The Philosophy and Mathematics of Set-Theoretic Truth

1.-2.Vorlesungen

Introduction

My aim in this course is to discuss the roles of the field of Set Theory together with evidence for the truth of set-theoretic assertions. This is of course a broad topic with a rich literature and I can't hope to survey all of the different approaches to these topics. Instead it is my hope to clarify the landscape through a discussion of Set Theory both as a mathematical theory in its own write, as a foundation for mathematics and as a theory of the set-concept, to classify the different sources for set-theoretic truth and to then focus in detail on one approach as provided by the *Hyperuniverse Programme*.

The different roles of Set Theory

In light of Set Theory's dramatic development as a mathematical discipline and its fundamental role as a foundation for mathematics it is easy to forget that it was born rather innocently out of a specific mathematical problem. Cantor was concerned with *sets of uniqueness* for Fourier Series, a topic still of interest to analysts. For our purposes it suffices to focus on the method that Cantor developed in order to solve the problem he was after, now known as the *iteration of the Cantor derivative.*

If X is a set of real numbers then we use X' to denote the *Cantor derivative of* X, i.e. the subset of X consisting of those points in X which are also limit points of X. Cantor observed that when replacing X by its derivative, new isolated (i.e. non-limit) points appear, and therefore X'', the derivative of X' may be strictly smaller than X'. Indeed, the process of iterating the derivative:

$$X \supseteq X' \supseteq X'' \supseteq \cdots \supseteq X^{'''\cdots} = X^{\omega}$$

need not terminate after the usual infinity of steps, demanding a further derivative $X^{\omega+1}$. What kind of entity is this $\omega + 1$? It was to answer this fundamental question that Cantor was led to the theory of *ordinal numbers* and this was the first major step towards the development of *Abstract Set Theory.*

Thus Set Theory emerged from a concrete question of mathematics. But today we see it as much more than a commentary regarding the Cantor derivative. The further development of Cantor's theory, followed by the resolution of the paradoxes through the introduction of the axioms of Zermelo Set Theory, reaching its mature form as the now-standard axioms ZFC, reveals its important role as the most successful foundation that we have for mathematics as a whole. And the further mathematical development of the subject establishes it as an important branch of mathematics in its own write.

So what is Set Theory? It is three things:

1. Set Theory is a branch of mathematics.

2. Set Theory provides a foundation for mathematics.

3. Set Theory is the study of the *set-concept*.

Points 1 and 2 above are commonly recoginised; it is however all too common to forget point 3. Clearly there is a *concept of set* independent of mathematics, as schoolchildren know: The famous Venn diagrams, illustrating the operations of union, intersection and difference of sets is (or at least was) taught in the public schools and readily understood at a young age. No mathematical experience is required to understand what the empty set is, how to form new sets through the singleton, union or difference operations, or even what the powerset of a given set denotes. These intuitions are part of the *intrinsic features* of the set-concept. It is important to remember this in discussions of set-theoretic truth, which I'll discuss next.

Sources for Set-Theoretic Truth

Corresponding to Set Theory's three distinct roles are three distinct sources of set-theoretic truth.

Set Theory is a branch of mathematics. From this one can derive a notion of *truth* in Set Theory (a special case of Maddy's *Thin Realism*) in which *truth* is dictated by what best advances the mathematical development of the subject. We can refer to this as *practice-based truth* in Set Theory. If an axiom of Set Theory yields dramatic consequences for shedding light on a particular area or for pointing the way to new and deep set-theoretic developments, then we take this as evidence that such an axiom is *true*.

But Set Theory also serves as a foundation for mathematics. What does this mean? Simply that the assertions of mathematics can be faithfully translated into assertions in the language of Set Theory and the theorems of mathematics then become derivable from the standard ZFC axioms of Set Theory. As long as mathematics can be "absorbed" into Set Theory in this way, and given that we already accept the axioms of ZFC as being *true*, the role of Set Theory as a foundation for mathematics is of no use in shedding new light on set-theoretic truth.

However, ZFC is not sufficient to answer all interesting problems of mathematics: there are *independent problems*. Not only questions of a literally set-theoretic form, such as Cantor's *Continuum Hypothesis*, but also questions such as the *Borel Conjecture*, the *Whitehead Problem* and the *Kaplansky Conjecture* have been shown to be *independent from ZFC*, i.e. neither provable nor refutable using the standard axioms. Such cases of independence in central mathematics, outside of Set Theory, continue to appear on a regular basis.

The latter raises an intriguing but so far unexplored possibility: Perhaps after a systematic study of the independence phenomenon across mathematics outside of Set Theory, identifying those areas of mathematics in which independence phenomena occur, and for each such area, identiffying those axioms of Set Theory which are most effective for resolving those independent questions, a pattern will emerge. This pattern may suggest that particular axioms of Set Theory are especially desirable for the resolution of independence across mathematics as a whole and therefore can be taken as *true* due to their role in the foundation of mathematics. The latter investigation I refer to as the *Independence Project* and its resulting notion of *truth* as *independence-based truth*, coming from the role of Set Theory in mathematics outside of Set Theory.

Taken together, both *practice-based truth* and *independence-based truth* serve as sources of *truth* as understood by the *Thin Realist* of Penelope Maddy:

"... doing good mathematics is the goal of mathematical practice, in set theory and elsewhere. A person can call this the search for truth if he likes, but if so (I say) then the grounding of this truth is in the goodness of the mathematics."

Maddy's *Thin Realist* does not presume a realm of objects (ontology) to which set-theoretic truth is to remain faithful, as would the Platonist (a *Robust Realist*). Instead, she considers *truth* in Set Theory to be determined by what is best for the development of Set Theory and mathematics as a whole as *good mathematics.*

Third and finally, Set Theory can be regarded as the study of the *set-concept.* This naturally leads us to the question: What set-theoretic assertions are derivable from inherent features of the *set-concept*? This is sometimes referred to as *intrinsic truth* in Set Theory, to contrast it with *extrinsic* sources coming from the practice of Set Theory, the nature of independence in mathematics outside of Set Theory or other sources (such as Gödel's notion of "success") also not regarded as derivable from the set-concept. How far does *intrinsic truth* take us? It has been commonly assumed that the standard axioms of ZFC express the limit of intrinsic truth, but a new approach, given by the *Hyperuniverse Programme (HP)*, challenges this assumption.

I'll now lay the foundations for the study of *intrinsic truth* by examining the set-concept in more detail as well as its key feature of *Maximality.*

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The Iterative Conception of Set

As Gödel put it, the *iterative conception* of set expresses the idea that a set is something obtainable from well-defined objects by iterated application of the powerset operation. In more detail: Sets are formed in *stages*, where only the empty set is formed at stage 0 and at any stage greater than 0, one forms collections of sets formed at earlier stages. (Said this way, a set is re-formed at every stage past where it is first formed, but this is OK.) Any set is formed at some least stage, after its elements have been formed. This conception excludes anomalies: We can't have $x \in x$, there is no set of all sets, there are no cycles $x_0 \in x_1 \in \cdots \in x_n \in x_0$ and there are no infinite sequences $\cdots \in x_n \in x_{n-1} \in x_{n-2} \in \cdots \in x_1 in x_0$, as there must be e a least stage at which one of the x_n's is formed. We also assume that there are infinite sets, so the iteration process leads to a limit stage ω , which is not 0 and is not a successor stage.

As Boolos pointed out, the iterative conception yields that the universe of sets is a model of the axioms of Zermelo Set Theory, i.e. ZFC without

Replacement and without the Axiom of Choice. The standard model for this theory is $V_{\omega+\omega}$.

Nevertheless, Replacement and Choice are included as part of the *standard axioms of Set Theory*, for very different reasons. The case for Choice is typically made on *extrinsic grounds*, citing its *fruitfulness* for the development of mathematics. I.e., Choice is regarded as "true" when contemplating Set Theory's role as a foundation for mathematics. It is not clear that Choice is derivable from the iterative conception (though Shoenfield and Tait have argued for this, Boolos against), nor from the practice of Set Theory as a branch of mathematics (though Ferreiros may have claimed this).

Replacement, on the other hand, is derivable from the concept of set. To see this, we need to extend the iterative conception to the stronger *maximal iterative conception*, also implicit in the set-concept.

Maximality and the iterative conception

The term *Maximal* is used in many different senses in Set Theory, what I have in mind here is a very specific use associated to the iterative conception (IC). Recall that according to the IC, sets appear inside levels indexed by the ordinal numbers, where each successor level $V_{\alpha+1}$ is the powerset of the previous. As Boolos explained, the IC alone takes no stand on how many levels there are (the *height* of the universe V) or on how fat the individual levels are (the *width* of V). However it is generally regarded as implicit in the set-concept that both of these should be *Maximal*:

Ordinal or height maximality: The universe V is "as tall as possible", i.e., the length of the ordinals is "as long as possible".

Powerset or width maximality: The universe V is "as wide (or thick) as possible", i.e., the powerset of each set is as "large as possible".

If we conjunct the IC with maximality we arrive at the MIC, the *maximal iterative conception*, also part of the set-concept but more of a challenge to explain than the simple IC. Let us proceed slowly with such an explanation.

There is implicitly a *comparative* aspect to *maximality*, as to be "as large as possible" can only mean "as large as possible within the realm of *possibilities*". Thus to explain ordinal and powerset maximality we need to compare our picture of the set-theoretic universe to other possible pictures.

Now what does this mean in the case of ordinal maximality? We need to consider the possibility of two pictures P and P^* where P^* *lengthens* P, i.e. the universe V depicted by P is a rank-initial segment of the universe V^* depicted by P^* (the ordinals of V form an initial segment of the ordinals of V^* and the powerset operations of these two universes agree on the sets in V). If P^* is a "lengthening" of P we equivalently say that P is a "shortening" of P^*.

For the case of *powerset maximality* we need to consider the possibility of pictures P and P^* of the universe where the universe V depicted by P is an inner model of the universe V^* depicted by P^* (V and V^* have the same ordinals and V is included in V^*). If P^* is a "thickening" of P we equivalently say that P is a "thinning" of P^*.

We can now begin to explain *ordinal maximality* (maximality in *height*) and *powerset maximality* (maximality in *width*).

If a picture P is *ordinal maximal* then any "property" of the universe V described by P also holds of some rank-initial segment of V. This is the typical formulation of *reflection*. (However we will see that *ordinal-maximality* is in fact much stronger than *reflection*.) Of course specifc realisations of *reflection* must specify exactly which "properties" are to be taken into account.

We can also begin to explain *powerset maximality*. A picture P of the universe is *powerset maximal* if any propertyöf the universe described by a thickening of P also holds of the universe described by some thinning of P. In the case of first-order properties this is called the *Inner Model Hypothesis*, or *IMH*.

Now let's step back a bit. Notice that our analysis of *maximality* is based on "thickenings" and "lengthenings" of a given universe in terms of pictures. Does it make sense to talk about pictures of universes V^* which "lengthen" or "thicken" the universe V of all sets? There are differences of opinion about this, which I'll take up next.

Actualism and Potentialism

Recall that in the *IC* we describe V, the universe of sets, via a process of iteration of the powerset operation. Does this process come to an end, or is it indefinite, always extendible further to a longer iteration? The former possibility, that there is a "limit" to the iteration process is referred to as *actualism* in the philosophy of Set Theory, and the latter view is called *potentialism.*

There is an ongoing debate regarding the suitability of *actualism* versus *potentialism.* One scholar who has spoken in favour of the latter is Geoffrey Hellman:

"The idea that any universe of sets can be properly extended (in height, not width) is extremely natural, endorsed by many mathematicians (e.g. MacLane, seemingly by Gödel, et. al.) ... As Maddy and others say, if it's possible that sets beyond some (putatively maximal) level exist, then they do exist ... Thus, if 'imaginable' (end) extensions of V are not incoherent, then they are possible, and then they are actual, and V wasn't really maximal after all. ... such extensions are always possible, so that the notion of a single fixed, absolutely maximal universe V of sets is really an incoherent notion."

And again:

"I have no earthly or heavenly idea what 'as high as possible' could mean, since the notion of a set domain that absolutely could not in logic be extended seems to me incoherent (or at any rate empty). As Putnam put it in his controversial paper, 'Mathematics without Foundations' (1967), 'Even God couldn't make a universe for Zermelo set theory that it would be impossible to extend.' And I agree, theology aside."

I endorse Hellman's view and will take a potentialist view of the universe of sets. (Indeed I need such a view to facilitate a satisfactory development of the Hyperuniverse Programme).

But recall that we have talked not only about "lengthenings" but also about "thickenings" of V. Correspondingly there can be two views on the possibility of "thickening" V, *width actualism* versus *width potentialism.* My personal view is much in favour of the latter, in a strong form that I call *radical potentialism.*

Radical potentialism can be described as follows. I sometimes refer to it by its nickname, *Skolem-worshipping*:

Let be begin with something less radical, *width potentialism*. From a "practice-based" viewpoint, it is obvious that we can thicken our pictures of universes. Let V be a universe, it could even be the "real V" belonging to a Platonist. Now even a Platonist has some idea of $V[G]$ where G is generic over V. Of course $V[G]$ does not exist because our "real V" has all the sets. But our Platonist can only picture $V[G]$ as a thickening: Just try to teach forcing and draw a picture of $V[G]$ without thickening a triangle depicting V to a wider triangle of the same height depicting $V[G]$! It is impossible. (Of course the Boolean universe V^B can be defined in V, but that's no picture!)

In *width potentialism*, any picture of the universe can be thickened, keeping the same ordinals, even to the extent of making ordinals countable. So for any ordinal α of V we can imagine how to thicken V to a universe where α is countable. So any ordinal is "potentially countable". But that does not mean that every ordinal *is* countable! There is a big difference between universes that we can imagine (where our $\aleph_1$ becomes countable) and universes we can "produce". So this "potential countability" does not threaten the truth of the powerset axiom in V.

The standard form of potentialism (in height) can be viewed as a process of lengthening as opposed to thickening. Once again, there is no model of ZFC "at the end" because there is no "end".

Now *radical potentialism* is in effect a unification of these two forms of potentialism. We allow V to be lengthened and thickened simultaneously. If we were to keep thickening to make every ordinal of V countable then after $\mathrm{Ord}(V)$ steps we are forced to also lengthen to reach a (picture of a) universe that satisfies ZFC. In that universe, the original V looks countable. But then we could repeat the process with this new universe until it is seen to be countable. The potentialist aspect is that we cannot end this process by taking the union of all of our pictures. In fact, whereas in the standard discussion of lengthenings there could be a debate about whether we can arrive at "the end", if we allow both lengthenings and thickenings, *potentialism is the only possibility*; actualism is ruled out because the union of our "universes" would not be a model of ZFC and would therefore *have to be* lengthened further! And again, the "potential countability of V" does not threaten the truth of the axioms of ZFC in V!

In *radical potentialsim* there is a huge wealth of pictures of V and some are "better" than others in the sense that some are better witnesses to maximality than others. The important question then arises: What first-order statements are common to all of these "better" universes? Is CH or its negation one of them?

But we are getting ahead of ourselves. *Radical potentialism* is not popular. Again we quote Geoffrey Hellman:

"I have a good idea, I think, about 'as thick as possible', since the notion of full power set of a given set makes perfect sense to me ... Granted that forcing extensions can be viewed as 'thickenings' of the cumulative hierarchy, as usually described, when we assert the standard Power Sets axiom, we implicitly build in bivalence, i.e. that either x belongs to y or it doesn't, i.e. we are in effect ruling forcing extensions or Boolean-valued generalizations as *non-standard* [my italics], i.e. 'full power set' is to be understood only in the standard way."

And further:

"Thus, to my way of thinking, there is an important disanalogy between 'all ordinals' ... and 'all subsets of a given set'. The latter is 'already relativized'; there is nothing implicit in the notion of 'subset' that allows for indefinite extensions, so long as we are speaking of 'subsets of a fixed, given set' ... In contrast, 'all ordinals' cries out for relativization (a point I find in Zermelo's [1930]); without it, it does allow for indefinite extensibility, by the very operations that we use to describe ordinals"

Fortunately for the uses of "thickenings" of V that I want to make, there is no need to impose any form of *width potentialism*, so I propose at this point that we adopt the position of *potentialism in height* with *actualism in width.*

5.Vorlesung

Maximality in Height and #-Generation

Recall that we take a potentialist view of the height of V, i.e., we allow ourselves the option of lengthening V to universes V^* which have V as a rank-initial segment. Of course we can also consider shortenings of V, replacing V

by one of its own rank-initial segments. Let us now make use of lengthenings and shortenings to formulate a *vertical maximality* principle for V, expressing the idea that the sequence of ordinals is "as long as possible".

Standard Lévy reflection tells us that a single first-order property of V with parameters will hold in some V_α which contains those parameters. It is natural to strengthen this to the simultaneous reflection of all first-order properties of V to some V_α, allowing arbitrary parameters from V_α. Thus we have reflected V to a V_α which is an elementary submodel of V.

Repeating this process leads us to an increasing, continuous sequence of ordinals $(\kappa_i \mid i < \infty)$, whee ∞ denotes the ordinal height of V, such that the models $(V_{\kappa_i} \mid i < \infty)$ form a continuous chain $V_{\kappa_0} \prec V_{\kappa_1} \prec \cdots$ of elementary submodels of V whose union is all of V.

Let C be the proper class consisting of the κ_i's. We can apply reflection to V with C as an additional predicate to infer that properties of (V, C) also hold of some $(V_\kappa, C \cap \kappa)$. But the unboundedness of C is a property of (V, C) so we get some $(V_\kappa, C \cap \kappa)$ where $C \cap \kappa$ is unbounded in κ and therefore κ belongs to C. As a corollary, properties of V in fact hold in some V_κ where κ belongs to C. It is convenient to formulate this in its contrapositive form: If a property holds of V_κ for all κ in C then it also holds of V.

Now note that for all κ in C, V_κ can be *lengthened* to an elementary extension (namely V) of which it is a rank-initial segment. By the contrapositive form of reflection of the previous paragraph, V itself also has such a lengthening V^*.

But this is clearly not the end of the story. For the same reason we can also infer that there is a continuous increasing sequence of such lengthenings $V = V_{\kappa_\infty} \prec V^*_{\kappa_{\infty+1}} \prec V^*_{\kappa_{\infty+2}} \prec \cdots$ of length the ordinals. For ease of notation, let us drop the *'s and write W_{κ_i} instead of $V^*_{\kappa_i}$ for $\infty < i$ and instead of V_{κ_i} for $i \leq \infty$. Thus V equals W_∞.

But which tower $V = W_{\kappa_\infty} \prec W_{\kappa_{\infty+1}} \prec W_{\kappa_{\infty+2}} \prec \cdots$ of lengthenings of V should we consider? Can we make the choice of this tower "canonical"?

Consider the entire sequence $W_{\kappa_0} \prec W_{\kappa_1} \prec \cdots \prec V = W_{\kappa_\infty} \prec W_{\kappa_{\infty+1}} \prec W_{\kappa_{\infty+2}} \prec \cdots$. The intuition is that all of these models resemble each other in the sense that they share the same first-order properties. Indeed by virtue of the fact that they form an elementary chain, these models all satisfy the same first-order sentences. But again in the spirit of "resemblance", it should be the case that any two pairs $(W_{\kappa_{i_1}}, W_{\kappa_{i_0}})$, $(W_{\kappa_{j_1}}, W_{\kappa_{j_0}})$ (with $i_0 < i_1$ and $j_0 < j_1$) satisfy the same first-order sentences, even allowing parameters which belong to both $W_{\kappa_{i_0}}$ and $W_{\kappa_{j_0}}$. Generalising this to triples, quadruples and n-tuples in general we arrive at the following situation:

$(*)$ V occurs in a continuous elementary chain $W_{\kappa_0} \prec W_{\kappa_1} \prec \cdots \prec V = W_{\kappa_\infty} \prec W_{\kappa_{\infty+1}} \prec W_{\kappa_{\infty+2}} \prec \cdots$ of length $\infty + \infty$, where the models W_{κ_i} form a *strongly-indiscernible chain* in the sense that for any n and any two increasing n-tuples $\vec{i} = i_0 < i_1 < \cdots < i_{n-1}$, $\vec{j} = j_0 < j_1 < \cdots < j_{n-1}$, the structures $W_{\vec{i}} = (W_{\kappa_{i_{n-1}}}, W_{\kappa_{i_{n-2}}}, \cdots, W_{\kappa_{i_0}})$ and $W_{\vec{j}}$ (defined analagously) satisfy the same first-order sentences, allowing parameters from $W_{\kappa_{i_0}} \cap W_{\kappa_{j_0}}$.

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We are getting closer to #-generation, Surely we can impose higher-order indiscernibility on our chain of models. For example, consider the pair of models $W_{\kappa_0} = V_{\kappa_0}$, $W_{\kappa_1} = V_{\kappa_1}$. Surely we would want that these models satisfy the same second-order sentences; equivalently, we would want $H(\kappa_0^+)^V$ and $H(\kappa_1^+)^V$ to satisfy the same first-order sentences. But as with the pair $H(\kappa_0)^V$, $H(\kappa_1)^V$ we would want $H(\kappa_0^+)^V$, $H(\kappa_1^+)^V$ to satisfy the same first-order sentences *with parameters*. How can we formulate this? For example, consider κ_0, a parameter in $H(\kappa_0^+)^V$ that is second-order with respect to $H(\kappa_0)^V$; we cannot simply require $H(\kappa_0^+)^V \vDash \varphi(\kappa_0)$ iff $H(\kappa_1^+)^V \vDash \varphi(\kappa_0)$, as κ_0 is the largest cardinal in $H(\kappa_0^+)^V$ but not in $H(\kappa_1^+)^V$. Instead we need to replace the occurence of κ_0 on the left side with a "corresponding" parameter on the right side, namely κ_1, resulting in the natural requirement $H(\kappa_0^+)^V \vDash \varphi(\kappa_0)$ iff $H(\kappa_1^+)^V \vDash \varphi(\kappa_1)$. More generally, we should be able to replace each parameter in $H(\kappa_0^+)^V$ by a "corresponding" element of $H(\kappa_1^+)^V$ and conversely, it should be the case that, to the maximum extent possible, all elements of $H(\kappa_1^+)^V$ are the result of such a replacement. It is natural to solve this parameter problem using embeddings. Thus we are led to the following.

Definition. A structure $N = (N, U)$ is called a # *with critical point* κ, or just a #, if the following hold:

- N is a model of ZFC^- (ZFC minus powerset) in which κ is the largest cardinal and κ is strongly inaccessible.

- (N, U) is amenable (i.e. $x \cap U \in N$ for any $x \in N$).

- U is a normal measure on κ in (N, U).

- N is iterable, i.e., all of the successive iterated ultrapowers starting with (N, U) are well-founded, yielding iterates (N_i, U_i) and Σ_1 elementary iteration maps $\pi_{ij} : N_i \to N_j$ where $(N, U) = (N_0, U_0)$.

We will use the convention that κ_i denotes the the largest cardinal of the i-th iterate N_i.

If N is a # and λ is a limit ordinal then $\mathrm{LP}(N_\lambda)$ denotes the union of the $(V_{\kappa_i})^{N_i}$'s for $i < \lambda$. (LP stands for "lower part".) $\mathrm{LP}(N_\infty)$ is a model of ZFC.

Definition. We say that a transitive model V of ZFC is *#-generated* iff for some # $N = (N, U)$ with iteration $N = N_0 \to N_1 \to \cdots$, V equals $\mathrm{LP}(N_\infty)$ where ∞ denotes the ordinal height of V.

#-generation fulfills our requirements for vertical maximality, with powerful consequences for reflection. L is #-generated iff $0^\#$ exists, so this principle is compatible with $V = L$. If V is #-generated via (N, U) then there are elementary embeddings from V to V which are canonically-definable through iteration of (N, U): In the above notation, any order-preserving map from the κ_i's to the κ_i's extends to such an elementary embedding. If $\pi : V \to V$ is any such embedding then we obtain not only the indiscernibility of the structures $H(\kappa_i^+)$, for all i but also of the structures $H(\kappa_i^{+\alpha}$ for any $\alpha < \kappa_0$ and more. Moreover, #-generation evidently provides the maximum amount of vertical reflection: If V is generated by (N, U) as $\mathrm{LP}(N_\infty)$ where ∞ is the ordinal height of V, and x is any parameter in a further iterate $V^* = N_{\infty^*}$ of (N, U), then any first-order property $\varphi(V, x)$ that holds in V^* reflects to $\varphi(V_{\kappa_i}, \bar{x})$ in N_j for all sufficiently large $i < j < \infty$, where $\pi_{j,\infty^*}(\bar{x}) = x$. This implies any known form of vertical reflection and summarizes the amount of reflection one has in L under the assumption that $0^\#$ exists, the maximum amount of reflection in L.

Thus #-generation tells us what lengthenings of V to look at, namely the initial segments of V^* where V^* is obtained by further iteration of a # that generates V. And it fully realises the idea that M should look exactly like closed unboundedly many of its rank initial segments as well as its "canonical" lengthenings of arbitrary ordinal height.

In summary, #-generation is the correct formalization of the principle of *vertical maximality*, and we shall refer to #-generated models as being *maximal in height*.

Maximality in Width and the IMH

Whereas in the case of *maximality in height* we can use height potentialism (i.e., the option of lengthening V to a taller universe) to arrive at an optimal criterion, the case of *maximality in width* is of a very different nature.

First, as we have adopted the perspective of width actualism, we cannot directly talk about "thickenings" of V in the same way that we used lengthenings of V to analyse height-maximality. To solve this problem we will invoke a logic called *V-logic* to formulate consistent theories which describe "thickenings" of V without actually having direct access to them.

Second, unlike in the case of height-maximality, we will not arrive at an optimal criterion. We will see that there are many distinct criteria for width-maximality, with no apparent "universal" such criterion. Moreover, to get a fair picture of maximality in both height and width, it is necessary to "synthesise" or "unify" width-maximality criteria with #-generation, the optimal height-maximality criterion.

A thorough analysis of the different possible width-maximality criteria, together with their synthesis with #-generation, with an aim towards arriving at an optimal criterion, is the principal aim of the *Hyperuniverse Programme.*

V-Logic

To motivate V-logic let's begin with a naive formulation of the *Inner Model Hypothesis (IMH):*

Naive IMH. If a first-order sentence holds in some "thickening" (outer model) of V then it holds in some inner model of V.

This formulation is "naive", as it fails to clarify the meaning of "thickening". Our main task now is to clarify the use of this word.

Before introducing V-logic let's discuss something a bit simpler, V_ω-logic. In V_ω-logic we have constant symbols $\bar{a}$ for $a \in V_\omega$ as well as a constant symbol $\bar{V}_\omega$ for V_ω itself (in addition to $\in$ other symbols of first-order logic). Then to the usual logical axioms and the rule of *Modus Ponens* we add the rules:

For $a \in V_\omega$: From $\varphi(\bar{b})$ for each $b \in a$ infer $\forall x \in \bar{a}\varphi(x)$.

From $\varphi(\bar{a})$ for each $a \in V_\omega$ infer $\forall x \in V_\omega\varphi(x)$.

Introducing the second of these rules generates new provable statements via proofs which are now infinite. The idea of V_ω-logic is to capture the idea of "model in which V_ω is standard". By the "ω-completeness theorem", the logically provable sentences of V_ω-logic are exactly those which hold in every model in which $\bar{a}$ is interpreted as a for $a \in V_\omega$ and $\bar{V}_\omega$ is interpreted as the (real, standard) V_ω. Thus a theory T in V_ω-logic is consistent in V_ω-logic iff it has a model in which V_ω is standard.

Now the set of logically-provable formulas (i.e. validities) in V_ω-logic, unlike in first-order logic is not arithmetical, i.e. it is not definable over the model V_ω. Instead it is definable over a larger structure, a *lengthening of* V_ω. Let me explain.

As proofs in V_ω-logic are no longer finite, they do not naturally belong to V_ω. Instead they belong to *the least admissible set* $(V_\omega)^+$ containing V_ω as an element, this is known to higher recursion-theorists as $L_{\omega_1^{ck}}$, where ω_1^{ck} is the least non-recursive ordinal. Something very nice happens: Whereas proofs in first-order logic belong to V_ω and therefore provability is Σ_1 definable over V_ω ("there exists a proof" is Σ_1), proofs in V_ω-logic belong to $(V_\omega)^+$ and provability is Σ_1 definable over $(V_\omega)^+$.

For our present purposes the point is that $(V_\omega)^+$ is a lengthening, not a thickening of V_ω and in this lengthening we can formulate theories which describe arbitrary models in which V_ω is standard. For example the existence of a real R such that (V_ω, R) satisfies a first-order property can be formulated as the consistency of a theory in V_ω-logic. As the structure (V_ω, R) can be regarded as a "thickening" of V_ω, we have described what can happen in thickenings of V_ω by a theory in $(V_\omega)^+$, a lengthening of V_ω. This is even more dramatic if we start not with V_ω but with $L_{\omega_1^{ck}}$ and introduce $L_{\omega_1^{ck}}$-*logic*, a logic for ensuring that the recursive ordinals are standard. Then in the lengthening $(L_{\omega_1^{ck}})^+$ of $L_{\omega_1^{ck}}$, the least admissible set containing $L_{\omega_1^{ck}}$, we can express the existence of a thickening of $L_{\omega_1^{ck}}$ in which a first-order statement holds, and such thickenings can contain new reals and more as elements.

V-logic is analogous to the above. It has the following constant symbols:

1. A constant symbol $\bar{a}$ for each set a in V.
2. A constant symbol $\bar{V}$ to denote the universe V.

Formulas are formed in the usual way, as in any first-order logic. To the usual axioms and rules of first-order logic we add the new rules:

$(*)$ From $\varphi(\bar{b})$ for all $b \in a$ infer $\forall x \in \bar{a}\varphi(x)$.

$(**)$ From $\varphi(\bar{a})$ for all $a \in V$ infer $\forall x \in \bar{V}\varphi(x)$.

This is the logic to describe models in which V is standard. The proofs of this logic appear in V^+, the least admissible set containing V as an element; this structure V^+ is a special lengthening of V of the form $L_\alpha(V)$, the α-th level of Gödel's L-hierarchy built over V. We refer to such lengthenings as *Gödel lengthenings.*

8.-9.Vorlesungen

The Inner Model Hypothesis

To illustrate how V-logic can be used to deal with specific criteria of width-maximality we now discuss the *IMH (Inner model hypothesis)*. Informally, this asserts the following:

IMH (informal version): Suppose that a first-order sentence holds in an outer model of V. Then it holds in an inner model of V.

By allowing ourselves the freedom to work not in ZFC but in its conservative extension GB (Gödel-Bernays class theory), we can easily express the notion of “inner model” used in the conclusion of the *IMH*, as we can quantify over transitive subclases of V which satisfy ZFC. More difficult is making sense of the notion of “outer model” used in the hypothesis.

As width-actualists we cannot talk directly about outer models or even about sets that do not belong to V. However using V-logic we can talk about them indirectly, as I'll now illustrate. Consider the theory in V-logic where we not only have constant symbols $\bar{a}$ for the elements of V and a constant symbol $\bar{V}$ for V itself, but also a constant symbol $\bar{W}$ to denote an "outer model" of V. We add the new axioms:

1. The universe is a model of ZFC (or at least the weaker KP, admissibility theory).
2. $\bar{W}$ is a transitive model of ZFC containing $\bar{V}$ as a subset and with the same ordinals as V.

So now when we take a model of our axioms which obeys the rules of V-logic, we get a universe modelling ZFC (or at least KP) in which $\bar{V}$ is interpreted correctly as V and $\bar{W}$ is interpreted as an outer model of V. Note that this theory in V-logic has been formulated without "thickening" V, indeed it is defined inside V^+, the least "admissible set containing V", a "Gödel lengthening" of V of the form $L_\alpha(V)$ for some ordinal α greater than the ordinals of V. This all makes sense because we have allowed ourselves to be height-potentialists and therefore can freely discuss arbitrary lengthenings of V.

So what does the *IMH* really say? It says the following:

IMH: Suppose that φ is a first-order sentence and the above theory, together the axiom "$\bar{W}$ satisfies φ" is consistent in V-logic. Then φ holds in an inner model of V.

In other words, instead of talking directly about "thickenings" of V (i.e. "outer models") we instead talk about the consistency of a theory formulated in V-logic and defined in V^+, a (mild) Gödel lengthening of V.

It is also worthwhile to briefly revisit height-maximality, where we introduced #-generation. Recall that this asserts the existence of a # for V, i.e., an iterable structure (N, U) which "generates" V. An important point is that a witness (N, U) to #-generation cannot be an element of V, just as $0^\#$ is not an element of Gödel's L. So again we have a "thickening" issue to deal with. Again this is solved in V-logic, but now with arbitrary Gödel lengthenings $L_\alpha(V)$:

#-generation: There is an (N, U) which through iteration for $\mathrm{Ord}(V)$-steps generates V and which in addition is iterable for α-steps for any ordinal α past $\mathrm{Ord}(V)$.

Once again we heavily use height-potentialism to make sense of ordinals past the height of V. #-generation becomes a criterion that is expressible in V-logic as follows: For any α let T_α be the theory in V-logic which asserts the above for the ordinal α. Then #-generation means that for each α, the theory T_α is consistent in V-logic. This time we cannot use a single theory, but must assert consistency of theories formulated in arbitrary Gödel-lengthenings of V.

So far we have worked with V, its lengthenings and its "thickenings" (via theories expressed in its lengthenings). We next come to an important step, which is to reduce this discussion to the study of certain properties of countable transitive models of ZFC, i.e., to the *Hyperuniverse*.

The Reduction to the Hyperuniverse

Of course it would be much more comfortable to remove the quotes in "thickenings" of V, as we could then dispense with the need to reformulate our intuitions about outer models via theories in V-logic. Indeed, if we were to have this discussion not about V but about a countable transitive ZFC model "little-V", then our worries evaporate, as genuine thickenings become available. For example, if P is a forcing notion in little-V then we can surely build a P-generic extension to get a little-$V[G]$. Of course we can't do this for V itself as in general we cannot construct generic sets for partial orders with uncountably many maximal antichains.

But the way we have analysed things with V-logic allows us to "reduce" our study of maximality criteria for V to a study of countable transitive models. As the collection of countable transitive models carries the name "Hyperuniverse", we are then led to what is known as the *Hyperuniverse Programme*.

I'll illustrate the reduction to the Hyperuniverse with the specific example of the IMH. Suppose that we formulate the IMH as above, using V-logic, and want to know what first-order consequences it has.

Claim. Suppose that a first-order sentence φ holds in all countable models of the IMH. Then it holds in all models of the IMH.

Here is the argument: Suppose that φ fails in some model V of the IMH, where V may be uncountable. Now notice that the IMH is first-order expressible in

V^+, a lengthening of V. But then apply the downward Löwenheim-Skolem theorem to obtain a countable little-V which satisfies the IMH, as verified in its associated little-V^+, and fails to satisfy φ. But this is a contradiction, as by hypothesis φ must hold in all *countable* models of the IMH. End of Proof.

So *without loss of generality*, when looking at first-order consequences of maximality criteria as formulated in V-logic via lengthenings of V, we can restrict ourselves to countable little-V's. The advantage of this is then we can dispense with the little-V-logic and the quotes in "thickenings" altogether, as by the Completeness Theorem for little-V-logic, consistent theories in little-V-logic do have models, thanks to the countability of little-V. Thus for a countable little-V, we can simply say:

IMH for little-V's: Suppose that a first-order sentence holds in an outer model of little-V. Then it holds in an inner model of little-V.

This is exactly the informal version of the IMH with which we began. Thus the informal and formal versions coincide on countable models.

The reduction to the Hyperuniverse is however not always so obvious; consider the case of #-generation, which uses not one theory but many. Thus to say that V is #-generated asserts the consistency of certain theories T_α in Gödel lengthenings $L_\alpha(V)$ for all α; with Löwenheim-Skolem we do indeed get a little-V with the property that for each α, the theory expressing the existence of an α-iterable generator for little-V is consistent. But the Completeness Theorem for little-V-logic then only gives us for each countable α the existence of an α-iterable generator and not a single generator that is α-iterable for all countable (and hence for all) α. We say that little-V is *weakly #-generated*. So the first-order consequences of #-generation coincide with the first-order consequences of *weak #-generation* for countable little-V's and not of full #-generation for countable little-V's; the latter indeed yields a larger theory.

The consistency of #-generation follows from the existence of 0#. But we haven't yet established the consistency of the IMH, which we discuss next.

Consistency of the IMH

In view of the *Reduction* described above, we can regard the IMH as a statement about countable transitive models of ZFC. So to establish its consistency we aim for the following result.

Theorem 1 *Assuming large cardinals there exists a countable transitive model M of ZFC such that if a first-order sentence φ holds in an outer model N of M then it also holds in an inner model of M.*

Proof. For any real R let $M(R)$ denote the least transitive model of ZFC containing R. We are assuming large cardinals so indeed such an $M(R)$ exists (the existence of just an inaccessible is sufficient for this). We will need the following consequence of large cardinals:

$(*)$ There is a real R such that for any real S in which R is recursive, the (first-order) theory of $M(R)$ is the same as the theory of $M(S)$.

One can derive $(*)$ from large cardinals as follows. Large cardinals yield Projective Determinacy (PD). A theorem of Martin is that PD implies the following *Cone Theorem*: If X is a projective set of reals closed under Turing-equivalence then for some real R, either S belongs to X for all reals S in which R is recursive or S belongs to the complement of X for all reals S in which R is recursive.

Now for each sentence φ consider the set $X(\varphi)$ consisting of those reals R such that $M(R)$ satisfies φ. This set is projective and closed under Turing-equivalence. By the cone theorem we can choose a real $R(\varphi)$ so that either φ is true in $M(S)$ for all reals S in which $R(\varphi)$ is recursive or this holds for $\sim \varphi$. Now let R be any real in which every $R(\varphi)$ is recursive; as there are only countably-many φ's this is possible. Then R witnesses the property $(*)$.

Now we claim that if N is an outer model of $M(R)$ satisfying ZFC and φ is a sentence true in N then φ is true in an inner model of $M(R)$. For this we need the following deep theorem of Jensen.

Coding Theorem. Let α be the ordinal height of N. Then N has an outer model of the form $L_\alpha[S]$ for some real S which satisfies ZFC and in which N is Δ_2-definable with parameters.

As R belongs to $M(R)$ it also belongs to N and hence to $L_\alpha[S]$ where S codes N as above. Also note that since α is least so that $M(R) = L_\alpha[R]$ models ZFC, it is also least so that $L_\alpha[S]$ satisfies ZFC and therefore $L_\alpha[S]$ equals $M(S)$.

Clearly we can choose S to be Turing above R (simply replace S by its join with R). But now by the special property of R, the theories of $M(R)$ and $M(S)$ are the same. As N is a definable inner model of $M(S)$, part of the theory of $M(S)$ is the statement "There is an inner model of φ which is Δ_2-definable with parameters" and therefore there is an inner model of $M(S)$ satisfying φ, as desired. $\Box$

Note that the model that we produce above for the IMH, $M(R)$ for some real R, is the minimal model containing the real R and therefore satisfies "there are no inaccessible cardinals". This is no accident:

Theorem 2 *Suppose that M satisfies the IMH. Then in M: There are no inaccessible cardinals and in fact there is a real R such that there is no transitive model of ZFC containing R.*

Proof. A theorem of Beller and David extends Jensen's Coding Theorem to say that any model M has an outer model of the form $M(R)$ for some real R, where as above $M(R)$ is the minimal transitive model of ZFC containing R. Now suppose that M satisfies the IMH and consider the sentence "There is no inaccessible cardinal". This is true in an outer model $M(R)$ of M and therefore in an inner model of M. It follows that there are no inaccessibles in M. The same argument with the sentence "There is a real R such that there is no transitive model of ZFC containing R" gives an inner model M_0 of M with this property for some real R; but then also M has this property as any transitive model of ZFC containing R in M would also give such a model in the $L[R]$ of M and therefore in M_0, as M_0 contains the $L[R]$ of M. $\Box$

It follows that if M satisfies the IMH then some real in M has no # and therefore boldface Π^1_1 determinacy fails in M.

10.-11.Vorlesungen

Synthesis

We introduced the IMH as a criterion for width maximality and #-generation as a criterion for height maximality. It is natural to see how these can be combined into a single criterion which recognises both forms of maximality. Note that the IMH implies that there are no inaccessibles yet #-generation implies that there are. So we cannot simply take the conjunction of these two criteria.

A #-generated model M satisfies the IMH# iff whenever a sentence holds in a #-generated outer model of M it also holds in an inner model of M.

Note that IMH# differs from the IMH by demanding that both M and M^*, the outer model, are #-generated (while the outer models considered in IMH are arbitrary). The motivation behind this requirement is to impose width maximality only with respect to those models which are height maximal.

Theorem 3 *Assuming that every real has a # there is a model satisfying IMH#.*

Proof. Let R be a real with the following property: Whenever X is a lightface and nonempty Π^1_2 set of reals, then X has an element recursive in R. We claim that any #-generated model M containing R as an element satisfies the IMH#.

Suppose that φ holds in M^*, a #-generated outer model of M. Let (m^*, U^*) be a # for M^*. Then the set X of reals S such that S codes such an (m^*, U^*) is a lightface Π^1_2 set. So there is such a real recursive in R and therefore in M. But then M has an inner model satisfying φ, namely any model generated by a # coded by an element of X in M. □

The original proof of the previous result used a Woodin cardinal with an inaccessible above. That proof seems to be needed for the consistency of variants of the IMH# which we'll discuss a bit later.

Corollary 4 *Suppose that φ is a sentence that holds in some V_κ with κ measurable. Then there is a transitive model which satisfies both the IMH# and the sentence φ.*

Proof. Let R be as in the proof of Theorem 3 and let U be a normal measure on κ. The structure $N = (H(\kappa^+), U)$ is a #; iterate N through a large enough ordinal ∞ so that $M = LP(N_\infty)$, the lower part model generated by N, has ordinal height ∞. Then M is #-generated and contains the real R. It follows that M is a model of the IMH#. Moreover, as M is the union of an elementary chain $V_\kappa = V^N_\kappa \prec V^{N_1}_{\kappa_1} \prec \cdots$ where φ is true in V_κ, it follows that φ is also true in M. □

Note that in Corollary 4, if we take φ to be any large cardinal property which holds in some V_κ with κ measurable, then we obtain models of the IMH# which also satisfy this large cardinal property. This implies the compatibility of the IMH# with arbitrarily strong large cardinal properties.

Thus we recognise that our perspective regarding the maximality of V in height and width changes as we discover and synthesise different mathematical criteria for expressing this feature. It is the aim of the Hyperuniverse Programme (HP) to formulate, analyse and synthesise the widest possible range of such criteria with an aim towards reaching consensus with an *optimal criterion*. This will no doubt take an enormous amount of work, both philosophical and mathematical, but the ultimate benefit will be that first-order consequences of such an optimal criterion can be regarded as consequences of the maximal iterative conception of set and therefore regarded as legitimate evidence for what is true beyond the axioms of ZFC.

Most of the mathematical work in the HP is yet to be done. Therefore what I will do in the remainder of this course is simply present a range of maximality criteria which are yet to be fully analysed and which give the flavour of how the HP is intended to proceed. These criteria are also referred to as *H-axioms*, formulated as properties of elements of the Hyperuniverse expressible within the Hyperuniverse.

H-Axioms: The Strong IMH

Our discussion of the IMH has been always with regard to sentences, without parameters. Stronger forms result if we introduce parameters.

First note the difficulties with introducing parameters into the IMH. For example the statement

"If a sentence with parameter ω_1^V holds in an outer model of V then it holds in an inner model"

is inconsistent, as the parameter ω_1^V could become countable in an outer model and therefore the above cannot hold for the sentence "ω_1^V is countable". If we however require that ω_1 is preserved then we get a consistent principle.

Theorem 5 *Let SIMH(ω_1) be the following principle: If a sentence with parameter ω_1 holds in an ω_1-preserving outer model then it holds in an inner model. Then the SIMH(ω_1) is consistent (assuming large cardinals).*

Proof. Again use PD to get a real R such that the theory of $M(S)$, the least transitive ZFC model containing S, is fixed for all S Turing above R. Now suppose that $\varphi(\omega_1)$ is a sentence true in an ω_1-preserving outer model M of $M(R)$, where ω_1 denotes the ω_1 of $M(R)$. Then as in the proof of consistency of the IMH, we can code M into $M(S)$ for some real S Turing above R, and moreover this coding is ω_1-preserving. As $\varphi(\omega_1)$ holds in a definable inner model of $M(S)$ and ω_1 is the same in $M(R)$ and $M(S)$, it follows that $M(R)$ also has an inner model satisfying $\varphi(\omega_1)$. $\square$

The above argument uses the fact that Jensen-coding is ω_1-preserving. It is however not ω_2-preserving unless CH holds, and therefore we have the following open question:

Q1. Let SIMH(ω_1, ω_2) be the following principle: If a sentence with parameters ω_1, ω_2 holds in an ω_1-preserving and ω_2-preserving outer model then it holds in an inner model. Then is the SIMH(ω_1, ω_2) consistent (assuming large cardinals)?

The SIMH(ω_1, ω_2) implies the failure of CH, as any model has a cardinal-preserving outer model in which CH is false and therefore there is an injection from ω_2 into the reals. Is there an analogue $M^*(R)$ of the minimal model $M(R)$ which does not satisfy CH? Is there a coding theorem which says that any outer model of $M^*(R)$ which preserves ω_1 and ω_2 has a further outer model of the form $M^*(S)$, also with the same ω_1 and ω_2? If so, then one could establish the consistency of the SIMH(ω_1, ω_2).

The most general from of the SIMH makes use of *absolute parameters*. A cardinal κ is *absolute* if some formula defines it in all outer models which preserve cardinals up to and including κ. Then SIMH($\leq \kappa$) for an absolute cardinal κ states that if a sentence with absolute cardinal parameters at most κ holds in an outer model which preserves cardinals up to and including κ then it holds in an inner model. The full SIMH (Strong Inner Model Hypothesis) states that this holds for every absolute cardinal parameter κ.

A simpler form of the SIMH is the *cardinal-preserving* SIMH, which uses full cardinal-preservation instead of the "local" cardinal-preservation of the SIMH. It says that if a sentence with cardinal-absolute parameters holds in a cardinal-preserving outer model then it holds in an inner model, where a cardinal is *cardinal-absolute* if some formula defines it in all cardinal-preserving outer models.

Q2. Is the cardinal-preserving SIMH consistent (assuming large cardinals)?

The SIMH criteria are closely related to strong forms of Lévy absoluteness. For example, define Lévy(ω_1) to be the statement that Σ_1 formulas with parameter ω_1 are absolute for ω_1-preserving outer models; this follows from the $SIMH(\omega_1)$ and is therefore consistent. But the consistency of Lévy(ω_1, ω_2), i.e. Σ_1 absoluteness with parameters ω_1, ω_2 for outer models which preserve these cardinals, is open.

The SIMH#

There are two possible syntheses of the strong forms of the IMH with #-generation, depending on whether one uses SIMH or its cardinal-preserving version. The consistency of either synthesis is open. The version with cardinal-preservation is formulated as follows:

V satisfies the *cardinal-preserving SIMH#* iff V is #-generated and whenever a first-order sentence φ with cardinal-absolute parameters holds in a cardinal-preserving, #-generated outer model then it holds in some inner model.

The version based on the SIMH uses only "local" versions of cardinal-preservation. I'll discuss only the case of SIMH#(ω_1), which says:

V is #-generated and whenever a first-order sentence $\varphi(\omega_1)$ with parameter ω_1 holds in an ω_1-preserving #-generated outer model then it holds in an inner model.

Theorem 6 *Assuming large cardinals, the SIMH#(ω_1) is consistent.*

Proof. Assume there is a Woodin cardinal with an inaccessible above. For each real R let $M^{\#}(R)$ be $L_\alpha[R]$ where α is least so that $L_\alpha[R]$ is #-generated. The Woodin cardinal with an inaccessible above implies enough projective determinacy to enable us to use Martin's Lemma to find a real R such that the theory of $M^{\#}(S)$ is constant for S Turing-above R. We claim that $M^{\#}(R)$ satisfies SIMH#(ω_1): Indeed, let M be a #-generated ω_1-preserving outer model of $M^{\#}(R)$ satisfying some sentence $\varphi(\omega_1)$. Let α be the ordinal height of $M^{\#}(R)$ (= the ordinal height of M). By a result of Jensen, M has a #-*generated* ω_1-preserving outer model W of the form $L_\alpha[S]$ for some real S

with $R \leq_T S$. Of course α is least so that $L_\alpha[S]$ is #-generated. So W equals $M^\#(S)$ and the ω_1 of W equals the ω_1 of $M^\#(R)$. By the choice of R, $M^\#(R)$ also has a definable inner model satisfying $\varphi(\omega_1)$. $\square$

However as with the SIMH(ω_1, ω_2), the consistency of SIMH#(ω_1, ω_2) is open.

12.-13.Vorlesungen

A Maximality Protocol

This protocol aims to organise the study of height and width maximality into three stages.

Stage 1. Maximise the ordinals (height-maximality).

Stage 2. Having maximised the ordinals, maximise the cardinals (and cofinalities).

Stage 3. Having maximised the ordinals and cardinals, maximise powerset.

Stage 1 is taken care of by #-generation. So we focus now on Stage 2, cardinal- (and cofinality-) maximisation. As cofinality-maximisation is a small variant of cardinal-maximisation I'll just talk about the latter.

We would like a criterion which says that for each cardinal κ, κ^+ is "as large as possible". To get started let's consider the case $\kappa = \omega$, so we want to maximise ω_1. The basic problem of course is the following:

Fact. V has an outer model V^* in which ω_1^V is countable.

But surely we would want something like: ω_1^L is countable. The reason for this is that ω_1^L, unlike ω_1^V in general, is "absolute" between V and its outer models.

Definition. Let p be a parameter in V and V^* an outer model of V. Then p is *absolute between V and V^** if there is a formula φ that defines p in both V and V^*.

So we could try the following:

CardMax1. Suppose that the ordinal α is absolute between V and an outer model V^*. Then if α is countable in V^* it is also countable in V.

It is natural to strengthen this from countability to an arbitrary cardinality. Let's say that p is *absolute between V and V^* relative to parameters in P* if there is a formula φ with parameters in P that defines p in both V and V^*.

CardMax2. Suppose that κ is an infinite cardinal and the ordinal α is absolute between V and the outer model V^* relative to the parameter κ. Then if α has cardinality at most κ in V^* it is has cardinality at most κ in V.

Proposition 7 *CardMax2 implies that κ^+ is greater than the κ^+ of HOD for each infinite cardinal κ.*

Proof. Suppose not and let V^* be an outer model of V in which the HOD of V equals the HOD of V^* and in which κ^+ is collapsed to ω. (Such a model can be obtained by coding the HOD of V into the GCH-pattern above κ^+ and then applying the homogeneous Lévy collapse of κ^+ to ω.) Then the κ^+ of V is absolute between V and V^* relative to the parameter κ, is countable in V^* but remains a cardinal in V, contradicting CardMax2. □

The conclusion of the Proposition is known to be consistent relative to a supercompact.

Q3. Is CardMax2 consistent?

And finally, we can introduce more parameters.

CardMax3. Suppose that the ordinal α is absolute between V and the outer model V^* relative to ordinal parameters less than α. Then if α is collapsed (i.e. is no longer a cardinal) in V^* it is also collapsed in V.

Proposition 8 *CardMax3 implies that κ^+ is inaccessible in HOD for each infinite cardinal κ.*

Proof. Suppose not and let κ^+ of V be λ^+ of HOD. Then follow the proof of Proposition 7 with the parameter λ to reach a contradiction. □

Recent work suggests that the conclusion of the previous Proposition is also consistent.

CardMax4. Suppose that the ordinal α is absolute between V and the outer model V^* relative to bounded subsets of α as parameters. Then if α is collapsed (i.e. is no longer a cardinal) in V^* it is also collapsed in V.

Theorem 9 *CardMax4 is inconsistent.*

Proof. By a theorem of Shelah, if κ is singular of uncountable cofinality then κ^+ equals κ^+ of HOD_x for some $x \subseteq \kappa$. Let α be the κ^+ of V. Now let V^* be an outer model of V in which HOD_x does not change and α is collapsed to ω with a homgeneous Lévy collapse. Then α is absolute between V and V^* relative to the parameters x, κ, collapsed in V^* but not in V, yielding a counterexample to CardMax4. □

OK, so we went too far. But perhaps we can explain our mistake: Our aim is to maximise cardinals. Now CardMax4 when κ is ω looks consistent. And having maximised ω_1 in this way we can then move on to CardMax4 for $\kappa = \omega_1$, but we should require that ω_1 is preserved when moving from V to its outer model V^*, as it has already been "maximised". In general what is missing is the requirement that the previous cardinal is preserved. So we arrive at:

CardMax. Suppose that κ is an infinite cardinal and the ordinal α is absolute between V and the outer model V^* relative to subsets of κ. Also suppose that κ remains a cardinal in V^*. Then if α has cardinality at most κ in V^* it also has cardinality at most κ in V.

Q4. Is CardMax consistent?

Stage 3: Having maximised the ordinals and cardinals, maximise powerset.

This is where we revisit the SIMH, but only in the context of cardinal-preservation.

Cardinal-preserving IMH with locally-absolute parameters. Suppose that V^* is an outer model of V with the same cardinals as V and p is absolute between V and V^*. Then if a sentence with p as parameter holds in V^* it also holds in an inner model of V.

Apparently weaker is to require more absoluteness of the parameters. A parameter p is *cardinal-absolute* if there is a parameter-free formula which has p as its unique solution in all outer models of V which have the same cardinals as V.

Cardinal-preserving IMH with cardinal-absolute parameters. Suppose that p is cardinal-absolute, V^* is an outer model of V with the same cardinals as V and φ is a sentence with parameter p which holds in V^*. Then φ holds in an inner model of V.

Q5. Is the cardinal-preserving IMH with locally-absolute cardinal parameters or with cardinal-absolute parameters consistent?

Note that either of these criteria implies a strong failure of CH.

Width Indiscernibility

An alternative to the Maximality Protocool (which ideally should be synthesised with it) is *Width Indiscernibility.* The motiviation is to provide a description of V in width analogous to its description in height as provided by #-generation.

Recall that with #-generation we arrive at the following:

$$V_0 \prec V_1 \prec \cdots \prec V = V_\infty \prec V_{\infty+1} \prec \cdots$$

where for $i < j$, V_i is a *rank-initial segment* of V_j. Moreover the models V_i form a collection of "indiscernible models" in a strong sense. This picture was the result of an analysis which began with "height reflection", starting with the idea that V must have unboundedly many rank-initial segments V_i which are elementary in V.

Analogously, we introduce "width reflection". We would like to say that V has proper inner models which are "elementary" in V. Of course this cannot literally be true, as if V_0 is an elementary submodel of V with the same ordinals as V then it is easy to see that V_0 equals V. Instead, we use elementary embeddings.

Width Reflection. For each ordinal α, there is a proper elementary submodel H of V such that $V_\alpha \subseteq H$ and H is *amenable*, i.e. $H \cap V_\beta$ belongs to V for each ordinal β.

Equivalently:

Width Reflection. For each ordinal α, there is a nontrivial elementary embedding $j : V_0 \to V$ with critical point at least α such that j is *amenable*, i.e. $j \restriction (V_\beta)^{V_0}$ belongs to V for each ordinal β.

Let's write $V_0 < V$ if there is a nontrivial amenable $j : V_0 \to V$, as in the second formulation of width reflection. This relation is transitive.

Proposition 10 *(a) If $V_0 < V$ then V_0 is a proper inner model of V.*
(b) Width Reflection is consistent relative to the existence of a Ramsey cardinal.

Proof. (a) This follows from Kunen's Theorem that there can be no nontrivial elementary embedding from V to V.
(b) Suppose that κ is Ramsey. Then it follows that any structure of the form $\mathcal{M} = (V_\kappa, \in, \ldots)$ has an unbounded set of indiscernibles, i.e. an unbounded subset I of κ such that for each n, any two increasing n-tuples from I satisfy the same formulas in $\mathcal{M}$. Now apply this to $\mathcal{M} = (V_\kappa, \in, <)$ where $<$ is a wellorder of V_κ of length κ. Let J be any unbounded subset of I such that $I \setminus J$ is unbounded and for any $\alpha < \kappa$, let $H(J \cup \alpha)$ denote the Skolem hull of $J \cup \alpha$ in $\mathcal{M}$. Then $H(J \cup \alpha)$ is an elementary submodel of V_κ and is not equal to V_κ because no element of $I \setminus J$ greater than α belongs to it. As V_κ contains all bounded subsets of κ it follows that $H(J \cup \alpha)$ is amenable. $\square$

A variant of the argument in (b) above yields the consistency of arbitrarily long finite chains $V_0 < V_1 < \cdots < V_n$. But obtaining infinite such chains seems more difficult, and even more ambitiously we can ask:

Q6. Is it consistent to have $V_0 < V_1 < \cdots < V$ of length $\mathrm{Ord} + 1$ such that the union of the V_i's equals V?

The latter would be a good start on the formulation of a consistent criterion of *Width Indiscernibility*, as an analogue for maximality in width to the criterion of maximality in height provided by #-generation.

14.-15.Vorlesungen

Omniscience

Although the Hyperuniverse Programme focuses on the maximality of V, it can also be used to study a related property, called *omniscience*. Although this property may not be "intrinsic" to the set-concept in the way that maximality is, it is nevertheless a natural and appealing feature for the universe to enjoy.

As we typically do in the Hyperuniverse Programme, we work with elements of the Hyperuniverse, i.e. with countable transitive models of ZFC. For convenience let V denote such a model (and not the universe of all sets). By $\mathrm{OMT}(V)$, the *outer model theory of* V we mean the set of sentences with arbitrary parameters from V which hold in all outer models of V. As usual, we can use V-logic to express this in $\mathrm{Hyp}(V)$, the least admissible set containing V as an element. Recall that $\mathrm{Hyp}(V)$ is also countable and of the form $L_\alpha(V)$ where α is least so that this set, the α-th level of the L-hierarchy built over V, satisfies Σ_1 replacement.

Recall the following version of Tarski's result on the undefinability of truth:

Proposition 11 *The set of sentences with parameters from V which hold in V is not (first-order) definable in V with parameters.*

Surprisingly, Mack Stanley showed however that $\mathrm{OMT}(V)$ can indeed be V-definable.

Theorem 12 *(M.Stanley) Suppose that in V there is a proper class of measurable cardinals, and indeed this class is Hyp(V)-stationary, i.e. Ord(V) is regular with respect to Hyp(V)-definable functions and this class intersects every club in Ord(V) which is Hyp(V)-definable. Then OMT(V) is V-definable.*

Proof. Using V-logic we can translate the statement that a first-order sentence φ (with parameters from V) holds in all outer models of V to the validity of a sentence φ^* in V-logic, a fact expressible over $\mathrm{Hyp}(V)$ by a Σ_1 sentence. Using this we show that the set of φ which hold in all outer models of V is V-definable.

As $\mathrm{Ord}(V)$ is regular with respect to $\mathrm{Hyp}(V)$-definable functions we can form a club C in $\mathrm{Ord}(V)$ such that for κ in C there is a Σ_1-elementary embedding from $\mathrm{Hyp}(V_\kappa)$ into $\mathrm{Hyp}(V)$ (with critical point κ, sending κ to $\mathrm{Ord}(V)$). Indeed C can be chosen to be $\mathrm{Hyp}(V)$-definable.

For any κ in C let φ^*_κ be the sentence of V_κ-logic such that φ holds in all outer models of V_κ iff φ^*_κ is valid (a Σ_1 property of $\mathrm{Hyp}(V_\kappa)$). By elementarity, φ^*_κ is valid iff φ^* is valid.

Now suppose that φ holds in all outer models of V, i.e. φ^* is valid. Then φ^*_κ is valid for all κ in C and since the measurables form a $\mathrm{Hyp}(V)$-stationary class, there is a measurable κ such that φ^*_κ is valid.

Conversely, suppose that φ^*_κ is valid for some measurable κ. Now choose a normal measure U on κ and iterate $(H(\kappa^+), U)$ for $\mathrm{Ord}(V)$ steps to obtain a wellfounded structure $(H^*, U*)$. (This structure is wellfounded, as for any admissible set A, any measure in A can be iterated without losing wellfoundedness for α steps, for any ordinal α in A.) Then H^* equals $\mathrm{Hyp}(V^*)$ for some $V^* \subseteq V$. By elementarity, the sentence $\varphi^*_{V^*}$ which asserts that φ holds in all outer models of V^* is valid. But as V^* is an inner model of V, φ also holds in all outer models of V.

Thus φ belongs to $\mathrm{OMT}(V)$ exactly if it belongs to $\mathrm{OMT}(V_\kappa)$ for some measurable κ, and this is first-order expressible. □

Are measurable cardinals needed for omniscience? Actually, M.Stanley was able to use just Ramsey cardinals, but as far as the consistency of omniscience we have the following:

Theorem 13 *(Radek and I) Suppose that κ is inaccessible and GCH holds. Then there is an omniscient model of the form $V_\kappa[G]$ where G is generic over V.*

I'll sketch the proof (the details will appear in a forthcoming paper).

Start with a ground model V satisfying GCH with a least inaccessible κ. Let $(X_i \mid i < \kappa)$ be a partition of X = the set of singular cardinals $< \kappa$ into κ-many unbounded pieces such that $X_i \cap i$ is empty for each i.

We are going to force over V with an *f-good iteration* of some length $\mu < \kappa^+$. Here, f is an injection from μ into X such that $\mathrm{Range}(f) \cap X_i$ is bounded in κ for each i and P is a full-support iteration of length μ which at stage i forces either with $\mathrm{Add}(f(i)^+, f(i)^{+++})$ or with $\mathrm{Add}(f(i)^{++}, f(i)^{++++})$. I.e., at stage i, P kills the GCH either at $f(i)^+$ or at $f(i)^+ +$, but not at both.

Note that each stage of the iteration is a forcing of size less than κ, so in a sense we are forcing over V_κ. But this is not an iteration over V_κ in the usual sense, as even though the stages of the iteration are of size less than κ, the length of the iteration can be greater than κ. It is a "non-monotone reverse Easton" iteration, in which the choice of cardinals at which to force does not increase as the iteration proceeds, but can jump up and down in a chaotic fashion. From the perspective of V_κ this is a new kind of "iterated hyperclass forcing".

A key lemma is that good iterations preserve cofinalities (see the forthcoming paper for details).

Now to prove the theorem we define a particular f-good iteration P of some length μ as the limit of f_i-good iterations P_i of lengths $\mu_i < \mu$ as follows:

At stage i, via bookkeeping look at the i-th sentence φ_i with parameters in $V_\kappa[G_i]$ and see if there is an f_i^*-good iteration P_i^* extending the f_i-good iteration P_i which forces that φ_i belongs to $\mathrm{OMT}(V_\kappa[G_i^*])$, where G_i^* denotes the P_i^*-generic. If so, then we take P_{i+1} to be P_i^* followed by the forcing to kill the GCH at α_i^+, where α_i is the least element of X_i greater than the elements of X_i used in the iteration P_i^* (i.e. greater than all elements of $X_i \cap \mathrm{Range}(f_i^*)$). And we define f_{i+1} to extend f_i^* by assigning the value α_i at the least ordinal outside the domain of f_i^*.

On the other hand if there is no such good iteration extending P_i then we instead just force to kill the GCH at α_i^{++} where α_i is the least element of X_i greater than the elements of X_i used in the iteration P_i.

We claim that $V_\kappa[G]$, where G is generic for the final good iteration P, the limit of the good iterations P_i for $i < \kappa$, is omniscient. Indeed, suppose that φ is a sentence with parameters from $V_\kappa[G]$ and suppose that φ belongs to $\mathrm{OMT}(V_\kappa[G]$. Then at some stage $i < \kappa$, $\varphi = \varphi_i$ and we extended P_i to a good iteration P_{i+1} forcing φ into $\mathrm{OMT}(V_\kappa[G_{i+1}])$, coding this by killing GCH at α_i^+; but then the largest cardinal in X_i where GCH fails in the final model is α_i^+, the successor (and not the double successor) of a singular cardinal.

Alternatively, if φ does not belong to $\mathrm{OMT}(V_\kappa[G])$ it also does not belong to $\mathrm{OMT}(V_\kappa[G_i])$ and we coded this by killing the GCH at α_i^{++}; this time the largest cardinal in X_i where GCH fails is a double successor. So in the final model $V_\kappa[G]$ we can definably recover the set of sentences φ which belong to $\mathrm{OMT}(V_\kappa[G])$, completing the proof.