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Negative solutions to Post's problem, II*

By Sy D. Friedman

This paper is an application of the techniques of modern set theory to problems in ordinal recursion theory. We concentrate on the global structure of the β-degrees for both admissible and inadmissible β and we show:

THEOREM 4 ($V = L$). *Let* $\beta = \aleph_{\omega_1}$ *and* $0'$ = *the* β-*degree of the complete* β-r.e. *set. Then the* β-*degrees greater than or equal to* $0'$ *are well-ordered by* $\leqq_\beta$ *with successor given by the* β-*jump.*

THEOREM 10 ($V = L$). *Let* $\beta = \aleph_{\omega_1} \cdot \omega$. *There is a well-ordered sequence* $e_0 < e_1 < \cdots$ *of* β-*degrees such that if* e *is an arbitrary* β-*degree then for some* γ, $e_\gamma \leqq e < e_{\gamma+1}$.

Moreover, if we assume the Generalized Continuum Hypothesis, the $\aleph_{\omega_1}$-degrees and the Turing degrees are not elementarily equivalent as partial orderings. If $V = L$ then the $\aleph_{\omega_1}$-jump is definable just in terms of the ordering of $\aleph_{\omega_1}$-degrees.

These results are in sharp contrast with earlier ones in ordinal recursior theory (see [9]), which tend to show that the β-degrees have a very ricl structure. Thus our work here shows that the broader point of view ob tained by considering inadmissible β has led to a structure theory of impor tance even for the admissible case.

In [8], Sacks and Simpson first established a connection between Gödel techniques in the study of L and ordinal recursion theory. This paper fu thers this idea by relating deep results of Jensen on the fine structure of to the structure of the β-degrees. The application of methods of combin torial set theory, initiated in [1], is also greatly extended as we make k use of techniques developed by Silver in [11] where he settles the singul cardinal problem at uncountable cofinalities.

Our earlier paper [2] is not a prerequisite for understanding the wo

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reported here, though we do presume a familiarity with the basic notions of β-recursion theory (as described in [1]). In [2], we applied Fodor's Theorem and the theory of stationary sets to establish the $\leqq_{w\beta}$-comparability of any two β-r.e. sets when $\beta = \aleph_{\omega_1}\cdot\omega$. In Section 1 of this paper similar methods are applied to establish Theorem 4.

However, the deeper results require an excursion into the fine structure of L, where the theory of master codes enables us to identify explicitly the degrees appearing in Theorems 4 and 10. This is carried out for Theorem 4 in Section 2. Theorem 10 requires the introduction of some further recursion-theoretic ideas and is dealt with in Section 3.

There are natural versions of the above results when $\aleph_{\omega_1}$ is replaced by any singular cardinal of uncountable cofinality. However these results do not hold when $\aleph_{\omega_1}$ is replaced by a singular cardinal of countable cofinality, as we will discuss in [3]. We end our paper by listing some further results and open questions concerning both the countable cofinality case and the finer structure of the β-degrees.

1. Stationary sets and the $\aleph_{\omega_1}$-degrees

We assume $V = L$. Our first goal is to prove that if $\alpha = \aleph_{\omega_1}$ then the α-degrees $\geqq 0'$ are well-ordered by $\leqq_\alpha$, where $0' = \alpha$-degree of the complete α-r.e. set. This provides an example of a first-order difference between the partial-orderings of $\aleph_{\omega_1}$-degrees and Turing degrees; indeed, the following sentence holds in the latter but not the former:

$$\forall\, d\ \exists\, e\ \exists\, f(e \nleqq d \cup f \ \text{ and } \ f \nleqq d \cup e)\ .$$

First-order differences in the language with jump were already discovered by Richard Shore (see [9]).

The proof technique used here is in fact imbedded in our earlier paper [2]. We continue to let α denote $\aleph_{\omega_1}$ (and to assume $V = L$).

LEMMA 1. *$C = \{\aleph_\beta \mid \beta < \omega_1\}$ has α-degree $0'$.*

Proof. Note that C is Π_1 over

$$L_\alpha\colon \gamma \in C \longleftrightarrow \gamma \geqq \omega \wedge \sim \exists\, \gamma' < \gamma\ \exists\, f(f\colon \gamma \xrightarrow{1\text{-}1} \gamma')\ .$$

So $C \leqq_\alpha 0'$. But every element of C is α-stable; i.e., $\gamma \in C \to L_\gamma$ is a Σ_1-elementary substructure of L_α. (This is proved in [8].). Thus, if $\phi(x)$ is a Σ_1 formula defining (over L_α) a complete α-r.e. set A then for $K, H \in L_\alpha$:

$$K \subseteqq A \leftrightarrow \exists\, \gamma \in C(K \subseteqq L_\gamma \wedge \forall\, \delta \in K\ ,\ L_\gamma \vDash \phi(\delta))\ ,$$
$$H \subseteqq L_\alpha - A \leftrightarrow \exists\, \gamma \in C(H \subseteqq L_\gamma \wedge \forall\, \delta \in H\ ,\ L_\gamma \vDash \sim\phi(\delta))\ .$$

So $A \leqq_\alpha C$. □

Every $A \subseteq \alpha$ of α-degree $\geqq_\alpha 0'$ has the same α-degree as an apparently much simpler set, its "cutoff" function.

Definition. For $A \subseteq \alpha$ define $f_A: \omega_\delta \to \alpha$, the cutoff function for A, by $f_A(\delta) = \beta$ if $A \cap \aleph_\delta$ is the βth element in the canonical well-ordering $<_{L_\alpha}$ of L_α.

Thus we see that for all δ, $f_A(\delta) < \aleph_{\delta+1}$. Note that $f_A \restriction X$ for X unbounded in ω_1 determines f_A completely. The idea of comparing two sets A, B by comparing the growth-rates of f_A, f_B has its beginnings in Jack Silver's work on the Singular Cardinal Problem [11] and is made very explicit in Karel Prikry's proof [7] of Silver's theorem. It is this idea which we use here.

LEMMA 2. *Suppose* $0' \leqq_\alpha A$, $0' \leqq_\alpha B$. *Then either* $A \leqq_\alpha B$ *or* $B \leqq_\alpha A$.

Proof. One of the sets $\{\delta \mid f_A(\delta) \leqq f_B(\delta)\}$, $\{\delta \mid f_A(\delta) \geqq f_B(\delta)\}$ must be a stationary subset of ω_1. We assume that the former set is and proceed to show that $A \leqq_\alpha B$.

It is enough to show that $f_A \restriction X$ can be computed from B for some unbounded $X \subseteq \omega_1$.

For any $\alpha < \beta$, let c_β be the $<_L$-least injection of β into cardinality (β). Define functions g, h on $\{\delta \mid f_A(\delta) \leqq f_B(\delta)\}$ by

$$g(\delta) = c_{f_B(\delta)+1}(f_A(\delta)) ,$$
$$h(\delta) = \text{least } \delta' \text{ such that } \quad g(\delta) < \aleph_{\delta'} .$$

Then $g(\delta) < \aleph_\delta$ for all δ, so δ limit $\to h(\delta) < \delta$. By Fodor's Theorem, choose a stationary $X \subseteq \omega_1$ and δ_0 such that $\delta \in X \to h(\delta) < \delta_0$.

But then $\delta \in X \to g(\delta) < \aleph_{\delta_0}$ so $g \restriction X$ is α-finite (since any bounded subset of α is a α-finite). We can now compute $f_A \restriction X$ from B by

$$f_A(\delta) = c^{-1}_{f_B(\delta)+1}(g(\delta)) , \quad \delta \in X . \qquad \square$$

Note the heavy use of uncountable cofinality here (in order to apply Fodor's Theorem). It is known that Lemma 2 fails when α is replaced by $\aleph_\omega$. (This is due to Leo Harrington and Bob Solovay independently and will be shown in [3].)

Our proof in fact shows:

LEMMA 3. *The α-degrees $\geqq_\alpha 0'$ are well-ordered by $\leqq_\alpha$.*

Proof. The proof of Lemma 2 shows that if $0' \leqq_\alpha A <_\alpha B$ then $\{\delta \mid f_A(\delta) < f_B(\delta)\}$ contains a closed unbounded subset. Now if $A_0 >_\alpha A >_\alpha \cdots$, each $A_i \geqq_\alpha 0'$, then $\{\delta \mid f_{A_{i+1}}(\delta) < f_{A_i}(\delta)\}$ contains a closed unbounded set for each so there must be δ such that $f_{A_{i+1}}(\delta) < f_{A_i}(\delta)$ for every i, as the countable intersection of closed unbounded sets is nonempty. Of course we have now contradicted the well-foundedness of the ordinals. □

It will be shown in [3] that the $\aleph_\omega$-degrees $\geqq 0'$ are not well-founded.

We next establish that the successor of each α-degree $\geqq_\alpha 0'$ is its α-jump. Our original proof of this fact made heavy use of the fine structure theory. Tony Martin later found a combinatorial proof. Subsequently we discovered a simpler combinatorial argument which we present here.

THEOREM 4. *The α-degrees $\geqq_\alpha 0'$ are well-ordered by $\leqq_\alpha$ with successor given by the α-jump.*

Proof. We have only to show that $0' \leqq_\alpha A <_\alpha B$ implies $A' \leqq_\alpha B$ where $A' = \alpha$-jump of A. Let $\varphi(x)$ be a Σ_1 formula which defines A' over $\langle L_\alpha, A\rangle$; i.e., $A' = \{x \mid \langle L_\alpha, A\rangle \models \varphi(x)\}$. For any $\gamma < \omega_1$ we let $A_\gamma = \{x \mid \langle L_{\aleph_\gamma}, A \cap \aleph_\gamma\rangle \models \varphi(x)\}$ and $f_\gamma = f_{A_\gamma}$, the cutoff function for A_γ. Then the sequence $\langle A_\gamma \mid \gamma < \omega_1\rangle$ is α-recursive in A. Thus for any $g\colon \omega_1 \to \omega_1$, the function f_g defined by $f_g(\delta) = f_{g(\delta)}(\delta)$ is also α-recursive in A (as any such g is α-finite).

We compare f_B to $\sup_\gamma f_\gamma$: If $\{\delta \mid f_B(\delta) < \sup_\gamma f_\gamma(\delta)\}$ is stationary then for some $g\colon \omega_1 \to \omega_1$ the set $\{\delta \mid f_B(\delta) < f_g(\delta)\}$ is stationary. The proof of Lemma 2 actually shows: If $\{\delta \mid f_B(\delta) \leqq h(\delta)\}$ is stationary and $h \leqq_\alpha A$ then $B \leqq_\alpha A$. Thus as we have assumed that $A <_\alpha B$ it must be the case that $\{\delta \mid \sup_\gamma f(\delta) \leqq f_B(\delta)\}$ contains a closed, unbounded set.

As in the proof of Lemma 2, for any $\beta < \alpha$ let c_β denote the $<_L$-least injection of β into cardinality (β). Now for limit $\delta < \omega_1$, $\aleph_\delta$ has countable cofinality. Thus for stationary many δ we may choose an uncountable $X_\delta \subseteqq \omega_1$ and $h(\delta) < \delta$ such that:

$$\gamma \in X_\delta \longrightarrow c_{f_B(\delta)+1}(f_\gamma(\delta)) < \aleph_{h(\delta)}\ .$$

Thus by Fodor's Theorem we may choose $\delta_0 < \omega_1$ such that for stationary many δ:

$$(*)\qquad \gamma \in X_\delta \longrightarrow c_{f_B(\delta)+1}(f_\gamma(\delta)) < \aleph_{h(\delta_0)}\ .$$

We let $g(\delta, \gamma) = c_{f_B(\delta)+1}(f_\gamma(\delta))$ and $S = \{\delta \mid (*) \text{ holds}\}$. For each $\beta < \alpha$ write $K(\beta) = y$ if y is the βth set in $<_L$. Then we see that for $\delta \in S$:

$$A' \cap \aleph_\delta = \bigcup_{\gamma \in X_\delta} K(c^{-1}_{f_B(\delta)+1}(g(\delta, \gamma)))\ .$$

As g, S and $\langle X_\delta \mid \delta \in S\rangle$ are α-finite and K is α-recursive this shows that $A' \leqq_\alpha B$. □

We note that there is an appropriate version of Theorem 4 if only the generalized continuum hypothesis (GCH) is assumed. In this case choose $A \subseteqq \alpha$ such that $\{\aleph_\delta \mid \delta < \omega_1\} \leqq_\alpha A$ and $2^{\aleph_\delta} \subseteqq L_{\aleph_{\delta+1}}[A]$ for each $\delta < \omega_1$. Then Theorem 4 holds when $0'$ is replaced by α-degree (A). Thus the GCH alone implies that there is a first-order difference between the $\aleph_{\omega_1}$-degrees and the Turing degrees as partial orderings.

Next we derive some information concerning the α-jump, assuming $V = L$. Following Carl Jockusch and David Posner, define an α-degree d to be *generalized low* if $d' = d \vee 0'$. If $0' \leqq d$ then d is *not* generalized low. However this is the only exception.

THEOREM 5. *For all d either $0' \leqq d$ or d is generalized low.*

Proof. Define $A \subseteq \alpha$ to be *hyper-regular* if $\langle L_\alpha, A \rangle$ is admissible. If A is hyper-regular then B is *A-hyper-regular* if $A \vee B$ is hyper-regular. The following claim generalizes a result of Richard Shore:

CLAIM. *If A is hyper-regular and B is not A-hyper-regular then $A' \leqq_\alpha B \vee A$.*

Proof of Claim. Choose a function $f: \gamma_0 \to \alpha$, $\gamma_0 < \alpha$, such that f is weakly α-recursive in $A \vee B$ and f has unbounded range. Also let $\varphi(x)$ be a Σ_1 formula such that $A' = \{x \mid \langle L_\alpha, A \rangle \models \varphi(x)\}$. As A is hyper-regular we can choose a function $h: \gamma_0 \to \gamma_0$ such that for all $\gamma < \gamma_0$,

$$A' \cap f(\gamma) = \{x \mid \langle L_{f(h(\gamma))}, A \cap f(h(\gamma)) \rangle \models \varphi(x)\} .$$

As h is α-finite this shows that $A' \leqq_\alpha A \vee B$. This proves the claim.

Now choose a set A in the α-degree d. If A is not hyper-regular then $0' \leqq_\alpha A$ by the claim. Otherwise $A' \leqq_\alpha 0' \vee A$ by the claim since $0'$ is not A-hyper-regular (Lemma 1). □

Stephen Simpson's work in [13] completes the picture by showing that every $d \geqq 0'$ is the α-jump of some generalized low degree. Now a theorem of Richard Shore in [10] shows that if A is hyper-regular then there are sets B, C which are α-r.e. in A such that $B \nleqq_\alpha A \vee C$, $C \nleqq_\alpha A \vee B$. These facts enable us to prove the following theorem:

THEOREM 6. *The α-jump operation on the α-degrees is first-order definable over the structure $\langle \alpha$-degrees, $\leqq_\alpha \rangle$.*

Proof. By our above remarks and Lemma 2 we have:

$$0' = \text{least } d \text{ such that } \quad \forall e \forall f (d \vee e \leqq_\alpha d \vee f \text{ or } d \vee f \leqq_\alpha d \vee e) .$$

Thus $\{0'\}$ is first-order definable over $\langle \alpha$-degrees, $\leqq_\alpha \rangle$. But then by Theorem 5:

$$e = d' \leftrightarrow [0' \nleqq_\alpha d \text{ and } e = d \vee 0'] \quad \text{or} \quad [0' \leqq_\alpha d \text{ and } e = \text{least degree} > d] .$$ □

We end this section by briefly mentioning how the above results extend to other singular cardinals of uncountable cofinality. Let β be such a cardinal. Then there is a least β-degree d such that the β-degrees $\geqq d$ are well-ordered by $\leqq_\beta$ (the least non-hyper-regular β-degree). Moreover for $A \subseteqq \beta$ one can make an appropriate definition of $A^{(\nu)}$, the νth iterate of the β-jump applied to A, and then $d = 0^{(\nu)}$ for some $\nu < \beta^+ =$ next cardinal after β. If $d \nleqq_\beta A$ then A is generalized low$_\nu$ (i.e., $A^{(\nu)} =_\beta A \vee 0^{(\nu)}$) and hyper-regular. $\{d\}$ is definable over $\langle \beta\text{-degrees}, \leqq_\beta \rangle$.

2. Master codes

Let β be a limit ordinal. We define the Σ_n *projectum of* β, ρ_n^β, and the Δ_n *projectum* of β, δ_n^β, by

$$\rho_n^\beta = \text{least } \gamma \text{ such that there is a 1-1 function from } \beta \text{ into } \gamma \text{ which is } \Sigma_n \text{ over } S_\beta$$

$$\delta_n^\beta = \text{least } \gamma \text{ such that there is a 1-1 function from } \beta \textit{ onto } \gamma \text{ which is } \Sigma_n \text{ over } S_\beta\,.$$

We also let $\rho_0^\beta = \delta_0^\beta = \beta$.

The S-hierarchy for L is defined in Devlin's book [1], page 82. It is often easier to work with this hierarchy as $S_\beta \cap \mathrm{On} = \beta$ for limit β. (For all β, $S_{\omega\cdot\beta} = J_\beta$ so the S-hierarchy is merely a "ramification" of the J-hierarchy.) Our definition of the projecta differs from Jensen's but is easily related to his definition and allows one to state cleanly the following characterization, which follows from Jensen's work:

THEOREM 7. (a) $\rho_n^\beta =$ *least* γ *such that there is a subset of* γ *which is* Σ_n *over* S_β *but not a member of* S_β.

(b) $\delta_n^\beta =$ *least* γ *such that there is a subset of* γ *which is* Δ_n *over* S_β *but not a member of* S_β.

(c) *For* $\delta \leqq \beta$ *let* $\Sigma_n^\beta\, cf(\delta) =$ *least* γ *such that there is a function from* γ *onto an unbounded subset of* δ *which is* Σ_n *over* S_β. *Then* $\delta_n^\beta = \max(\rho_n^\beta, \Sigma_n^\beta cf(\rho_{n-1}^\beta))$.

The notion of a Σ_n master code is the key in Jensen's fine structure theory. $A \subseteqq \beta$ is a Σ_n *master code* for β if:

(a) $A \subseteqq \rho_n^\beta$; (b) A is Σ_n over S_β; (c) Let $\mathfrak{A} = \langle S_{\rho_n^\beta}, A\rangle$. Then $B \subseteqq \rho_n^\beta$ is Σ_1 over $\mathfrak{A}$ if and only if B is Σ_{n+1} over S_β.

Jensen showed that Σ_n master codes always exist. As in [4], choose a canonical Σ_n master code for β, C_n^β, and let $\mathfrak{A}_n^\beta = \langle S_{\rho_n^\beta}, C_n^\beta\rangle$. It follows from (c) that $B \subseteqq \rho_n^\beta$ is Σ_m over $\mathfrak{A}_n^\beta$ if and only if B is Σ_{n+m} over S_β.

The notion of a Δ_n master code occurs more rarely (though it was considered by Jockusch and Simpson [5] in case $\rho_n^\beta = \omega$). $A \subseteq \beta$ is a Δ_n *master code* for β if:

(a) $A \subseteq \delta_n^\beta$, (b) A is Δ_n over S_β. (c) If $\mathfrak{A} = \langle S_{\delta_n^\beta}, A\rangle$ then $B \subseteq \delta_n^\beta$ is Σ_1 over $\mathfrak{A}$ if and only if B is Σ_n over S_β.

Again, (c) implies that $B \subseteq \delta_n^\beta$ is Σ_m over $\mathfrak{A}$ if and only if B is Σ_{m+n-1} over S_β. Δ_n master codes do not always exist. They do if $\delta_n^\beta = \omega$ or if $\rho_{n-1}^\beta = \delta_n^\beta$ (in this case C_{n-1}^β is a Δ_n master code for β). The case that concerns us here is when $\delta_n^\beta = \aleph_{\omega_1}$ and the results below show that a Δ_n master code exists for β if and only if $\Sigma_n^\beta cf(\rho_{n-1}^\beta) = \omega_1$. In general a Δ_n master code exists for β if and only if $\Sigma_n^\beta cf(\rho_{n-1}^\beta) = \Sigma_n^\beta cf(\delta_n^\beta)$.

As in the previous section let α denote $\aleph_{\omega_1}$ and assume $V = L$. We will describe a procedure for dissecting the α-degrees $\geqq 0'$. We first establish some basic facts about master codes which are subsets of α, *α-master codes.* Define A to be an α-master code for β if A is a Σ_n or Δ_n master code for β for some n and $A \subseteq \alpha$.

PROPOSITION 8. (a) *If $\rho_n^\beta = \alpha$ then all Σ_n master codes for β have the same α-degree.*

(b) *If $\delta_n^\beta = \alpha$ then all Δ_n master codes for β have the same α-degree.*

(c) *If $\beta_1 < \beta_2$ then all α-master codes for β_1 are α-recursive in all α-master codes for β_2.*

(d) *If $\rho_{n-1}^\beta = \alpha$ then the Σ_{n-1} master codes for β and the Δ_n master codes for β have the same α-degree.*

(e) *If $\alpha = \delta_n^\beta$ and if D is a Δ_n master code for β, C a Σ_n master code for β, then $C =_\alpha \alpha$-jump(D).*

(f) *If $\rho_n^\beta = \alpha$ and C_1 is a Σ_n master code for β, C_2 a Σ_{n+1} master code for β, then $C_2 =_\alpha \alpha$-jump(C_1).*

Proposition 8 gives us the following picture: Say that a limit ordinal $\beta \geqq \alpha$ is *projectible into α* if $\rho_n^\beta = \alpha$ for some n. List the limit ordinals $\beta \geqq \alpha$ which are projectible into α in order: $\alpha = \beta_0 < \beta_1 < \cdots$ and list the α-degrees of α-master codes in $\leqq_\alpha$-increasing order $0 = d_1 < d_2 < \cdots$. If β is projectible into α let $n(\beta) =$ least n such that $\rho_n^\beta = \alpha$. Now let $\beta = \beta_\nu$, $n = n(\beta)$ and $\lambda = \omega \cdot \nu$. Then if $\delta_n^\beta = \alpha$ and there is a Δ_n master code for β, we have $d_\lambda = \alpha$-degree (any Δ_n master code for β) and $d_{\lambda+m} = \alpha$-degree (any Δ_{n+m} master code for β). This occurs exactly when $\Sigma_n^\beta cf(\rho_{n-1}^\beta) = \omega_1$. Otherwise $d_{\lambda+m} = \alpha$-degree (any Σ_{n+m} master code for β). Finally, for all ν, $d_{\nu+1} = \alpha$-jump(d_ν).

Our ultimate aim is to show that the d_ν's, $\nu \geqq 2$, exhaust the α-degrees $\geqq_\alpha 0' = d_2$. We show how this is done after proving Proposition 8.

Proof of Proposition 8. (a) If $A \subseteqq L_\alpha$ then $\{x \in L_\alpha | x \subseteqq A\}$ and $\{x \in L_\alpha | x \cap A = \emptyset\}$ are both Σ_1 over $\langle L_\alpha, A\rangle$. Therefore if C_1, C_2 are Σ_n master codes for β, $\rho_n^\beta = \alpha$ then $\{x \in L_\alpha | x \subseteqq C_1\}$ and $\{x \in L_\alpha | x \cap C_1 = \emptyset\}$ are Σ_1 over $\langle L_\alpha, C_1\rangle$, hence Σ_{n+1} over β, hence Σ_1 over $\langle L_\alpha, C_2\rangle$. So $C_1 \leqq_\alpha C_2$.

(b) Similar to (a).

(c) If $\beta_1 < \beta_2$ then any α-master code C_1 for β_1 is β_2-finite; therefore $\{x \in L_\alpha | x \subseteqq C_1\}$ and $\{x \in L_\alpha | x \cap C_1 = \emptyset\}$ are β_2-finite, hence Δ_1 over S_{β_2} and therefore Δ_1 over $\langle L_\alpha, C\rangle$ for any α-master code C for β_2.

(d) As we have noted, if $\rho_{n-1}^\beta = \alpha$ then any Σ_{n-1} master code for β is also a Δ_n master code for β. The result now follows from (b).

(e) We have $C \leqq_\alpha \alpha\text{-jump}(D)$ as C is Σ_1 over $\langle L_\alpha, D\rangle$ and α-jump(D) has the largest α-degree of any set Σ_1 over $\langle L_\alpha, D\rangle$. For the converse ($\alpha$-jump ($D$) $\leqq_\alpha C$), it suffices to show that both of the sets $\{x \in L_\alpha | x \subseteqq \alpha\text{-jump}(D)\}$ and $\{x \in L_\alpha | x \cap \alpha\text{-jump}(D) = \emptyset\}$ are Σ_{n+1} over S_β. The latter set is actually Π_1 over $\langle L_\alpha, D\rangle$, hence Π_n over S_β. To complete the proof it suffices to show that $\{x \in L_\alpha | x \subseteqq \alpha\text{-jump}(D)\}$ is Σ_2 over $\langle L_\alpha, D\rangle$.

CLAIM. *If $B \subseteqq \alpha$, B is Σ_1 over $\langle L_\alpha, D\rangle$, then $B^* = \{x \in L_\alpha | x \subseteqq B\}$ is Σ_2 over $\langle L_\alpha, D\rangle$.*

Proof of Claim. If $\langle L_\alpha, D\rangle$ is admissible then B^* is actually Σ_1 over $\langle L_\alpha, D\rangle$. Otherwise choose $\gamma < \alpha$, $f: \gamma \to \alpha$ unbounded such that f is Σ_1 over $\langle L_\alpha, D\rangle$. Let $\varphi(x)$ be a Σ_1 formula such that $y \in B \leftrightarrow \langle L_\alpha, D\rangle \models \varphi(y)$. Then

$$x \subseteqq B \leftrightarrow \exists \text{ sequence } \langle x_\beta \mid \beta < \gamma\rangle \in L_\alpha \text{ such that } x = \textstyle\bigcup_\beta x_\beta \text{ and}$$
$$\forall \beta < \gamma\ \forall y \in x_\beta \langle L_{f(\beta)}, D \cap f(\beta)\rangle \models \varphi(y)\ .$$

This is true since any bounded subset of α is a member of L_α. This gives a Σ_2 over $\langle L_\alpha, D\rangle$ definition of B^*.

(f) Follows from (d) and (e). □

Our method of showing that every α-degree $\geqq_\alpha 0'$ is one of the d_ν's, $\nu \geqq 2$ is to prove for each ν that for all $d \geqq 0'$:

$(*)_\nu$ $$d > d_{\nu'} \quad \text{for all} \quad \nu' < \nu \to d \geqq d_\nu\ .$$

Given this, argue that for all $d \geqq 0'$ there is ν such that $d = d_\nu$ as follows: Let ν be least so that $d \ngeqq d_\nu$. If $d \neq d_\nu$, for all $\nu' < \nu$ then $d > d_{\nu'}$ for all $\nu' < \nu$ so by $(*)_\nu$, $d \geqq d_\nu$, a contradiction.

The proof of $(*)_\nu$ breaks into cases. The case $\nu = 2$ is trivial. The successor case is handled by Theorem 4 and Proposition 8 as for any ν, $d_{\nu+1} = \alpha\text{-jump}(d_\nu)$. Our next result handles the limit case.

THEOREM 9. *Suppose $\rho_{n-1}^\beta > \rho_n^\beta = \alpha$ and let C be a Σ_n master code for β.*

(a) *If $\Sigma_n^\beta cf(\rho_{n-1}^\beta) = \omega_1$ then $\delta_n^\beta = \alpha$, there is a Δ_n master code D for β and for all $A \subseteqq \alpha$:*

$$A \text{ is } \beta\text{-finite or } D \leqq_\alpha A \vee 0' .$$

(b) *If $\Sigma_n^\beta cf(\rho_{n-1}^\beta) \neq \omega_1$ then for all $A \subseteqq \alpha$:*

$$A \text{ is } \beta\text{-finite or } C \leqq_\alpha A \vee 0' .$$

For then suppose λ is a limit ordinal, $\lambda = \omega \cdot \nu$. Then if (a) above holds with $\beta = \beta_\nu$, $n = n(\beta_\nu)$, we have $d_\lambda = \alpha$-degree of a Δ_n master code for β_ν; if (b) holds then $d_\lambda = \alpha$-degree of a Σ_n master code for β_ν. In either case for all $A \subseteqq \alpha$, A is β-finite or $d_\lambda \leqq_\alpha A \vee 0'$. But A β_ν-finite implies A is α-recursive in some α-master code for some $\beta' < \beta_\nu$ and hence $A \leqq_\alpha d_{\nu'}$ for some $\nu' < \lambda$. This demonstrates $(*)_\lambda$.

The proof of Theorem 9 necessitates the introduction of some technical notions associated with Skolem functions. Let $\mathfrak{A} = \langle S_\beta, A\rangle$ be an amenable structure (i.e., $A \cap S_\gamma \in S_\beta$ for all $\gamma < \beta$). If $p \in S_\beta$ then a Σ_1^p *Skolem function* for $\mathfrak{A}$ is a partial function $h(i, x)$ which is Σ_1 over $\mathfrak{A}$ with parameter p with the property that whenever $\varphi(x, y, z)$ is $\Sigma_1(\mathfrak{A})$ then

$$\mathfrak{A} \models \forall x(\exists y\varphi(x, y, p) \longrightarrow \exists i\varphi(x, h(i, x), p)) .$$

If $p = 0$ then we say that h is a Σ_1 Skolem function for $\mathfrak{A}$. Σ_1^p Skolem functions are easily constructed (see [0], p. 88). Also we define $\mathfrak{A}^* = (\beta, A)^* =$ least γ such that there is a $\Sigma_1(\mathfrak{A})$ function $f\colon \beta \xrightarrow{1\text{-}1} \gamma$ and $p(\mathfrak{A}) = p(\beta, A) =$ least p such that there is such an f which is Σ_1 over $\mathfrak{A}$ with parameter p.

We are interested in using a $\Sigma_1^{p(\beta,A)}$ Skolem function h to take Skolem hulls and then to analyze the nature of these Skolem hulls after they are transitively collapsed. If $\gamma < (\beta, A)^*$ then let $H_\gamma = h[\omega \times \gamma]$ and $\pi\colon \langle H_\gamma, \varepsilon\rangle \simeq \langle S_{\gamma'}, \varepsilon\rangle$. Then γ is (β, A)-*pseudostable* (or $\mathfrak{A}$-*pseudostable*) if $\gamma \notin H_\gamma$. It can be easily checked that γ (β, A)-pseudostable implies γ is a γ'-cardinal. We also let $\gamma_A =$ least δ such that $\pi[A \cap H_\gamma]$ is definable over S_δ.

In many contexts it is desirable to have a bound on γ_A (in terms of γ). The reason for this is that if γ is (β, A)-pseudostable and $B \subseteqq (\beta, A)^*$ is Σ_1 over $\mathfrak{A}$ in some parameter in $\gamma \cup \{p(\beta, A)\}$, then $B \cap \gamma$ is Σ_1-definable over $\langle S_{\gamma'}, \pi[A \cap H_\gamma]\rangle$ and thus a bound on γ_A gives a bound on where $B \cap \gamma$ is constructed. In case $(\beta, A)^* = \alpha$ this allows us to estimate the growth rate of f_B, which is useful in view of Lemma 2.

We describe now an important possible bound on γ_A. For $\gamma < (\beta, A)^*$, let $\hat{\gamma} =$ greatest δ such that $S_\delta \models \gamma$ is a cardinal ($\hat{\gamma} = \gamma$ if there is no such δ). Then A is *collapsible* on β (or $\mathfrak{A}$ is *collapsible*) if $\gamma_A \leqq \hat{\gamma}$ for all sufficiently large (β, A)-pseudostable $\gamma < (\beta, A)^*$.

Examples. (a) If A is Σ_1 over S_β then A is collapsible. For, choose $\gamma_0 < (\beta, A)^*$ such that A is Σ_1 over S_β with parameter $q \in h[\omega \times \gamma_0]$. Then ($\gamma \geqq \gamma_0$, γ is β, A-pseudostable) $\rightarrow A \cap h[\omega \times \gamma]$ is definable over $h[\omega \times \gamma] \rightarrow \gamma' \geqq \gamma_A$. But $S_{\gamma'} \models \gamma$ is a cardinal.

(b) If A is a Δ_n (Σ_{n-1}) master code for μ and $\beta = \delta_n^\mu$ ($\beta = \rho_{n-1}^\mu$) then A is a collapsible predicate on β.

Proof. Choose $q \in S_\mu$ so that both A, $\beta - A$ are $\sum_n$ over S_μ with parameter q. Let k be Σ_n^q Skolem function for S_μ and choose $\gamma_0 < \beta$ so that $k \cap (\omega \times \beta) \times \beta$ is Σ_1 over $\langle S_\beta, A\rangle$ with some parameter $r \in S_{\gamma_0}$. Then for $\gamma \geqq \gamma_0$, $h[\omega \times \gamma]$ is closed under $k \cap (\omega \times \beta) \times \beta$ and so $k[\omega \times \gamma] \cap \beta = h[\omega \times \gamma] = H_\gamma$. Now if γ is β, A-pseudostable let $K_\gamma = k[\omega \times \gamma]$ and $\tau: \langle K_\gamma, \varepsilon\rangle \cong \langle S_{\gamma''}, \varepsilon\rangle$. Then $\tau \supseteqq \pi: \langle H_\gamma, \varepsilon\rangle \cong \langle S_{\gamma'}, \varepsilon\rangle$, $\gamma'' \geqq \gamma'$ and $S_{\gamma''} \models \gamma$ is a cardinal. But $A \cap K_\gamma$ is Δ_n over K_γ so $\pi[A \cap H_\gamma]$ is Δ_n over $S_{\gamma''}$. □

Proof of Theorem 9. Let $\mathfrak{A} = \langle S_{\rho_{n-1}^\beta}, C_{n-1}^\beta\rangle$ and $\gamma_0 = \Sigma_1 cf(\mathfrak{A})$. Choose an $\mathfrak{A}$-recursive order-preserving function f from γ_0 onto an unbounded subset of ρ_{n-1}^β such that $\gamma < \gamma_0$, γ limit $\rightarrow \mathfrak{A}_\gamma = \langle S_{f(\gamma)}, C_{n-1}^\beta \cap f(\gamma)\rangle$ is amenable. Let s_δ^γ be the first $\mathfrak{A}_\gamma$-pseudostable greater than $\aleph_\delta$ (for all $\gamma < \gamma_0$) and let s_δ be the first $\mathfrak{A}$-pseudostable greater than $\aleph_\delta$. Then $s_\delta = \bigcup \{s_\delta^\gamma \mid \gamma < \gamma_0\}$.

First assume that $\gamma_0 < \alpha$.

Recall that the proof of Lemma 2 shows: If $\{\delta \mid f_A(\delta) \leqq g(\delta)\}$ is stationary and $g \leqq_\alpha B$ then $A \leqq_\alpha B$. As $\mathfrak{A}$ is collapsible, whenever $C \subseteqq \alpha$ is Σ_1 over $\mathfrak{A}$ then $C \cap \aleph_\delta$ is definable over $S_{\hat{s}_\delta}$ for δ sufficiently large. Thus if $A \subseteqq \alpha$ and $f_A(\delta) \geqq s_\delta$ for stationary many δ then $C_n^\beta \leqq_\alpha A$. Otherwise $f_A(\delta) < s_\delta$ for stationary many δ and A is $\mathfrak{A}$-recursive.

(a) Assume $\gamma_0 = \omega_1$. Let $D = \langle s_\delta^\delta \mid \delta < \omega_1\rangle$.
Then A Σ_1 over $\langle S_\alpha, D\rangle \rightarrow A$ Σ_1 over $\mathfrak{A}$. Conversely: By Jensen's extension of embeddings lemma ([1], page 100) $C_{n-1}^\beta \cap f(\gamma)$ is a Σ_{n-1} master code for some ordinal η such that $\rho_{n-1}^\eta = f(\gamma)$, and hence is a collapsible predicate on $f(\gamma)$. It follows that if $\phi(x)$ is Σ_1 then $\{x < \aleph_\delta \mid \mathfrak{A}_\delta \models \phi(x)\}$ is definable over S_{η_δ} where $\eta_\delta = \hat{s}_\delta^\delta$. Therefore the equivalence:

$$\mathfrak{A} \models \phi(x) \leftrightarrow \exists\, \delta[x < \aleph_\delta \wedge \mathfrak{A}_\delta \models \phi(x)]$$

shows that any subset of α which is Σ_1 over $\mathfrak{A}$ is Σ_1 over $\langle S_\alpha, D\rangle$. So D is a Δ_1 master code for $\mathfrak{A}$ and hence a Δ_n master code for β.

Now if $A \subseteqq \alpha$ and $f_A(\delta) < s_\delta$ for stationary many δ then either $f_A(\delta) < s_\delta^\delta$ for stationary many δ or $f_A(\delta) \geqq s_\delta^\delta$ for stationary many δ. In the former case Fodor's Theorem implies that for some fixed $\delta_0 < \omega_1$, $f_A(\delta) < s_\delta^{\delta_0}$ for stationary many δ so A is ρ_{n-1}^β-finite since the sequence $\langle s_\delta^{\delta_0} \mid \delta < \omega_1\rangle$ is. In the latter case $D \leqq_\alpha A \vee 0'$. Case (a) is complete.

(b) If $\gamma_0 \neq \omega_1$ then $f_A(\delta) < s_\delta$ for stationary many δ implies that there is a fixed $\delta_0 < \omega_1$ such that $f_A(\delta) < s_\delta^{\delta_0}$ for stationary many δ. Again A is ρ_{n-1}^β-finite.

To complete the proof of Theorem 9 we must treat the case $\gamma_0 > \alpha$. Then it is still true that $f_A(\delta) \geqq s_\delta$ for stationary many δ implies $C_n^\beta \leqq_\alpha A \vee 0'$. So assume that $f_A(\delta) < s_\delta$ for stationary many δ. Let $p = p^{(\rho_{n-1}^\beta \cdot C_{n-1}^\beta)}$ and let h be a Σ_1^p Skolem function for $\mathfrak{A}$. For each δ such that $f_A(\delta) < s_\delta$ choose $g(\delta) < \delta$ so that $f_A(\delta) \in h[w \times \aleph_{g(\delta)}]$ and $n_\delta < \omega$, $y_\delta < \aleph_{g(\delta)}$ so that if $x_\delta = (n_\delta, y_\delta)$ then $h(x_\delta) = f_A(\delta)$. Then by Fodor's Theorem there is $\delta_0 < \omega_1$ such that $X = \{\delta \mid g(\delta) < \delta_0\}$ is stationary. Then $\{x_\delta \mid \delta \in X\}$ is α-finite and as $\gamma_0 > \alpha$, $h \mid \{x_\delta \mid \delta \in X\}$ and hence A is ρ_{n-1}^δ-finite. □

Thus we have established:

THEOREM. *For any* $A \subseteqq \alpha$, $0' \leqq_\alpha A$ *has the same* α*-degree as an* α*-master code.*

3. The structure of the β-degrees

We continue to assume $V = L$ and let α denote $\aleph_{\omega_1}$. Now also let β denote a limit ordinal such that:

(i) $\beta > \alpha$, (ii) $\beta^* = \alpha$, (iii) $\Sigma_1 cf\beta < \beta^*$.

Typical such β's are $\alpha \cdot \omega$, $\alpha \cdot \omega_1$, $\alpha \cdot \omega_2$.

We develop a method of deducing structural properties of the β-degrees from the results of Section 2. If e, f are β-degrees, define $e \leqq_{w\beta} f$ if $E \leqq_{w\beta} f$ for some $E \in e$. Our main result is:

THEOREM 10. *There is a well-ordered sequence* $e_0 <_\beta e_1 <_\beta \cdots$ *of* β*-degrees of order type* $\aleph_{\omega_1+1}$ *such that*:

(i) *For all* γ, $e_{\gamma+1} \leqq_{w\beta} e_\gamma$.

(ii) *If* e *is an arbitrary* β*-degree then there is a unique* γ *such that* $e_\gamma \leqq_\beta e <_\beta e_{\gamma+1}$. *Also* $E \leqq_{w\beta} e_\gamma$ *for every* $E \in e$.

Thus the β-degrees are "nearly" well-ordered. It follows from the second part of (ii) that there do not exist subsets of β which are incomparable with respect to $\leqq_{w\beta}$.

Our proof of Theorem 10 depends upon choosing special representatives of the α-degrees. In case $\Sigma_1 cf\beta = \omega_1$, all the degrees of Theorem 10 arise as the β-degrees of specially chosen subsets of α. Moreover this method is of use to us in the case $\Sigma_1 cf\beta \neq \omega_1$. For now we only assume that β satisfies conditions (i), (ii), (iii) listed above.

Definition. For $B, C \subseteqq \beta$ we say that $B \leqq_{f\beta} C$ if for some β-r.e. sets

W_0, W_1:

$$x \subseteq B \leftrightarrow \exists z, w[\langle x, z, w\rangle \in W_0 \wedge z \subseteq C \wedge w \subseteq \beta - C],$$
$$y \subseteq \beta - B \leftrightarrow \exists z, w[\langle y, z, w\rangle \in W_1 \wedge z \subseteq C \wedge w \subseteq \beta - C]$$

where x, y, z, w vary over *finite* subsets of β.

Definition. If $A \subseteq \alpha$ then

$$N_\beta(A) = \{(x, y) \mid x, y \in S_\beta, x \subseteq A, y \subseteq \alpha - A\}$$

and

$$N_\alpha(A) = \{(x, y) \mid x, y \in S_\alpha, x \subseteq A, y \subseteq \alpha - A\}.$$

LEMMA 11. *For every $A \subseteq \alpha$ there is $A^* \subseteq \alpha$ such that $A =_\alpha A^*$ and $N_\beta(A^*) \leqq_{\beta f} N_\alpha(A^*)$.*

Proof. Let $p \in S_\beta$ be a parameter such that there are $f: \beta \xrightarrow{1\text{-}1} \alpha$ and $g: \Sigma_1 cf\beta \to \beta$ unbounded which are Σ_1 over S_β with parameter p. Let $\gamma_0 = \Sigma_1 cf\beta$ and choose a Σ_1^p Skolem function $h(i, x)$ for S_β and approximation $h^\gamma(i, x)$ such that for $\gamma < \gamma_0$, h^γ is a Σ_1^p Skolem function for $S_{g(\gamma)}$. Finally define $s_\delta^\gamma = h^\gamma[\omega \times \aleph_\delta] \cap \aleph_{\delta+1}$ and $s_\delta = \bigcup_{\gamma<\gamma_0} s_\delta^\gamma =$ the first β, ϕ-pseudostable greater than $\aleph_\delta$. Note that $h[\omega \times \alpha] = S_\beta$.

Key Fact. If $x \in h^\gamma[\omega \times \aleph_\delta]$ and $x \subseteq \alpha$ then $f_x(\delta') < s_{\delta'}^{\gamma+1}$ for all $\delta' \geqq \delta$, where $f_x =$ cutoff function for x. (Proof: $x \cap \aleph_{\delta'} \in h^\gamma[\omega \times \aleph_{\delta'}]$ and so $f_x(\delta') \leqq s_{\delta'}^\gamma < s_{\delta'}^{\gamma+1}$.)

We assume that A is not β-finite.

Case 1. $\Sigma_1 cf\beta = \gamma_0 = \omega_1$. Then the proof of Theorem 9 shows that $X = \{\delta \mid f_A(\delta) \geqq s_\delta^\delta\}$ is stationary. Define:

$$A^* = \{\gamma \mid \aleph_\delta \leqq \gamma \leqq f_A(\delta) \text{ for some } \delta \in X\}.$$

Case 2. $\Sigma_1 cf\beta = \gamma_0 \neq \omega_1$. Then the proof of Theorem 9 shows that $X = \{\delta \mid f_A(\delta) \geqq s_\delta\}$ is stationary. Define:

$$A^* = \{\gamma \mid \aleph_\delta \leqq \gamma \leqq f_A(\delta) \text{ some } \delta \in X\}.$$

We show now that A^* works. Clearly $A^* =_\alpha f_A =_\alpha A$. Note that for $y \subseteq \alpha$, $y \in h^\gamma[\omega \times \aleph_\delta]$:

$$y \cap [\aleph_\delta, \aleph_{\delta+1}) \neq \varnothing \longrightarrow y \cap [\aleph_\delta, s_\delta^\gamma) \neq \varnothing.$$

Therefore $y \subseteq \alpha - A^* \to y = y_1 \cup y_2$ where y_1, y_2 are β-finite and $y_1 \subseteq \bigcup_{\delta \notin X} [\aleph_\delta, \aleph_{\delta+1})$, y_2 bounded in α. Also for $x \subseteq \alpha$, $x \in h^\gamma[\omega \times \aleph_\delta]$:

$$x \text{ bounded in } [\aleph_\delta, \aleph_{\delta+1}) \longrightarrow x \subseteq [\aleph_\delta, s_\delta^\gamma).$$

Therefore $x \subseteq A^* \to x = x_1 \cup x_2$ where x_1, x_2 are β-finite, for some γ $x \in h^\gamma[\omega \times \alpha]$ and

$$x_1 \subseteq \bigcup_{\delta \in X} [\aleph_\delta, s_\delta^\gamma) , \quad x_2 \text{ bounded in } \alpha .$$

Thus $N_\beta(A^*) \leqq_{f\beta} N_\alpha(A^*)$. □

Note that for $A, B \subseteq \alpha$, $A \leqq_\alpha B \rightarrow A^* \leqq_\beta B$.

We now apply Lemma 11 to the master code degrees $d_1, d_2, \cdots, d_\gamma, \cdots$ which we defined in Section 2. Let d_{γ_1} be the α-degree of a Σ_1 master code for β. Choose canonical representatives $D_{\gamma_1}, D_{\gamma_1+1}, \cdots$ of $d_{\gamma_1}, d_{\gamma_1+1}, \cdots$ respectively and define:

$$\hat{e}_0 = 0 ,$$
$$\hat{e}_{1+\gamma} = \beta\text{-degree } (D^*_{\gamma_1+\gamma}) .$$

Also let $E_0 = \varnothing$ and $E_{1+\gamma} = D^*_{\gamma_1+\gamma}$. Then $E_\gamma \in \hat{e}_\gamma$ for all γ and $E_{\gamma+1} =_\alpha \alpha\text{-jump}(E_\gamma)$ for $\gamma \geqq 1$.

To understand the relationship between $\hat{e}_\gamma$ and $\hat{e}_{\gamma+1}$ we must discuss the weak β-jump.

Definition. Let e, f be β-degrees. Then f is the *weak β-jump* of e if f is the largest β-degree which is $\leqq_{w\beta} e$.

LEMMA 12. (a) *f is the weak β-jump of e iff $f \leqq_\beta g$ for some $g \leqq_{w\beta} e$ and $e' = \beta\text{-jump}(e) \leqq_{w\beta} f$.*

(b) *$\hat{e}_{\gamma+1}$ = the weak β-jump of $\hat{e}_\gamma$ for all γ.*

Proof. (a) Choose representatives E, F of e, f, respectively. It suffices to show that $E' \leqq_{w\beta} F$, $G \leqq_{w\beta} E \rightarrow G \leqq_\beta F$. Choose e_1, e_2 so that

$$x \in G \leftrightarrow \{e_1\}^E_\beta(x) \quad \text{diverges} ,$$
$$x \notin G \leftrightarrow \{e_2\}^E_\beta(x) \quad \text{diverges} .$$

Also choose a β-recursive $f(K, e)$ such that $\{f(K, e)\}^E_\beta(0)$ diverges $\leftrightarrow \forall x \in K$, $\{e\}^E_\beta(x)$ diverges. Then

$$K \subseteq G \leftrightarrow \{f(K, e_1)\}^E(0) \quad \text{diverges} ,$$
$$H \subseteq \beta - G \leftrightarrow \{f(H, e_2)\}^E_\beta(0) \quad \text{diverges} ,$$

so β-finite neighborhood questions about G can be answered using finite neighborhood information on E'. But $E' \leqq_{w\beta} F$ so $G \leqq_\beta F$.

(b) Recall that $N_\beta(E_\gamma) \leqq_{f\beta} N_\alpha(E_\gamma)$. We claim that $\beta\text{-jump}(E_\gamma) \leqq_{w\beta} E_{\gamma+1}$. In case $\gamma = 0$ this is clear as E_1 is a Σ_1 master code for β. Otherwise note that any subset of α which is Σ_1 over S_β is Σ_1 over $\langle S_\alpha, E_\gamma \rangle$ so we can write:

$$\begin{aligned}(e, x) \in \beta\text{-jump}(E_\gamma) &\leftrightarrow \exists z, w \in S_\beta[\langle x, z, w \rangle \in W_e \wedge z \subseteq E_\gamma \wedge w \subseteq \beta - E_\gamma] \\ &\leftrightarrow \exists z, w \in S_\alpha[\langle x, z, w \rangle \in W_{f(e)} \wedge z \subseteq E_\gamma \wedge w \subseteq \alpha - E_\gamma]\end{aligned}$$

where f is β-recursive. This last predicate is Σ_1 over $\langle S_\alpha, E_\gamma \rangle$. Thus β-jump$(E_\gamma) \leqq_{f\beta} \alpha\text{-jump}(E_\gamma) =_\alpha E_{\gamma+1}$. So $\beta\text{-jump}(E_\gamma) \leqq_{w\beta} E_{\gamma+1}$.

Now define: $\hat{E}_\gamma = \{\langle \delta, e, x\rangle \mid \{e\}_\alpha^{E_\gamma}(x)$ converges by stage $\aleph_\delta\} \subseteq \omega_1 \times \alpha \times \alpha$. (Here, $\{e\}_\alpha^E$ denotes the eth function partial α-recursive in E.) Then $\hat{E}_\gamma \leqq_{w\alpha} E_\gamma$ so $\hat{E}_\gamma \leqq_\alpha \alpha$-jump$(E_\gamma)$. Also α-jump$(E_\gamma) \leqq_\alpha \hat{E}_\gamma$. For $K \in S_\alpha$,

$$\begin{aligned} K \subseteq \alpha\text{-jump}(E_\gamma) &\leftrightarrow \forall x \in K\,,\quad (x = \langle x_0, x_1\rangle \text{ and } \{x_0\}_\alpha^{E_\gamma}(x_1) \text{ converges}) \\ &\leftrightarrow \exists f \in S_\alpha\,,\quad (f\colon \omega_1 \longrightarrow \alpha \wedge \textstyle\bigcup \text{Range} f = K \wedge \forall x \in f(\delta), \\ &\qquad \{x_0\}_\alpha^{E_\gamma}(x_1) \text{ converges by stage } \aleph_\delta, \\ &\qquad \text{for every } \delta < \omega_1) \\ &\leftrightarrow \exists f \in S_\alpha\,,\quad (f\colon \omega_1 \longrightarrow \alpha \wedge K = \textstyle\bigcup \text{Range} f \wedge \forall \delta < \omega_1, \\ &\qquad \forall x \in f(\delta),\ \langle \delta, x_0, x_1\rangle \in \hat{E}_\gamma)\,. \end{aligned}$$

And, for $H \in S_\alpha$,

$$\begin{aligned} H \subseteq \alpha - (\alpha\text{-jump}(E_\gamma)) &\leftrightarrow \forall x \in H\,,\quad (\{x_0\}_\alpha^{E_\gamma}(x_1) \text{ diverges}) \\ &\leftrightarrow \forall x \in H\,,\quad (\omega_1 \times \{x_0\} \times \{x_1\} \subseteq \alpha - \hat{E}_\gamma)\,. \end{aligned}$$

So α-jump$(E_\gamma) =_\alpha \hat{E}_\gamma$.

Now we have $E_{\gamma+1} =_\alpha \alpha$-jump$(E_\gamma) =_\alpha \hat{E}_\gamma$ so $E_{\gamma+1} \leqq_\beta \hat{E}_\gamma$ (since for any $B \subseteq \alpha$, $E_{\gamma+1} \leqq_\alpha B \rightarrow E_{\gamma+1} \leqq_\beta B$). But β-jump$(E_\gamma) \leqq_{w\beta} E_{\gamma+1}$ so the hypotheses of part (a) are now satisfied and $\hat{e}_{\gamma+1} = \beta$-degree$(E_{\gamma+1}) =$ weak β-jump$(\hat{e}_\gamma)$. □

For all inadmissible β and all β-degrees e, weak β-jump (weak β-jump $(e)) = \beta$-jump(e). Thus in this case it is appropriate to refer to weak β-jump as the "β-half-jump."

We now have all the ingredients needed to provide a proof of Theorem 10 in the case $\Sigma_1 cf\beta = \omega_1$. For such a β let $e_\gamma = \hat{e}_\gamma$ for each $\gamma \geqq 0$. As $\Sigma_1 cf\beta = \omega_1$ there is a *tame* $\Sigma_1(S_\beta)$ bijection $f\colon \beta \leftrightarrow \alpha$; i.e., for each $\delta < \alpha$, $f^{-1}[\delta]$ is β-finite and the sequence $\langle f^{-1}[\delta] \mid \delta < \alpha\rangle$ is Σ_1 over S_β.

Now let $B \subseteq \beta$ and consider $f[B] \subseteq \alpha$. Then by the proof of Theorem 9 either $f[B]$ is β-recursive or $E_1 \leqq_\alpha f[B]$. In the former case, B is β-recursive and so $0 = e_0 \leqq \beta$-degree$(B) \leqq e_1$ and $B \leqq_{w\beta} e_0$. In the latter case choose γ so that $E_\gamma =_\alpha f[B]$. Then $N_\alpha(f[B]) \leqq_{w\beta} B$ as f is tame and so we have

$$N_\beta(E_\gamma) \leqq_{f\beta} N_\alpha(E_\gamma) \leqq_{f\beta} N_\alpha(f[B]) \leqq_{w\beta} B\,;$$

therefore $E_\gamma \leqq_\beta B$. Also $B \leqq_{f\beta} f[B] \leqq_{w\beta} E_\gamma$ so $B \leqq_{\omega\beta} E_\gamma$. This proves Theorem 10 in this case.

Note. There is an easier proof of Theorem 10 in case $\Sigma_1 cf\beta = \omega_1$: Define a β-degree d to be *regular* if $\langle S_\beta, D\rangle$ is amenable for some $D \in d$. One can show that in case $\Sigma_1 cf\beta = \omega_1$, the regular β-degrees are well-ordered and for any $A \subseteq \beta$ there is a regular β-degree d such that $A \leqq_{\omega\beta} d$ and $d \leqq_\beta \beta$-degree(A). Moreover if d is regular, then weak β-jump(d) is the least regular β-degree greater than d. However the proof that we have given shows that the degrees e_γ have representatives contained in α and also provides us with the objects needed to analyze the β-degrees when $\Sigma_1 cf\beta \neq \omega_1$.

Finally we establish Theorem 10 in the case $\Sigma_1 cf\beta \neq \omega_1$. Recall the degrees $\hat{e}_\gamma$, obtained by taking the β-degrees of specially chosen representatives of the α-degrees $\geqq$ α-degree (Σ_1 master code for β). In our present case ($\Sigma_1 cf\beta \neq \omega_1$), the conclusion of Theorem 10 does not hold if we simply take $e_\gamma = \hat{e}_\gamma$; other natural β-degrees arise. For any limit ordinal γ define λ so that $E_\gamma \in \hat{e}_\gamma$ is an α-master code for λ and let $\mathfrak{A}_\gamma = \langle S_{\rho^\lambda_{n-1}}, C^\lambda_{n-1}\rangle$ where $n = n(\lambda) =$ least m such that ρ^λ_m equals α, $C^\lambda_{n-1} = \Sigma_{n-1}$ master code for λ. Then γ is *masterful* if $\Sigma_1 cf\,\mathfrak{A}_\gamma = \gamma_0 = \Sigma_1 cf\beta$. We shall define a certain β-degree $\hat{f}_\gamma$ for masterful γ.

LEMMA 13. *γ masterful $\rightarrow$ $\Sigma_1^{\mathfrak{A}_\gamma}$-cofinality$(\gamma) = \gamma_0$.*

Proof. Note that $\gamma = \omega\cdot$ordertype $\{\beta' < \rho^\lambda_{n-1} \mid \beta'$ is projectible into $\alpha\}$.

(a) If $\alpha =$ largest ρ^λ_{n-1}-cardinal then $P = \{\beta' < \rho^\lambda_{n-1} \mid \beta'$ is projectible into $\alpha\}$ is unbounded in ρ^λ_{n-1} unless ρ^λ_{n-1} is of the form $\beta' + \omega$. In the former case

$$\begin{aligned}\Sigma_1^{\mathfrak{A}_\gamma}\text{-cofinality}(\gamma) &= \Sigma_1^{\mathfrak{A}_\gamma}\text{-cofinality(ordertype of } P)\\ &= \Sigma_1^{\mathfrak{A}_\gamma}\text{-cofinality}(\rho^\lambda_{n-1}) = \gamma_0\end{aligned}$$

and in the latter case

$$\Sigma_1^{\mathfrak{A}_\gamma}\text{-cofinality}(\gamma) = \omega = \Sigma_1^{\mathfrak{A}_\gamma}\text{-cofinality}(\rho^\lambda_{n-1}) = \gamma_0 .$$

(b) If $\kappa =$ next ρ^λ_{n-1}-cardinal after α then $\gamma = \kappa$. There exists a parameter $p \in S_{\rho^\lambda_{n-1}}$ such that if h is a Σ_1^p Skolem function for $\mathfrak{A}_\gamma$ then $h[\omega \times \alpha] = S_{\rho^\lambda_{n-1}}$. Let $f: \gamma_0 \to \rho^\lambda_{n-1}$ be unbounded and Σ_1 over $\mathfrak{A}_\gamma$. Then define $g: \gamma_0 \to \kappa$ by $g(\delta) = \sup(h^{f(\delta)}[\omega \times \alpha] \cap \kappa)$, where $h^{f(\delta)}$ is the interpretation of a Σ_1^p definition of h inside $\langle S_{f(\delta)}, C_{n-1} \cap f(\delta)\rangle$ (if this structure is amenable then $h^{f(\delta)}$ is a Σ_1^p Skolem function for it). The function g is Σ_1 over $\mathfrak{A}_\gamma$ and unbounded since $\bigcup_\delta h^{f(\delta)} = h$ and $\kappa \subseteq h[\omega \times \alpha]$. □

Now let $f_\gamma: \gamma_0 \to \gamma$ be cofinal, continuous, increasing and $\Sigma_1(\mathfrak{A}_\gamma)$. Also choose $f: \gamma_0 \to \beta$ cofinal, continuous, increasing and $\Sigma_1(S_\beta)$. Define:

$$F_\gamma = \{\langle f(\gamma'), \delta\rangle \mid \gamma' < \gamma_0 \wedge \delta \in E_{f_\gamma(\gamma')}\} \subseteqq S_\beta .$$

(Thus F_γ is obtained by "spreading out" the sequence $\langle E_{f_\gamma(\gamma')} \mid \gamma' < \gamma_0\rangle$ cofinally in S_β.) F_γ is the $\Sigma_1(S_\beta)$ union of:

$$F_\gamma^{\gamma'} = \{\langle f(\gamma'), \delta\rangle \mid \delta \in E_{f_\gamma(\gamma')}\} , \quad \gamma' < \gamma_0$$

and for $\beta' < \beta$, $S_{\beta'} \cap F_\gamma^{\gamma'} = \varnothing$ for sufficiently large $\gamma' < \gamma_0$.

LEMMA 14. *For all masterful γ,*

$$N_\beta(F_\gamma) = \{(x, y) \mid x, y \in S_\beta \wedge x \subseteq F_\gamma \wedge y \subseteq S_\beta - F_\gamma\}$$
$$\leqq_{w\beta} N_\beta(F_\gamma) \cap \{(x, y) \mid x \textit{ and } y \textit{ are contained in } f[\gamma_0] \times \delta, \textit{ for some } \delta < \alpha\}\ .$$

Proof. We employ the Skolem function h introduced in the proof of Lemma 11. Then for all $\gamma' < \gamma_0$ there is a stationary set $X_{\gamma'} \subseteq \omega_1$ such that if $y \subseteq \alpha - E_{f_\gamma(\gamma')}$, $y \in h[\omega \times \aleph_\nu]$ then $y = y_1 \cup y_2$ where y_1, y_2 are β-finite, $y_1 \subseteq \bigcup_{\nu' \notin X_{\gamma'}} [\aleph_{\nu'}, \aleph_{\nu'+1})$ and $y_2 \subseteq \aleph_\nu$. It follows that if $H \subseteq S_\beta - F_\gamma$, $H \in h[\omega \times \aleph_\nu]$ then $H = H_1 \cup H_2$ where H_1, H_2 are β-finite,

$$H_1 \subseteq \bigcup_{\gamma' < \gamma_0} \left(\{f(\gamma')\} \times \bigcup_{\nu' \notin X_{\gamma'}} [\aleph_{\nu'}, \aleph_{\nu'+1})\right) \quad \text{and} \quad H_2 \subseteq f[\gamma_0] \times \aleph_\nu\ .$$

Also if $x \subseteq \alpha$, $x \in h[\omega \times \aleph_\nu]$, then $x \subseteq E_{f_\gamma(\gamma')}$ if and only if $x \subseteq \bigcup_{\nu' \in X_{\gamma'}} [\aleph_{\nu'}, \aleph_{\nu'+1})$, $|x \cap \aleph_{\nu'+1}| \leqq \aleph_{\nu'}$ for each $\nu' < \omega_1$ and $x \cap \aleph_\nu \subseteq E_{f_\gamma(\gamma')}$. It follows that if $K \subseteq S_\beta$, $K \in h[\omega \times \aleph_\nu]$ then $K \subseteq F_\gamma$ if and only if $K \subseteq \bigcup_{\gamma' < \gamma_0} \left(\{f(\gamma')\} \times \bigcup_{\nu' \in X_{\gamma'}} [\aleph_{\nu'}, \aleph_{\nu'+1})\right)$, $|K \cap \{f(\gamma')\} \times \aleph_{\nu'+1}| \leqq \aleph_{\nu'}$ for $\nu' < \omega_1$, $\gamma' < \gamma_0$ and $K \cap f[\gamma_0] \times \aleph_\nu \subseteq F_\gamma$. These two facts suffice to prove the lemma. □

We are ready to define the e_γ's in this case. For γ masterful let $\hat{f}_\gamma = \beta$-degree(F_γ). Then we set, for $\gamma = 0$ or a limit, $n \in \omega$:

$$e_{\gamma+n} = \hat{e}_{\gamma+n}\ , \qquad \gamma \textit{ not } \text{masterful}$$
$$e_\gamma = \hat{f}_\gamma\ , \quad e_{\gamma+n+1} = \hat{e}_{\gamma+n} \qquad \gamma \text{ masterful}\ .$$

Our next lemma is the key lemma toward understanding the degrees e_γ, γ masterful. Let $g_\gamma: \gamma_0 \to \rho^\lambda_{n-1}$ be cofinal, continuous, increasing and $\Sigma_1(\mathfrak{A}_\gamma)$. In case $\gamma_0 > \omega$ we also assume that $C^\lambda_{n-1} \cap g_\gamma(\gamma')$ is a regular subset of $g_\gamma(\gamma')$ for each $\gamma' < \gamma_0$. If $\gamma_0 = \omega$ then there is a Σ_{n-1} master code D^λ_{n-1} for λ such that $g_\gamma(\gamma') \cap D^\lambda_{n-1}$ is finite (and hence collapsible) for each $\gamma' < \omega$. Finally let $\bar{C} = C^\lambda_{n-1}$ if $\gamma_0 > \omega$, $\bar{C} = D^\lambda_{n-1}$ if $\gamma_0 = \omega$ and set $s(\nu, \gamma') =$ least $g_\gamma(\gamma')$, $\bar{C} \cap g_\gamma(\gamma')$-pseudostable greater than $\aleph_\nu$, for each $\nu < \omega_1$, $\gamma' < \gamma_0$.

UNIFORMITY LEMMA. (a) *If $s \leqq_{w\beta} B$ then $F_\gamma \leqq_\beta B$.*

(b) *If $t: \omega_1 \times \gamma_0 \to \alpha$, $t \leqq_{w\beta} B$ and for each $\gamma' < \gamma_0$, $\{\nu \mid s(\nu, \gamma') < t(\nu, \gamma')\}$ is stationary then $s \leqq_{w\beta} B$.*

Proof. (a) If $A \subseteq \alpha$ let $f_A: \omega_1 \to \alpha$ denote the cutoff function for A (as defined in Section 1). For $\gamma' < \gamma_0$, let $G_{\gamma'} = \bigcup_{\gamma'' < \gamma'} \{\gamma''\} \times E_{f_\gamma(\gamma'')} \subseteq \alpha$ and let $g_{\gamma'} = f_{G_{\gamma'}}$. Then $g_{\gamma'}$ is ρ^λ_{n-1}-finite so there is an ordinal $h(\gamma') < \gamma_0$ such that $g_{\gamma'}(\nu) < s(\nu, h(\gamma'))$ for sufficiently large $\nu < \omega_1$. By Lemma 14 it suffices to show that $g \leqq_{w\beta} B$ where $g(\nu, \gamma') = g_{\gamma'}(\nu)$. But by the way we have defined $G_{\gamma'}$, it actually suffices to show that $g \mid X \leqq_{w\beta} B$ where for unboundedly many $\gamma' < \gamma_0$, $\{\nu \mid \langle \nu, \gamma' \rangle \in X\}$ is unbounded.

We repeat now the argument of Lemma 2. For each $\gamma' < \gamma_0$ and $\nu < \omega_1$ let $m_\nu^{\gamma'}$ be the $<_J$-least injection of $s(\nu, h(\gamma'))$ into $\aleph_\nu$ and choose a stationary set $X_{\gamma'} \subseteq \omega_1$ such that $Y_{\gamma'} = \{m_\nu^{\gamma'}(g_{\gamma'}(\nu)) \mid \nu \in X_{\gamma'}\}$ is a bounded subset of α. As $\gamma_0 \neq \omega_1$ there is an unbounded $Y \subseteq \gamma_0$ such that $\bigcup_{\gamma' \in Y} Y_{\gamma'}$ is bounded in α. But then $g \mid X \leqq_{f\alpha} s$ where $X = \{\langle \nu, \gamma' \rangle \mid \nu \in X_{\gamma'} \wedge \gamma' \in Y\}$. Since $s \leqq_{w\beta} B$ we get $g \mid X \leqq_{w\beta} B$.

(b) Note that for all $\nu < \omega_1$, $\gamma' < \gamma_0$, $s \mid \nu \times \gamma' \in S_\eta$ where $\eta =$ largest η' such that $S_{\eta'} \models s(\nu, \gamma')$ is a cardinal (use the fact that $\bar{C} \cap g_\gamma(\gamma')$ is a collapsible predicate on $g_\gamma(\gamma')$). Now define $\tilde{s}(\nu, \gamma') = \eta$ if $s \restriction \nu \times \gamma'$ is the ηth set in the canonical well-ordering $<_J$. Thus we can assume that for each $\gamma' < \gamma_0$, $\{\nu \mid \tilde{s}(\nu, \gamma') < t(\nu, \gamma')\}$ is stationary and it suffices to produce $X \subseteq \omega_1 \times \gamma_0$ such that $\tilde{s} \restriction X \leqq_{w\beta} B$, and for unboundedly many $\gamma' < \gamma_0$, $\{\nu \mid \langle \nu, \gamma' \rangle \in X\}$ is unbounded in ω_1. Now simply repeat the argument in the second paragraph of part (a) to get such an X. □

COROLLARY TO PROOF. *Suppose* $t: \omega_1 \times \gamma_0 \to \alpha$, $u: \omega_1 \times \gamma_0 \to \alpha$ *and for each* $\gamma' < \gamma_0$, $\{\nu \mid t(\nu, \gamma') < u(\nu, \gamma')\}$ *is stationary. Then* $t \restriction X \leqq_{f\alpha} u$ *for some* X *such that for unboundedly many* $\gamma' < \gamma_0$, $\{\nu \mid (\nu, \gamma') \in X\}$ *is stationary.*

The Uniformity Lemma and its corollary are key steps in completing our proof. We first establish the relationship between $\hat{e}_\gamma$ and $\hat{f}_\gamma$ for masterful γ.

LEMMA 15. *If* γ *is masterful then* $\hat{e}_\gamma =$ *weak* β*-jump* $\hat{f}_\gamma$.

Proof. We use Lemma 12(a). Let E_γ and F_γ be as defined earlier. If $k: \beta \leftrightarrow \alpha$ is a β-recursive bijection then $E_\gamma \leqq_\beta k[F_\gamma]$ since F_γ (and hence $k[F_\gamma]$) is not ρ_{n-1}^λ-finite. As $k[F_\gamma] \leqq_{w\beta} F_\gamma$ it only remains to show β-jump$(F_\gamma) \leqq_{w\beta} E_\gamma$ in order to establish our lemma.

Note that the sequence $\langle E_{f_\gamma(\gamma')} \mid \gamma' < \gamma_0 \rangle$ is $\Sigma_1(\mathfrak{A}_\gamma)$ and therefore so is the sequence $\langle F_\gamma \cap f(\gamma') \mid \gamma' < \gamma_0 \rangle$. It follows that $N_\beta(F_\gamma)$ is Σ_1 over $\mathfrak{A}_\gamma$ and since

$$\langle e, x \rangle \in \beta\text{-jump}(F_\gamma) \leftrightarrow \exists \langle z, w \rangle \in N_\beta(F_\gamma) \quad [\langle x, z, w \rangle \in W_e^\beta]$$

we see that β-jump(F_γ) is Σ_1 over $\mathfrak{A}_\gamma$. But E_γ is a Σ_n master code for λ (a Σ_1 master code for $\mathfrak{A}_\gamma$) and therefore β-jump$(F_\gamma) \leqq_{f\beta} k[\beta\text{-jump}(F_\gamma)] \leqq_\alpha E_\gamma$. Thus β-jump$(F_\gamma) \leqq_{w\beta} E_\gamma$. □

We now complete our proof. First choose an *increasing* β-recursive sequence $\langle K_\gamma \mid \gamma' < \gamma_0 \rangle$ of β-finite sets such that β-cardinality $(K_{\gamma'}) = \alpha$ for all γ' and $S_\beta = \bigcup_\gamma K_{\gamma'}$ (using the facts that $\beta^* = \alpha$ and $\gamma_0 = \Sigma_1 cf\beta$). For each $\gamma' < \gamma_0$ let $k_{\gamma'}$ be the $<_J$-least bijection from $K_{\gamma'}$ onto α.

Now choose $B \subseteq S_\beta$ and let $B_{\gamma'} = k_{\gamma'}[B \cap K_{\gamma'}]$. Define $g: \gamma_0 \to \aleph_{\omega_1+1}$ by the property: $B_{\gamma'} =_\alpha E_{g(\gamma')}$. Let $\gamma = \sup g[\gamma_0]$.

Case 1. $\gamma = g(\gamma')$ for some γ'. If $\gamma = 0$ let $\lambda = \beta$ and $n = 1$. Otherwise, let E_γ be a Δ_n master code for λ. Let $\delta = \Sigma_1 cf\,\mathfrak{A}^\lambda_{n-1}$ and choose $s: \omega_1 \times \delta \to \alpha$ to be Σ_n over S_λ and such that $\lim_{\delta' \to \delta} s(\nu, \delta') =$ least ρ^λ_{n-1}, C^λ_{n-1}-pseudostable $> \aleph_\nu$ for each $\nu < \omega_1$. For each $\gamma' < \gamma_0$, if $j_{\gamma'} =$ cutoff function for $B_{\gamma'}$ then $\{\nu \mid j_{\gamma'}(\nu) < s(\nu, \delta')\}$ is stationary for some $\delta' < \delta$. It follows from the corollary to the Uniformity Lemma that if we let $j(\nu, \gamma') = j_{\gamma'}(\nu)$ then $j \restriction X \leqq_{w\beta} E_\gamma$ for some $X \subseteq \omega_1 \times \gamma_0$ such that for unboundedly many $\gamma' < \gamma_0$, $\{\nu \mid (\nu, \gamma') \in X\}$ is stationary. Thus since $\langle K_{\gamma'} \mid \gamma' < \gamma_0\rangle$ is increasing, $j \leqq_{w\beta} E_\gamma$ and so $B \leqq_{w\beta} E_\gamma$. Also $E_\gamma \leqq_\beta B$ since for some $\gamma' < \gamma_0$, $E_\gamma \leqq_\alpha B_{\gamma'}$ (so $N_\beta(E_\gamma) \leqq_{f\beta} N_\alpha(E_\gamma) \leqq_{f\beta} N_\alpha(B_{\gamma'}) \leqq_{w\beta} B$).

Case 2. Otherwise. Choose λ so that E_γ is a master code for λ and let n be least so that $\rho^\lambda_n = \alpha$. Let $j_{\gamma'} =$ the cutoff function for $B_{\gamma'}$ and $j(\nu, \gamma') = j_{\gamma'}(\nu)$. Also define $\delta = \Sigma_1 cf(\mathfrak{A}^\lambda_{n-1})$ and choose $s: \omega_1 \times \delta \to \alpha$ to be $\Sigma_1(\mathfrak{A}^\lambda_{n-1})$ and such that $\lim_{\delta' \to \delta} s(\nu, \delta') =$ the least ρ^λ_{n-1}, C^λ_{n-1}-pseudostable greater than $\aleph_\nu$. As each $B_{\gamma'}$ is ρ^λ_{n-1}-finite, for each $\gamma' < \gamma_0$ there is $\delta' < \delta_0$ such that $\{\nu \mid j(\nu, \gamma') < s(\nu, \delta')\}$ is stationary. But it cannot be the case that for a fixed $\delta' < \delta$, $\{\nu \mid j(\nu, \gamma') < s(\nu, \delta')\}$ is stationary for unboundedly many $\gamma' < \gamma_0$. Thus if $\delta < \alpha$ we must have $\delta = \gamma_0$. And, the argument at the end of the proof of Theorem 9 shows that $\delta < \alpha$. Thus γ is masterful.

As $\gamma_0 \neq \omega_1$ there is no Δ_n master code for λ and thus E_γ is a Σ_n master code for λ. Now for each $\gamma' < \gamma_0$ choose $h_1(\gamma')$ so that $\{\nu \mid s(\nu, \gamma') < j(\nu, h_1(\gamma'))\}$ contains a closed unbounded set (by the preceding paragraph). The Uniformity Lemma (b) is satisfied by $t(\nu, \gamma') = j(\nu, h_1(\gamma'))$ and thus $F_\gamma \leqq_\beta B$. But the above argument applies equally well to F_γ as to B, so for each $\gamma' < \gamma_0$ choose $h_2(\gamma')$ so that $\{\nu \mid s(\nu, \gamma') < l(\nu, h_2(\gamma'))\}$ contains a closed, unbounded set where $l(\nu, \gamma') = l_{\gamma'}(\nu)$ and $l_{\gamma'}$ is the cutoff function for $k_{\gamma'}[F_\gamma \cap K_{\gamma'}]$. Then for each $\gamma' < \gamma_0$ there is $h_3(\gamma')$ such that $\{\nu \mid j(\nu, \gamma') < l(\nu, h_3(\gamma'))\}$ is stationary so the corollary to the Uniformity Lemma applies to show that for some "large" X, $j \restriction X \leqq_{f\alpha} l$. But $j \leqq_{f\beta} j \restriction X$ and $l \leqq_{w\beta} F_\gamma$ so $j \leqq_{w\beta} F_\gamma$. Thus $B \leqq_{w\beta} F_\gamma$. □

We have established the conclusion of Theorem 10 when $\Sigma_1 cf\beta \neq \omega_1$.

4. Further results and open questions

As we have earlier mentioned, there are incomparable $\aleph_\omega$-degrees above $0'$ as well as infinite descending sequences of $\aleph_\omega$-degrees. However the degrees in these examples are of sets Δ_2 over $L_{\aleph^+_\omega}$ where $\aleph^+_\omega =$ next admissible after $\aleph_\omega$. Thus we propose:

Problem 1. Show that there are incomparable $\aleph_\omega$-degrees above $0'$ and infinite descending sequences of $\aleph_\omega$-degrees above $0'$ constructed (in L) before $\aleph_\omega^+$.

Harrington has shown that if $\Sigma_1 cf\beta = \Sigma_1^\beta cf(\beta^*) < \beta^*$ then incomparable β-r.e. degrees exist. We have extended this to show that for any such β there are incomparable β-degrees between e_γ and $e_{\gamma+1}$ for all γ (where e_γ is as in Theorem 10). These results will be presented in [3]. However the case $\Sigma_1 cf\beta \neq \Sigma_1^\beta cf(\beta^*)$ remains unsettled.

Problem 2. Show that if $\Sigma_1 cf\beta$, $\Sigma_1^\beta cf(\beta^*) < \beta^*$, $\Sigma_1 cf\beta \neq \Sigma_1^\beta cf(\beta^*)$ then there exist incomparable β-r.e. degrees.

Our last problem concerns the optimality of Theorem 10. A positive solution shows that $\{e_\gamma | \gamma < \aleph_{\omega_1+1}\}$ is definable over the β-degrees as a partial ordering. For then this collection consists of all β-degrees e such that for all β-degrees f, either $f \leqq e$ or $e \leqq f$.

Problem 3. Let $\langle e_\gamma | \gamma < \aleph_{\omega_1+1}\rangle$ be defined as in Theorem 10. Show that if $e_\gamma < f < e_{\gamma+1}$, then there is a g incomparable f, $e_\gamma < g < e_{\gamma+1}$.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASS.

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