Projective Singletons

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Assuming large cardinals, the structure of Π^1_n-singletons for odd n is well understood: using the prewellordering property of the pointclass Π^1_n, it follows that they are prewellordered by Δ^1_n-reducibility. But for even n, Π^1_n the analysis of Π^1_n-singletons requires new techniques.

For $n = 2$ one has:

Theorem 1. ([2]) There is a Π^1_2-singleton R such that $0 <_L R <_L 0^\#$. Moreover, there are two such Π^1_2-singletons of incomparable L-degree.

The proof of this result makes essential use of L-coding ([1], [4]). To generalize this result to level 4, it is necessary to understand the correct analogues of L-reducibility and $0^\#$ for the pointclass Π^1_4, and to develop an appropriate generalized coding method.

The correct analogue of $0^\#$ at level 4 is $M_2^\#$, the least mouse with two Woodin cardinals and a sharp above. The correct analogue of L at level 4 is M_2, the iterable extender model with 2 Woodin cardinals that results after iterating the sharp at the top of $M_2^\#$ to infinity. And the correct analogue of L-reducibility is M_2-reducibility: $R <_4 S$ iff R belongs to M_2^S, the canonical iterable extender model with 2 Woodin cardinals containing S.

Next we make a few comments about Woodin cardinals. There are two equivalent definitions of this concept. An inaccessible cardinal δ is *Woodin* iff either of the following equivalent properties holds:

1. For any $f : \delta \to \delta$ there is $\kappa < \delta$ closed under f and an embedding $\pi : V \to M$ with critical point κ which is f*-strong*, i.e., such that $V_{\pi(f)(\kappa)}$ is

contained in M.
2. For any $A \subseteq \delta$, there is $\kappa < \delta$ which is *A-strong up to δ*: for any $\alpha < \delta$ there is $\pi : V \to M$ with critical point κ such that $A \cap \alpha = \pi(A) \cap \alpha$.

A set S of extenders (on V) is a *weak Woodin witness for* δ iff δ is Woodin via the first definition above, using ultrapower embeddings given by extenders in S. S is a *strong Woodin witness for* δ iff δ is Woodin via the second definition, using ultrapower embeddings given by extenders in S.

We shall also need the following technical fact, which follows directly from the results of [5].

Theorem 2. Assume that there are two Woodin cardinals with a measurable above them. Suppose that x is a real in W, an inner model which is (ω_1+1-) iterable with respect to a countable set of extenders strongly witnessing the Woodinness of two cardinals. Then W contains all the reals of M_2^x.

We are now ready to discuss the analogue of Theorem 1 for Π^1_4. First we need a class forcing technique in the context of Woodin cardinals.

Theorem 3. Suppose that $W = L^E$ is a good extender model (like M_2, with sufficient Condensation and $\square$, and with the properties that the class of extenders on the E-sequence is sufficient to witness Woodinness and is closed under cutbacks to cardinals of W). Then there is a real x which is class-generic but not set-generic over W such that every Woodin cardinal of W remains Woodin in $W[x]$. Moreover Woodinness in W has a W-definable weak witness, consisting of extenders on the E-sequence which have cardinal (of W) true length and lift to $W[x]$.

By combining the techniques used in the proofs of Theorems 1 and 3, we can produce a real x with the following properties:

1. x is class generic over M_2, x belongs to $L[M_2^\#]$ but not to M_2 and M_2, $M_2[x]$ have the same cofinalities.
2. For some formula φ, x is the unique real such that $(M_2 \restriction \alpha)[x] \vDash \varphi(x)$ for each ordinal α.
3. The Woodinness of the two Woodin cardinals of M_2 can be weakly witnessed by a set of extenders $T \in M_2$, each element of which is on the E-sequence of M_2, has M_2-cardinal true length and lifts to $M_2[x]$.

It can be shown that property 2 is Π^1_4 in any real which codes $M_2 \restriction \delta_1$, where δ_1 is the smaller Woodin cardinal of M_2. It follows that R is a Π^1_4-singleton relative to any such real.

Thus we have obtained an analogue of Theorem 1 at level 4 provided we can show that $M_2^\#$ does not belong to M_2^x, the canonical iterable extender model with 2 Woodin cardinals containing x. It is clear that the real $M_2^\#$ does not belong to $M_2[x]$, as the latter has the same cardinals as M_2. But this does not suffice, as $M_2[x]$ and M_2^x might be different models. (In the Π^1_2 singleton result, there is no distinction between L^x and $L[x]$.)

By Theorem 2 it suffices to show that the model $M_2[x]$ is $(\omega_1+1\text{-})$ iterable with respect to the extenders in a countable strong witness T^* to Woodinness in $M_2[x]$, for then $M_2[x]$ contains all reals of M_2^x and therefore the real $M_2^\#$ cannot belong to M_2^x. We take T^* to be the set of all liftings to $M_2[x]$ of extenders on the M_2-sequence which have M_2-cardinal true length. By Property 3 above, this is a weak witness to Woodinness in $M_2[x]$; T^* is in fact a strong witness: Suppose not, and choose δ to be Woodin in M_2 and $A \subseteq \delta$ such that no $\kappa < \delta$ is A-strong up to δ via extenders in T^*. For each $\kappa < \delta$ let $f(\kappa)$ be several M_2-cardinals past the supremum of those $\alpha < \delta$ such that κ is A-strong up to α via extenders in T^*. As T^* is a weak witness to the Woodinness of δ in $M_2[x]$, we can choose $\kappa < \delta$ closed under f and an extender $E \in T^*$ such that E witnesses f-strength. In $\mathrm{Ult}(M_2[x], E)$, $E(f)(\kappa)$ is several M_2-cardinals past the supremum of those $\alpha < \delta$ such that κ is $E(A)$-strong up to α via extenders in $E(T^*)$. Let F be obtained from E by cutting its length back by one M_2-cardinal. Then F belongs to $\mathrm{Ult}(M_2[x], E)$ and witnesses that κ is $E(A)$-strong up to its length. But F is the lifting of a cutback to an M_2-cardinal of an extender on the extender sequence of M_2 and therefore of $\mathrm{Ult}(M_2, E)$, by coherence. Since such cutbacks are also on the extender sequence of $\mathrm{Ult}(M_2, E)$, F belongs to $E(T^*)$, a contradiction.

Finally we argue that $M_2[x]$ is iterable with respect to the extenders in T^*. It suffices to show that any iteration $\langle N_i[x], E_i^* \mid i < \lambda\rangle$ of $M_2[x]$ via liftings of extenders from M_2 of M_2-cardinal length is a lifting to $M_2[x]$ of an iteration $\langle N_i, E_i \mid i < \lambda\rangle$ of M_2. This is clear by induction, provided the lifted extender E_i^* is applied to the model $N_j[x]$, when E_i is applied to N_j, and the resulting ultrapower embedding of $N_j[x]$ via E_i^* lifts the corresponding ultrapower embedding of N_j via E_i. By definition, j is least so that the critical point of E_i is less than the true length of E_j. So for the first of these properties it suffices to show that E_i and E_i^*, the lifting of E_i, have the same critical point and true length. The fact that E_i^* is a lifting of E_i implies that E_i and E_i^* have the same critical point and that the true length of E_i^* is at most that of E_i. But since x preserves cardinals over N_i and the true length of E_i is a cardinal of N_i, it follows that the true length of E_i^* cannot be smaller than the true length of E_i. So E_i^* is indeed applied to $N_j[x]$, as desired. The second property follows from the special nature of the forcing P that produces x. Indeed, using density-reduction for P, any lifting E^* of an extender E from a model N to $N[x]$, where x is P-generic over N, gives rise to an ultrapower embedding of $N[x]$ which lifts the ultrapower embedding given by E.

In summary we have:

Theorem 4. There is a real R such that $0 <_4 R <_4 M_2^\#$ and R is a Π^1_4-singleton an any real coding $M_2 \restriction \delta_1$, where δ_1 is the smaller Woodin cardinal of M_2. Moreover there are two such reals which are $<_4$-incomparable.

The natural analogue of Theorem 3 also holds at higher even levels, where δ_1 is the least Woodin cardinal of M_{2n}. At level ω however, one needs a code for the entire M_ω, and not just $M_\omega \restriction \delta_1$.

References

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