# Generic Saturation

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Assuming that ORD is $\omega+\omega$-Erdös we show that if a class forcing amenable to $L$ (an $L$-forcing) has a generic then it has one definable in a set-generic extension of $L[O^\#]$. In fact we may choose such a generic to be *periodic* in the sense that it preserve the indiscernibility of a final segment of a periodic subclass of the Silver indiscernibles, and therefore to be *almost codable* in the sense that it is definable from a real which is generic for an $L$-forcing (and which belongs to a set-generic extension of $L[O^\#]$). This result is best possible in the sense that for any countable ordinal $\alpha$ there is an $L$-forcing which has generics but none periodic of period $\leq \alpha$. However, we do not know if an assumption beyond $ZFC$ + "$O^\#$ exists" is actually necessary for these results.

Let $P$ denote a class forcing definable over an amenable ground model $\langle L, A\rangle$ and assume that $O^\#$ exists.

**Definition.** $P$ is *relevant* if $P$ has a generic definable in $L[O^\#]$. $P$ is *almost relevant* if $P$ has a generic definable in a set-generic extension of $L[O^\#]$.

**Remark.** The reverse Easton product of Cohen forcings $2^{<\kappa}$, $\kappa$ regular is relevant. So are the Easton product and the full product, provided $\kappa$ is restricted to the successor cardinals. See Chapter 3, Section Two of Friedman [97]. Of course any set-forcing (in $L$) is almost relevant.

**Definition.** $\kappa$ is $\alpha$-Erdös if whenever $C$ is CUB in $\kappa$ and $f : [C]^{<\omega} \longrightarrow \kappa$ is regressive (i.e., $f(a) < \min(a)$) then $f$ has a homogeneous set of ordertype $\alpha$.

**Definition.** Let $\mathcal{A} = \langle T, \epsilon, \ldots\rangle$ be transitive (in a countable language). $I \subseteq \mathrm{ORD}(T)$ is a *good set of $\Sigma_1$ indiscernibles* for $\mathcal{A}$ if $\gamma \in I \longrightarrow I-\gamma$ is a set of $\Sigma_1$ indiscernibles for $\langle \mathcal{A}, \alpha\rangle_{\alpha<\gamma}$.

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**Fact.** $\kappa$ is $\alpha$-Erdös iff whenever $\mathcal{A} = \langle T, \epsilon, \ldots \rangle$ is transitive (in a countable language), $\kappa \subseteq \mathrm{ORD}(T), C$ CUB in $\kappa$ then there exists $I \subseteq C$, ordertype $(I) = \alpha$ such that $I$ is a good set of $\Sigma_1$ indiscernibles for $\mathcal{A}$.

**Theorem 1.** *Suppose $P$, defined over $\langle L, A \rangle$, has a generic $G$ and there is a good set $X$ of $\Sigma_1$ indiscernibles for $\langle L[O^\#, G], \epsilon, G, A \rangle$ of ordertype $\omega + \omega$ such that $\alpha \in X \longrightarrow \alpha$ is $\Sigma_1 -$stable in $O^\#, G, A$ (i.e., $\langle L_\alpha[O^\#, G], \epsilon, G \cap L_\alpha, A \cap L_\alpha \rangle$ is $\Sigma_1-$elementary in $\langle L[O^\#, G], \epsilon, G, A \rangle$). Then $P$ is almost relevant.*

**Corollary 2.** *Suppose $P$ has a generic and ORD is $\omega + \omega$-Erdös. Then $P$ is almost relevant.*

**Remark.** If $\{\kappa | \kappa$ is $\alpha$-Erdös$\}$ is stationary then it follows that ORD is $\alpha$-Erdös.

The proof of Theorem 1 provides a stronger conclusion which we describe next.

**Definition.** $P$ is *codable* if $P$ has a generic $G$ definable over $L[R]$, $R$ a real generic over $L$, $R \in L[O^\#]$. $P$ is *almost* codable if $P$ has a generic $G$ definable over $L[R], R$ a real generic over $L, R$ in a set-generic extension of $L[O^\#]$.

These notions can be alternatively described in terms of indiscernibility-preservation:

**Definition.** Let $I = \langle i_\gamma | \gamma \in \mathrm{ORD} \rangle$ be the increasing enumeration of the Silver indiscernibles. For any ordinals $\lambda_0, \lambda (\lambda > 0)$ define $I_{\lambda_0, \lambda} = \{i_\alpha | \alpha$ of the form $\lambda_0 + \lambda \cdot \beta, \beta \in \mathrm{ORD}\}$. $P$ is $\lambda_0, \lambda$-*periodic* if there is a $P$-generic $G$ such that $I_{\lambda_0, \lambda}$ is a class of indiscernibles for $\langle L[G], \epsilon, G, A \rangle$. $P$ is *almost* $\lambda_0, \lambda$-*periodic* if it is $\lambda_0, \lambda$-periodic in a set-generic extension of $V$.

**Proposition 3.**

(a) If $A = \emptyset, P$ $L$-definable without parameters then $P$ is codable iff $P$ is almost $\lambda_0, \lambda$-periodic for some $\lambda_0, \lambda$.

(b) $P$ is almost codable iff $P$ is almost $\lambda_0, \lambda$-periodic for some $\lambda_0, \lambda$.

Proof. (a) For the "only if" direction, see Chapter 5, Section Two of Friedman [97]. For the "if" direction, build a tree in $L[O^\#]$, a branch through which produces a real coding a generic witnessing $\lambda_0, \lambda$-periodicity for some countable $\lambda_0, \lambda$. Then this tree has a branch in $L[O^\#]$, proving that $P$ is codable. Part (b) is similar (and does not need the assumption of (a) since for any $A$ there exists some $\lambda_0$ such that $I_{\lambda_o,1}$ is a class of indiscernibles for $\langle L, A\rangle$). $\dashv$

**Remark.** It follows that in Theorem 1 and Corollary 2, if $A = \emptyset, P$ is $L-$definable without parameters then “almost relevant” can be replaced by “relevant”.

The standard examples of relevant class forcing are in fact $0, 1$-periodic.

**Periodicity Conjecture.** If $P$ has a generic then $P$ is almost $\lambda_0, \lambda$-periodic for some countable $\lambda$.

Our proof of Theorem 1 establishes the Periodicity Conjecture, under the extra hypothesis that ORD is $\omega + \omega$-Erdös:

**Theorem 4.** *Suppose $P$ satisfies the hypothesis of Theorem 1. Then $P$ is almost $\lambda_0, \lambda$-periodic for some countable $\lambda$. Thus if ORD is $\omega+\omega$-Erdös then the Periodicity Conjecture is true.*

The Periodicity Conjecture cannot be strengthened.

**Theorem 5.** *Suppose $\alpha, \beta$ are ordinals, $\beta$ countable. Then there is an $L$-forcing $P$ such that $P$ has a generic but $P$ is not almost $\lambda_0, \lambda$-periodic for $\lambda_0 < \alpha$ or for $\lambda < \beta$.*

Proof of Theorem 4. Fix a $P$-generic $G$ as in the hypothesis of Theorem 4; we shall construct another $P$-generic $G^*$ such that for some $\lambda_0$ and countable $\lambda$, $I_{\lambda_0,\lambda}$ is a class of indiscernibles for $\langle L[G^*], \epsilon, G^*, A\rangle$. Let $X$ be a good set of $\Sigma_1$ indiscernibles for $\langle L[O^\#, G], \epsilon, G, A\rangle$ of ordertype $\omega + \omega$ such that $\alpha \in X \longrightarrow \alpha$ is $\Sigma_1-$Stable in $O^\#, G, A$ and $L_\alpha$ contains the parameters defining $P$ in $\langle L, A\rangle$.

Select a canonical enumeration of the $\langle L, A\rangle$-definable open dense subclasses of $P$: Thus let $\langle D_n | n \in \omega\rangle$ be a sequence of predicates where each $D_n(x, \alpha_1 \dots \alpha_n)$ is definable over $\langle L, A\rangle$ such that for each $\alpha_1 < \dots < \alpha_n$ in ORD, $\{x \in L | D_n(x, \alpha_1 \dots \alpha_n)\}$ is an open dense subclass of $P$ and every open dense subclass of $P$ is of this form for some $n$, for some $\alpha_1 < \dots < \alpha_n$ in $I =$(Silver) indiscernibles. We may also assume that $\{\langle n, x, \vec{\alpha}\rangle | D_n(x, \vec{\alpha})\}$ is definable over $\langle L, A\rangle$ relative to a satisfaction predicate for $\langle L, A\rangle$. For $\alpha_1 < \dots < \alpha_n$ in ORD we abuse notation and write $D(\alpha_1 \dots \alpha_n)$ for $\{x \in L | D_n(x, \alpha_1, \dots \alpha_n)\}$. Also let $D^*(\alpha_1 \dots \alpha_n) = \cap\{D(\vec{\beta}) | \vec{\beta} \subseteq \vec{\alpha}\}$.

Now we construct an $\omega$-sequence of terms with Silver indiscernible parameters which we will use to define $G^*$.

For $j_0 \in X$ choose the least $t_{j_0}(\vec{k}_0(j_0), j_0, \vec{k}_1(j_0))$ in $D(j_0) \cap G$, where $t_{j_0}$ is a Skolem term for $L, \vec{k}_0(j_0) < j_0 < \vec{k}_1(j_0)$ is an increasing sequence of Silver indiscernibles. By the good-indiscernibility of $X$, $t_{j_0} = t_0$, $\vec{k}_0(j_0) = \vec{k}_0$ are fixed. Thus we can write $t_0(\vec{k}_0, j_0, \vec{k}_1(j_0)) \in D(j_0) \cap G$ for $j_0 \in X$. By the $\Sigma_1$–stability in $O^\#, G, A$ of the elements of $X$ we have: $j_0 < j_1$ in $X \longrightarrow \vec{k}_1(j_0) < j_1$.

Next for $j_0 < j_1$ in $X$ choose the least $t_{j_0,j_1}(\vec{k}_0^1(j_0,j_1), j_0, \vec{k}_1^1(j_0,j_1), j_1, \vec{k}_2^1(j_0,j_1))$ in $D^*(\vec{k}_0, j_0, \vec{k}_1(j_0), j_1, \vec{k}_1(j_1)) \cap G$. By the good-indiscernibility of $X$ we can write the above term with Silver indiscernible parameters as $t_1(\vec{k}_0^1, j_0, \vec{k}_1^1(j_0), j_1, \vec{k}_2^1(j_0,j_1))$. However, we want to argue that $\vec{k}_2^1(j_0,j_1)$ can be chosen independently of $j_0$. To arrange this, first note that $t_{j_0,j_1}(\vec{k}_0^1(j_0,j_1), j_0, \vec{k}_1^1(j_0,j_1), j_1, \vec{k}_2^1(j_0,j_1)) = t_{j_0,j_1}(\vec{k}_0^1(j_0,j_1), j_0, \vec{k}_1^1(j_0,j_1), j_1, \vec{k}_{2,0}^1(j_0,j_1), \vec{\infty})$ where the latter is independent of the choice of the Silver indiscernibles $\vec{\infty}$ above $\vec{k}_{2,0}^1(j_0,j_1)$ and where $(\vec{k}_0^1(j_0,j_1)$, $\vec{k}_1^1(j_0,j_1), \vec{k}_{2,0}^1(j_0,j_1))$ is the least sequence of *ordinals* such that this term with parameters belongs to $D^*(\vec{k}_0, j_0, \vec{k}_1(j_0), j_1, \vec{k}_1(j_1)) \cap G \cap L_{\min \vec{\infty}}$. By the good-indiscernibility of $X$ we can write this as $t_1(\vec{k}_0^1, j_0, \vec{k}_1^1(j_0), j_1, \vec{k}_{2,0}^1(j_0,j_1), \vec{\infty})$. Note that $(\vec{k}_0^1, \vec{k}_1^1(j_0), \vec{k}_{2,0}^1(j_0,j_1))$ is definable in $\langle L[G], G, A\rangle$ from $\vec{\infty}, \vec{k}_0, j_0, \vec{k}_1(j_0), j_1, \vec{k}_1(j_1)$ and therefore $\vec{k}_{2,0}^1(j_0,j_1)$ is definable in $\langle L[G], G, A\rangle$ from $\vec{\infty}, \vec{k}_1(j_1)$ and parameters $\leq j_1$.

**Lemma 6.** $\vec{k}_{2,0}^1(j_0,j_1)$ is independent of $j_0$.

Proof. Enumerate the first $\omega+1$ elements of $X$ in increasing order as $j_0 < j_1 < \cdots < j = (\omega+1)$st element of $X$ and for any $m, n$ let $\vec{k}(j_n, j)$ $(m)$ denote the $m^{\text{th}}$ element of $\vec{k}_{2,0}^1(j_n, j)$. If the Lemma fails then for some fixed $m, \vec{k}(j_0,j)(m) < \vec{k}(j_1,j)(m) < \ldots$ forms an increasing $\omega$-sequence of Silver indiscernibles with supremum $\ell \in I$. By the remark immediately preceding this Lemma, $\ell$ has cofinality $\leq j$ in $L[G]$. By Covering between $L$ and $L[G], \ell$ has cofinality $< (j^+$ in $L[G])$ in $L$. This contradicts the following.

**Claim.** $j^+$ in $L[G] = j^+$ in $L$.

Proof of Claim. If not then in $L[G]$ there is a CUB $C \subseteq j$ such that $C$ is almost contained in each CUB constructible $D \subseteq j$. But $I \cap j$ is the intersection of countably many such $D$ and therefore as $j$ is regular (in $L[G, O^\#]$) we get that $C$ is almost contained in $I$; so $O^\#$ belongs to $L[G]$, contradiction. This proves the Claim and hence the Lemma. $\dashv$

Thus we can write $t_1(\vec{k}_0^1, j_0, \vec{k}_1^1(j_0), j_1, \vec{k}_2^1(j_1)) \in D^*(\vec{k}_0, j_0, \vec{k}_1(j_0), j_1, \vec{k}_1(j_1)) \cap G$ for $j_0 < j_1$ in $X$. By modifying the term $t_1$ we may assume that $\vec{k}_1^1(j_0) = \vec{k}_2^1(j_0)$ for $j_0 \neq \min(X)$. Also we can assume that $\vec{k}_0 \subseteq \vec{k}_0^1, \vec{k}_1(j_0) \subseteq \vec{k}_1^1(j_0)$ for $j_0 \in X$ and moreover that the structure $\langle \vec{k}_1^1(j_0), < \rangle$ with a unary predicate for $\vec{k}_1(j_0)$ has isomorphism type independent of $j_0 \in X$.

We obtain $t_2$ in a similar way: thus,

$$\begin{aligned}&t_2(\vec{k}_0^2, j_0, \vec{k}_1^2(j_0), j_1, \vec{k}_1^2(j_1), j_2, \vec{k}_1^2(j_2)) \in \\ &D^*(\vec{k}_0^1, j_0, \vec{k}_1^1(j_0), j_1, \vec{k}_1^1(j_1), j_2, \vec{k}_1^1(j_2)) \cap G\end{aligned}$$

for $j_0 < j_1 < j_2$ in $X$ and $\vec{k}_0^1 \subseteq \vec{k}_0^2, \vec{k}_1^1(j_0) \subseteq \vec{k}_1^2(j_0)$, $\langle \vec{k}_1^2(j_0), < \rangle$ with unary predicates for $\vec{k}_1^1(j_0), \vec{k}_1(j_0)$ has isomorphism type independent of $j_0$. Continue in this way to define $t_n(\vec{k}_0^n, j_0, \vec{k}_1^n(j_0), \dots, j_n, \vec{k}_1^n(j_n))$ for each $n$ and for $j_0 < \dots < j_n$ in $X$. (The analogous version of Lemma 6 uses the first $\omega + n$ elements of $X$.)

Let $i_{\lambda_0} = \min X$ and $\lambda = \text{ordertype}\,(\bigcup_n \vec{k}_1^n(j_0))$ for $j_0 \in X$, an ordinal independent of the choice of $j_0$.

We may assume that $\lambda$ is a limit ordinal and in a generic extension where $\lambda_0$ is countable we may arrange that $\bigcup_n \vec{k}_0^n = I \cap i_{\lambda_0}$. Also note that $I - i_{\lambda_0}$ is a class of indiscernibles for $\langle L, A \rangle$. Now in $V[g]$, where $g$ is a Lévy collapse of $i_{\lambda_0}$ to $\omega$, carry out the above construction, arranging that $\bigcup_n \vec{k}_0^n = i_{\lambda_0}$. For any Silver indiscernible $i_\delta$ define $\vec{k}_1^n(i_\delta) \subseteq I \cap (i_\delta, i_{\delta+\lambda})$ so that $\langle I \cap (i_\delta, i_{\delta+\lambda}), < \rangle$ with a predicate for $\vec{k}_1^n(i_\delta)$ is isomorphic to $\left\langle \bigcup_n \vec{k}_1^n(j_0), < \right\rangle$ with a predicate for $\vec{k}_1^n(j_0)$, for $i_{\lambda_0} < j_0 \in X$. Define:

$$\begin{aligned}G^* = \{p \in P | p \text{ is extended by some } &t_n(\vec{k}_0^n, i_{\lambda_1}, \vec{k}_1^n(i_{\lambda_1}), \dots \\ &i_{\lambda_n}, \vec{k}_1^n(i_{\lambda_n})) \text{ where } \lambda_0 \leq \lambda_1 < \dots < \lambda_n \\ &\text{are of the form } \lambda_0 + \lambda \cdot \alpha, \alpha \in \text{ORD}\}.\end{aligned}$$

Using the indisciernibility of $I - i_{\lambda_0}$ in $\langle L, A \rangle$ we see that $G^*$ is compatible and meets every $\langle L, A \rangle$-definable open dense class on $P$. Thus $P$ is $\lambda_0, \lambda$-periodic in $V[g]$. Note that $\lambda$ is countable in $V$. This proves Theorem 4. ⊣

**Remark.** The proof of Theorem 4 only made use of a weaker hypothesis: Define $X$ to be a good set of $\Sigma_1$ *n-indiscernibles* for $\mathcal{A} = \langle T, \epsilon, \dots \rangle$ if $\gamma \in X \longrightarrow X - \gamma$ is $\Sigma_1$ indiscernible for $\mathcal{A}$ *for n-tuples*. Our proof only used the existence of $X_1 \supseteq X_2 \supseteq \dots$ such that each $X_n$ is a good set of $\Sigma_1$ $n$-indiscernibles for $\langle L[O^\#, G], \epsilon, G, A \rangle$ of ordertype at least $\omega + \omega$ such that $\alpha \in X_n \longrightarrow \alpha$ is $\Sigma_1$−stable in $O^{\#}, G, A$. This hypothesis is weaker in terms of consistency strength than the hypothesis stated in Theorem 4.

Proof of Theorem 5. We employ here the techniques of Friedman [90] and Friedman [94]. In the former, an $L$-definable forcing is constructed so as to have a unique generic, which can be considered to be a real. In Friedman [97], Chapter 5, Section Two it is shown that there exist reals $R$ such that $I^R$ = Silver indiscernibles for $L[R]$ is equal to Even $(I) = \{i_{2\alpha} | \alpha \in \text{ORD}\}$. By combining the latter construction with the construction of Friedman [90] one obtains an $L$-definable forcing $Q$ with a unique generic real $R$, such that $I^R =$ Even $(I)$.

Now suppose that $\alpha$ is an $L$-countable ordinal. Define an iterated class forcing as follows: $P_0 = \{0\}$. $P_{\beta+1} = P_\beta * P(\beta)$ where $P(\beta)$ applies the forcing $Q^{R_\beta} = (Q$ relativized to $R_\beta)$ over the model $L[R_\beta]$, where $R_\beta =$ the $P_\beta$-generic real. (Thus if $R_{\beta+1} =$ the $P_{\beta+1}$-generic real we get $I^{R_\beta, R_{\beta+1}} =$ Even $(I^{R_\beta})$.) For limit $\lambda \leq \alpha$ let $P_\lambda =$ Inverse limit $\langle P_\beta | \beta < \lambda \rangle$ and $R_\lambda =$ Join of $\langle R_\beta | \beta < \lambda \rangle$ using the $L$-least counting of $\lambda$.

By Friedman [94], the $P_\beta$'s preserve cofinalities and ZFC. And $P_\beta$-generics exist, using the methods of Friedman [97], Chapter 3, Section Two. The forcing $P_\alpha$ adds a real $R$ such that $I^R = \{i_{2\alpha\gamma} | \gamma \in \text{ORD}\}$ (and has a unique generic). If $\alpha$ is not countable in $L$, first apply a Lévy collapse of $\alpha$ and then perform the above construction to obtain $P_\alpha$. The generic is no longer unique (as the Lévy collapse is not) but it is the case that for any generic real $R$, $I^R = \{i_{2\alpha\gamma} | \gamma \in \text{ORD}\} - (\alpha + 1)$. To prove Theorem 5: Choose $\alpha$ to be the $\beta$ of the statement of that theorem; then a $P_\alpha$-generic exists (as $\alpha$ is countable) and $P_\alpha$ is not almost $\lambda_0, \lambda$-periodic for $\lambda < \beta$. To rule out the case $\lambda_0 < \alpha^* = (\alpha$ of Theorem 5), add a Cohen set to $(\alpha^*)^+$ of $L$, after forcing with $P_\alpha$. This proves Theorem 5. $\dashv$

**Questions.** (a) Is the Periodicity Conjecture provable in the theory ZFC $+O^{\#}$ exists?

(b) Suppose that whenever $P$ is an $L$-forcing with a generic $G$ such that $\langle V[G], G \rangle \models$ ZFC then there is such a $G$ definable in a set-generic extension of $V$. Does $O^{\#}$ exist?

(c) For which $\alpha$ countable in $L[O^{\#}]$ does there exist an $L$-forcing $P$ with a unique generic $G$, such that $\alpha$ is countable in $L[G]$?

## References

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