THE JOURNAL OF SYMBOLIC LOGIC
Volume 48, Number 3, Sept. 1983

SOME RECENT DEVELOPMENTS IN HIGHER RECURSION THEORY[1]

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Abstract. In recent years higher recursion theory has experienced a deep interaction with other areas of logic, particularly set theory (fine structure, forcing, and combinatorics) and infinitary model theory. In this paper we wish to illustrate this interaction by surveying the progress that has been made in two areas: the global theory of the κ-degrees and the study of closure ordinals.

§1. Global theory of the κ-degrees.

A. Basics. Degree theory on ω has a natural generalization to any limit ordinal. We confine ourselves here to cardinals only. We also assume $V = L$ (although GCH suffices for most of what follows). If κ is a cardinal and $A, B \subseteq L_\kappa$ we write $A \leq_\kappa B$ (A is *κ-recursive in* B) if for any $C \subseteq L_\kappa$

$$C \Sigma_1 \text{ over } \langle L_\kappa, A\rangle \to C \Sigma_1 \text{ over } \langle L_\kappa, B\rangle.$$

This relation is transitive. When $\kappa = \omega$, $A, B \subseteq \omega$ then this definition reduces to Turing reducibility. If for any $A \subseteq L_\kappa$ we define $A^* = \{\langle x, y\rangle \mid x, y \in L_\kappa \text{ and } x \subseteq A, y \cap A = \varnothing\}$ then it is easy to see that A^* is $\Sigma_1\langle L_\kappa, A\rangle$. In fact:

$$A \leq_\kappa B \text{ if and only if } A^* \text{ is } \Sigma_1\langle L_\kappa, B\rangle.$$

Intuitively: $A \leq_\kappa B$ if the question of whether a member of L_κ (a *κ-finite* set) is a subset of or is disjoint from A can be answered in a κ-RE in B way.

$A \subseteq L_\kappa$ is *κ-RE in* $B \subseteq L_\kappa$ if A is $\Sigma_1\langle L_\kappa, B\rangle$. The existence of a universal Σ_1 predicate for $\langle L_\kappa, B\rangle$ implies the existence of a universal κ-RE in B set B' (*κ-jump* of B). Thus B' is κ-RE in B and whenever A is κ-RE in B there is $e \in L_\kappa$ such that

$$x \in A \leftrightarrow \langle e, x\rangle \in B', \text{ for all } x \in L_\kappa.$$

An interesting fact in κ-recursion theory is that A κ-RE in B, $(L_\kappa - A)$ κ-RE in $B \nrightarrow A \leq_\kappa B$ (in general).

Finally we discuss the κ-degrees. The relation on $A, B \subseteq L_\kappa$ given by ($A \leq_\kappa B$ and $B \leq_\kappa A$) is an equivalence relation. The κ-degrees are the equivalence classes of this equivalence relation. The κ-degree of A, κ-deg(A), is the equivalence class

Received July 14, 1981.

[1]This paper accompanies an invited lecture given by the author at the Spring 1981 Meeting of the ASL in Milwaukee, Wisconsin. The preparation of this paper was supported by NSF Grant #MCS 7906084.

0022-4812/83/4803-0013/$02.40

to which A belongs. If d_1, d_2 are κ-degrees then we write $d_1 \leq d_2$ if $\forall A \in d_1 \forall B \in d_2(A \leq_\kappa B)$. If $d = \kappa\text{-deg}(A)$ then $d' = \kappa\text{-deg}(A')$. 0 denotes κ-deg($\emptyset$) and therefore $0'$ denotes the largest κ-degree of a κ-RE (in $\emptyset$) set.

Just as in the case $\kappa = \omega$ (see Simpson [77]) there are two types of κ-degree theory for $\kappa > \omega$: local and global. In local κ-degree theory the emphasis is on the κ-degrees $\leq 0'$. The techniques used are generalizations of the priority method. Work of Sacks-Simpson, Lerman and Shore has shown that these degrees have a rich structure. See Shore [77] for a survey of these results.

Global κ-degree theory is concerned with the structure of the κ-degrees as a whole, emphasizing results which relativize to an arbitrary κ-degree. The methods here arise from combinatorial set theory. We shall discuss global κ-degree theory by considering the following two properties:

$$(*) \qquad \forall d \exists d_0 > d \exists d_1 > d (d_0, d_1 \text{ are } \leq\text{-incomparable}),$$

$$(**) \qquad \forall d \forall d_0 > d \exists d_1 > d (d_0, d_1 \text{ are } \leq\text{-incomparable}).$$

We will show that these properties do hold in the κ-degrees when κ is regular, but not if κ is singular of uncountable cofinality. (*) holds in case κ is singular of cofinality ω, but the situation regarding (**) is unknown. (We conjecture that (**) fails in this case.)

We note that the above division into three cases is reminiscent of the singular cardinals problem. Indeed, it was Silver's solution to the singular cardinals problem at uncountable cofinalities (Silver [74]) that led to the first results in global κ-degree theory for singular κ.

B. κ-Regular. We can establish (**) in this case by a simple application of Cohen forcing over L_κ.

THEOREM 1. *(**) holds if κ is regular.*

PROOF. We fix D, $D_0 \subseteq L_\kappa$, $D \leq_\kappa D_0$, $D_0 \nleq_\kappa D$. Our goal is to produce $D_1 \subseteq L_\kappa$, $D \leq_\kappa D_1$, D_0 and D_1 $\leq$ κ-incomparable. Let $\mathscr{P} = 2^{<\kappa}$ = conditions of size $<\kappa$ for adding a new subset of κ. Introduce a forcing language for adding a set $\mathscr{P}$-generic over $\langle L_\kappa, D_0\rangle$. There exists $G \subseteq \kappa$ which is $\mathscr{P}$-generic over $\langle L_\kappa, D_0\rangle$ since we are only concerned with intersecting κ-many dense open subsets of $\mathscr{P}$, and $\mathscr{P}$ is $<\kappa$-closed.

CLAIM. (a) $G \nleq_\kappa D_0$. (b) $D_0 \nleq_\kappa G \vee D$.

PROOF OF CLAIM. (a) Let $\mathscr{D}_A = \{p \in \mathscr{P} \mid \text{for some } x \in A,\ p \Vdash x \notin G \text{ or for some } x \notin A,\ p \Vdash x \in G\}$, for any $A \subseteq L_\kappa$. Cleary $\mathscr{D}_A$ is dense. Thus $G \cap \mathscr{D}_A \neq \emptyset$ if A is definable over $\langle L_\kappa, D_0\rangle$. This proves that G is not definable over $\langle L_\kappa, D_0\rangle$.

(b) As $D_0 \nleq_\kappa D$ it must be that either $\{x \in L_\kappa | x \subseteq D_0\}$ or $\{x \in L_\kappa \mid x \cap D_0 = \emptyset\}$ is *not* κ-RE in D. Without loss of generality, assume the former. Now suppose $p \in \mathscr{P}$ and $\phi(x, D, \underset{\sim}{G})$ is a Σ_1^p formula. Then $\{x \in L_\kappa | \exists q \leq p\ q \Vdash \phi(x, D, \underset{\sim}{G})\}$ is κ-RE in D so cannot equal $\{x \in L_\kappa \mid x \subseteq D_0\}$. Thus $\{p \in \mathscr{P} \mid \text{For some } x \subseteq D_0\ p \Vdash {\sim}\phi(x, D, \underset{\sim}{G}) \text{ or for some } x \nsubseteq D_0\ p \Vdash \phi(x, D, \underset{\sim}{G})\}$ is dense. So D_0 is not κ-recursive in $G \vee D$, since ϕ is arbitrary. $\dashv_{\text{Claim}}$.

Finally we set $D_1 = G \vee D$. This proves (**). $\dashv$

The above argument does not use the hypothesis $V = L$. The regularity of κ is used to obtain $\mathscr{P}$-generic sets.

C. κ singular of uncountable cofinality. In this case (*) fails in a strong sense. The following result is contained in Friedman [81B] in the case $\kappa = \aleph_{\omega_1}$:

THEOREM 2. *There is a κ-degree d such that the κ-degrees $\geq$ d are well-ordered with successor given by κ-jump.*

We describe here the main idea in the proof. Choose a closed unbounded $D \subseteq \kappa$ consisting of cardinals, ordertype$(D) = \mathrm{cf}(\kappa) < \kappa$. Let $d = \kappa$-deg(D). We show that the κ-degrees $\geq d$ are well-ordered by $\leq_\kappa$.

The κ-degree of $A \geq_\kappa D$ is determined by the growth rate of its "cutoff" function, f_A. For $x \in L_\kappa$ let $|x| = <_L$-rank of x, where $<_L$ is the canonical Δ_1 well-ordering of L. Also let $\kappa_\gamma = \gamma$th member of D. Then f_A: $\mathrm{cf}(\kappa) \to \kappa$ is defined by:

$$f_A(\gamma) = |A \cap \kappa_\gamma|.$$

Note that $f_A(\gamma) < \kappa_\gamma^+$ for all γ and that A is determined by $f_A \restriction X$ for any unbounded $X \subseteq \mathrm{cf}(\kappa)$.

MAIN LEMMA. *If $f_A(\gamma) \leq f_B(\gamma)$ for stationary many $\gamma < \mathrm{cf}(\kappa)$ then $A \leq_\kappa B \vee D$.*

We can now establish that the κ-degrees $\geq d$ are well-ordered using this lemma. Indeed, suppose $A \geq_\kappa D$, $B \geq_\kappa D$. Then either $f_A(\gamma) \leq f_B(\gamma)$ for stationary many γ or $f_A(\gamma) > f_B(\gamma)$ for a closed unbounded set of γ's. The lemma implies that either $A \leq_\kappa B$ or $B \leq_\kappa A$. Moreover, suppose $A_0 >_\kappa A_1 >_\kappa A_2 >_\kappa \cdots$ and each $A_i \geq_\kappa D$. Then there are closed unbounded sets $C_0, C_1, \ldots$ so that

$$f_{A_i}(\gamma) > f_{A_{i+1}}(\gamma) \text{ for } \gamma \in C_i.$$

But $\bigcap_i C_i \neq \varnothing$ so we have $f_{A_0}(\gamma) > f_{A_1}(\gamma) > \cdots$ for some γ. This contradicts the well-foundedness of ORD.

The conclusion of the Main Lemma is established by constructing a function g: $\kappa \to \kappa$, $g \leq_\kappa B \vee D$, so that $g \circ f_A$ has bounded range on some unbounded $X \subseteq \mathrm{cf}(\kappa)$. The equation $f_A(\gamma) = g^{-1} \circ (g \circ f_A(\gamma))$ then shows that $f_A \restriction X \leq_\kappa B \vee D$ (as $(g \circ f_A) \restriction X \in L_\kappa$). But as was indicated earlier, $A \leq_\kappa f_A \restriction X$ for any unbounded $X \subseteq \mathrm{cf}(\kappa)$.

The combinatorial ingredient needed to construct g is Fodor's theorem. This type of argument originated in combinatorial proofs of Silver's theorem (the GCH holds at a singular cardinal of uncountable cofinality provided it holds below it). In fact Silver's theorem can be derived as a corollary of Theorem 2, as all that is needed for its proof is the assumption of the GCH at stationary many cardinals less than κ.

A refinement of the Main Lemma proves the statement in Theorem 2 concerning the κ-jump. The full version of Theorem 2 in Friedman [81B] also provides canonical representatives of the κ-degrees $\geq d$ in terms of Jensen's master codes, assuming $V = L$. (More on this in subsection E below.) Moreover there is a natural choice of d (assuming only GCH): $d =$ least κ-degree which contains an unbounded $D \subseteq \kappa$ of ordertype $< \kappa$.

D. κ singular of cofinality ω. The techniques of subsection C cannot be used here, as the combinatorics of stationary sets do not apply to ω. In fact, an argument due to Harrington and Solovay (independently) shows that (*) holds in this case. We give here an indication of Solovay's argument, which is based on the combinatorial notion of scale (his argument will appear in Friedman [82]).

Define D and f_A as in C. Thus D has ordertype ω and $f_A(n) < \kappa_n^+$ for each n, where $\kappa_n = n$th member of D.

DEFINITION. A *quasi-scale* is a sequence $\langle A_\alpha \mid \alpha < \kappa^+\rangle$ of subsets of κ such that:

(a) $\alpha_1 < \alpha_2 \to f_{A_{\alpha_2}}$ eventually dominates $f_{A_{\alpha_1}}$.

(b) For any $g\colon \omega \to \kappa$, $g(n) < \kappa_n^+$ for all n, there is $\alpha < \kappa^+$ s.t. f_{A_α} eventually dominates g.

Quasi-scales are easily constructed.

To any $A \subseteq \kappa$ Solovay associates a function $h_A\colon \omega \to \kappa$ such that $h_A(n) < \kappa_n^+$ for all n and whenever $B \leq_\kappa A \vee D$, h_A eventually dominates f_B. Thus the strategy for obtaining $\leq_\kappa$-incomparable $A \geq_\kappa D$, $B \geq_\kappa D$ is to guarantee that f_A dominates h_B infinitely often, and f_B dominates h_A infinitely often.

It is not hard to see how quasi-scales can be used to do this. First, there must be $n_0 \in \omega$ so that for all $\gamma < \kappa_{n_0}^+$ there are unboundedly many $\alpha < \kappa^+$ so that $f_{A_\alpha}(n_0) > \gamma$. Otherwise for each n let $g(n) < \kappa_n^+$ be a counterexample. Then for at most κ-many α can we have $f_{A_\alpha}(n) > {}^{r}g(n)$ for some n. This contradicts (b) in the definition of quasi-scale.

We can choose an unbounded $X_0 \subseteq \kappa^+$ and $\gamma_0 < \kappa_{n_0}^+$ so that $\alpha \in X_0 \to h_A(n_0) = \gamma_0$ (by regularity of κ^+). Then choose an unbounded $Y_0 \subseteq \kappa^+$ so that $\alpha \in Y_0 \to f_{A_\alpha}(n_0) > \gamma_0$. Thus we have guaranteed that $A \in \{A_\alpha | \alpha \in Y_0\}$, $B \in \{A_\alpha | \alpha \in X_0\}$ implies that $f_A(n_0) > h_B(n_0)$. Now we can iterate this procedure to produce unbounded $Y_1 \subseteq Y_0$, $X_1 \subseteq X_0$ and $n_1 > n_0$ so that $A \in \{A_\alpha \mid \alpha \in Y_1\}$, $B \in \{A_\alpha \mid \alpha \in X_1\}$ $\to h_A(n_1) < f_B(n_1)$. Continue with $Y_2, Y_3, \ldots, X_2, X_3, \ldots, n_2, n_3, \ldots$. If $h_A(n_{2\kappa+1}) = h_{A_\alpha}(n_{2\kappa+1})$ for $\alpha \in Y_{2\kappa+1}$, $f_A(n_{2\kappa}) = f_{A_\alpha}(n_{2\kappa})$ for $\alpha \in Y_{2\kappa}$ and if $h_B(n_{2\kappa}) = h_{A_\alpha}(n_{2\kappa})$ for $\alpha \in X_{2\kappa}$, $f_B(n_{2\kappa+1}) = f_{A_\alpha}(n_{2\kappa+1})$ for $\alpha \in X_{2\kappa+1}$, then f_A dominates h_B infinitely often, f_B dominates h_A infinitely often, so $A \vee D$, $B \vee D$ are $\leq_\kappa$-incomparable. Moreover h_A is defined so that for any infinite $S \subseteq \omega$, the limit of functions of the form $h_A \upharpoonright S$ is also of the form $h_A \upharpoonright S$ (similarly for f_A). Thus we can obtain A, B meeting the above hypotheses. The above construction easily relativizes to any $E \geq_\kappa D$. So we have established (*).

Harrington's proof of this result is based on the Σ_1-compactness of the fragment of $\mathscr{L}_{\infty\omega}$ based on $L_\alpha(D)$, where α = least D-admissible ordinal. (This compactness theorem is considered in more detail in §2C.) Harrington's proof makes it clear that A, B may be chosen to be Δ_2 over $L_\alpha(D)$. By relativization one obtains:

THEOREM 3 (HARRINGTON, SOLOVAY). *Suppose d is a κ-degree containing an unbounded $D \subseteq \kappa$ of ordertype $<\kappa$. Then there exist $\leq_\kappa$-incomparable $D_0 \geq_\kappa D$, $D_1 \geq_\kappa D$ which are Δ_2 over $L_\alpha(D)$, α = least D-admissible ordinal.*

The status of (**) in this case is open. We conjecture that it fails. A verification of this conjecture would show that the three cases we have considered differ according to the status of (*) and (**).

E. A canonical jump hierarchy through the κ-degrees. Jensen's master codes provide a natural way to iterate the κ-jump for $(\kappa^+)^L$ steps, for any cardinal κ. If $V = L$ then this provides a *cofinal* jump hierarchy through the κ-degrees. When $\kappa = \omega$ this was discussed in Simpson [80].

For simplicity of notation assume $V = L$. The subsets of κ appear at various ordinal stages $\alpha < \kappa^+$ in the usual L-hierarchy. If a new subset of κ appears at level α (that is, it belongs to $L_{\alpha+1} - L_\alpha$) then it must belong to $\Delta_n(L_\alpha) - L_\alpha$ for some n. A Δ_n master code of α is a member of $\Delta_n(L_\alpha) - L_\alpha$ which "encodes" all subsets of κ in $\Delta_n(L_\alpha) - L_\alpha$.

DEFINITION. $D \subseteq \kappa$ is a Δ_n *master code* for α if for all $A \subseteq \kappa$

$$A \text{ is } \Delta_n(L_\alpha) \leftrightarrow A \text{ is } \Delta_1\langle L_\kappa, D\rangle.$$

In case $\Delta_n(L_\alpha) - L_\alpha$ contains a subset of κ, one would like to select a Δ_n master code for α as a canonical representative. Unfortunately Δ_n master codes do not necessarily exist in this case. (In the special case $\kappa = \omega$ this problem does not arise.) A more complete discussion can be found in Friedman [81B].

Notice that any two Δ_n master codes for α have the same κ-degree. Moreover, if D is a Δ_n master code for α then $D' = \kappa$-jump (D) is a Δ_{n+1} master code for α. Thus a cofinal jump hierarchy through the κ-degrees is obtained by listing the κ-degrees of master codes (that is, Δ_n master codes for α for some $n < \omega$, some $\alpha < \kappa^+$) in their natural order: if D_1 is a Δ_{n_1} master code for α_1 and D_2 is a Δ_{n_2} master code for α_2 then $D_1 \leq_\kappa D_2$ iff $\alpha_1 < \alpha_2$ or ($\alpha_1 = \alpha_2$ and $n_1 \leq n_2$). We let 0^γ denote the γth κ-degree in this sequence. This definition relativizes, thus providing a meaning for d^γ for any κ-degree d.

The question arises: is this the correct jump hierarchy? Or equivalently: for γ limit is 0^γ the least natural upper bound for 0^α, $\alpha < \gamma$? When $\kappa = \omega$, Hodes [80] provides strong evidence that the answer is "yes". When $\kappa = \aleph_{\omega_1}$ the following result definitively settles the question.

THEOREM 4 ($\kappa = \aleph_{\omega_1}$). *For γ limit 0^γ = least upper bound $\{0^\alpha \mid \alpha < \gamma\}$. Thus the κ-degrees $\geq 0'$ are exactly the degrees 0^γ, $\gamma \geq 1$.*

. This is the main result of Friedman [81B]. A thorough study of the master codes is required. There is a version for any singular cardinal κ of uncountable cofinality: If d is the least κ-degree with the property of Theorem 2 then any κ-degree $\geq d$ is of the form d^γ for some γ.

It would be interesting to obtain a positive answer to our question in the remaining cases (κ regular, uncountable and κ singular of cofinality ω). There are complexities arising in these cases which are not present when $\kappa = \omega$, due to the fact that ordinals between κ and κ^+ can have different cofinalities when $\kappa > \omega$. Hodes [80] uses Steel forcing. A generalization of Steel forcing to uncountable regular cardinals appears in Friedman [81E]. This may allow the adaptation of Hodes' arguments to the uncountable regular case. The appropriate tool when κ is singular of cofinality ω would be Barwise compactness, as developed in Friedman [81D].

§2. Closure ordinals.

A. Basics. Let T be a set theory, such as KP, ZF$^-$, Z, ZF. For any set x we can define the *T-closure ordinal* of x, $\alpha_T(x)$, as the least ordinal α such that some transitive model of T contains x (as a member) and has ordinal height α. Given a cardinal κ and a theory T we are particularly interested in characterizing the T-closure ordinals of subsets of κ.

We shall focus on the theories $\mathrm{KP}_n = \Sigma_n$-admissibility and ZF. Notice that in this case it is possible to provide a clearer definition of $\alpha_T(x)$: it is the least α such that $L_\alpha(x) \models T$.

If $\kappa = \omega$ then the T-closure ordinals of subsets of κ were characterized in Sacks [76] for the case $T = \mathrm{KP}_n$ and in David [81][2] for the case $T = \mathrm{ZF}$. The case where $\kappa > \omega$ (and $V = L$) is treated in Friedman [81C, D] when $T = \mathrm{KP}_n$ for regular κ and when $T = \mathrm{KP}_n$ or ZF for singular κ. A characterization of the ordinals of the form $\alpha_{\mathrm{ZF}}(x)$, $x \subseteq \kappa$, is not yet known for regular $\kappa > \omega$.[3]

Sacks uses his method of pointed perfect forcing, while David's proof is based on a very sophisticated use of almost disjoint set forcing from Jensen [75]. Friedman [81C, D] uses almost disjoint set forcing in the regular case, Barwise compactness in the singular cofinality ω case and Theorem 4 above in the singular uncountable cofinality case.

Our goal in subsection B is to describe the various forcing methods used. In addition to those mentioned above, Steel [78] developed a very elegant forcing method which can be used to give a simpler proof of Sacks' characterization of the $\alpha_{\mathrm{KP}}(x)$, $x \subseteq \omega$. Friedman [81E] adapted Steel's method to uncountable regular cardinals, simplifying the proof of the characterization in Friedman [81C] of the $\alpha_{\mathrm{KP}}(x)$, $x \subseteq \kappa$, when κ is regular.

In C we consider the way in which model-theoretic ideas apply to the study of closure ordinals. The first application of this kind can be found in H. Friedman-Jensen [68] where Barwise's compactness theorem is used to prove Sacks' characterization of the $\alpha_{\mathrm{KP}}(x)$, $x \subseteq \omega$. Later Grilliot and Simpson found an even simpler argument (see Keisler [71], page 58). A proof of the same result based on Barwise's study of $\mathrm{HYP}(\mathscr{M})$ (see Barwise [75]) can be found in Friedman [81A]. This proof generalizes to all regular κ (see Friedman [81E]). And, the Grilliot-Simpson proof is adapted to singular cardinals of cofinality ω in Friedman [81D].

B. Forcing methods. We begin with a description of Sacks' proof of:

THEOREM 5 (SACKS [76]). *If $\alpha > \omega$ is a countable admissible ordinal then for some $R \subseteq \omega$:*

(a) *α is the least R-admissible ordinal (i.e., $\alpha_{\mathrm{KP}}(R) = \alpha$).*

(b) *If $S \leq_h R$ then either $R \leq_h S$ or the least S-admissible ordinal is less than α.*

In (b), $\leq_h$ refers to hyperarithmetic reducibility.

Of course Theorem 5(a) solves the closure ordinal problem for KP in the countable case. The statement (b) is a bonus obtained from the special type of forcing Sacks uses.

It is easiest to describe Sacks' proof of Theorem 5 if we assume $L_\alpha \models$ every set is countable (α is *locally countable*). The desired $R \subseteq \omega$ is produced by hyperarithmetically-pointed perfect set forcing over L_α. A condition in this forcing is a perfect tree $T \subseteq 2^{<\omega}$ = all finite strings of 0's and 1's such that $T \in L_\alpha$ and T is hyperarithmetic in any infinite branch f through T. A condition T_1 is stronger than a condition T_2 if $T_1 \subseteq T_2$. If G is generic then $\bigcap G$ is an infinite branch f_G through $2^{<\omega}$.

[2]David's result was proved independently by Beller; see A. Beller, R. Jensen and P. Welch, ***Coding the universe,*** Cambridge University Press, London, 1982.

[3]There is now such a characterization: see R. David and S. Friedman, *Uncountable ZF-ordinals*, ***Recursion theory*** (American Mathematical Society Summer Institute, Ithaca, New York, 1982), Proceedings of Symposia in Pure Mathematics, American Mathematical Society, Providence, R.I. (to appear).

Given any $S \in L_\alpha$, $S \subseteq \omega$, there is such a condition T such that $S \leq_h T$. Thus if G is generic over L_α then $S \leq_h f_G$ for any $S \in L_\alpha$. So hyperarithmetically pointed perfect set forcing over L_α produces a hyperarithmetic upper bound for the reals in L_α. The assumption that α is locally countable guarantees that any ordinal $\beta < \alpha$ is the ordertype of a well-ordering of ω which belongs to L_α. Therefore no ordinal less than α can be admissible relative to f_G, whenever G is generic over L_α (as all reals hyperarithmetic in f_G belong to the least admissible set containing f_G).

Sacks shows that this forcing also preserves the admissibility of α, thus establishing Theorem 5(a). His key lemma for accomplishing this is the *fusion lemma* which provides a weak form of countable closure. This allows one to bound any Σ_1-function in the generic extension by one in L_α, thereby establishing the admissibility of α relative to any generic real f_G.

Theorem 5(b) in the locally countable case comes about due to the fact that the conditions used are branching. This enables Sacks to show: If $\tau(f_G)$ is a term denoting a real in the generic extension and T is a condition, then T can be strengthened to $T' \subseteq T$ so that either $\tau(f)$ is the same for all infinite branches f through T' or $\tau(f)$ uniquely determines f (as an infinite branch through T'). A consequence of this is the fact that if G is generic and $g \leq_h f_G$ then either $g \in L_\alpha$ or $f_G \leq_h g \vee f$ for some $f \in L_\alpha$. Now if $\alpha = \alpha_{\mathrm{KP}}(f)$ for some $f \in L_\alpha$ then Theorem 5(b) is easy. Otherwise no g of lower hyperdegree than f_G can obey $\alpha_{\mathrm{KP}}(g) = \alpha$ (as this implies $f_G \leq_h g \vee f$ for some $f \in L_\alpha$. But clearly $f \leq_h g$). The use of perfect conditions to guarantee minimality was inspired by Spector's construction of a minimal Turing degree.

We now turn to the case where α is not locally countable. One can establish Theorem 5(a) by first generically extending L_α to $L_\alpha[A]$, $A \subseteq \alpha$, so that $L_\alpha[A]$ is now locally countable. Then by a careful use of pointed perfect forcing over $L_\alpha[A]$ a solution R to Theorem 5(a) can be produced. By a more elaborate argument R can be chosen to be a $\leq_h$-minimal upper bound for the reals in $L_\alpha[A]$. Theorem 5(b) requires more work. It is necessary to make $A \subseteq \alpha$ as above using branching conditions so that subsequent pointed perfect forcing will produce a minimal solution R to $\alpha = \alpha_{\mathrm{KP}}(R)$. Thus in some sense A is made up of minimal collapsing maps from the α-cardinals to ω.

One of the advantages to this type of forcing is that it adapts naturally to a number of other situations by altering the notion of pointedness used. If one defines $R \leq_n S$ iff R is a member of the least Σ_n-admissible set containing S then forcing with $\leq_n$-pointed perfect conditions establishes:

THEOREM 6 (SACKS [76]). *Let $n \geq 1$. If $\alpha > \omega$ is a countable Σ_n-admissible ordinal, then for some $R \subseteq \omega$:*

(a) *α is the least Σ_n-admissible ordinal relative to R (i.e., $\alpha = \alpha_{\mathrm{KP}_n}(R)$).*

(b) *If $S \leq_n R$ then either $R \leq_n S$ or $\alpha_{\mathrm{KP}_n}(S) < \alpha$.*

This solves the closure ordinal problem for the theories KP_n, in the countable case.

A disadvantage of pointed perfect forcing is its complexity. If one is willing to drop property (b) in Theorem 5 then simpler proofs can be given, both by forcing and by model theory. We now turn to the simplest of these forcing proofs, which is easily adapted to treating the uncountable regular case as well.

Steel forcing. Steel developed his method of forcing with tagged trees in Steel [78] to study definability in analysis. An additional benefit of his method is that it provides a very elegant solution to the closure ordinal problem for KP in the countable case.

Friedman [81E] describes a generalization of this method to any regular cardinal and shows how it can be used to characterize the ordinals $\alpha_{\mathrm{KP}}(x)$, $x \subseteq \kappa$, when κ is regular. This generalization suggests a modification in Steel's definition, which is what we present here.

We fix an infinite regular cardinal κ and an admissible ordinal α of cardinality κ. The forcing $\mathscr{P}_\kappa$ described below is designed to arrange that if G is $\mathscr{P}_\kappa$-generic over L_α then G produces $T \subseteq \kappa$ so that $\alpha_{\mathrm{KP}}(T) = \alpha$.

A condition in $\mathscr{P}_\kappa$ is a pair (T, h) where T is a size $< \kappa$ subtree of $\kappa^{<\omega}$ and h is a tagging function on T. A *tag* is something of the form ∞ or (β, γ, δ) where $\beta < \alpha$, $\gamma < \kappa$, $\delta < \gamma$. *A tagging function* h: $T \to$ Tags must obey the following conditions:

(a) $h(\emptyset) = \infty$.

(b) If $h(\sigma) = (\beta, \gamma, \delta)$ and $\tau \in T$ is an immediate extension of σ then either (i) $h(\tau) = (\beta', \gamma', \delta')$ where $\beta' < \beta$ or (ii) $h(\tau) = (\beta, \gamma, \delta')$ some $\delta' < \gamma$.

The idea of this forcing is that if $G = (T_G, h_G)$ is generic then T_G is a tree such that $\alpha = \sup\{\kappa\text{-rank}(\sigma) \mid \sigma \in T_G\}$.

κ-rank is defined as follows: For any tree T and $\sigma \in T$, $|\sigma|_\kappa = 0$ iff σ has fewer than κ immediate extensions on T. Otherwise $|\sigma|_\kappa = \sup\{\beta + 1 \mid \sigma$ has κ-many immediate extensions $\tau \in T$ s.t. $|\tau|_\kappa = \beta\}$. The *κ-rank* of σ is the ordinal $|\sigma|_\kappa$. It may happen that not every $\sigma \in T$ gets assigned a κ-rank by the above definition. In this case it is convenient to write $|\sigma|_\kappa = \infty$.

It is easily checked that if $G = \langle T_G, h_G\rangle$ is $\mathscr{P}_\kappa$-generic over L_α then for $\sigma \in T_G$, $h_G(\sigma) = (\beta, \gamma, \delta)$ iff $|\sigma|_\kappa = \beta$ and $h_G(\sigma) = \infty$ iff $|\sigma|_\kappa = \infty$. This implies that $\alpha_{\mathrm{KP}}(T_G) \geq \alpha$, as the inductive definition of $|\sigma|_\kappa$, $\sigma \in T_G$, can be carried out inside any admissible set containing T_G as a member.

The proof that $\alpha_{\mathrm{KP}}(T_G) \leq \alpha$ is based on Steel's retagging lemma. This roughly states that as far as sentences of rank $< \beta$ are concerned, $\beta < \alpha$, the above forcing is unchanged by restriction to those conditions $\langle T, h\rangle$ where $\sigma \in T \to h(\sigma) = \infty$ or (β', γ, δ) for some $\beta' < \beta$. Thus the original class forcing now becomes a set forcing provided $\beta^{<\kappa} \in L_\alpha$ for each $\beta < \alpha$. The fact that T_G preserves the admissibility of α now follows from the elementary fact that set forcing preserves admissibility.

Note that the condition $\beta < \alpha \to \beta^{<\kappa} \in L_\alpha$ is automatic when $\kappa = \omega$. Moreover $\mathscr{P}_\omega$-generics over L_α exist since L_α is countable. This establishes Theorem 5(a). The main result of Friedman [81C] establishes that L_α must obey this condition if $\alpha = \alpha_{\mathrm{KP}}(x)$ for some $x \subseteq \kappa$ (assuming $V = L$). We say that α is *$< \kappa$-admissible* if α is admissible, cofinality$(\alpha) \geq \kappa$ and $\beta < \alpha \to \beta^{<\kappa} \in L_\alpha$. A lemma of Friedman [81C] shows that if α is $< \kappa$-admissible then L_α is admissible relative to the function $\beta \mapsto \beta^{<\kappa}$. As $\mathscr{P}_\kappa$ is a $<\kappa$-closed forcing we have established the following result.

THEOREM 7 (FRIEDMAN [81C]). *Assume $V = L$ and κ is regular. Then $\alpha = \alpha_{\mathrm{KP}}(x)$ for some $x \subseteq \kappa$ iff $\alpha < \kappa^+$ and α is $< \kappa$-admissible.*

The proof of necessity in this theorem requires a thorough analysis of $< \kappa$-admissibility in terms of Jensen's Σ_n-projecta.

It is not known if generalized Steel forcing can be used in the uncountable case to prove the analog of Theorem 6(a). The argument used in the proof of Theorem 8 is based on almost disjoint set forcing, which we discuss momentarily.

THEOREM 8 (FRIEDMAN [81C]). *Let $n \geq 1$. Assume $V = L$ and κ regular. Then $\alpha = \alpha_n(x)$ for some $x \subseteq \kappa$ iff $\alpha < \kappa^+$ and α is $<\kappa$-Σ_n admissible* (*i.e., α is Σ_n admissible, cofinality* $(\alpha) \geq \kappa$ *and L_α is closed under $\beta \mapsto \beta^{<\kappa}$*).

Analogs of Theorems 5(b) and 6(b) can also be established. One must use pointed perfect trees on κ. The theory of (iterated) perfect set forcing on ω_1 was developed in Kanamori [80] and applied in its pointed version by Sacks-Slaman [82].

Almost disjoint set forcing. Recently, David [81] settled the closure ordinal problem for ZF, in the countable case. His work is based on Jensen [75], which is a highly elaborate use of almost disjoint set forcing. This forcing technique has its origins in Jensen-Solovay [70].

Almost disjoint forcing makes it possible to code large sets by small ones. We illustrate with the simplest case. Suppose X is a subset of ω_1. We show how to generically construct a real $R \subseteq \omega$ so that $X \in V[R]$. First select a sequence $\langle R_\alpha | \alpha < \omega_1 \rangle$ of infinite almost disjoint subsets of ω; i.e., $\alpha \neq \beta \rightarrow R_\alpha \cap R_\beta$ is finite. This is easily done by first selecting a sequence $\langle R'_\alpha | \alpha < \omega_1 \rangle$ of distinct subsets of ω and then letting $R_\alpha = \{n | n$ codes a finite initial segment of the characteristic function of $R'_\alpha\}$. Our forcing is designed so that if R is generic then $X = \{\alpha | R \cap R_\alpha$ is finite$\}$. So $X \in V[R]$.

A condition is a pair $\langle s, F \rangle$ where s is a finite subset of ω and F is a finite subset of X. We say that $\langle s', F' \rangle$ is at least as strong as $\langle s, F \rangle$, $\langle s', F' \rangle \leq \langle s, F \rangle$, if $s' \supseteq s$, $F' \supseteq F$ and $(s' - s) \cap R_\alpha = \emptyset$ for all $\alpha \in F$. If G is generic let $R_G = \bigcup \{s | \langle s, F \rangle \in G$ for some $F\}$. Then $\alpha \in X \leftrightarrow R_G \cap R_\alpha$ is finite, by a density argument.

This forcing is very gentle as it obeys the countable chain condition (any collection of pairwise incompatible conditions is countable). Indeed, for $\langle s, F \rangle$ and $\langle s', F' \rangle$ to be incompatible one must have $s \neq s'$.

This method extends easily to allow one to code a subset of κ^+ by a subset of κ, whenever κ is regular. In Jensen [75] it is shown that if $0^\#$ does not exist then one can also do a similar coding when κ is singular. Then assuming GCH, Jensen puts all of these codings together simultaneously to code a given class $A \subseteq$ ORD by a real R. Thus any model of GCH can be generically extended to one of the form $L[R]$, $R \subseteq \omega$.

There are formidable difficulties in Jensen's construction. The main lemma shows that for any successor cardinal κ the forcing can be broken up into two pieces, one "below" κ which obeys the $\leq\kappa$-chain condition, and the other "above" κ which is κ-distributive. This allows one to show that cardinals are preserved and that the ZF axioms hold in the generic extension.

We now turn to David's result.[4]

THEOREM 9 (DAVID [81]).[4] *If L_α is a countable model of* ZF *then for some real R, $\alpha_{\mathrm{ZF}}(R) = \alpha$.*

David's basic strategy is to build a predicate $A \subseteq \alpha$ which "destroys" models

[4]This was proved independently by Beller.

of ZF of the form L_β, $\beta < \alpha$, and then to code A into a real using Jensen's methods. The exact property that one wants A to have is:

(*) If $\gamma < \beta$ and $L_\beta[A \cap \gamma] \models$ ZF then $L_\beta[A \cap \gamma] \models \gamma$ is not a cardinal.

Now if $R \subseteq \omega$ codes A then $L_\beta[R] \not\models$ ZF whenever β is not an α-cardinal. To see this, suppose $L_\beta[R] \models$ ZF and $\kappa < \beta < \kappa^+$, κ an α-cardinal. The details of Jensen coding show that if $\gamma = (\kappa^+)^{L_\beta[R]}$ then $A \cap \gamma \in L_\beta[R]$. By (*) we have a contradiction. To arrange that $L_\beta[R] \not\models$ ZF when β is an α-cardinal, David first uses a comparatively simple forcing to add a class A_0 to L_α so that

$$\langle L_\beta[A_0 \cap \beta], A_0 \cap \beta\rangle \not\models \text{Replacement for all } \alpha\text{-cardinals } \beta.$$

Then by applying Jensen coding a real R_0 is produced so that $A_0 \cap \beta$ is definable over $L_\beta[R_0]$ (and so $L_\beta[R_0] \not\models$ ZF) for all $L_\alpha[R_0]$-cardinals β. Then the construction of A obeying (*) is done over the model $L_\alpha[R_0]$.

Building A to satisfy (*) so that $L_\alpha[A] \models$ ZF necessitates delving into Jensen's construction in Jensen [75]. The reason is that the only way to arrange (*) for ordinals between κ and κ^+ is to first know that (*) has been arranged for ordinals of α-cardinality κ^+ and to have a code for $A \cap (\kappa^+, \kappa^{++})$ as a subset of κ^+. It is this type of "backwards" induction that forces one to build a coding of A into ω simultaneously with the construction of A. Fortunately Jensen's technique blends nicely with the construction of A.

The uncountable regular case of the closure ordinal problem for ZF remains open.[5] The main problem is to develop the fine structure analysis which isolates the correct necessary condition for an ordinal α of regular cardinality $\kappa > \omega$ to be the ZF-closure ordinal of a subset of κ. The singular case is treated in Part C.

C. Model-theoretic methods. H. Friedman-Jensen [68] established a connection between infinitary logic and the closure ordinal problem for KP. We shall sketch here a proof of Theorem 5(a) based on a strengthening of Barwise's compactness theorem (Barwise [69]).

The Barwise compactness theorem deals with fragments of $\mathscr{L}_{\omega_1\omega}$, infinitary logic with countable conjunctions and disjunctions but only finite strings of quantifiers. We can think of formulas as being coded by sets in some natural way, and thereby any countable admissible set A gives rise to a fragment of $\mathscr{L}_{\omega_1\omega}$, $\mathscr{L}_A = \{\phi \in L_{\omega_1\omega} \mid \text{code}(\phi)$ belongs to $A\}$. In what follows we shall identify a formula with its code.

Barwise's compactness theorem states that if $\Phi \subseteq \mathscr{L}_A$ is a $\Sigma_1\langle A, \varepsilon\rangle$ collection of sentences and any $\Phi_0 \subseteq \Phi$, $\Phi_0 \in A$, has a model, then Φ has a model. We shall need a refinement of this result which concerns the special case that KP $\subseteq \Phi$.

A nonstandard admissible set is a model $\langle M, E\rangle$ of KP which is not well-founded. In this case let $M' = \{m \in M \mid E$ is well-founded below $m\}$. Then the *standard part* of $\langle M, E\rangle$ is the unique transitive set T such that $\langle T, \varepsilon\rangle \simeq \langle M', E \restriction M'\rangle$. A theorem in Barwise [75] states that T is admissible.

If KP $\subseteq \Phi$ then Barwise's compactness theorem has the following refinement (also due to Barwise): Under the same hypotheses as before, Φ has a model M

[5]See footnote 3, above.

such that the standard part of M has ordinal height $\leq \mathrm{ORD}(A)$. This is proved by a sort of type-omitting argument.

We are now prepared to present a model-theoretic proof of Theorem 5(a). Let α be a countable admissible ordinal and consider the following collection Φ of sentences in $\mathscr{L}_{L_\alpha}$:

(a) $\mathrm{KP} + \underset{\sim}{R} \subseteq \omega$.

(b) $V = L(\underset{\sim}{R})$.

(c) $\forall b$ (b an ordinal $\to L_b(\underset{\sim}{R})$ is inadmissible).

(d) Diagram(L_α).

Some explanation is necessary for (d). Introduce names $\underset{\sim}{x}$ for each $x \in L_\alpha$. Then Diagram(L_α) consists of all sentences of the form $\forall z(z \in \underset{\sim}{x} \leftrightarrow \bigvee_{y \in x} z = \underset{\sim}{y})$. These sentences guarantee that L_α is contained in the standard part of any model of Φ.

By the refined compactness theorem stated earlier, as Φ is $\Delta_1\langle L_\alpha, \varepsilon\rangle$ there is $M \models \Phi$ such that $T =$ standard part (M) has ordinal height $\leq \alpha$. In fact $T \cap \mathrm{ORD} = \alpha$ by axioms (d).

Now $R = (\underset{\sim}{R})^M$ belongs to T, so the admissibility of T implies that $L_\alpha(R)$ is admissible. But axiom (c) guarantees that $L_\beta(R)$ is inadmissible for $\beta < \alpha$. This proves Theorem 5(a).

A similar argument will work in the uncountable case provided one has the above refined version of Barwise compactness for $\mathscr{L}_{L_\alpha}$. Friedman [81D] characterizes exactly when this holds, assuming $V = L$. The cardinality of $\alpha = \kappa$ must be cofinal with ω and there must be a *tame* 1-1 function $f\colon L_\alpha \to \kappa$ (i.e., $f^{-1}[\gamma] \in L_\alpha$ for all $\gamma < \kappa$). Finally, if there is a greatest α-cardinal then it must have cofinality ω.

THEOREM 10 (FRIEDMAN [81D]). *Assume $V = L$ and κ a cardinal cofinal with ω. Then $\alpha = \alpha_{\mathrm{KP}}(x)$ for some $x \subseteq \kappa$ iff*

(i) $\kappa < \alpha < \kappa^+$,

(ii) *there is a tame* 1-1 *function $f\colon L_\alpha \to \kappa$, and*

(iii) *if $\gamma =$ the largest α-cardinal then* $\mathrm{cof}(\gamma) = \omega$.

The "only if" direction is based on a fine structure analysis of the above properties. A related result appears in Magidor-Shelah-Stavi [81].

To give some idea why properties (i), (ii), (iii) yield a compactness theorem for $\mathscr{L}_{L_\alpha}$ note that (ii) allows one to write a given collection Φ of sentences as a countable union $\bigcup_n \Phi_n$ where each $\Phi_n \in L_\alpha$. Then a Henkin construction can be performed in ω steps to build a model of Φ. Condition (iii) comes into play when it is necessary to consistently choose disjuncts for each of the disjunctions in Φ_n.

Though Barwise compactness is not available at uncountable regular cardinals, a model-theoretic proof is nonetheless possible. Moreover an interesting connection with Scott rank emerges from this approach, which is based on Barwise's theory of admissible sets with urelements.

If $\mathscr{M}$ is a structure of finite similarity type then Barwise [75] discusses admissible sets *above* $\mathscr{M}$, where the elements of $|\mathscr{M}|$ are treated as urelements, objects different from $\varnothing$ but without elements. $A_{\mathscr{M}}$ is admissible above $\mathscr{M}$ if $|\mathscr{M}| \in A_{\mathscr{M}}$ and $A_{\mathscr{M}}$ satisfies Δ_0-separation and Δ_0-bounding when the basic operations of the structure $\mathscr{M}$ on $|\mathscr{M}|$ are taken as primitive. Barwise shows that there is a least admissible set above $\mathscr{M}$, HYP($\mathscr{M}$), and that in case $O(\mathscr{M}) = \text{HYP}(\mathscr{M}) \cap \text{ORD}$ is greater than ω, HYP($\mathscr{M}$) is just $L_{O(\mathscr{M})}(\mathscr{M})$ (defined just as in the usual L-hierarchy but with $\mathscr{M}$ put in at the bottom).

The usefulness of HYP($\mathscr{M}$) for the study of closure ordinals is that it provides us with a new technique for solving the equation $\alpha = \alpha_{\text{KP}}(R)$, $R \subseteq \omega$. Indeed, if $\alpha = O(\mathscr{M})$ for some structure $\mathscr{M}$ then by forcing over HYP($\mathscr{M}$) with finite conditions $p\colon F \to^{1:1} |\mathscr{M}|$, $F \subseteq \omega$, one produces a generic structure $\mathscr{M}_G$ on ω such that $\mathscr{M}_G \simeq \mathscr{M}$. Moreover $L_\alpha(\mathscr{M}_G)$ is admissible since we are forcing over the admissible structure HYP($\mathscr{M}$) with a set of conditions $\mathscr{P} \in \text{HYP}(\mathscr{M})$. Thus $\alpha = \alpha_{\text{KP}}(R)$ where $R \subseteq \omega$ codes $\mathscr{M}_G$.

The natural candidate for $\mathscr{M}$ so that $O(\mathscr{M}) = \alpha$ is the tree $\mathscr{T}$ produced by the variant of Steel forcing described in subsection B. The fact that $O(\mathscr{T}) \geq \alpha$ follows from the fact that each $\beta < \alpha$ occurs as the ω-rank of a node of $\mathscr{T}$. To argue that $O(\mathscr{T}) \leq \alpha$ we make use of the general result of Ressayre and Schlipf below (see Barwise [75], page 143). A structure $\mathscr{M}$ is *α-recursively saturated* if $\mathscr{M} \models \exists x\, \Lambda\Phi(x)$ for any $\Delta_1(L_\alpha)$ collection of formulas $\Phi(x)$ of $\mathscr{L}_{L_\alpha}$ such that $\mathscr{M} \models \exists x \Lambda\Phi_0(x)$ for all $\Phi_0 \subseteq \Phi$, $\Phi_0 \in L_\alpha$.

THEOREM (RESSAYRE, SCHLIPF). *$O(\mathscr{M})$ = least α such that $\mathscr{M}$ is α-recursively saturated.*

Now it is easy to determine the possible $\mathscr{L}_{L_\alpha}$-types consistent with the structure $\mathscr{T}$: A complete 1-type is determined by specifying the level on $\mathscr{T}$, the ω-rank (either $\beta < \alpha$ or ∞) and the number of immediate extensions of the same ω-rank. It follows easily that $\mathscr{T}$ is α-recursively saturated and hence $O(\mathscr{T}) \leq \alpha$.

We can use these ideas to give a new proof of "sufficiency" in Theorem 7. The appropriate generalization of HYP($\mathscr{M}$) is $<\kappa$-HYP($\mathscr{M}$), obtained by first closing $|\mathscr{M}|$ under $<\kappa$-sequences and then applying the HYP operation. This gives the least $<\kappa$-admissible set above $\mathscr{M}$. Given a $<\kappa$-admissible $\alpha < \kappa^+$, the tree $\mathscr{T}$ constructed for α in subsection B obeys $O_{<\kappa}(\mathscr{T}) \geq \alpha$ as any $\beta < \alpha$ is the κ-rank of a node on $\mathscr{T}$ ($O_{<\kappa}(\mathscr{M}) = {<\kappa}\text{-HYP}(\mathscr{M}) \cap \text{ORD}$). In the converse direction, one shows $O_{<\kappa}(\mathscr{T}) \leq \alpha$ by using the α-recursive saturation of $\mathscr{T}$ for the language $\mathscr{L}_{\infty\omega_1} \cap L_\alpha$. And finally, $\alpha = \alpha_{\text{KP}}(x)$ where $x \subseteq \kappa$ codes the generic collapse of the structure $\mathscr{T}$ via a $<\kappa$-closed set forcing over $<\kappa$-HYP($\mathscr{T}$).

A more complete discussion of these ideas can be found in Friedman [81E]. We close our brief description here by mentioning its implications for the theory of Scott rank. Nadel [74] shows that for any structure $\mathscr{M}$, the Scott rank of $\mathscr{M} \leq O(\mathscr{M})$. He calls a structure *tall* if equality holds. Given an admissible ordinal α, a tall structure of Scott rank α is obtained by considering the linear ordering $\alpha + \alpha \cdot \eta$ (η = ordertype of the rationals) or the tree $\mathscr{T}$ defined (for countable α) in subsection B. For any regular κ, the ordinal $O_{<\kappa}(\mathscr{M})$ provides an upper bound on the $<\kappa$-Scott rank of $\mathscr{M}$, or equivalently the $\mathscr{L}_{\infty\kappa}$-rank of $\mathscr{M}$. The generalized Steel trees then provide examples of *$<\kappa$-tall* structures of any given $<\kappa$-admissible $\mathscr{L}_{\infty\kappa}$-rank α (where $\mathscr{M}$ is $<\kappa$-tall if $\mathscr{L}_{\infty\kappa}\text{-rank}(\mathscr{M}) = O_{<\kappa}(\mathscr{M})$).

Lastly, we turn to the remaining singular cases of our closure ordinal problem. Assume $V = L$. For the singular, uncountable cofinality case it will be convenient to specify a specific example, $\aleph_{\omega_1}$. Then Theorem 4 tells us that if $\alpha = \alpha(X) > \aleph_{\omega_1}$ for some $X \subseteq \aleph_{\omega_1}$ then $\alpha = \alpha(D)$ for some master code $D \subseteq \aleph_{\omega_1}$.

LEMMA (JENSEN [72A]). *Suppose $\kappa < \beta$ and there is a $\Delta_n(L_\beta)$ subset of κ which is not a member of L_β. Then there is a $\Delta_n(L_\beta)$ bijection $f\colon \beta \leftrightarrow \kappa$.*

Thus if $D \subseteq \kappa$ is a Δ_n master code for β then there is a well-ordering W of κ of ordertype β which is $\Delta_n(L_\beta)$ and hence $\leq_\kappa D$. It follows that $\alpha(D) > \beta$. Combining this with Theorem 4 we obtain the following result. Call α a ZF-*ordinal* if $L_\alpha \models$ ZF.

THEOREM 11 (FRIEDMAN [81D]). *Assume $V = L$ and $\kappa = \aleph_{\omega_1}$, $n \geq 1$. Then $\alpha = \alpha_{\mathrm{KP}_n}(x)$ for some $x \subseteq \kappa$ iff α is a successor Σ_n-admissible ordinal and $L_\alpha \models \kappa$ is the largest cardinal. $\alpha = \alpha_{\mathrm{ZF}}(x)$ for some $x \subseteq \kappa$ iff α is successor* ZF-*ordinal and $L_\alpha \models$ The supremum of the* ZF-*ordinals has cardinality κ.*

Another way of expressing the content of Theorem 11 is: If $V = L$ and $x \subseteq \aleph_{\omega_1}$ then $x \in L_{\alpha_{\mathrm{KP}}}(x)$. A similar statement is true for $\aleph_\omega$: If $V = L$ and $x \subseteq \aleph_\omega$ then $x \in L_{\alpha_{\mathrm{KP}_2}}(x)$. This latter fact is established via the application of an effective version of Jensen's covering lemma, Devlin Jensen [74]. The effective version states that for $x \subseteq \kappa$, if $L_\alpha(x)$ is Σ_2 admissible, κ is a cardinal and $L_\alpha(x) \models \kappa$ is singular then $x \in L_\alpha$ (assuming $V = L$). It is proved by showing that Jensen's arguments in Devlin-Jensen [74] can be performed inside a Σ_2 admissible set. As for $\aleph_{\omega_1}$ we can now state the solution to the closure ordinal problem for $\aleph_\omega$ when $T = \mathrm{KP}_n$, $n \geq 2$, or ZF.

THEOREM 12 (FRIEDMAN [81D]). *Assume $V = L$ and $\kappa = \aleph_\omega$, $n \geq 2$. Then $\alpha = \alpha_{\mathrm{KP}_n}(x)$ for some $x \subseteq \kappa$ iff α is a successor Σ_n-admissible ordinal and $L_\alpha \models \kappa$ is the largest cardinal. $\alpha = \alpha_{\mathrm{ZF}}(x)$ for some $X \subseteq \kappa$ iff α is a successor* ZF-*ordinal and $L_\alpha \models$ The supremum of the* ZF-*ordinals has cardinality κ.*

Friedman [81D] also treats other singular cardinals, using the effective covering lemma.

§3. Some open questions.

1) Assume $V = L$, κ regular and uncountable. Is there a nice characterization of the degrees 0^λ for limit λ in terms of the 0^α, $\alpha < \lambda$?

2) Assume $V = L$ and let $T = \mathrm{KP}_n$ or ZF, κ a cardinal. Which sequences of ordinals $\langle \alpha_\gamma | \gamma < \gamma_0 \rangle$ less than κ^+ appear as the first γ_0 ordinals α s.t. $L_\alpha(x) \models T$, for some $x \subseteq \kappa$? Progress was made when $T = \mathrm{KP}_n$, $\kappa = \omega$, $\gamma_0 < \omega_1$ by Jensen [72B] and when $T = \mathrm{ZF}$, $\kappa = \omega$, $\gamma_0 < \omega_1$ by David [81]. If $\gamma_0 = \kappa^+$ then class forcing as in David [81] is necessary.

3) Are there minimal solutions to the closure ordinal problem for ZF, or for KP when $\kappa = \aleph_\omega$?

4) Can model theory be used to prove closure ordinal results for KP_n, $n > 1$? For ZF?

5) Is (**) true when $\kappa = \aleph_\omega$? In particular, given a degree d, is there a degree $e > d$ such that e, d' are incomparable?

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