The Completeness of Isomorphism

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In classical descriptive set theory, analytic equivalence relations (i.e., Σ_1^1 equivalence relations with parameters) are compared under the relation of Borel reducibility (for example, see [5]). An important subclass of the Σ_1^1 equivalence relations are the isomorphism relations, i.e., the restrictions of the isomorphism relation on countable structures (viewed as an equivalence relation on reals coding such structures) to the models of a sentence of the infinitary logic $L_{\omega_1\omega}$. Scott's Theorem implies that the equivalence classes of any isomorphism relation are Borel, and therefore no isomorphism relation can be complete (under Borel reducibility) within the class of Σ_1^1 equivalence relations as a whole, some of which contain non-Borel equivalence classes. (This is clarified below.)

The picture is different in the computable setting. It is shown in [2] that isomorphism on computable structures (viewed as an equivalence relation on natural numbers coding such structures), indeed on computable trees, is complete for Σ_1^1 equivalence relations under the natural analogue of Borel-reducibility for equivalence relations on numbers: E_0 is reducible to E_1 iff for some computable $f : \mathbb{N} \to \mathbb{N}$, $E_0(m, n)$ iff $E_1(f(m), f(n))$ for all m, n.

In this article we survey the situation for classes of structures between the class of computable structures and the class of arbitrary countable structures. Our aim is to determine in which cases isomorphism is complete and in which cases it is not.

Work has also been done by considering not arbitrary isomorphisms, but isomorphisms of a restricted type (such as computable or hyperarithmetic isomorphism). For this I refer the reader to [3].

Section 1. Classes of structures

To discuss classes of structures intermediate between the class of computable structures and the class of arbitrary countable structures we make use of the L-hierarchy. We fix a computable first-order language and consider structures for that language with universe ω. Assume $V = L$; thus every structure is definable over L_α for some infinite countable ordinal α.

For pairs (α, n) where α is an infinite countable ordinal, $0 < n \in \omega$, define:

$X(\alpha, n) = $ all reals (subsets of ω) which are Δ_n definable over L_α

Also when α is a countable ordinal greater than ω we define:

$X(\alpha, 0) = $ all reals (subsets of ω) which are elements of L_α

Now fix α, n as above and let E be an equivalence relation on reals which is Σ_1^1 with parameter from $X(\alpha, n)$. We say that E is *complete on* $X(\alpha, n)$ iff whenever F is another such equivalence relation there exists a function f from reals to reals sending $X(\alpha, n)$ into $X(\alpha, n)$ such that for $x, y \in X(\alpha, n)$:

$$E(x, y) \text{ iff } F(f(x), f(y)),$$

where f is Hyp (i.e. Δ_1^1) in a parameter from $X(\alpha, n)$.

Note that isomorphism (viewed as an equivalence relation on reals coding countable structures) is a parameter-free Σ_1^1 equivalence relation.

Main Question. For which α, n is isomorphism complete on $X(\alpha, n)$?

Section 2: When isomorphism is complete

The basic positive result from [2] reads as follows.

Theorem 1 *([2]) Isomorphism is complete on $X(\omega, 1)$, the set of computable reals.*

Roughly speaking, the proof goes as follows. Suppose that $E(m, n)$ is a Σ_1^1 equivalence relation on computable reals with a computable parameter; we can translate E into a Σ_1^1 equivalence relation E' on natural numbers without parameter. By Kleene's Representation Theorem choose a computable sequence $(T(m, n) \mid m, n \in \omega)$ of computable trees such that $E'(m, n)$ iff $T(m, n)$ is illfounded. Using "rank-saturated" trees (see [1]) we can assume that the isomorphism type of $T(m, n)$ depends only on the rank of $T(m, n)$ (which is ∞ if $T(m, n)$ is illfounded). The main trick is to ensure that this rank depends only on the E'-equivalence classes of m, n. Then by defining $T^*(m)$ to be the "join" of the $T(m, n)$, $n \in \omega$, we obtain: $E'(m_0, m_1)$ iff $T^*(m_0)$ is isomorphic to $T^*(m_1)$. For the details see [2]. Now using a Hyp function which takes a computable real to a Turing-index for it, we obtain the desired Hyp reduction of E to isomorphism on computable structures.

Now the above clearly relativises to a real parameter. Say that isomorphism is *complete on the p-computable reals* (where p is a real parameter) iff whenever E is a Σ_1^1 equivalence relation with a p-computable parameter there is a Hyp function f with p-computable parameter sending p-computable reals to p-computable structures such that for p-computable x, y: $E(x, y)$ iff $f(x), f(y)$ are isomorphic.

Corollary 2 *For any parameter p, isomorphism is complete on the set of p-computable reals.*

This reduces the *Main Question* to the cases where $n = 0$, using the following fine-structural fact (see [6] or [4]).

Theorem 3 *For any α, n, $X(\alpha, n)$ either equals $X(\alpha, 0)$ or equals the set of p-computable reals for some real p.*

The reason for this is that if $X(\alpha, n)$ does not equal $X(\alpha, 0)$ then there is a real which is Δ_n over L_α but does not belong to L_α; then there is a "canonical" such real called the "Δ_n master code" for L_α which serves as the parameter p in the conclusion of the theorem.

We can reduce our *Main Question* even further. For example, consider $X(\omega + 1, 0)$, the set of arithmetical reals. There is a Hyp function which takes an arithmetical real to an arithmetical code for it and this reduces the completeness of isomorphism on $X(\omega + 1, 0)$ to its completeness on $X(\omega, 1)$, the content of Theorem 1. More generally, suppose that $X(\alpha, 0)$ is distinct from $X(\beta, 0)$ for each $\beta < \alpha$ (an assumption we can make without loss of generality) and that for some real p in L_α, α is less than the least p-admissible ordinal ω_1^p; then there is a Hyp in p function which send the reals of $X(\alpha, 0)$ injectively into ω, thereby reducing the completeness of isomorphism on $X(\alpha, 0)$ to its completness on the p-computable reals, Corollary 2. Thus we have the completeness of isomorphism on $X(\alpha, n)$ in all cases except when $n = 0$ and one of the following holds:

1. α is admissible but not the limit of admissibles.
2. α is a limit of admissibles.

We now show that isomorphism is *not* complete on $X(\alpha, 0)$ in the second of these cases.

Section 3: When isomorphism is not complete

First we need to clarify why isomorphism on arbitrary countable structures is not complete for Σ_1^1 equivalence relations on arbitrary reals.

Proposition 4 *There is a Σ_1^1 equivalence relation E on reals with an equivalence class which is not Borel (i.e., not Hyp with a real parameter).*

Proof. Let X be a Σ_1^1 set of reals which is not Borel. Define E by: $E(x, y)$ iff $x, y \in X$ or $x = y$. Then X is an equivalence class of E. $\square$

Theorem 5 *(Scott, see [5]) For any countable structure $\mathcal{A}$, the set of (codes for) countable structures which are isomorphic to $\mathcal{A}$ is Borel.*

Proof. Let φ be the Scott sentence of $\mathcal{A}$, i.e., the canonical sentence of $L_{\omega_1\omega}$ whose countable models are exactly those isomorphic to $\mathcal{A}$. This set of models is Borel, as the set of countable models of any sentence of $L_{\omega_1\omega}$ is Borel. $\square$

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Corollary 6 *Isomorphism on countable structures is not complete for Σ_1^1 equivalence relations (under Borel, i.e. Hyp in a real parameter, reducibility).*

Proof. A Borel reduction from a Σ_1^1 equivalence relation E to another such equivalence relation F takes non-Borel equivalence classes to non-Borel equivalence classes. $\square$

Now suppose that we replace the set of all reals by some subset $X(\alpha, 0)$ of the reals; what do we need to know about α for the above argument to still work?

Proposition 7 *Suppose that α is a limit of admissibles. Let $\mathcal{A}$ be a countable structure with code in L_α. Then the set of codes for countable structures isomorphic to $\mathcal{A}$ is Hyp with parameter in L_α.*

Proof. The canonical Scott sentence φ for $\mathcal{A}$ belongs to the second admissible set containing x whenever x is a real coding $\mathcal{A}$. As α is a limit of admissibles, φ is coded by a real in L_α. It follows that the set of countable structures isomorphic to $\mathcal{A}$, i.e., the set of countable models of φ, is Hyp with parameter in L_α. $\square$

Corollary 8 *Isomorphism is not complete on $X(\alpha, 0)$ when α is a limit of admissibles.*

Proof. Let X be the set of reals which code linear orders which have infinite descending chains. Then X is Σ_1^1 and not Borel. Now $X \cap L_\alpha$ is the set of reals in L_α which code linear orders which have infinite descending chains in L_α, using the fact that α is a limit of admissibles. Thus $X \cap L_\alpha$ is Σ_1^1 but not Δ_1^1 in L_α. And if B is a Hyp set of reals with parameter in L_α then $B \cap L_\alpha$ is Δ_1^1 in L_α, so it follows that X and B disagree on the reals of L_α. Now as before consider the equivalence relation $E(x, y)$ iff $x \in X$ or $x = y$; this equivalence relation is not reducible to isomorphism on $X(\alpha, 0)$ as its restriction to L_α has an equivalence class which is not Δ_1^1 in L_α but the intersection with L_α of the equivalence classes of isomorphism are each Δ_1^1 in L_α. $\square$

Successor admissibles?

We are left with cases of $X(\alpha, 0)$ when α is a successor admissible, i.e. an admissible ordinal which is not the limit of admissibles. Note the following.

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Proposition 9 *Suppose that α is a successor admissible. Then either $X(\alpha, 0)$ equals $X(\beta, 0)$ where β is a limit of admissibles or the reals of $X(\alpha, 0)$ are exactly those which are hyperarithmetic in p for some fixed real p.*

Proof. If L_α thinks that $\aleph_1$ exists then $X(\alpha, 0)$ equals $L(\beta, 0)$ where β is the $\aleph_1$ of L_α. Otherwise we may choose a real p in L_α which codes the supremum of the admissibles less than α and then the reals of L_α are exactly those which are hyperarithmetic in p. $\square$

Thus the only remaining cases are relativisations to a real parameter of the following.

Open question. Is isomorphism complete on $X(\omega_1^{ck}, 0)$, the set of hyperarithmetic reals?

Recall that this asks the following: Suppose that E is a Σ_1^1 equivalence relation on reals. Is there a Hyp function f which takes reals to countable structures such that $E(x, y)$ iff $f(x), f(y)$ are isomorphic, whenever x, y are hyperarithmetic? The proof methods of Theorem 1 and Corollary 8 do not appear to cover this case.

Section 4: A variant

There is a strengthening of Corollary 8 for the case of $X(\alpha, 0)$ when α is a limit of limits of admissibles.

Let E_1 be the equivalence relation $E_1(x, y)$ iff for sufficiently large n, $(x)_n = (y)_n$, where $(x)_n$ is the n-th "column" of x via some computable pairing funcion $\langle \cdot, \cdot \rangle$ on the natural numbers: $(x)_n(m) = x(\langle m, n \rangle)$ for all m. E_1 is a Hyp equivalence relation.

Theorem 10 *Let α be a limit of limits of admissibles. Then E_1 is not reducible to isomorphism on structures with codes in $X(\alpha, 0)$, the set of reals in L_α, via a Hyp function with parameter from $X(\alpha, 0)$.*

Proof. Suppose that there were such a reduction f with parameter p in L_α and choose a limit of admissibles $\alpha_0 < \alpha$ so that p belongs to L_{α_0}.

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Let M denote L_α, M_0 denote L_{α_0} and let $(z_n \mid n \in \omega) \in M$ be generic for the ω-product of Sacks forcing over M_0. Define x_n so that $(x_n)_k$ is the 0-real for $k < n$ and is z_k otherwise. The x_n's are pairwise E_1-equivalent so the $f(x_n)$'s are pairwise isomorphic. Choose a permutation π of ω in M which is Cohen-generic over $M_0[x_0]$. Let x be the code for the structure obtained from $f(x_0)$ by applying π. Then the structure coded by x is isomorphic to the structures coded by the $f(x_n)$'s. Now choose a real y in $M_0[x]$ so that $f(y)$ is isomorphic to the structure coded by x; this is possible as $M_0[x]$ is elementary in V for Σ_1^1 statements, using the fact that α_0 is a limit of x-admissibles. Then y is E_1-equivalent to x_0 and therefore some z_n is a component of y. But then $M_0[x]$, a Cohen-generic extension of M_0, contains a real which is Sacks-generic over M_0, a contradiction. $\square$

References

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