Completeness and Iteration in Modern Set Theory

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Set theory entered the modern era through the work of Gödel and Cohen. This work provided set-theorists with the necessary tools to analyse a large number of mathematical problems which are unsolvable using only the traditional axiom system ZFC for set theory. Through these methods, together with their subsequent generalisation into the context of large cardinals, set-theorists have had great success in determining the axiomatic strength of a wide range of ZFC-undecidable statements, not only within set theory but also within other areas of mathematics.

Through this work a very attractive picture of the universe of sets is starting to emerge, a picture based upon the existence of inner models satisfying large cardinal axioms. In this article I shall argue for the correctness of this picture, using the principles of *completeness* and *iteration*.

1. Constructibility.

Gödel ([4]) provided an interpretation of ZFC whose structure can be thoroughly analysed. The universe L of *constructible sets* consists of all sets which appear within the hierarchy

$L_0 = \emptyset$
$L_{\alpha+1} =$ The set of definable subsets of L_α
$L_\lambda = \bigcup\{L_\alpha \mid \alpha < \lambda\}$ for limit λ
$L = \bigcup\{L_\alpha \mid \alpha \in \text{ORD}\}$.

This hierarchy differs from the von Neumann hierarchy of V_α's in that only *definable* subsets are considered at successor stages, as opposed to arbitrary subsets. By restricting the power set operation in this way, one brings the notion of set much closer that of ordinal number, and can achieve as clear an understanding of arbitrary sets as one has of ordinal numbers. As an example, one can show that in L the set of reals and the set of countable ordinals have the same cardinality.

Jensen ([5]) went one step further, by dividing the transition from L_α to $L_{\alpha+1}$ into ω intermediate levels $L_\alpha \subseteq L^1_\alpha \subseteq L^2_\alpha \subseteq \cdots \subseteq L_{\alpha+1}$. The power of Jensen's idea, which led to his *fine structure theory for* L, is that these new successor levels L^{n+1}_α consist of sets which can be enumerated in a way analogous to that in which the Σ^0_{n+1}-definable sets of arithmetic can be recursively enumerated using an oracle for the n-th Turing jump $0^{(n)}$. What is perhaps surprising is that this "ramification" of levels can be used to establish new results concerning the structure of the constructible universe as a whole. For example, Jensen shows that the following combinatorial principle holds in L:

The $\square$ *Principle.* To every limit ordinal α that is not a regular cardinal, one can assign an unbounded subset X_α of α with ordertype less than α, such that if $\bar\alpha$ is a limit of elements of X_α then $X_{\bar\alpha} = X_\alpha \cap \bar\alpha$.

Every known proof that this principle holds in L makes use of some version of the fine structure theory. Indeed, Jensen's theory is so powerful that one has the impression that any question in combinatorial set theory (not implying the consistency of ZFC) can be resolved under the assumption $V = L$.

L is the least *inner model*, i.e. transitive class containing all ordinals, in which the axioms of ZFC hold. Can we construct larger inner models which admit a similar Gödel-Jensen analysis?

2. Completeness.

It will be convenient to work now, not with the usual theory ZFC, but with the Gödel-Bernays theory of classes GB. This theory is no stronger than ZFC, but allows us to discuss classes which may not necessarily be definable. For an inner model M, a class A *belongs to* M iff $A \cap x$ belongs to M for every set x in M.

$V = L$ (i.e., the statement that every set is constructible) is not a theorem of GB: The forcing method allows us to consistently enlarge L to models $L[G]$ where G is a set or class that is *P-generic over L* for some *L-forcing P*, i.e., some partial ordering P that belongs to L. Thus it is consistent with GB that there are inner models larger than L.

Assume now that generic extensions of L do exist, and let us see what implications this has for the nature of the set-theoretic universe. For this purpose we introduce the notion of *CUB-completeness*.

Definition. A class of ordinals is CUB (closed and unbounded) iff it is a proper class of ordinals which contains all of its limit points. A class X of ordinals is *large* iff it contains a CUB subclass.

Largeness is not absolute: It is possible that a class X belonging to L is not large but becomes large after expanding the universe by forcing.

Definition. A class X is *potentially large* iff it is large in a generic extension of the universe.

Now we pose the following question: Can the universe be complete with respect to the largeness of classes that belong to L? That is, can the universe be *CUB-complete over* L in the sense that every class which belongs to L and is potentially large is already large? Using the fact that Jensen's $\Box$ Principle holds in L, we have the following.

Theorem 1 ([2]). There exists a sequence X_n, $n \in \omega$ of classes such that:
1. Each X_n belongs to L and indeed the relation "α belongs to X_n" is definable in L.
2. $X_n \supseteq X_{n+1}$ for each n and each X_n is potentially large.
3. If each X_n is large then the universe is CUB-complete over L.

Thus we have the following picture: Let n be least so that X_n is not large, if such a finite n exists, and $n = \infty$ otherwise. If n is finite then n can be increased by going to a generic extension of the universe, further increased by going to a further generic extension, and so on. The only alternative is that the universe be CUB-complete over L.

Can there be many different ways of making the universe CUB-complete over L? The next result says not.

Theorem 2. If the universe is CUB-complete over L then there is a smallest inner model $L^\#$ which is CUB-complete over L.

Thus $L^\#$ is the "canonical" completion of L with respect to largeness of classes that belong to L.

What is $L^\#$? This model is *not* a generic extension of L, but rather a new kind of extension, which can be defined in terms of the concept of *rigidity*: An *embedding of* L is an elementary embedding $\pi : L \to L$ which is not the identity. We say that L is *rigid* iff there is no such embedding.

Fact. The universe is CUB-complete over L iff L is not rigid. If this is the case then $L^\#$ is the smallest inner model to which an embedding of L belongs.

A more explicit description of $L^\#$ is the following: Let $\pi : L \to L$ be an embedding of L and let α be the least ordinal such that $\pi \restriction \alpha$ is not an element of L. Then $L^\# = L[\pi \restriction \alpha]$ for any choice of π.

$L^\#$ is usually written as $L[0^\#]$, where $0^\#$ is a special set of integers, and the hypothesis that L is not rigid is usually written as "$0^\#$ exists".

The hypothesis that $0^\#$ exists not only completes the universe with regard to the largeness of classes that belong to L, but also solves another mystery: It can be shown that P-generics cannot exist simultaneously for all L-forcings P. How can we decide whether or not a P-generic should exist for a given P? The next result leads us to a good criterion, under the assumption that $0^\#$ exists.

Theorem 3 ([1]). Assume slightly more than GB (precisely: ORD is $\omega + \omega$-Erdős). If $0^\#$ exists, P is an L-forcing which is definable in L (without parameters) and there exists a P-generic, then there exists a P-generic definable in $L[0^\#]$.

Thus the inner model $L[0^\#]$ is *saturated* with respect to L-definable forcings. The existence of P-generics for L-definable forcings P is thereby resolved by the assumption that $0^\#$ exists: P has a generic iff it has one definable in $L[0^\#]$.

3. Iteration.

The above discussion leads us to the existence of $0^\#$. We can go further: $0^{\#\#}$ relates to the model $L[0^\#]$ in the same way as $0^\#$ relates to L, and its existence follows from the CUB-completeness of the universe with respect to $L[0^\#]$. Indeed, through iteration of a suitable "# operation", we are led to models much larger than L, which satisfy strong large cardinal axioms.

We have said that the existence of $0^\#$ is equivalent to the non-rigidity of L, i.e., to the existence of an embedding of L. Let us use this as a basis for generalisation. Suppose that M is a non-rigid inner model and let $\pi : M \to M$ be an embedding of M. Let κ be the *critical point of* π, i.e., the least ordinal such that $\pi(\kappa) \neq \kappa$. (For technical reasons we assume that M is of the form L^A for some class of ordinals A and that π *respects* A in the sense that $\pi(A \cap \kappa) = A \cap \pi(\kappa)$, $H^M_\kappa = L^A_\kappa$ and $H^M_{\pi(\kappa)} = L^A_{\pi(\kappa)}$.) For some least ordinal $\alpha = \alpha(\pi)$, the restriction $\pi \restriction \alpha$ is not an element of M. Normally this ordinal is κ^+ of M. We then define the # (or *extender*) *derived from* π to be the restriction $E_\pi = \pi \restriction \kappa^+$ of M. A # *for* M is a # derived from some embedding $\pi : M \to M$. Thus M has a # iff M is non-rigid.

A # *iteration* is a sequence $M_0, M_1, \ldots$ of inner models where

$M_0 = L$
$M_{i+1} = M_i[E_i]$, where E_i is a # for M_i
M_λ for limit λ is the "limit" of $\langle M_i \mid i < \lambda \rangle$.

The type of model that arises through such an iteration is called an *extender model* and is of the form $L[E]$ where $E = \langle E_\alpha \mid \alpha \in \mathrm{ORD} \rangle$ is a sequence of extenders.

How large an extender model can we produce through #-iteration? A #-iteration is *maximal* if it cannot be continued to a larger extender model. An extender model is *maximal* if it is the final model of a maximal #-iteration. Do such models exist, and if so, how large are they?

Theorem 4. There exists a maximal extender model, unless there is an inner model with a superstrong cardinal.

Superstrength is a very strong large cardinal property, more than sufficient to carry out most applications of large cardinals in combinatorial and descriptive set theory. Thus if we are interested in showing that there are inner models which satisfy useful large cardinal axioms, we may without harm assume that there is a maximal extender model.

Now we turn to the question of largeness for maximal extender models. There are two ways in which an extender model M can be maximal: Either there is no # for M, i.e. M is rigid, or M is not enlarged through the addition of a # for itself. The latter possibility also leads to an inner model with a superstrong cardinal.

Theorem 5. Suppose that M is a non-rigid, maximal extender model. Then in M there is a superstrong cardinal.

Thus to obtain an inner model with a superstrong cardinal, it suffices to build a maximal extender model and then argue that it is not rigid.

But first we must address a central problem in the theory of extender models: How can we ensure that our maximal extender models satisfy the Gödel and Jensen properties, GCH and $\Box$? Define an extender model to be *good* iff it satisfies GCH and $\Box$. The construction of *good* maximal extender models is far more difficult than the construction of arbitrary maximal extender models. However using work of Steel ([7]) and Schimmerling-Zeman ([6]) we do have:

Theorem 6. Assume slightly more than GB (precisely: ORD is subtle). Then there is a *good* maximal extender model, unless there is an inner model with a Woodin cardinal.

Thus if we can argue for the non-rigidity of good maximal extender models, we obtain an inner model with a Woodin cardinal, a property still strong enough to carry out many applications of large cardinals to combinatorial and descriptive set theory.

4. Completeness, again.

Using CUB-completeness we argued that L is non-rigid. Can this argument be generalised to good maximal extender models?

Theorem 7. Suppose that the universe is CUB-complete over a good maximal extender model. Then there is an inner model with a measurable cardinal.

Measurable cardinals are much stronger than $0^{\#}$, but still far weaker than Woodin cardinals. To go further, we need to consider a variant of CUB-completeness.

Definition. A class of ordinals C is CUB$^+$ iff C is CUB and for each limit cardinal α in C, $C \cap \alpha^+$ is CUB in α^+. X is large$^+$ iff X contains a CUB$^+$ subclass. X is *potentially large*$^+$ iff X is large$^+$ in a generic extension of the universe.

Now we repeat what we did earlier for L, with CUB-completeness replaced by CUB$^+$-completeness.

Theorem 8. Suppose that M is a good maximal extender model. Then there exists a sequence X_n, $n \in \omega$ of classes such that:
1. Each X_n belongs to M.
2. $X_n \supseteq X_{n+1}$ for each n and each X_n is potentially large$^+$.
3. If each X_n is large$^+$ then for some CUB class C: (α^+ of M) $< \alpha^+$ for α in C.

Thus if the universe is CUB$^+$-complete over a good maximal extender model M, it follows that α^+ of M is less than α^+ for α belonging to a CUB class. Now we apply the following refinement of Theorem 6 (see [7]).

Theorem 9. Assume slightly more than GB (again: ORD is subtle). Unless there is an inner model with a Woodin cardinal, there is a good maximal extender model M with the following property: For no CUB class C is (α^+ of M) $< \alpha^+$ for α in C.

Putting this all together:

Theorem 10. Assume slightly more than GB (again: ORD is subtle). Suppose that the universe is CUB$^+$-complete with respect to good maximal extender models. Then there is an inner model with a Woodin cardinal.

In the sense of Theorem 10, completeness and iteration can be used to argue in favour of the existence of inner models with large cardinals.

5. Speculations beyond One Woodin Cardinal

Further work of Andretta, Jensen, Neeman and Steel suggests that a good maximal extender model should exist, unless there is an inner model with many Woodin cardinals. But there have been serious obstacles to extending this work up to the level of a superstrong cardinal.

Perhaps the difficulties come from definability. The good extender models M that have been constructed until now satisfy:

($*$) M is a definable inner model (in the language of class theory).

Property ($*$) is not used in our above discussion of CUB- and CUB$^+$-completeness, nor in many applications of inner models, and may have to be sacrificed if one is to reach the level of a superstrong cardinal.

One way to approach this question is to ask: Assume that a superstrong cardinal exists. What kinds of inner models with a superstrong cardinal can one construct?

Conjecture. Suppose that there is a superstrong cardinal. Then:
(a) There is an inner model of GCH which has a superstrong cardinal.
(b) There need not be a definable such inner model.

If true, this conjecture implies that one should look not for "canonical" inner models for large cardinals, but rather a family of inner models, any of which could serve as a good approximation to the universe of all sets.

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