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The Π^1_2-Singleton Conjecture: An Introduction

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Solovay conjectured that there is a Π^1_2-Singleton R such that $0 <_L R <_L 0^{\#}$. My purpose here is to provide a setting for this conjecture and to give an idea of its proof. This article is intended for the non–specialist; I will do my best to explain all the basic notions.

I will begin with the following result of Cohen:

1) There can be a nonconstructible real.

By this, I mean that "$\exists R \subseteq \omega (R \notin L)$" is consistent with ZFC. It is natural to ask for something stronger:

2) Can there be a *definable* nonconstructible real?

This was answered affirmatively by McAloon, who showed using Cohen's methods that $R = \{n | 2^{\aleph_n} = \aleph_{n+1}\}$ can be nonconstructible. For our present purposes a more definitive result is due to Silver and Solovay:

3) If there is a measurable cardinal then $0^{\#} = Thy\langle L, \aleph_1, \aleph_2, \ldots \rangle$ is a definable nonconstructible real.

By this we mean that the set of Gödel numbers of formuls $\phi(x_1 \ldots x_n)$ in the language of set theory such that $\langle L, \epsilon \rangle \models \phi(\aleph_1, \ldots, \aleph_n)$ is the definable, nonconstructible real $0^{\#}$. When we write $\aleph_1, \aleph_2 \ldots$ we mean the first ω-many uncountable cardinals of $V=$ the real world, not of $L =$ the constructible universe. In fact if there is a measurable cardinal then $\aleph_1^L, \aleph_2^L, \ldots$ are all countable ordinals.

It is not difficult to generate other examples of definable, non-constructible reals:

4) $\omega - 0^{\#}$ is definable, nonconstructible.

5) $0^{\#\#} = Thy\langle L[0^{\#}], \aleph_1, \aleph_2, \ldots \rangle$ is definable, nonconstructible.

We would like to eliminate examples as in 4), by modding out by a suitable equivalence relation. Define $R \leq_L S$ iff $R \in L[S], R <_L S$ iff $(R \leq_L S, S \not\leq_L R), R =_L S$ iff $(R \leq_L S, S \leq_L R)$. Then $=_L$ is an equivalence relation.

Solovay's conjecture addresses the following question: Are $0^{\#}, 0^{\#\#}, 0^{\#\#\#}, \ldots$ the only "canonical" nonconstructible reals, up to $=_L$? Specifically: Is $0^{\#}$ the $\leq_L$-least "canonical" nonconstructible real? We must be careful with the word "canonical". If we simply take this to mean "definable" we have a counterexample:

6) There is a real $R \leq_L 0^{\#}$ which is Cohen generic over L. Now $0^{\#}$ is not $=_L$-equivalent

to any real Cohen generic over L. Thus $R_o = (L[0^\#]$-least real Cohen generic over L) is a definable, nonconstructible real and $0^\# \not\leq_L R_o$.

But R_o is not really "canonical", because to define R_o we need to refer to $0^\#$ and $0^\#$ is not constructible from R_o. Thus we are looking for reals defined in a more absolute way, as follows:

Definition R is a *Solovay Singleton* if for some formula $\phi(x)$ with ordinal parameters, R is the unique real R such that $L[R] \models \phi(R)$.

Now $0^\#$ is a Solovay Singleton of a special type:

Definition A Π^1_2-*formula* is a formula of the form $\forall S \exists T \psi$ where ψ is arithmetical and S, T range over reals. R is a Π^1_2-*Singleton* if R is the unique solution to a Π^1_2 formula.

Levý–Shoenfield absoluteness implies that:

7) R is a Π^1_2-Singleton $\Longleftrightarrow$ R is a Solovay Singleton via a formula $\phi(x)$ that is Π_1 in the Lévy hierarchy and has *no* ordinal parameters.

Thus a Π^1_2-Singleton is the simplest type of Solovay Singleton that could be nonconstructible.

8) $0^\#$ is a Π^1_2-Singleton.

Solovay's Π^1_2-Singleton Conjecture There is a Π^1_2-Singleton R, $0 <_L R <_L 0^\#$.

Results

Theorem 1 There is a Π^1_2-Singleton $R, 0 <_L R <_L 0^\#$.

A question related to the Π^1_2-Singleton Conjecture is due to Kechris. A set X is Π^1_2 if $X = \{R | \phi(R)\}$ where ϕ is a Π^1_2 formula.

Question Does every nonempty countable Π^1_2 set contain a Π^1_2-singleton?

Theorem 2 There is a nonempty countable Π^1_2 set containing no Π^1_2-singleton.

On the other hand we have:

Theorem (Harrington–Kechris) If X is a nonempty countable Π^1_2 set then X contains a Solovay Singleton, defined using parameters $\aleph_1, \dots \aleph_n$ for some n.

Theorem 2′ For each n there is a nonempty countable Π^1_2 set X_n such that no element of X_n is a Solovay Singleton defined using parameters $\aleph_1, \dots, \aleph_n$. Thus the Harrington–Kechris result is best possible.

On the Proof of Theorem 1

R arises as the generic real for an L-definable forcing $\mathcal{P}$. This is a *class* forcing: $\mathcal{P}$ is not a set. $\mathcal{P}$ is built out of three types of forcings:

1) Jensen Coding: This enables R to code a class of information.

2) Backwards Easton Forcing: This is used to add certain CUB subsets to L-inaccessible cardinals. Such a forcing is an iteration $\mathcal{P}_o * \mathcal{P}_1 * \mathcal{P}_2 * \cdots * \mathcal{P}_\alpha * \ldots$ through the ordinals with the restriction that the support of any condition is bounded in any L-inaccessible.

3) Set Forcing: This helps to make R a Π^1_2-Singleton, via a trick of Solovay.

The existence of $0^\#$ entails the existence of a canonical CUB class of indiscernibles I for $\langle L, \epsilon\rangle$ such that $L = \Sigma_1$ Skolem hull (I) in $\langle L, \epsilon\rangle$. Our desired $\mathcal{P}$-generic $G \subseteq \mathcal{P}$ arises in a very natural way from I : $G = \{p | p(i_i \ldots i_n) \leq p$ for some $i_1 \ldots i_n \in I\}$ where $(i_1, \ldots i_n) \longmapsto p(i_1 \ldots i_n)$ is a $\Sigma_1(L, \epsilon)$ procedure.

Our goal is to show that there is only *one* $\mathcal{P}$-generic. This is where the Backwards Easton forcing comes in. There is a method of adding CUB sets (in a Backwards Easton fashion) for the purpose of "killing" guesses $(i_1 \ldots i_n)$ at an n-tuple of indiscernibles. No correct quess $(i_1 \ldots i_n) \in I^n$ can be killed. Our forcing is set up so that the generic G will kill any guess $(i, \ldots i_n)$ which via the $\Sigma_1(L, \epsilon)$ procedure above produces information $p(i_1 \ldots i_n)$ contradicting G. So there can only be our one generic G, as another generic H would have to kill the correct guesses $(i_1 \ldots i_n) \in I^n$ which produce $p(i_1 \ldots i_n) \notin H$.

The other key ingredient in the proof is the Recursion Theorem. To create the forcing $\mathcal{P}$ we need a Σ_1 index for the procedure $(i_1 \ldots i_n) \longmapsto p(i_1 \ldots i_n)$ in order to know which guesses $(i_1 \ldots i_n)$ to kill. But this procedure in turn cannot be defined explicitly without knowing the forcing $\mathcal{P}$! This circularity is dealt with by using the Recursion Theorem to obtain the desired index.

A similar technique can be used to establish Theorem 2.

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