Equivalence Relations in Set Theory, Computation Theory and Complexity Theory

Definable Equivalence relations constitute a popular topic in *Classical Descriptive Set Theory*

Some examples:

• (*Non-turbulent*): Isomorphism relations for classes of countable linear orders, groups, graphs, fields, trees, Boolean algebras

• (*Turbulent*): Conjugacy of homeomorphisms of the unit square
Conjugacy of ergodic, measure-preserving transformations
Unitary equivalence of unitary operators
Conformal equivalence of Riemann surfaces

These are analytic (Σ^1_1 with parameters) equivalence relations on Polish spaces (think of the reals)

Equivalence Relations in Set, Computation and Complexity Theory

Such equivalence relations are compared using *Borel reducibility*:

E_0 *is Borel reducible to* E_1 iff there is a Borel function $f : X_0 \to X_1$ such that

$$xE_0y \text{ iff } f(x)E_1f(y)$$

Recent work: Three new contexts for this study

1. Effective Theory of Borel reducibility: E_0, E_1 are now Σ^1_1 *without parameters* and f is Hyp ($=$ Hyperarithmetic $= \Delta^1_1$ without parameters $=$ effectively Borel).

Equivalence Relations in Set, Computation and Complexity Theory

2. Σ^1_1 equivalence relations on the natural numbers $\mathcal{N}$: The reduction $f : \mathcal{N} \to \mathcal{N}$ is required to be Hyp. Key examples: Isomorphism relations on the *computable* models of a Hyp theory (Hyp classes of computable groups, computable graphs, computable fields, $\cdots$)

3. NP equivalence relations on the set Σ of finite strings: The reduction $f : \Sigma \to \Sigma$ is required to be Polytime computable. Key examples: Isomorphism relations on classes of *finite structures* which are invariant (closed under isomorphism) and Polytime-definable (finite linear orders, finite vector spaces over a fixed finite field, finite Abelian groups, finite connected graphs with a fixed bound on the degree, $\cdots$)

Equivalence Relations in Other Contexts: Effective Borel reducibility

Effective Theory of Borel Reducibility

Joint work with Katia Fokina and Asger Törnquist (postdocs at the KGRC)

First a review of the classical, non-effective theory. We focus on the case of *Borel* (not arbitrary analytic) equivalence relations

Object of study: $\mathcal{B}$ = Degrees of Borel equivalence relations under Borel reducibility

The Effective Theory of Borel Equivalence Relations

Work of Silver and of Harrington-Kechris-Louveau identifies an interesting initial segment of $\mathcal{B}$:

Theorem

$\mathcal{B}$ has an initial segment

$1 < 2 < \cdots < \omega < =_R < E_0$

where:

$n =$ Borel equivalence relations with exactly n classes
$\omega =$ Borel equivalence relations with exactly $\aleph_0$ classes
$=_R$ is $({}^{\omega}\omega, =)$ (equality on reals)
E_0 is the equivalence relation xE_0y iff $x(n) = y(n)$ for all but finitely many n

In fact: Any Borel equivalence relation is Borel equivalent to one of the above or lies strictly above E_0 under Borel reducibility.

The Effective Theory of Borel Equivalence Relations

Question: What happens if we replace “Borel” by “Hyp”? We define:

If E and F are Hyp equivalence relations on the reals then
E is *Hyp reducible to* F, written $E \leq_H F$, iff
For some Hyp function f, $x \; E \; y$ iff $f(x) \; F \; f(y)$

$\leq_H$ is reflexive and transitive
$E \equiv_H F$ iff $E \leq_H F$ and $F \leq_H E$ (equivalence relation)
$[E]_H =$ the equivalence class of E under $\equiv_H$

Object of study: $\mathcal{H} =$ Degrees of Hyp equivalence relations on the reals under Hyp reducibility

There are some surprises!

The Effective Theory of Borel Equivalence Relations

Again we have degrees

$1 < 2 < \cdots < \omega < =_R < E_0$

defined as follows:

n is represented by $xE^n y$ iff $x(0) = y(0) < n - 1$ or
$x(0), y(0) \geq n - 1$
ω is represented by $xE^\omega y$ iff $x(0) = y(0)$
$=_R$, E_0 are as before:
xRy iff $x = y$
$xE_0 y$ iff $x(n) = y(n)$ for all but finitely many n

Proposition

There are Hyp equivalence relations strictly between 1 and 2!

The Effective Theory of Borel Equivalence Relations

Explanation:

Let E be a Hyp equivalence relation. Recall that the $\mathcal{H}$-degree n is represented by the equivalence relation E^n where:

xE^ny iff $x(0) = y(0) < n - 1$ or $x(0), y(0) \geq n - 1$

Fact 1. E^n is Hyp reducible to E iff at least n distinct E-equivalence classes contain Hyp reals

Proof. Suppose that E^n Hyp reduces to E via the Hyp function f. Each of the n equivalence classes of E^n contains a Hyp real; let $x_0, \ldots, x_{n-1}$ be Hyp, pairwise E^n-inequivalent reals. Then the reals $f(x_i)$, $i < n$, are Hyp, pairwise E-inequivalent reals.
Conversely, if $y_0, \ldots, y_{n-1}$ are Hyp, pairwise E-inequivalent reals then send the E^n-equivalence class of x_i to the real y_i; this is a Hyp reduction of E^n to E. □

Fact 2. E is Hyp reducible to E^2 iff E has at most 2 equivalence classes.

Proof. If E is Hyp reducible to E^2 then E has at most 2 equivalence classes because E^2 has only 2 equivalence classes. Conversely, suppose that the equivalence classes of E are A_0 and A_1. We may assume that A_0 has a Hyp element x. Then A_0 is Hyp as it consists of those reals E-equivalent to x and A_1 is Hyp as it consists of those reals not E-equivalent to x. Now we can reduce E to E^2 by choosing E^2-inequivalent Hyp reals y_0, y_1 and sending the elements of A_0 to y_0 and the elements of A_1 to y_1. $\square$

So to get a Hyp equivalence relation between 1 and 2 we need only find one with two equivalence classes but with all Hyp reals in just one class. This follows from a classical fact from Hyp theory:

Fact 3. There are nonempty Hyp sets of reals which contain no Hyp element.

Proof. Let A be the set of non-Hyp reals. Then A is Σ^1_1 and therefore the projection of a Π^0_1 subset P of Reals $\times$ Reals. P is nonempty. A Hyp real $h = (h_0, h_1)$ in P would give a Hyp real h_0 in A, contradiction. $\square$

Now we ask a harder question: Are there incomparable degrees between 1 and 2?

To answer this we prove:

Theorem

There exists Hyp sets of reals A, B such that for no Hyp function F do we have $F[A] \subseteq B$ or $F[B] \subseteq A$.

Given this Theorem, define E_A to be the equivalence relation with equivalence classes A and $\sim A$ (the complement of A); define E_B similarly. Note that the sets A, B contain no Hyp reals, else there would be a constant Hyp function F mapping one of them into the other. So a Hyp reduction of E_A to E_B would have to send the elements of $\sim A$ (which contains Hyp reals) to elements of $\sim B$, and therefore the elements of A to elements of B, contradicting the Theorem. Similarly there is no Hyp reduction of E_B to E_A.

It remains to prove:

Theorem

There exists Hyp sets of reals A, B such that for no Hyp function F do we have $F[A] \subseteq B$ or $F[B] \subseteq A$.

Proof Sketch. First we quote a result of Harrington. For reals a, b and a recursive ordinal α we say that a is α-*below* b iff a is recursive in the α-jump of b.

Fact. For any recursive ordinal α there are Π^0_1 singletons a, b such that a is not α-below b and b is not α-below a.

Now using Barwise Compactness, find a nonstandard ω-model M of ZF^- with standard ordinal ω_1^{CK} in which are there are Π_1^0 singletons a, b such that for all recursive α, a is not α-below b and b is not α-below a (i.e., a and b are Hyp incomparable.)
Let a, b be the unique solutions in M to the Π_1^0 formulas φ_0, φ_1, respectively.
The desired sets A, B are $\{x \mid \varphi_0(x)\}$ and $\{x \mid \varphi_1(x)\}$.
If F were a Hyp function mapping A into B, then it would send the element a of A to an element $F(a)$ of $B \cap M$;
but then $F(a)$ must equal b and therefore b is Hyp in a, contradicting the choice of a, b. $\square$

The Effective Theory of Borel Equivalence Relations

For the remainder of this talk, fix A, B as in the Theorem: there is no Hyp function F such that $F[A] \subseteq B$ or $F[B] \subseteq A$.

Using A, B we can also get incomparable Hyp equivalence relations between n and $n+1$ for any finite n, by considering E_A, E_B where the equivalence classes of E_A are A together with a split of $\sim A$ into n classes, each of which contains a Hyp real (similarly for E_B).

We now consider Hyp equivalence relations with infinitely many equivalence classes.

The Effective Theory of Borel Equivalence Relations

Recall the Silver and Harrington-Kechris-Louveau dichotomies:

Theorem

(a) (Silver) A Borel equivalence relation is either Borel reducible to ω or Borel reduces $=_R$.
(b) (H-K-L) A Borel equivalence relation is either Borel reducible to $=_R$ or Borel reduces E_0.

How effective are these results?
Harrington's proof of (a) and the original proof of (b) show:

Theorem

(a) A Hyp equivalence relation is either Hyp reducible to ω or Borel reduces $=_R$.
(b) A Hyp equivalence relation is either Hyp reducible to $=_R$ or Borel reduces E_0.

The sets A, B can be used to show that the Silver and Harrington-Kechris-Louveau dichotomies are *not* fully effective:

Theorem

(a) There are incomparable Hyp equivalence relations between ω *and* $=_R$*.*
(b) There are incomparable Hyp equivalence relations between $=_R$ *and* E_0*.*

The Effective Theory of Borel Equivalence Relations

Proof Sketch of (a): Consider the relations

$E_A(x, y)$ iff ($x \in A$ and $x = y$) or ($x, y \notin A$ and $x(0) = y(0)$)

E_B: The same, with A replaced by B

Now E^ω Hyp reduces to E_A by $n \mapsto (n, 0, 0, \ldots)$.
Also E_A Hyp reduces to $=_R$ via the map $G(x) = x$ for $x \in A$, $G(x) = (x(0), 0, 0, \ldots)$ for $x \notin A$ (same for B)

There is no Hyp reduction of E_A to E_B:
If F were such a reduction then let C be $F^{-1}[\sim B]$.
As $\sim B$ is Hyp, C is also Hyp and therefore $A \cap C$ is also Hyp.
But $A \cap C$ must be countable as F is a reduction.
So if $A \cap C$ were nonempty it would have a Hyp element, contradicting the fact that A has no Hyp element.
Therefore F maps A into B, which is impossible by the choice of A, B. By symmetry, there is no Hyp reduction of E_B to E_A.

The Effective Theory of Borel Equivalence Relations

The overall picture of the degrees is the following:
Call a degree *canonical* if it is one of $1 < 2 < \cdots < \omega <=_R< E_0$.

For any two canonical degrees $a < b$ there is a rich collection of degrees which are above a, below b and incomparable with all canonical degrees in between.

However at least one nice things happens: If a degree is above n for each finite n, then it is also above ω.

The Effective Theory of Borel Equivalence Relations

Some remaining Open Questions:
1. If a Hyp equivalence relation is Borel reducible to E_0 must it also be Hyp reducible to E_0? (This is true for finite n, ω, $=_R$.)
2. Are there any nodes other than 1? I.e., is there a Hyp equivalence relation with more than one equivalence class which is comparable with all Hyp equivalence relations under Hyp reducibility?
3. Is there a minimal degree? Are there incomparables above each degree?

There is also a jump operation, which requires further study.

Analytic equivalence relations on the reals

So far we have considered only Borel equivalence relations.
But there are many interesting analytic (Σ^1_1 with parameters) equivalence relations which are not Borel:

Let T be any theory in first-order logic (or any sentence of the infinitary logic $\mathcal{L}_{\omega_1\omega}$). Then the isomorphism relation on the countable models of T is an analytic equivalence relation which need not be Borel.

But there are many analytic equivalence relations which are not reducible to such an isomorphism relation;
an example is E_1, the equivalence relation on R^ω defined by:

$$\vec{x} E_1 \vec{y} \text{ iff } \vec{x}(n) = \vec{y}(n) \text{ for almost all } n$$

Note that E_1 is even Hyp.

Σ^1_1 equivalence relations on the natural numbers

A motivating question for the study of Σ^1_1 equivalence relations on the natural numbers was the following:

Question. Is every Σ^1_1 equivalence relation on the natural numbers reducible to isomorphism on a Hyp class of *computable* structures?

The reducibility we use is: $E_0 \leq_H E_1$ iff there is a Hyp function $f : \mathcal{N} \to \mathcal{N}$ such that $m E_0 n$ iff $f(m) E_1 f(n)$.
(We say that E_0 is *Hyp-reducible* to E_1.)

Theorem

(Fokina-Friedman-Knight-Montalban et.al.) Every Σ^1_1 equivalence relation on $\mathcal{N}$ is Hyp-reducible to isomorphism on computable trees.

This answers the above Question positively.

Σ^1_1 equivalence relations on the natural numbers

Theorem

(Fokina-Friedman-Knight-Montalban et.al.) Every Σ^1_1 equivalence relation on $\mathcal{N}$ is Hyp-reducible to isomorphism on computable trees.

Proof Sketch: Let E be a Σ^1_1 equivalence relation on $\mathcal{N}$ and choose a computable $f : \mathcal{N}^2 \rightarrow$ *Computable Trees* such that $\sim mEn$ iff $f(m, n)$ is well-founded.

Now associate to pairs m, n computable trees $T(m, n)$ so that:

$T(m, n)$ is isomorphic to $T(n, m)$
mEn implies that $T(m, n)$ is isomorphic to the "canonical" non-well-founded computable tree
$\sim mEn$ implies that $T(m, n)$ is isomorphic to the "canonical" computable tree of rank α, where α is least so that $f(m', n')$ has rank at most α for all $m' \in [m]_E$, $n' \in [n]_E$.

Σ^1_1 equivalence relations on the natural numbers

Now to each n associate the tree T_n gotten by gluing together the $T(n,i)$, $i \in \omega$.
If mEn then T_m is isomorphic to T_n as they are obtained by gluing together isomorphic trees.
And if $\sim mEn$ then T_m, T_n are not isomorphic as they are obtained by gluing together trees which on some component are non-isomorphic. $\square$

Σ_1^1 equivalence relations on the natural numbers

It can be shown that the isomorphism relation on computable trees (and therefore any Σ_1^1 equivalence relation on $\mathcal{N}$) Hyp-reduces to the isomorphism relation on each of the following Hyp classes:

1. Computable graphs
2. Computable torsion-free Abelian groups
3. Computable Abelian p-groups for a fixed prime p
4. Computable Boolean Algebras
5. Computable linear orders
6. Computable fields

In the classical setting, the analogue of 2 is an open problem and the analogue of 3 is false!

Σ^1_1 equivalence relations on the natural numbers

Fokina and I show that the structure of Σ^1_1 equivalence relations on $\mathcal{N}$ under Hyp reducibility as a whole is very rich: it embeds the partial order of Σ^1_1 sets under Hyp many-one reducibility.

But it is not known if there is a single isomorphism relation on computable structures which is neither Hyp nor complete under Hyp-reducibility!

However we do have:

Σ^1_1 equivalence relations on the natural numbers

Theorem

(Fokina-Friedman) Every Σ^1_1 equivalence relation is Hyp bi-reducible to a bi-embeddability relation on computable structures.

The proof is based on the analagous result in the non-effective setting:

Theorem

(Friedman-Motto Ros) Every analytic equivalence relation on the reals is Borel bi-reducible to a bi-embeddability relation on countable structures.

NP equivalence relations on finite strings

The motivation for this topic is the following:

Borel reducibility allows us to compare isomorphism relations on Borel classes of countable structures. Is there an analogous reducibility for "nice" classes of *finite* structures?

Fix a finite language $\mathcal{L}$
Identify n with $n = \{0, 1, \ldots, n-1\}$ for finite n
Finmod = $\mathcal{L}$-structures with universe n for some finite n

Goal: Compare isomorphism relations on *nice* subclasses of Finmod

NP equivalence relations on finite strings

Examples:

1. Finite Linear orders
2. Finite vector spaces over a fixed finite field
3. Finite fields
4. Finite linear orders with a unary relation
5. Finite Abelian groups
6. Finite cyclic groups
7. Finite groups with a fixed number of generators
8. Finite connected graphs with a fixed bound on the degree
9. Finite graphs with a fixed bound on the degree
10. Finite groups
11. Finite graphs

Except for 6,7,8: Above examples are first-order

Examples 6,7,8 belong to P (recognisable in polynomial time)

Nice subclass of *Finmod* = *Invariant Polytime subclass*

NP equivalence relations on finite strings

If $\mathcal{C}_0$, $\mathcal{C}_1$ are invariant Polytime classes then $\mathcal{C}_0$ is *reducible* to $\mathcal{C}_1$ iff there is a Polytime function F such that

$$M_0 \simeq N_0 \text{ iff } F(M_0) \simeq F(N_0)$$

$\mathcal{C}$ is *complete* iff all invariant Polytime classes are reducible to it

NP equivalence relations on finite strings

Analogies

Let Mod denote the class of (countable) models with universe ω

Nice (invariant Borel) subclasses of Mod $\approx$
Nice (invariant Polytime) subclasses of Finmod

$\simeq$ on a nice subclass of Mod is Σ^1_1
$\simeq$ on a nice subclass of Finmod is NP

$\simeq$ on a nice subclass of Mod need not be Borel
$\simeq$ on a nice subclass of Finmod need not be in P? (Maybe P$=$ NP!)

However, not assuming P$\neq$ NP:

There are many inequivalent nice subclasses of Mod
There are indeed many inequivalent nice subclasses of Finmod!

For NP isomorphism relations, my reducibility is finer than the usual one.

NP equivalence relations on finite strings

Potential Reducibility

$\mathcal{C}$ a nice subclass of Finmod.
$\mathcal{C}(n)$ = the set of models in $\mathcal{C}$ with universe m for some $m \leq n$
$\#_{\mathcal{C}}$ is defined by:
$\#_{\mathcal{C}}(n) = \#$ of isomorphism classes of models in $\mathcal{C}(n)$

Proposition

Suppose that $\mathcal{C}_0$, $\mathcal{C}_1$ are nice subclasses of Finmod and $\mathcal{C}_0$ is reducible to $\mathcal{C}_1$. Then $\#_{\mathcal{C}_0}$ is bounded by $\#_{\mathcal{C}_1} \circ p$ for some polynomial p.

We say that $\mathcal{C}_0$ is *potentially reducible* to $\mathcal{C}_1$ iff the above conclusion holds.
So: *reducible* implies *potentially reducible*.

NP equivalence relations on finite strings

Proposition

Suppose that $\mathcal{C}_0$, $\mathcal{C}_1$ are nice subclasses of Finmod and $\mathcal{C}_0$ is reducible to $\mathcal{C}_1$. Then $\#_{\mathcal{C}_0}$ is bounded by $\#_{\mathcal{C}_1} \circ p$ for some polynomial p.

Proof: Suppose that $F : \mathcal{C}_0 \to \mathcal{C}_1$ is in Polytime, $M_0 \simeq N_0$ iff $F(M_0) \simeq F(M_1)$. Let p be a polynomial such that if $M \in \mathcal{C}_0$ has size at most n then $F(M)$ has size at most $p(n)$. Then $\#_{\mathcal{C}_0}(n)$ is at most $\#_{\mathcal{C}_1}(p(n))$. $\square$

Examples: (1) If $\#_{\mathcal{C}_1}$ is polynomially-bounded and $\mathcal{C}_0$ is reducible to $\mathcal{C}_1$ then $\#_{\mathcal{C}_0}$ is also polynomially-bounded.
So LOU (finite linear orders with a unary relation) does *not* reduce to finite cyclic groups, finite fields or finite vector spaces.
(2) Every $\mathcal{C}$ is potentially reducible to LOU, because LOU has the maximum number of isomorphism types (up to a polynomial).

NP equivalence relations on finite strings

Proposition

There are nice subclasses $\mathcal{C}_0$, $\mathcal{C}_1$ of Finmod such that for no polynomial p is $\#_{\mathcal{C}_0}$ bounded by $\#_{\mathcal{C}_1} \circ p$ or vice-versa. So $\mathcal{C}_0$, $\mathcal{C}_1$ are incomparable with respect to potential reducibility and therefore incomparable with respect to reducibility.

Proof sketch: An increasing $f : \omega \to \omega$ is *time constructible* iff the set of pairs $(1^n, 1^{f(n)})$, $n \in \omega$ is computable in Polytime. A *pointed linear order* is a linear order together with a single constant. Now choose f to be time constructible and to grow very fast, and let $\mathcal{C}_i$ consist of all pointed linear orders of size $f(2n+i)$ for some n. Then $\#_{\mathcal{C}_0}(f(2n))$ is $\sum_{k \leq n} f(2k)$, $\#_{\mathcal{C}_1}(f(2n)) = \sum_{k<n} f(2k+1)$ and for any polynomial p, $\sum_{k \leq n} f(2k)$ is greater than $p(\sum_{k<n} f(2k+1))$ for large n. So $\mathcal{C}_0$ is not potentially reducible to $\mathcal{C}_1$. Similarly, $\mathcal{C}_1$ is not potentially reducible to $\mathcal{C}_0$. $\square$

NP equivalence relations on finite strings

A similar argument shows:

Theorem

(Buss-Chen-Flum-Friedman-Müller) There is a strong embedding from $(TC, \leq^*)$ into the isomorphism relations on invariant Polytime classes of pointed linear orders under (potential) reducibility, where TC is the class of time-constructible functions and $f \leq^* g$ iff $Range(f)$ is almost contained in $Range(g)$.

It follows that there are infinite chains and antichains in the (potential) reducibility ordering. Also note that the isomorphism relations used above are not only in NP but in fact in P.

NP equivalence relations on finite strings

Do reducibility and potential reducibility coincide? We have:

Theorem

(Buss-Chen-Flum-Friedman-Müller) (a) Assume P$=\#$P. Then reducibility and potential reducibility coincide.
(b) Assume N2EXP $\cap$ co-N2EXP $\neq$ 2EXP. Then reducibility and potential reducibility are distinct.

NP equivalence relations on finite strings

Finally, we consider NP equivalence relations as a whole.
As before, E_0 is *reducible* to E_1 iff there is a Polytime $f : \Sigma \to \Sigma$ such that xE_0y iff $f(x)E_1f(y)$.

Open Questions: (1) Is every NP equivalence relation reducible to an isomorphism relation on an invariant Polytime class?
(2) Is there a maximal NP equivalence relation under the above reducibility?

Regarding Question 1 there is some progress:

NP equivalence relations on finite strings

Proposition

(Buss-Chen-Flum-Friedman-Müller) Assume that the Polytime hierarchy does not collapse. Then not every NP equivalence relation reduces to an isomorphism relation on an invariant Polytime class.

Proof. SAT can be turned into an NP equivalence relation:
xEy iff $x = y$ or $x, y \in$ SAT.
Then a reduction of E to graph isomorphism (which is maximal among isomorphism relations) would imply that graph isomorphism is NP-complete.
It is known that the latter implies that the Polytime hierarchy collapses. □

Final remark

Another interesting context for equivalence relations:

Computably enumerable equivalence relations
First studied by Professor Ershov in 1971!

Further work:
Visser (1980), Bernardi-Sorbi (1983), Lachlan (1987), Nies (1994)

Question: Is there a computable F such that:
ZFC $\vdash \varphi \leftrightarrow \psi$ iff
PA $\vdash F(\varphi) \leftrightarrow F(\psi)$?

Congratulations to Professor Ershov on his 70th birthday!