Provable Π^1_2-Singletons

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Abstract

In this note we describe a subtheory T of $ZFC + 0^\#$ exists such that T is consistent with $V = L$ and there is a T-provable Π^1_2-singleton R, $0 <_L R <_L 0^\#$.

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In Friedman [90] we constructed a Π^1_2-singleton R such that $0 <_L R <_L 0^\#$. An open question is whether such a Π^1_2-singleton can be ZFC-provable, in the sense that $ZFC \vdash \phi$ has at most one solution, where ϕ is a Π^1_2 formula characterizing R. In this note we observe that the construction from Friedman [90] can be used to obtain a T-provable Π^1_2-singleton R, $0 <_L R <_L 0^\#$, where T is a theory consistent with $V = L$ contained as a subtheory of $ZFC + 0^\#$ exists. T has the same consistency strength as $ZFC+$ there exists an n-subtle cardinal for each n.**

First we recall a definition from Friedman [90]. For $i_1 < \cdots < i_{n+1}, n \geq 1$ define $I(i_1, \cdots, i_{n+1}) = \{i < i_1 | i$ is L-inaccessible and i, i_1 satisfy the same Σ_1 properties in $L_{i_{n+1}}$ with parameters from $i \cup \{i_2, \cdots, i_n\}\}$. An *acceptable guess* is such a sequence $(i_1, \cdots, i_{n+1})$ where i_1 is L-inaccessible and $1 \leq k < \ell \leq n \longrightarrow i_k \in I(i_\ell, \cdots, i_{n+1})$.

Now we say that an acceptable guess $(i_1, \cdots, i_{n+1})$ is *good* if in addition $I(i_1, \cdots, i_{n+1})$ is stationary in i_1. We refer to n as the *length* of the guess $(i_1, \cdots, i_{n+1})$.

T is the theory ZFC together with the single sentence: "There are arbitrarily long good guesses." T is a subtheory of $ZFC + 0^\#$ exists since any increasing sequence of Silver indiscernibles $(i_1, \cdots, i_{n+1})$, where $n \geq 1$ and i_1 is regular, is

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a good guess. (In fact $\mathrm{I}(i_1, \cdots, i_{n+1})$ is CUB in i_1 in this case.) Also note that T follows from the existence, for each n, of a cardinal κ such that any regressive function on n-tuples from κ has a homogeneous set X containing an α such that $X \cap \alpha$ is stationary in α, together with $n-2$ larger ordinals. And if T is true then it is true in L.

Now recall that in Friedman [90] a Π^1_2-singleton R is constructed so as to "kill" acceptable guesses $(i_1, \cdots, i_{n+1})$ such that $i_{n+1} < (i_1^+)^L$ and $p(i_1, \cdots, i_{n+1})_0$ contradicts R. Here, $p(i_1, \cdots, i_{n+1})$ is a $\Sigma_1(L)$-procedure which assigns a forcing condition to the guess $(i_1, \cdots, i_{n+1})$ and $p(i_1, \cdots, i_{n+1})_0$ is the "real part" of $p(i_1, \cdots, i_{n+1})$, consisting of a function from $(2^{<\omega})^{<\omega}$ into perfect trees. R is in fact a set of finite sequences of finite sequences of 0's and 1's and is determined by the $p(i_1, \cdots, i_{n+1})_0$ where $p(i_1, \cdots, i_{n+1})$ belongs to the generic class. A simple requirement that we may impose on the procedure $p(i_1, \cdots, i_{n+1})$ is that $p(i_1, \cdots, i_{n+1})$ must decide which of the first n elements of $(2^{<\omega})^{<\omega}$ belongs to R, for some fixed (constructible) ω-listing of $(2^{<\omega})^{<\omega}$.

An acceptable guess $(i_1, \cdots, i_{n+1})$ is killed by adding a CUB subset to i_1 disjoint from $I(i_1, \cdots, i_{n+1})$. The Π^1_2 formula characterizing R implies that R kills all acceptable guesses $(i_1, \cdots, i_{n+1})$ such that $i_{n+1} < (i_1^+)^L$ and $p(i_1, \cdots, i_{n+1})$ forces a false membership fact about R. Now suppose T holds and that $R \neq S$ were both solutions to our Π^1_2 formula. Choose n so that R and S differ on the membership of one of the first n elements of $(2^{<\omega})^{<\omega}$ and let $(i_1, \cdots, i_{n+1})$ be a good guess. By a Skolem hull argument we may assume that $i_{n+1} < (i_1^+)^L$. Then either R or S must kill $(i_1, \cdots, i_{n+1})$ since $p(i_1, \cdots, i_{n+1})$ decides membership of the first n elements of $(2^{<\omega})^{<\omega}$. But goodness means that $I(i_1, \cdots, i_{n+1})$ is stationary, a contradiction.

So T proves that our Π^1_2 formula characterizing R has at most one solution.

Reference

Friedman [90] The Π^1_2-Singleton Conjecture, Journal of the AMS.