot{\alpha}_n$ equals some β_k. The *semiproper game (for P below p)* is defined in the same way, but with "ordinals" replaced with "countable ordinals". P is *proper (semiproper)* iff for each $p \in P$, II has a winning strategy in the proper (semiproper) game for P below p.

Properness (semiproperness) can be equivalently formulated in terms of the existence of generics over countable models.

Definition. Let P be a forcing. For any countable set M, q is (M, P)*-generic (semigeneric)* iff for every name $\sigma \in M$ for an ordinal (countable ordinal), q forces that σ equals some ordinal of M.

Lemma 10 *P is proper (semiproper) iff for sufficiently large cardinals λ there is a closed unbounded set of $M \in [H_\lambda]^{\aleph_0}$ such that each $p \in M$ has an extension which is (M, P)-generic (semigeneric).*

The *Proper forcing axiom PFA (the semiproper forcing axiom SPFA)* is the assertion that if P is a proper (semiproper) forcing and $\mathcal{D}$ a collection of $\aleph_1$-many dense subsets of P then there is a compatible $G \subseteq P$ which intersects each element of $\mathcal{D}$.

Lemma 11 *Suppose that P_α is a countable support iteration of forcings $\langle \dot{Q}_\beta \mid \beta < \alpha \rangle$ such that $P_\alpha \restriction \beta$ forces $\dot{Q}_\beta$ to be proper for each $\beta < \alpha$. Then P_α is proper.*

Definition. κ is λ*-supercompact*, where λ is a cardinal $\geq \kappa$, iff there is an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and $M^\lambda \subseteq M$. κ is *supercompact* iff κ is λ-supercompact for all λ.

Remark. Supercompactness is a first-order property, as the λ-supercompactness of κ can be witnessed by an embedding of the form $j_U : V \to M_U$ where U is a normal measure on $P_\kappa \lambda$.

Theorem 12 *If κ is supercompact then there is a proper forcing extension in which κ equals $\aleph_2$ and PFA holds.*

Proof. We need the following Lemma.

Lemma 13 *Suppose that κ is supercompact. Then there is a function $f : \kappa \to V_\kappa$ such that for every set x and every cardinal $\lambda \geq \kappa$ such that $x \in H_{\lambda^+}$ there is a $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$, $M^\lambda \subseteq M$ and $j(f)(\kappa) = x$. f is called a* Laver function *on κ.*

Proof. Assume that the Lemma fails. For each $f : \kappa \to V_\kappa$ let λ_f be the least cardinal $\geq \kappa$ such that some $x \in H_{\lambda_f^+}$ witnesses that f is not a *Laver function* for κ, i.e., such that $j(f)(\kappa) \neq x$ for every $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and $M^\lambda \subseteq M$. Let ν be greater than all of the λ_f's and let $j : V \to M$ witness the ν-supercompactness of κ.

Now inductively define $f : \kappa \to V_\kappa$ as follows: If $f \restriction \alpha$ is not a Laver function for α then let λ be least so that some $x \in H_{\lambda^+}$ witnesses this and choose $f(\alpha) = x_\alpha$ to be such an x; otherwise set $f(\alpha) = 0$.

Now consider $x = j(f)(\kappa)$. By the definition of f and the elementarity of j, x witnesses the failure of f to be a Laver function in M. As $M^\nu \subseteq M$, x also witnesses the failure of f to be a Laver function in V and λ_f is defined the same way in M as in V. This is a contradiction, as $j(\kappa) > \lambda_f$ and $j(f)(\kappa) = x$. $\square$ (Lemma 13)

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Now we prove the Theorem. Let $f : \kappa \to V_\kappa$ be a Laver function. Construct a countable support iteration P_κ of $\langle \dot{Q}_\alpha \mid \alpha < \kappa \rangle$ as follows. At stage α, if $f(\alpha)$ is a pair $(\dot{P}, \dot{\mathcal{D}})$ of P_α-names such that $\dot{P}$ is proper and $\dot{\mathcal{D}}$ is a γ-sequence of dense subsets of $\dot{P}$ for some $\gamma < \kappa$ then set $\dot{Q}_\alpha = \dot{P}$; otherwise let $\dot{Q}_\alpha$ be the trivial forcing.

Let G be P_κ-generic. As P_κ is proper, $\aleph_1$ is preserved. Each P_α, $\alpha < \kappa$, has size less than κ and the iteration is performed with countable support; it follows that P_κ is κ-cc and therefore κ is preserved.

We claim that in $V[G]$, if P is proper and $\mathcal{D} = \langle D_\alpha \mid \alpha < \gamma \rangle$, $\gamma < \kappa$, is a sequence of dense subsets of P then there is a compatible subset of P which intersects each D_α. Let $\dot{P}$ and $\dot{\mathcal{D}}$ be P_κ-names for P and $\mathcal{D}$. Choose λ to be much larger than P and let $j : V \to M$ have critical point κ with $j(\kappa) > \lambda$, $M^\lambda \subseteq M$ and $j(f)(\kappa) = (\dot{P}, \dot{\mathcal{D}})$. We can assume that V_λ^M is very elementary in M and therefore $V_\lambda^{M[G]} = V_\lambda^{V[G]}$ is very elementary in $M[G]$; it follows that P is not only proper in $V[G]$, but also in $M[G]$.

Now consider the iteration $j(P_\kappa)$ in M, which is a countable support iteration of length $j(\kappa)$ using the Laver function $j(f)$. As $j(f)(\kappa) = (\dot{P}, \dot{\mathcal{D}})$ and $\dot{P}$ is proper in $M[G]$, it follows that the forcing $\dot{P}$ is used at stage κ in the $j(P_\kappa)$ iteration in M. So we can write $j(P_\kappa) = P_\kappa * \dot{P} * \dot{R}$ for some $\dot{R}$. If $H * K$ is generic for $\dot{P} * \dot{R}$ over $V[G]$, then in $V[G * H * K]$ we can extend j to an elementary embedding $j^* : V[G] \to M[G * H * K]$. H is P-generic over $V[G]$ and therefore meets each D_α, $\alpha < \gamma$. Let $E = \{j^*(p) \mid p \in H\}$. Then E belongs to $M[G * H * K]$ and is a compatible set of conditions in $j^*(P)$ that meets each dense set in $j^*(\mathcal{D})$. By elementarity it follows that in $V[G]$ there is a compatible set of conditions in P which meets each dense set in $\mathcal{D}$, as desired.

It now follows that $V[G]$ is a model of PFA as $\aleph_1 < \kappa$. Also note that P_κ collapses each $\gamma < \kappa$ to ω_1 as Coll (ω_1, γ) is countably-closed, and therefore proper, and for each $\alpha < \gamma$, the set of conditions $f \in$ Coll (ω_1, γ) with $\alpha \in$ Range (f) is dense. So κ is the ω_2 of $V[G]$. $\Box$

The iteration lemma for proper forcing has an analogue for semiproper forcing. There is a notion of *revised* countable support iteration that preserves semiproperness, and therefore one has:

Theorem 14 *If κ is supercompact then there is a semiproper forcing extension in which κ equals $\aleph_2$ and SPFA holds.*

SPFA implies an apparently stronger axiom. A forcing P is *stationary-preserving* iff each stationary subset of ω_1 remains stationary in P-generic extensions. *Martin's maximum MM* is the assertion that if P is stationary-preserving and $\mathcal{D}$ a collection of $\aleph_1$-many dense subsets of P then there is a compatible $G \subseteq P$ which intersects each element of $\mathcal{D}$.

9.Vorlesung

Theorem 15 *SPFA implies MM.*

Proof. In fact SPFA implies that every stationary-preserving forcing is semiproper, as we now show.

Let X be a set of countable elementary submodels of $H^*_\lambda = (H_\lambda, \in, <)$ (where $<$ is a wellordering of H_λ). We write $X^\perp$ for $\{M \in [H_\lambda]^{\aleph_0} \mid M \prec H^*_\lambda$ and $N \notin X$ for every countable N that satisfies $M \prec N \prec H^*_\lambda$ and $N \cap \omega_1 = M \cap \omega_1\}$. A *nice chain in* H^*_λ is a sequence $\langle M_\alpha \mid \alpha < \theta\rangle$ of countable elementary submodels of H^*_λ such that $\alpha < \beta \to M_\alpha \in M_\beta$ and M_λ is the union of M_α, $\alpha < \lambda$, for limit λ.

Lemma 16 *(Main Lemma) Assume SPFA and let $\omega_1 \leq \kappa < \lambda$ with λ regular and sufficiently large. Let $Y \subseteq [H_\kappa]^{\aleph_0}$ be stationary and $X = \{M \in [H_\lambda]^{\aleph_0} \mid M \cap H_\kappa \in Y\}$ (the "lifting" of Y to H_λ). Then there exists a nice chain $\langle M_\alpha \mid \alpha < \omega_1\rangle$ in H^*_λ such that $M_\alpha \in X \cup X^\perp$ for every α.*

We now prove the Theorem using the Main Lemma. Assume SPFA and suppose Q is a stationary-preserving forcing. Choose κ large enough so that any Q-names for a countable ordinal is equivalent to one in H_κ. Choose a condition p in Q and define $Y = \{M \in [H_\kappa]^{\aleph_0} \mid$ There exists no (M, Q)-semigeneric $q \leq p\}$. Choose $\lambda > \kappa$ to be regular and let $X = \{M \in [H_\lambda]^{\aleph_0} \mid M \cap H_\kappa \in Y\}$ be the lifting of Y to H_λ. By the choice of κ, $X = \{M \in [H_\lambda]^{\aleph_0} \mid$ There exists no (M, Q)-semigeneric $q \leq p\}$.

By the Main Lemma, there is a nice chain $\langle M_\alpha \mid \alpha < \omega_1\rangle$ in H^*_λ such that $M_\alpha \in X \cup X^\perp$ for each $\alpha < \omega_1$. We claim that $S = \{\alpha < \omega_1 \mid M_\alpha \in X\}$ is nonstationary. Let G be Q-generic, p in G. Let $\dot\delta_\xi$, $\xi < \omega_1$, enumerate all names for countable ordinals in $\bigcup_{\alpha<\omega_1} M_\alpha$. Then $C = \{\alpha < \omega_1 \mid M_\alpha \cap \omega_1 = \alpha$ and $\dot\delta_\xi \in M_\alpha$, $\dot\delta^G_\xi < \alpha$ for all $\xi < \alpha\}$ is closed unbounded. And for each $\alpha \in C$, there exists $q \in G$ below p which forces each $\dot\delta_\xi \in M_\alpha$ to equal some ordinal in M_α and is therefore (M_α, Q)-semigeneric. So S is nonstationary in $V[G]$ and therefore nonstationary in V.

It follows that there is a nice chain $\langle M_\alpha \mid \alpha < \omega_1\rangle$ in H^*_λ such that $M_\alpha \in X^\perp$ for each $\alpha < \omega_1$. Let $\mu > \lambda$ be sufficiently large. Choose a countable $M \prec (H_\mu, \in, <, Q, \langle M_\alpha \mid \alpha < \omega_1\rangle)$ (where $<$ is a wellordering of H_μ) with $p \in M$. Set $\delta = M \cap \omega_1$. Then $M \cap H_\lambda \supseteq M_\delta$ and $\delta = M_\delta \cap \omega_1$; since M_δ belongs to $X^\perp$ we have $M \cap H_\lambda \notin X$. So by the definition of X, there exists an (M, Q)-semigeneric q below p. So for each $p \in Q$, there is a closed unbounded set of countable $M \prec H^*_\mu = (H_\mu, \in, <)$ and an (M, Q)-semigeneric $q \leq p$. It follows by taking a diagonal intersection that for a closed unbounded set of countable $M \prec H^*_\mu$, there is an (M, Q)-semigeneric below any $p \in M$, establishing the semiproperness of Q.

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Proof of the Main Lemma. Let P be the forcing that adds a nice ω_1-chain through $X \cup X^\perp$ using nice countable chains$\langle M_\alpha \mid \alpha \leq \gamma\rangle$ through $X \cup X^\perp$, ordered by end-extension.

For each $\gamma < \omega_1$ the set D_γ of conditions in P of length at least γ is dense: Let G be generic for collapsing H_λ to ω_1 with countable conditions. Then in $V[G]$, there is a nice ω_1-chain through $[H_\lambda^V]^{\aleph_0}$ with union H_λ^V, and $X \subseteq [H_\lambda^V]^{\aleph_0}$ is stationary. It follows that in $V[G]$ there are nice chains through X of any countable length, and therefore such chains exist also in V. It then follows that any condition in P can be extended to any countable length and therefore each D_γ is dense.

We show that P is semiproper. Let $\mu > \lambda$ be sufficiently large and $M \prec H_\mu^* = (H_\mu, \in, <)$, M countable (where $<$ is a wellordering of H_μ). Let p belong to $P \cap M$. We show that there is a $q \leq p$ which is (M, P)-semigeneric. First note that there is a countable N, $M \prec N \prec H_\mu^*$, such that $N \cap \omega_1 = M \cap \omega_1$ and $N \cap H_\lambda \in X \cup X^\perp$: This is clear if $M \cap H_\lambda$ belongs to $X^\perp$; otherwise choose a countable N', $M \cap H_\lambda \subseteq N' \prec H_\lambda^*$, such that $N' \cap \omega_1 = M \cap \omega_1$ and N' belongs to X. Let N be the least elementary submodel of H_μ^* containing $M \cup (N' \cap H_\kappa)$. Then $N' \cap H_\kappa = N \cap H_\kappa$ so N is as desired.

Now we find the desired (M, P)-semigeneric below p. Choose N as above. We can build a descending ω-sequence of conditions $p_n = \langle M_\alpha \mid \alpha \leq \gamma_n\rangle \in N$ below p such that the union of the M_{γ_n}'s equals $N \cap H_\lambda$ and every name in N for a countable ordinal is forced by some p_n to be an ordinal in N. Define q to be $\langle M_\alpha \mid \alpha < \gamma = \sup_n \gamma_n\rangle$ together with $M_\gamma = N \cap H_\lambda$. Then q is a condition below p which is (N, P)-semigeneric and therefore also (M, P)-semigeneric.

Finally, apply SPFA to obtain a nice chain of length ω_1 using the semiproperness of P. $\square$

We return to saturated ideals.

Theorem 17 *MM implies that the ideal of nonstationary subsets of ω_1 is ω_2-saturated.*

Proof. Assume MM and let $\{A_i \mid i \in W\}$ be a maximal collection of stationary subsets of ω_1 such that $A_i \cap A_j$ is nonstationary for distinct i, j. We show that for some $W_0 \subseteq W$ of size at most ω_1, $\{A_i \mid i \in W_0\}$ is also maximal.

Let P be the 2-step iteration $Q * R$ where Q adds a surjection $f : \omega_1 \to W$ using countable conditions $q : \alpha \to W$, $\alpha < \omega_1$, and R adds a closed unbounded subset to $\triangledown_{\alpha<\omega_1} A_{f(\alpha)} = \{\alpha \mid \alpha \in A_{f(\beta)}$ for some $\beta < \alpha\}$ using countable closed subsets r of $\triangledown_{\alpha<\omega_1} A_{f(\alpha)}$, ordered by end-extension. Then P is stationary-preserving: Suppose that $S \subseteq \omega_1$ is stationary. Then $S \cap A_i$ is stationary for some $i \in W$ by the maximality of $\{A_i \mid i \in W\}$. Forcing with Q preserves the stationarity of $S \cap A_i$ as Q is ω-closed. And forcing with R preserves the stationarity of any stationary subset of $\triangledown_{\alpha<\omega_1} A_{f(\alpha)}$ and therefore the stationarity of $S \cap A_i$.

Now for each $\alpha < \omega_1$ the set D_α of conditions (q, r) in P such that $\alpha \in \mathrm{dom}(q) \cap \max(r)$ is dense and therefore by MM there is a compatible $G \subseteq P$ which intersects each D_α. Then $\bigcup\{q \mid (q, r) \in G$ for some $r\}$ is a function $f : \omega_1 \to W$ and $\bigcup\{r \mid (q, r) \in G$ for some $q\}$ is a closed unbounded subset C of $\triangledown_{\alpha<\omega_1} A_{f(\alpha)}$. It follows that $\{A_i \mid i \in \text{Range }(f)\}$ is maximal, as any stationary subset of $\triangledown_{\alpha<\omega_1} A_{f(\alpha)}$ has stationary intersection with some single $A_{f(\alpha)}$. □

11.Vorlesung

The tree property

A *tree* is a partial ordering $T = (T, \leq_T)$ with the property that for each $t \in T$, $T_t =$ the set of $\leq_T$-predecessors of t is well-ordered by $\leq_T$. The *α-th level of* T is $T_\alpha = \{t \in T \mid T_t$ is well-ordered by $\leq_T$ with ordertype $\alpha\}$. The *height* of T is the supremum of $\{\alpha + 1 \mid T_\alpha$ is nonempty$\}$.

Let κ be an infinite regular cardinal. T is a *κ-tree* iff T has height κ and for $\alpha < \kappa$, T_α has cardinality less than κ. A κ-tree T is *κ-Aronszajn* iff it has no κ-branch, i.e., there is no subset of T well-ordered by $\leq_T$ with ordertype κ.

κ has the *tree property* iff there is no κ-Aronszajn tree. $\aleph_0$ has the tree property as by König's Lemma, a finitely branching tree of height ω must have an infinite branch. But ω_1 does not have the tree property:

Theorem 18 *There is an ω_1-Aronszajn tree.*

Proof. We construct a an ω_1-tree T whose elements are bounded, increasing, well-ordered sequences of rational numbers, ordered by end-extension. It is clear that such a tree has no ω_1-branch, as that would give an increasing sequence of rationals of length ω_1, which is impossible.

We construct the α-th level T_α of T by induction on $\alpha < \omega_1$. We inductively maintain the following property:

$(*)$ T_α is countable and if x belongs to T_β, $\beta < \alpha$ and q is a rational greater than $\sup(x)$ then x is extended by some $y \in T_\alpha$ with $\sup(y) < q$.

T_0 consists only of the empty sequence (we take $\sup(\emptyset)$ to be $-\infty$. To define $T_{\alpha+1}$ from T_α, simply extend each $x \in T_\alpha$ with each rational $q > \sup(x)$. It is clear that property $(*)$ is preserved. If α is a limit ordinal then for each x in some T_β, $\beta < \alpha$, and each rational $q > \sup(x)$, we extend x to $x_1 \subseteq x_2 \subseteq \cdots$ so that $\sup(x_n) < q$ for each n and the levels of the x_n's are cofinal in α; then put the resulting sequence $\bigcup_n x_n$ into T_α. It follows that T_α is countable and that for each $x \in \bigcup_{\beta<\alpha} T_\beta$ and $q > \sup(x)$, x has an extension y in T_α with $\sup(y) \leq q$; by choosing q' between q and $\sup(x)$ we can in fact guarantee $\sup(y) < q$, which gives $(*)$ for α. $\Box$

The previous proof generalises. For an infinite cardinal λ, let Q_λ be the set of $< \lambda$ sequences of ordinals less than λ, ordered lexicographically. Then λ can be order-preservingly embedded into any interval of Q_λ. Now the cardinality of Q_λ is $\lambda^{<\lambda}$; if this is λ, then we can replace the rationals by Q_λ in the previous proof, obtaining:

Theorem 19 *If $\lambda^{<\lambda} = \lambda$ then there is a λ^+-Aronszajn tree. In particular if GCH holds and λ is regular, there is a λ^+-Aronszajn tree.*

The consistency strength of the existence of an uncountable κ with the tree property is that of a weakly compact:

Theorem 20 *(1) If κ is weakly compact then κ has the tree property.*
(2) In L, κ has the tree property iff κ is weakly compact.
(3) If κ has the tree property then κ is weakly compact in L.

Proof. (1) It suffices to show that any κ-tree T with universe κ has a κ-branch. If T is a κ-tree on κ then the statement that T has no κ-branch is a Π^1_1 statement about the structure $(H_\kappa, \in, T)$. As κ is weakly compact, it is Π^1_1 reflecting, which implies that for some $\alpha < \kappa$, $T|\alpha = \bigcup_{\beta<\alpha} T_\beta$ has no α-branch. But this is impossible, as the T-predecessors of any element of T_α form an α-branch through $T|\alpha$.
(2) This uses the fine structure theory and will not be proved here.
(3) Sketch: If κ is not weakly compact in L then by 2 there is a κ-tree T in L with no κ-branch in L. Now build another κ-tree T^* in L with the property that any κ-branch through T^* gives rise to a constructible κ-branch through T; it follows that κ does not have the tree property. $\square$

Can ω_2 have the tree property? By the above results, we will need to use a weakly compact cardinal and kill CH to obtain the consistency of this. The following characterisation of weak compactness in terms of elementary embeddings will prove useful.

Proposition 21 *κ is weakly compact iff κ is strongly inaccessible and for every transitive model M of ZF$^-$ such that κ belongs to M, M is $< \kappa$-closed and M has size κ there is an elementary embedding $j : M \to N$, N transitive, with critical point κ.*

12.Vorlesung

Theorem 22 *Suppose that κ is weakly compact. Then in some forcing extension, $\kappa = \omega_2$, $2^{\aleph_0} = \aleph_2$ and ω_2 has the tree property.*

Proof. Consider the following "mixed support" iteration $P = \langle P_\alpha \mid \alpha < \kappa \rangle$. For each $\alpha < \kappa$, $P_{\alpha+1} = P_\alpha * Q_\alpha$, where Q_α is a P_α-name for the product ω-Cohen $\times$ ω_1-Cohen. For limit α we take all $p = \langle (p(\beta)_0, p(\beta)_1) \mid \beta < \alpha \rangle$ in the inverse limit of the P_β, $\beta < \alpha$, such that for all but finitely many $\beta < \alpha$, $p(\beta)_0$ is trivial and for all but countably many $\beta < \alpha$, $p(\beta)_1$ is trivial. For $p \in P$ write $(p)_0$ for $\langle p(\beta)_0 \mid \beta < \text{length } (p) \rangle$ and $(p)_1$ for $\langle p(\beta)_1 \mid \beta < \text{length } (p) \rangle$ (where length (p) denotes the strict supremum of the support of p). Thus $(p)_0$ is finitely supported and $(p)_1$ is countably supported.

At stage $\alpha < \kappa$, Q_α collapses α to ω_1 as P_α adds α reals and Q_α adds an ω_1-Cohen set. P is κ-cc: If X is a maximal antichain in P then for some

$\alpha < \kappa$ of uncountable cofinality, $X \cap P_\alpha$ is a maximal antichain in P_α and since P_α is a direct limit, $X \cap P_\alpha$ is in fact a maximal antichain in P.

Note that it is dense for $p \in P$ to have the property that for each $\alpha <$ length (p), if $p(\alpha)_0$ is not the trivial name then it is forced by $p \restriction \alpha$ to be equal to some particular ω-Cohen condition. This is proved for P_α by induction on α; the successor case is easy, and as $(p)_0$ is finitely supported, the case where α is a limit ordinal is trivial.

Also note that any condition in P is equivalent to a condition p in P with the property that for each $\alpha <$ length (p), the trivial condition in P_α forces $p(\alpha)$ to belong to Q_α. This is because we can replace the P_α name $p(\alpha)$ by a name which is forced by the trivial condition to equal $p(\alpha)$ if $p(\alpha)$ belongs to Q_α and is forced to be the trivial condition of Q_α otherwise.

Let P^* be the dense set of conditions in P with the above two properties. We show that P^*, and therefore also P, preserves ω_1: Suppose that $\dot{f}$ is forced to be a function from ω into ω_1. Given a condition p we will find an extension of p which forces a countable bound on the range of $\dot{f}$. Extend p to a condition q_1 which decides a value of $\dot{f}(0)$ and let p_1 be obtained from q_1 by setting $p_1(\alpha)_0$ to be $p(\alpha)_0$ for $\alpha <$ length (p) and to be the trivial name for α in [length (p), length (q_1)). Extend p_1 to a condition q_2 which decides a different value of $\dot{f}(0)$ and obtain p_2 from q_2 by setting $p_2(\alpha)_0$ to be $p(\alpha)_0$ for $\alpha <$ length (p) and to be the trivial name for α in [length (p), length (q_2)). Continue this construction as long as possible, taking greatest lower bounds at countable limit stages. In fact this construction terminates at some countable stage, as the collection of $(q_i)_0$'s forms an antichain in the finite support iteration of ω-Cohen, and any such antichain is countable. The result is a condition q extending p which forces a bound on $\dot{f}(0)$. Now repeat this for $\dot{f}(1)$, $\dot{f}(2)$, etc., resulting in an extension of p which forces a bound on $\dot{f}$.

So in $V[G]$, where G is P-generic, κ equals ω_2 and there are ω_2 reals. Suppose that T were an ω_2-Aronszajn tree in $V[G]$. Let $\dot{T}$ be a name for T. As κ is weakly compact, there is an elementary embedding $j : M \to N$ with critical point κ where $\dot{T}$ belongs to M and M, N are transitive ZF^- models. Then $\dot{T}$ belongs to N and therefore T belongs to $N[G]$. As T has no cofinal branch in $V[G]$, it has none in $N[G]$.

Now the forcing $j(P)$ is the mixed support iteration of ω-Cohen and ω_1-Cohen in N, of length $j(\kappa)$. The forcing $j(P)$ factors as $P * Q$ where Q is the mixed support iteration of ω-Cohen and ω_1-Cohen defined in N^P, indexed on the interval $[\kappa, j(\kappa))$. Choose H to be Q^G-generic over $N[G]$; then the embedding $j : M \to N$ lifts to $j^* : M[G] \to N[G][H]$. As T is an initial segment of the tree $j^*(T)$, it follows that T has a cofinal branch in $N[G][H]$. However this contradicts the following Claim.

13.Vorlesung

Claim. The forcing Q^G for adding H over $N[G]$ adds no cofinal branch through T.

Proof of Claim. Let C be generic over $N[G]$ for Coll (ω_1, ω_2), the forcing which collapses ω_2 using countable conditions. Then T has no cofinal branch in $N[G][C]$: Suppose that $\dot{B}$ were a name for such a branch. Build an infinite binary tree of conditions p_s, $s \in 2^{<\omega}$, in Coll (ω_1, ω_2) and an ω-sequence $\alpha_1 < \alpha_2 < \cdots$ less than κ such that for distinct s and t of length n, p_s, p_t force different elements of T to belong to $\dot{B}$ at level α_n. Then as there are ω_2 reals in $N[G]$, this gives ω_2 different elements of the α-th level of T, where α is the supremum of the α_n's, contradicting the fact that T is an ω_2-tree.

To prove the Claim, it suffices to show that Q^G does not add a cofinal branch through T over the ground model $N[G][C]$. Suppose that $p \in Q^G$ forces $\dot{B}$ to be such a branch and let $\langle \alpha_i \mid i < \omega_1 \rangle$ be an increasing sequence in $N[G][C]$ cofinal in $\omega_2^{N[G]}$. As in the proof that P preserves ω_1, form a decreasing sequence of conditions p_i, $i < \omega_1$ in Q^G as follows: Extend p to a condition q_1 which decides which element of T_{α_0} belongs to $\dot{B}$ and let p_1 be obtained from q_1 by setting $p_1(\alpha)_0$ to be $p(\alpha)_0$ for $\alpha <$ length (p) and to be the trivial name for α in [length (p), length (q_1)). Extend p_1 to a condition q_2 which decides which element of T_{α_1} belongs to $\dot{B}$ and obtain p_2 from q_2 by setting $p_2(\alpha)_0$ to be $p(\alpha)_0$ for $\alpha <$ length (p) and to be the trivial name for α in [length (p), length (q_2)). Continue this construction for ω_1 stages, taking greatest lower bounds at countable limit stages. By a Δ-system argument, there is an uncountable $S \subseteq \omega_1$ such that for any α, β in S, $(q_\alpha)_0, (q_\beta)_0$ are compatible. But this gives a cofinal branch through T in $N[G][C]$, contradiction. □

The previous proof generalises to show that if $\lambda > \omega$ is regular and $\kappa > \lambda$ is weakly compact, then in some forcing extension, λ^+ has the tree property: Use the length κ iteration of ω-Cohen $\times$ λ-Cohen, with finite support on the ω-Cohen forcings and $< \lambda$ support on the λ-Cohen forcings. (For this argument, (ω, finite) could be replaced with $(\bar{\lambda}, < \bar{\lambda})$ for any regular $\bar{\lambda} < \lambda$.)

Can the successor of a singular cardinal have the tree property? We provide a postitive answer using strongly compact cardinals.

Definition. κ is *λ-strongly compact* iff it is the critical point of an elementary embedding $j : V \to M$ such that any $X \subseteq M$ of cardinality λ is a subset of some $Y \in M$ of M-cardinality $< j(\kappa)$. We say that κ is *strongly compact* iff it is λ-strongly compact for every λ.

Note that λ-supercompactness easily implies λ-strong compactness, as in that case $j(\kappa)$ is greater than λ and M is closed under λ-sequences, so we may take Y to equal X.

14.Vorlesung

Lemma 23 *κ is λ-strongly compact iff for any set I, any κ-complete filter on I generated by at most λ sets can be extended to a κ-complete ultrafilter on I.*

Proof. Suppose that κ is λ-strongly compact, witnessed by $j : V \to M$, and let X be a collection of λ-many sets on I which generate a κ-complete filter $\mathcal{F}$. Choose $Y \supseteq j[X]$ in M of M-cardinality $< j(\kappa)$. Then $j(\mathcal{F})$ is a $j(\kappa)$-complete filter in M and $j(\mathcal{F}) \cap Y$ is a subset of $j(\mathcal{F})$ in M of M-cardinality less than $j(\kappa)$. So we may choose $a \in \bigcap(j(\mathcal{F}) \cap Y)$. Define an ultrafilter $\mathcal{U}$ by: $A \in \mathcal{U}$ iff $A \subseteq I$ and $a \in j(A)$. Then $\mathcal{U}$ is a κ-complete ultrafilter extending $\mathcal{F}$. Conversely, consider the κ-complete filter $\mathcal{F}$ on $P_\kappa\lambda$ generated by the sets $\{x \mid \alpha \in x\}$ for $\alpha < \lambda$. Extend $\mathcal{F}$ to a κ-complete ultrafilter $\mathcal{U}$ and let $j : V \to M = V^{P_\kappa\lambda}/\mathcal{U}$ be the ultrapower of V by $\mathcal{U}$. If $X = \{[f_\alpha] \mid \alpha < \lambda\} \subseteq M$, define $G(x) = \{f_\alpha(x) \mid \alpha \in x\}$. Then $X \subseteq [G]$ and $M \vDash \text{card}([G]) < j(\kappa)$. □

Theorem 24 *If $\lambda_0 < \lambda_1 < \cdots$ is an ω-sequence with supremum λ and each λ_n is λ^+-strongly compact then λ^+ has the tree property.*

Proof. Let T be a λ^+-tree. We assume that the α-th level T_α of T is the set $\lambda \times \{\alpha\}$. For each n let $T_{\alpha,n}$ be $\lambda_n \times \{\alpha\}$.

Claim. There is an unbounded $D \subseteq \lambda^+$ and $n \in \omega$ such that whenever $\alpha < \beta$ belong to D, there are $a \in T_{\alpha,n}$ and $b \in T_{\beta,n}$ with $a <_T b$.

Proof of Claim. Using the fact that λ_0 is λ^+-strongly compact extend the filter of subsets of T with complement of size at most λ to a countably complete ultrafilter $\mathcal{U}$. For $\alpha < \lambda^+$ define $n_\alpha \in \omega$ as follows: For $x \in T$ at some level greater than α choose $p^x_\alpha \in T_\alpha$ below x and let n^x be least so that p^x_α belongs to T_{α,n^x}. By the countable completeness of $\mathcal{U}$ there is some $n_\alpha \in \omega$ such that $X_\alpha = \{x \in T \mid n^x = n_\alpha\}$ belongs to $\mathcal{U}$.

Now choose an unbounded $D \subseteq \lambda^+$ such that n_α is some fixed n for $\alpha \in D$. If we take $\alpha < \beta$ in D then $X_\alpha \cap X_\beta$ contains some x and then $p^x_\alpha <_T p^x_\beta$ belong to $T_{\alpha,n}$, $T_{\beta,n}$, respectively. $\square$ (*Claim*)

Now let D, n be as in the Claim and choose $\mathcal{V}$ to be a λ_n^+-complete ultrafilter on λ^+ containing D and all final segments of λ^+. Choose any $\alpha \in D$. For every $\beta > \alpha$ in D find $a(\beta) \in T_{\alpha,n}$ and $b(\beta) \in T_{\beta,n}$ such that $a(\beta) <_T b(\beta)$. Using the λ_n^+-completeness of $\mathcal{V}$, find $a_\alpha \in T_{\alpha,n}$ and $\xi_\alpha < \lambda_n$ such that for a set of β's in $\mathcal{V}$, $a_\alpha = a(\beta)$ and $b(\beta) = (\xi_\alpha, \beta)$. For an unbounded $D' \subseteq D$, the ordinal ξ_α has a fixed value ξ for $\alpha \in D'$. Now the collection $\{a_\alpha \mid \alpha \in D'\}$ is a branch through T, because if α_1, α_2 belong to D' then for some β (indeed for a set of β's in $\mathcal{V}$) both a_{α_1} and a_{α_2} are below (ξ, β). $\square$

Magidor and Shelah also showed that in fact $\aleph_{\omega+1}$ can have the tree property. For this they needed to assume the consistency of an ω-sequence of cardinals $\kappa < \lambda_0 < \lambda_1 < \cdots$ with κ the critical point of $j : V \to M$, $j(\kappa) = \lambda_0$, M closed under $\mu = (\sup_n \lambda_n)^+$ sequences, with each λ_n being μ-supercompact.

15.Vorlesung

Jónsson cardinals

A structure $\mathcal{A}$ of cardinality κ for a countable language is a *Jónsson structure* iff it has no proper elementary submodel of cardinality κ. We say that κ is a *Jónsson cardinal* iff there is *no* Jónsson structure of cardinality κ. We do *not* assume here that κ is regular.

Using Skolem functions it is easy to show that κ is a Jónsson cardinal iff $\kappa \to [\kappa]^{<\omega}_\kappa$, i.e., whenever $F : [\kappa]^{<\omega} \to \kappa$ there is $H \subseteq \kappa$ of cardinality κ such that the range of F on $[H]^{<\omega}$ is a proper subset of κ.

We show that measurable cardinals are Jónsson.

Definition. We write $\kappa \to (\kappa)^{<\omega}_{\lambda}$ for the following: For any $F : [\kappa]^{<\omega} \to \lambda$ there is $H \subseteq \kappa$ of cardinality κ such that F is constant on $[H]^n$ for each n. κ is *Ramsey* iff $\kappa \to (\kappa)^{<\omega}_{\lambda}$ for all $\lambda < \kappa$.

Theorem 25 *(a) Measurable cardinals are Ramsey.*
(b) Ramsey cardinals are Jónsson.

Proof. (a) Suppose κ is measurable with nonprincipal, κ-complete, normal ultrafilter $\mathcal{U}$. We prove by induction on $n < \omega$ that for any $F_n : [\kappa]^n \to \lambda$, $\lambda < \kappa$, there is a set H_n in $\mathcal{U}$ such that F_n is constant on $[H_n]^n$. For $n = 1$ this is clear by the κ-completeness of $\mathcal{U}$. Suppose the result holds for n and $F_{n+1} : [\kappa]^{n+1} \to \lambda$. For each $\alpha < \kappa$ define $G^{\alpha}_n : [(\alpha, \kappa)]^n \to \lambda$ by $G^{\alpha}_n(x) = F_{n+1}(\{\alpha\} \cup x)$. By induction there is some $\beta_\alpha < \lambda$ such that G^{α}_n is constant with value β_α on $[H^{\alpha}_n]^n$ for some H^{α}_n in $\mathcal{U}$. By the κ-completeness of $\mathcal{U}$ there is a fixed $\beta < \lambda$ such that G^{α}_n is constant on $[H^{\alpha}_n]^n$ with value β for all α in some set $H \in \mathcal{U}$. It follows that F_{n+1} is constant on $[H_{n+1}]^{n+1}$, where $H_{n+1} \in \mathcal{U}$ is the intersection with H of the diagonal intersection of the H^{α}_n, $\alpha \in H$.

Then if $F : [\kappa]^{<\omega} \to \lambda$, $\lambda < \kappa$, we can choose $H_n \in \mathcal{U}$ for each n such that $F_n = F \restriction [\kappa]^n$ is constant on $[H_n]^n$; it follows that F is constant on $[H]^n$ for each n, where H is the intersection of the H_n's.

(b) Suppose that κ is Ramsey and $F : [\kappa]^{<\omega} \to \kappa$. Consider the structure $\mathcal{A} = (\kappa, <, F_1, F_2, \ldots)$ where F_n is the restriction of F to $[\kappa]^n$. Using Ramseyness we can get $I \subseteq \kappa$ of cardinality κ such that for each n, all increasing n-tuples from I realise the same type in $\mathcal{A}$. (Apply Ramseyness to $F : [\kappa]^{<\omega} \to 2^{\aleph_0}$ where $F(x)$ describes the type of x in $\mathcal{A}$.) Now let $i_0 < i_1$ be the first two elements of I. Then for $x \in [I \setminus \{i_0, i_1\}]^{<\omega}$, $F(x)$ cannot equal i_0; otherwise, by the choice of I, $F(x)$ would also have to equal i_1, contradicting the fact that F is a function. So the range of F on $[I \setminus \{i_0, i_1\}]^{<\omega}$ is not all of κ, proving Jónssonness. □

Mitchell showed that all Jónsson cardinals are Ramsey in the Dodd-Jensen core model, and therefore these two large cardinal notions have the same consistency strength.

The next result shows that a Jónsson cardinal can be singular.

Theorem 26 *Suppose that κ is measurable. Then in a forcing extension, κ is a singular Jónsson cardinal.*

Proof. Use Prikry Forcing: Conditions are pairs (s, A) where $s \in [\kappa]^{<\omega}$ and $A \in \mathcal{U}$, where $\mathcal{U}$ is a normal measure on κ. The condition (t, B) extends (s, A) iff t end-extends s, $B \subseteq A$ and $t \setminus s \subseteq A$. Prikry forcing preserves cardinals and gives κ cofinality ω.

Now suppose that $(s, A) \Vdash \dot{F} : [\kappa]^{<\omega} \to \kappa$. We find $(s, B) \leq (s, A)$ which forces Range $(\dot{F} \restriction [B]^{<\omega}) \neq \kappa$. Let $\langle R_i \mid i < \omega_1 \rangle$ be a partition of κ into ω_1 disjoint pieces.

For $s, t \in [\kappa]^{<\omega}$ write $s < t$ for $\max(s) < \min(t)$. Now for each $t \in [\kappa]^{<\omega}$ with $s < t$ consider the partition $F_t : [\kappa]^{<\omega} \to \omega_1$ defined by: $F_t(u) = i + 1$ iff for some $B \in \mathcal{U}$, $(t, B) \Vdash \dot{F}(u) \in R_i$; $F_t(u) = 0$ if otherwise undefined. (Note that F_t is single-valued.) Using the proof of Theorem 25(b), for each t with $s < t$ choose $A_t \in \mathcal{U}$ such that for each n, F_t is constant on $[A_t]^n$, and denote this constant value by $G_n(t)$. Let B_0 be the diagonal intersection of the A_t, i.e., the set of $\alpha < \kappa$ such that α belongs to A_t for each t with $\max(t) < \alpha$; then for each t, F_t is constant with value $G_n(t)$ on $[B_0 \setminus (\max(t) + 1)]^n$. Now choose $B_1 \in \mathcal{U}$ such that for each n, G_n is constant on $[B_1]^n$ and let B the intersection of B_1 with B_0. Then F_t can take on only countably many values for t in $[B]^{<\omega}$ and therefore (s, B) forces a countable bound on $\{i < \omega_1 \mid \text{Range } (\dot{F} \restriction [B]^{<\omega}) \cap R_i \neq \emptyset\}$. In particular, (s, B) forces that Range $(\dot{F} \restriction [B]^{<\omega})$ is not all of κ. □

Mitchell showed that if κ is a singular Jónsson cardinal then κ is measurable in some inner model, and therefore the existence of a singular Jónsson cardinal is equiconsistent with that of a measurable cardinal.

16.Vorlesung

Can a small cardinal be Jónsson? $\aleph_0$ is obviously not a Jónsson cardinal. The next result implies that neither is any $\aleph_n$, n finite.

Theorem 27 *If κ is not a Jónsson cardinal then neither is κ^+.*

Proof. Assume that κ is not a Jónsson cardinal and let $F : [\kappa]^{<\omega} \to \kappa$ witness this. For any $\alpha \in [\kappa, \kappa^+)$ we can use a bijection between κ and α

to get $F_\alpha : [\alpha]^{<\omega} \to \alpha$ which is surjective when restricted to $[A]^{<\omega}$ for any $A \subseteq \alpha$ of cardinality κ. Define $G : [\kappa^+]^{<\omega} \to \kappa^+$ by $G(\alpha_1, \dots, \alpha_n) = 0$ if $\alpha_n < \kappa$, $G(\alpha_1, \dots, \alpha_n) = F_{\alpha_n}(\alpha_1, \dots, \alpha_{n-1})$, otherwise. Then if $A \subseteq \kappa^+$ has cardinality κ^+ it follows that the range of G on $[A]^{<\omega}$ contains α for unboundedly many $\alpha < \kappa^+$, and therefore the range of G is all of κ^+. □

Can $\aleph_\omega$ be a Jónsson cardinal? The answer is unknown. However there are some results about the failure of certain regular cardinals to be Jónsson.

Theorem 28 *If λ is a regular Jónsson cardinal then stationary reflection holds for λ.*

Corollary 29 *The successor of a regular cardinal is not Jónsson.*

Proof of Theorem 28. Let λ be a regular Jónsson cardinal and choose M to be elementary in some large $H(\theta)$ so that $\lambda \in M$, $M \cap \lambda$ has cardinality λ but λ is not a subset of M. We show that each stationary $S \subseteq \lambda$ belonging to M reflects, i.e., has a stationary proper initial segment. By the elementarity of M this suffices.

First note that $S \setminus M$ is stationary. Otherwise, let E be cub in λ, $E \cap S \subseteq M$. In M we can split S into λ-many disjoint stationary subsets, so there is in M a function $f : S \to \lambda$ such that S_α, the preimage of $\{\alpha\}$ under f, is stationary for each $\alpha < \lambda$. Choose $\alpha \notin M$. Since $S_\alpha \subseteq S$, $E \cap S_\alpha$ is a nonempty subset of M. But if β belongs to $E \cap S_\alpha$, it then follows that $\alpha = f(\beta)$ belongs to M, contradiction.

So choose $\delta \in S \setminus M$ such that $\delta = \sup(M \cap \delta)$. Define β_δ to be $\min(M \setminus \delta)$. Then $\delta < \beta_\delta$ and β_δ is a limit ordinal of uncountable cofinality. We show that $S \cap \beta_\delta$ is stationary in β_δ: If not, then since S and β_δ are in M, M contains a cub subset C of β_δ which is disjoint from S. As $M \cap \delta$ is cofinal in δ, for any $\alpha < \delta$ there is $\beta \in M$ with $\alpha < \beta < \delta$. Since $M \vDash C$ is unbounded in β_δ, there is $\gamma \in M \cap C$ with $\beta < \gamma$. By choice of β_δ, γ must be less than δ. We have shown that δ is a limit point of C and therefore belongs to C; this contradicts our assumption that S and C are disjoint. □

17.Vorlesung

Theorem 30 $\aleph_{\omega+1}$ *is not a Jónsson cardinal.*

Proof. We use the following result of Shelah:

Lemma 31 *There exists an infinite $I \subseteq \omega$ and $F \subseteq \prod_{n\in I} \aleph_n$ such that:*
(i) F is wellordered by $\leq^*$ in length $\aleph_{\omega+1}$, where $\leq^*$ denotes the eventual domination order.
(ii) F is $\leq^*$-cofinal in $\prod_{n\in I} \aleph_n$.

For each n choose a structure $\mathcal{A}_n$ with universe $\aleph_n$ for a countable language with no proper substructure of cardinality $\aleph_n$, using the fact that $\aleph_n$ is not Jónsson. Choose $\mathcal{A}$ to be the least elementary submodel of $(H(\aleph_{\omega+2}), \in, <)$ (where $<$ is a wellordering of $H(\aleph_{\omega+2})$) of cardinality $\aleph_{\omega+1}$ which contains $\aleph_{\omega+1}$ as a subset and F as well as each $\mathcal{A}_n$ as elements. We show that $\mathcal{A}$ has no proper elementary submodel of cardinality $\aleph_{\omega+1}$ which contains F and the $\mathcal{A}_n$'s as elements, proving that $\aleph_{\omega+1}$ is not Jónsson.

Suppose that B were the universe of a proper elementary submodel $\mathcal{B}$ of $\mathcal{A}$ of cardinality $\aleph_{\omega+1}$ containing F and the $\mathcal{A}_n$'s. As $\mathcal{A}$ is the least elementary submodel of itself containing $\aleph_{\omega+1}$ as a subset and F as well as each $\mathcal{A}_n$ as elements, it follows that $B \cap \aleph_{\omega+1}$ is unbounded in $\aleph_{\omega+1}$.

If $B \cap \aleph_n$ is unbounded in $\aleph_n$ for infinitely many $n \in I$ then as each $\mathcal{A}_n$ witnesses that $\aleph_n$ is not Jónsson, B must contain $\aleph_\omega$ as a subset. It follows that $B \cap \aleph_{\omega+1}$ is an initial segment of $\aleph_{\omega+1}$ and therefore equals $\aleph_{\omega+1}$. Therefore B is the universe of $\mathcal{A}$, contradiction.

So it must be that for large enough $n \in I$, $g(n) = \sup(B \cap \aleph_n)$ is less than $\aleph_n$. We may choose $f \in F$ such that $g <^* f$; as B is cofinal in $\aleph_{\omega+1}$ and F is wellordered by $\leq^*$ in length $\aleph_{\omega+1}$, we may in fact choose $f \in F \cap \mathcal{B}$. But then $f(n)$ belongs to $B \cap \aleph_n$ for each n and for large enough n, $f(n)$ is greater than $g(n) = \sup(B \cap \aleph_n)$, contradiction. $\Box$