

News about HOD

Gödel invented two definable inner models: L and HOD

HOD = The inner model of hereditarily ordinal-definable sets

Both L and HOD satisfy AC

L satisfies GCH but HOD might not

However HOD is “close to V ” and L might not be

How “close” is HOD to V ?

News about HOD

Four ways for HOD to be “close” to V :

Genericity: V is a generic extension of HOD

Weak Covering: For arbitrarily large cardinals α ,
 $\alpha^+ = (\alpha^+ \text{ of HOD})$

Rigidity: There is no nontrivial elementary embedding from HOD to HOD

Large Cardinal Witnessing: Any large cardinal property witnessed in V is witnessed in HOD

The purpose of this talk is to report on some recent results concerning these four properties.

News about HOD: Genericity

An old and beautiful result of Vopenka:

Theorem

(Vopenka) Any set of ordinals is generic over HOD.

In fact the entire V is generic over HOD. To explain this I need to introduce the *Stability Predicate*.

Let SL denote the class of strong limit cardinals

α in SL is n -Admissible if $(H(\alpha), \text{SL} \cap \alpha)$ satisfies Σ_n Replacement

For $\alpha < \beta$ in SL, α is n -Stable in β if $(H(\alpha), \text{SL} \cap \alpha)$ is Σ_n -elementary in $(H(\beta), \text{SL} \cap \beta)$

The *Stability Predicate* S consists of all triples (α, β, n) where α is n -stable in β and β is n -Admissible.

News about HOD: Genericity

The Stability Predicate S is definable and therefore (HOD, S) is a model of ZFC.

Theorem

V is generic over (HOD, S) .

The forcing used to prove this is definable over (HOD, S) .
I strongly doubt that S is HOD-definable in general or even that V must be generic over HOD for a HOD-definable forcing.

Actually V is generic over the inner model $(L[S], S)$ called the *Stable Core* which can be strictly smaller than HOD.
The Stable Core is a very useful tool for understanding HOD.

News about HOD: Weak Covering

One of the milestones of core model theory is:

Weak Covering at Singulars: If α is a singular cardinal then $\alpha^+ = (\alpha^+ \text{ of "the core model"})$

Does one have Weak Covering for HOD? Unfortunately:

Theorem

(Cummings, me, Golshani) It is consistent that $\alpha^+ > (\alpha^+ \text{ of HOD})$ for every infinite cardinal α .

But there may be more to the story.

If we want to have a supercompact cardinal κ as well then the best we can get is $\alpha^+ > (\alpha^+ \text{ of HOD})$ for a club of $\alpha < \kappa$. And Woodin has conjectured that one cannot improve this to all $\alpha < \kappa$.

News about HOD: Rigidity

An easy Corollary of the genericity of V over the Stable Core $\mathbb{S} = (L[S], S)$ is the following.

Proposition

There is no nontrivial V -definable elementary embedding from \mathbb{S} to itself.

Proof. Let V be generic over \mathbb{S} via the definable forcing \mathbb{P} . Then fix a formula φ that (with parameters) defines in V a nontrivial embedding from \mathbb{S} to itself, and let α be the least ordinal which some condition in \mathbb{P} forces to be the critical point of such an embedding. But the ordinal α is \mathbb{S} -definable and therefore cannot be the critical point of any embedding from \mathbb{S} to itself, contradiction. \square

It follows that also (HOD, S) is rigid for V -definable embeddings.

News about HOD: Rigidity

But what about embeddings which are not V -definable?

The previous proof took advantage of the fact that the embeddings were just as definable as they were elementary.

The Enriched Stable Core

I won't bore you with the definition, but there is a richer form S^* of the Stability Predicate S which is also first-order definable and can be used to get a better rigidity result.

For simplicity, work in Morse-Kelley and note that this theory is strong enough to build an “ L -hierarchy” over V and therefore a notion of V -constructible class.

News about HOD: Rigidity

Theorem

The Enriched Stable Core $(L[S^], S^*)$ is rigid for V -constructible embeddings and therefore so is (HOD, S^*) .*

But it is still unknown whether HOD without the predicate S^* is rigid with respect to V -constructible embeddings.

The long-standing open conjecture is that HOD is in fact rigid for arbitrary embeddings.

News about HOD: Large Cardinal Witnessing

Here the news is pretty bad.

Theorem

(Cheng, me and Hamkins) It is consistent that there are supercompacts but none in HOD. One can even add to this that no supercompact is weakly compact in HOD.

But this doesn't quite end the story:

1. Maybe if there are very large cardinals like extendibles then there must be supercompacts in HOD. In other words it could be that HOD does witness large cardinal properties, but with a certain loss of strength.
2. Maybe the Stable Core does a better job of *Large Cardinal Witnessing*. If so, then the Stable Core is a better choice of “canonical” inner model than HOD.

News about HOD: Summary

We looked at four ways that HOD could be “close” to V (of course they all hold if $V = \text{HOD}$):

Genericity: V is a generic extension of HOD

Holds with HOD replaced by (HOD, S) ; open otherwise.

Weak Covering: For arbitrarily large cardinals α ,
 $\alpha^+ = (\alpha^+ \text{ of HOD})$

Fails but maybe holds if there are supercompacts.

Rigidity: There is no nontrivial elementary embedding from HOD to HOD

Holds with HOD replaced by (HOD, S^*) for V -constructible embeddings, open otherwise.

Large Cardinal Witnessing: Any large cardinal property witnessed in V is witnessed in HOD

Fails, but maybe holds allowing a drop in strength from V to HOD.

Final Comment

HOD is better than L because it is closer to V .

But L satisfies GCH!

Perhaps the biggest challenge is to get models that have the advantages of both:

HUGE Open Problem: Is there an inner model that satisfies any of *Genericity*, *Weak Covering*, *Rigidity*, or *Large Cardinal Witnessing* and also satisfies GCH?

If you can do that then you can have my job.

The End