New Σ^1_3 Facts

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Assume that $0^\#$ exists and that M is an inner model of ZFC, $0^\# \notin M$. Then of course M is not Σ^1_3-correct: the true Σ^1_3 sentence "$0^\#$ exists" is false in M. In this article we use a result about L-definable partitions (which may be of independent interest) to show that in fact this effect can be achieved by forcing over M. We work in Morse-Kelly class theory.

Theorem 1 *Assume that $0^\#$ exists. There exists an ω-sequence of true Σ^1_3 sentences $\langle \varphi_n \mid n \in \omega \rangle$ such that if M is an inner model, $0^\# \notin M$:*

(a) φ_n is false in M for some n.

(b) For each n, some generic extension of M satisfies φ_n.

Moreover if $M = L[R]$, R a real then these generic extensions can be taken as inner models of $L[R, 0^\#]$.

The above result is based on the next result, concerning L-definable partitions.

Theorem 2 *There exists an L-definable function n : L-Singulars $\to \omega$ such that if M is an inner model, $0^\# \notin M$:*

(a) For some n, $M \models \{\alpha \mid n(\alpha) \leq n\}$ is stationary.

(b) For each n there is a generic extension of M in which $0^\#$ does not exist and $\{\alpha \mid n(\alpha) \leq n\}$ is non-stationary.

Remark "Stationary in M" means: intersects every M-definable (with parameters) CUB.

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Proof: We define $n(\alpha)$. Let $\langle C_\alpha \mid \alpha\ L\text{-singular}\rangle$ be an L-definable $\Box$-sequence: C_α is CUB in α, otC_α = ordertype $C_\alpha < \alpha$ and $\bar\alpha \in \lim C_\alpha \to C_{\bar\alpha} = C_\alpha \cap \bar\alpha$. If otC_α is L-regular then $n(\alpha) = 0$. Otherwise $n(\alpha) = n(otC_\alpha) + 1$.

(a) is clear, as otherwise there is a $CUB\ C \subseteq L$-regulars amenable to M, contradicting that Covering Theorem and the hypothesis that $0^\#$ does not belong to M.

Now we prove (b). Fix $n \in \omega$. In M let P consist of closed, bounded $p \subseteq$ ORD such that $\alpha \in p \to \alpha$ L-regular or $n(\alpha) \geq n+1$, ordered by $p \leq q$ iff p end extends q.

We claim that P is ∞-distributive in M. Suppose that $p \in P$ and $\langle D_\alpha \mid \alpha < \kappa\rangle$ is a definable sequence of open dense subclasses of P, κ regular. We wish to find $q \leq p$, $q \in D_\alpha$ for all $\alpha < \kappa$. Let $C = \{\beta \mid \beta$ a strong limit cardinal, for all $\alpha < \kappa : r \in V_\beta \to \exists s \leq r(s \in V_\beta,\ s \in D_\alpha)\}$, a CUB class of ordinals. It suffices to show that $C \cap \{\beta \mid n(\beta) \geq n+1\}$ has a closed subset of ordertype $\kappa + 1$, for then p can be successively extended κ times meeting the D_α's, to conditions with maximum in $\{\beta \mid n(\beta) \geq n+1\}$; the final condition (at stage κ) extends p and meets each D_α.

Lemma 3 *Suppose $m \geq n$, α is regular and C is a closed set of ordinals greater than α^{+m} of ordertype $\alpha^{+m}+1$ (where $\alpha^{+0} = \alpha$, $\alpha^{+(k+1)} = (\alpha^{+k})^+$). Then $C \cap \{\beta \mid n(\beta) \geq n\}$ has a closed subset of ordertype $\alpha^{+(m-n)}+1$.*

Proof of Lemma 3: By induction on n. Suppose $n = 0$. Let $\beta = \max C$. Then β is singular and hence singular in L. So C_β is defined and $\lim(C_\beta \cap C)$ is a closed set of ordertype $\alpha^{+m}+1$ consisting of L-singulars. So $\lim(C_\beta \cap C) \subseteq C \cap \{\gamma \mid n(\gamma) \geq 0\}$ satisfies the lemma.

Suppose the lemma holds for n and let $m \geq n$, C a closed set of ordertype $\alpha^{+(m+1)}+1$ consisting of ordinals greater than $\alpha^{+(m+1)}$. Let $\beta = \max C$. Then C_β is defined and $D = \lim(C_\beta \cap C)$ is a closed set of ordertype $\alpha^{+(m+1)}+1$. Let $\bar\beta = (\alpha^{+m} + \alpha^{+m} + 1)$st element of D. Then $\bar D = \{otC_\gamma \mid \gamma \in D$, $(\alpha^{+m}+1)$st element of $D \leq \gamma \leq \bar\beta\}$ is a closed set of ordertype $\alpha^{+m}+1$ consisting of ordinals greater than α^{+m}. By induction there is a closed $\bar D_0 \subseteq \bar D \cap \{\gamma \mid n(\gamma) \geq n\}$ of ordertype $\alpha^{+(m-n)}+1$. But then $D_0 = \{\gamma \in D \mid otC_\gamma \in \bar D_0\}$ is a closed subset of $C \cap \{\gamma \mid n(\gamma) \geq n+1\}$ of ordertype $\alpha^{+(m-n)}+1$. As $\alpha^{+(m-n)} = \alpha^{+((m+1)-(n+1))}$ we are done. $\dashv$ (Lemma 3)

By the lemma, $C \cap \{\beta \mid n(\beta) \geq n\}$ has arbitrary long closed subsets for any n, for any CUB $C \subseteq$ ORD. It follows that P is ∞-distributive. Now to prove (b), we apply the forcing P to M, producing C witnessing the nonstationarity of $\{\alpha \mid n(\alpha) \leq n\}$, and then follow this with the forcing to code $\langle M, C\rangle$ by a real, making C definable. Of course this will not produce $0^{\#}$ as every successor to a strong limit cardinal is preserved in the coding. $\dashv$

We also note that in Theorem 2 the generic extension can be formed in $L[R, 0^{\#}]$ in the case $M = L[R]$, R a real, using the fact that in $L[R, 0^{\#}]$, generics can be constructed for P (an "Amenable" forcing) and for Jensen coding (see [99, Friedman]).

Proof of Theorem 1: We use David's trick (see [98, Friedman]). Let φ_n be the Σ^1_3 sentence: $\exists R \forall \alpha (L_\alpha[R] \models ZF^- \rightarrow L_\alpha[R] \models \beta$ a limit cardinal $\rightarrow$ β L-regular or $n(\beta) \geq n)$. By Theorem 2(b) and cardinal collapsing (to guarantee that limit cardinals β are either L-regular or satisfy $n(\beta) \geq n$), M has a generic extension $L[R] \models \beta$ a limit cardinal $\rightarrow \beta$ L-regular or $n(\beta) \geq n$ (inside $L[S, 0^{\#}]$ if $M = L[S]$, S a real). By David's trick we can in fact obtain φ_n in $L[R]$. $\dashv$

Question Can the generic extensions in Theorem 1(b) be taken to have the same cofinalities as M, in case M satisfies GCH?

References

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