Answer to a Question of Wayne Richter

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Wayne Richter posed the following question: Suppose that α is least so that for some $\kappa < \alpha$, L_κ and L_α have the same Σ_n theory. Then is L_κ a Σ_n elementary submodel of L_α?

The answer is Yes. We give the proof here for the case $n = 2$.

Say that $x \in L_\alpha$ is a Σ_2 *singleton* if $\{x\}$ is Σ_2 definable in L_α. We shall show that $H = \{x \in L_\alpha \mid x \text{ is a } \Sigma_2 \text{ singleton}\}$ is a Σ_2 elementary submodel of L_α. Given this claim, we prove the desired result as follows: $L_\alpha \vDash \omega_1$ does not exist, due to the leastness of α. Therefore H is transitive and must equal either L_κ or L_α. But if $H = L_\alpha$ then κ is a Σ_2 singleton and therefore the Σ_2 theory of $L_\alpha =$ the Σ_2 theory of L_κ is Π_2 definable over L_α, a contradiction. So $H = L_\kappa$ and therefore L_κ is Σ_2 elementary in L_α.

Now we prove that H is Σ_2 elementary in L_α. First we show that if A is nonempty and Σ_2 definable over L_α then A has an element in H. Let β be least so that some element of A belongs to the Σ_1 hull (in L_α) of $\beta + 1$ ($=$ Σ_1 hull of $\{\beta\}$, since $L_\alpha \vDash \omega_1$ does not exist) and either β is Σ_1 stable (i.e., L_β is Σ_1 elementary in L_α) or $\beta = 0$. Such a β exists since every element of L_α belongs to the Σ_1 hull of $\beta + 1$ for such a β.

If $\beta = \kappa$ then there cannot be a Σ_1 stable greater than κ (as any such would have the same Σ_2 theory as L_α, contradicting the leastness of α) and therefore κ is the unique β such that β is Σ_1 stable and A contains an element of the Σ_1 hull of $\beta + 1$. So in this case κ is a Σ_2 singleton and therefore so is each element of the Σ_1 hull of $\{\kappa\}$ = the Σ_1 hull of $\kappa + 1$, including some element of A.

If $\beta \neq \kappa$ then let β^* be the least Σ_1 stable greater than β, if it exists, and α otherwise. Choose a Σ_2 sentence φ which is true in L_{β^*} but false in L_β, using the leastness of α if $\beta^* < \alpha$ and the fact that $\beta \neq \kappa$ if $\beta^* = \alpha$. Then

β is the unique Σ_1 stable such that $L_\beta \models \sim \varphi$ and A contains an element of the Σ_1 hull of $\beta + 1 =$ the Σ_1 hull of $\{\beta\}$. So again β is a Σ_2 singleton and therefore so is each element of the Σ_1 hull of $\{\beta\}$, including some element of A.

Thus we have shown that H contains an element of each nonempty Σ_2 definable subset of L_α. If A is Σ_2 definable over L_α with parameters $x_1, \ldots, x_n \in H$, then H must contain $(x, x_1, \ldots, x_n)$ for some $x \in A$ and since the x_i's are Σ_2 singletons, H must also contain x. So we have shown that H is Σ_2 elementary in L_α, as desired.

The above proof generalises to arbitrary n, using the fact that the relation $x \in \Sigma_{n-1}$ hull of $\{\beta\}$ is Σ_n definable.

Remarks. (a) Regarding the case $n = 2$: κ must be the largest Σ_1 stable, as any larger Σ_1 stable would yield the same Σ_2 theory as α, in contradiction to the leastness of α. And α is not Σ_1 admissible, as otherwise there would be an $\bar{\alpha} < \alpha$ such that $L_{\bar{\alpha}}$ satisfies all Π_2 sentences true in L_α (since this set of sentences belongs to L_α); but then $L_{\bar{\alpha}}$, L_α have the same Σ_2 theory, again contradicting the leastness of α. So in fact there is a bijection between ω and α which is Σ_1 definable over L_α with parameter κ. (b) A similar proof shows: Suppose that k, n are positive and let α be least so that there are k ordinals $\beta < \alpha$ such that L_β has the same Σ_n theory as L_α. Then L_β is Σ_n elementary in L_α for such β. There is also a version for infinite k.