J.E. Fenstad, R.O. Gandy, G.E. Sacks (Eds.)
GENERALIZED RECURSION THEORY II

Negative Solutions to Post's Problem, I

Sy D. Friedman
Department of Mathematics
University of Chicago
Chicago, Illinois 60637

§ 0. Introduction

For background in β-Recursion Theory, see [2] and our earlier paper in this volume. In [2], [3] the following version of Post's Problem is solved for a large class of ordinals β:

(*) There are β-r.e. sets A, B s.t. $A \not\leq_{w\beta} B$, $B \not\leq_{w\beta} A$.

It was conjectured in [2] that (*) holds for arbitrary limit ordinals β. It is the purpose of this note to exhibit a failure of (*) for some primitive-recursively closed β. The results of [2], [3] imply that such a β must be strongly inadmissible and for such a β, $\beta^* = \Sigma_1$ projectum of β must be singular with respect to β-recursive functions.

Thus the priority method can be applied to many but not all limit ordinals. We are not at present able to determine exactly for which ordinals (*) holds, but make the

<u>Conjecture</u> (*) holds if and only if either β is weakly admissible or β^* is regular with respect to β-recursive functions.

Thus, we feel that the positive results of [2], [3] are best possible. A conceptual explanation for our Conjecture is as follows: Define $K \subseteq \beta$ to be <u>β^*-finite</u> if K is β-finite of β-cardinality less than β^*. Then our Conjecture says that (*) holds if and only if β cannot be written as the β-recursive union of β^*-finitely many β^*-finite sets.

A key ingredient in our proof is a use of stationary sets and Fodor's Theorem much in the way Silver used them in his work ([7]) on the Generalized Continuum Hypothesis at singular cardinals of uncountable cofinality. We have found Prikry's proof ([5]) of Silver's Theorem extremely useful.

§1. Statement of Theorem and Preliminaries

Fix $\beta = \omega^{\text{th}}$ primitive-recursively closed ordinal greater than $\aleph^L_{\omega_1}$. Let $f: \omega \to \beta$ be defined by $f(n) = n^{\text{th}}$ primitive-recursively closed ordinal greater than $\aleph^L_{\omega_1}$. Then f is β-recursive, so β is strongly inadmissible and $\Sigma_1 \text{cf}\, \beta = \omega$. It now follows that $\beta^* = \aleph^L_{\omega_1}$ and thus β^* is singular with respect respect to the β-finite function $d: \omega^L_1 \to \beta^*$ given by $d(\alpha) = \aleph^L_\alpha$. Fix $C = \{\langle e, x\rangle \mid \{e\}(x)\downarrow\}$, a complete β-r.e. set.

Theorem. If A is β-r.e. then either $A =_\beta \emptyset$ or $C \leq_{w\beta} A$.

As B β-r.e. implies that $B \leq_{f\beta} C$, the Theorem shows that (*) fails for β. Moreover, any β-recursive set is β-reducible to C using only finite neighborhood conditions on C and thus β-reducible to any set A s.t. $C \leq_{w\beta} A$. So if $\underset{\sim}{d}$ is a β-r.e. degree then $\underset{\sim}{0} < \underset{\sim}{d} \to \underset{\sim}{0}^{1/2} \leq \underset{\sim}{d}$. In a future paper we shall exhibit a primitive-recursively closed ordinal where $\underset{\sim}{0}, \underset{\sim}{0}^{1/2}$ and $\underset{\sim}{0}'$ are the only β-r.e. degrees.

We end this section by reducing our Theorem to a lemma. This lemma has as its forerunner a theorem of Simpson ([8], page 71) who established it when $\beta = (\aleph^L_\omega)^+$, the first admissible greater than $\aleph^L_\omega$:

Main Lemma. If $A \subseteq \beta^*$ is β-r.e. then either A is β-finite or $C \leq_{w\beta} A$.

Proof of Theorem from Lemma. Let $A \subseteq \beta$ be β-r.e. If $A \cap f(n)$ is not β-finite for some n, then an application of the Lemma shows that $C \leq_{w\beta} A \cap f(n) \leq_\beta A$ so we are done. Otherwise, let $K: L_\beta \xrightarrow{1-1} \beta^*$ be β-recursive and define $\ell(n) = K(A \cap f(n))$. Then $\ell : \omega \to \mathcal{X}^L_{\omega_1}$ and as ℓ is constructible, ℓ is β-finite. But then $K^{-1}\ \ell$ is a β-recursive function listing $A \cap f(0)$, $A \cap f(1), \ldots$. From this it is easily seen that $A =_\beta \emptyset$. $\dashv$

§2. Proof of Main Lemma.

Let $A \subseteq \beta^*$ be β-r. e. There is an injection $L_\beta \xrightarrow{1-1} \beta^*$ which is Σ_1 over L_β with parameter $\mathcal{X}^L_{\omega_1}$. This implies that there is a complete β-r. e. set $C^* \subseteq \beta^*$ which is Σ_1 over L_β with parameter $\mathcal{X}^L_{\omega_1}$ and that A is Σ_1 definable over L_β with parameter of the form $p = \langle \mathcal{X}^L_{\omega_1}, p_0 \rangle$, $p_0 \in L_{\mathcal{X}^L_{\omega_1}}$. We will show that either A is β-finite or $C^* \leq_{w\beta} A$.

Let $h(i, x)$ be a Σ^p_1 Skolem Function for L_β; i.e., h is a partial function from $\omega \times L_\beta$ into L_β, h is Σ_1 over L_β with parameter p and if $\varphi(x, y)$ is a Σ_1 formula with parameter p, then for some i and all $x \in L_\beta$,

$$L_\beta \models \exists y \varphi(x, y) \longrightarrow h(i, x) \text{ is defined and } L_\beta \models \varphi(x, h(i, x)) .$$

Now fix $\lambda_0 < \omega_1^L$ such that $p_0 \in L_{\lambda_0}$. If $\lambda_0 \leq \lambda < \omega_1^L$ then $h[\omega \times \mathcal{X}^L_\lambda]$ is a Σ^p_1-elementary substructure of L_β and so $C^* \cap h[\omega \times \mathcal{X}^L_\lambda]$ is Σ^p_1 Definable over $h[\omega \times \mathcal{X}^L_\lambda]$. The function h has a natural approximation $h_n = (h)^{L_{f(n)}}$ and then h_n is a Σ^p_1 Skolem Function for $L_{f(n)}$.

Now for each $\lambda \geq \lambda_0$, $n < \omega$, $h_n[\omega \times \mathcal{X}^L_\lambda] \cap \mathcal{X}^L_{\lambda+1}$ is an ordinal. Call it S^n_λ. Then $\mathcal{X}^L_\lambda < S^1_\lambda < S^2_\lambda < \ldots$ and if $S_\lambda = \bigcup_n S^n_\lambda$ then $S_\lambda = h[\omega \times \mathcal{X}^L_\lambda] \cap \mathcal{X}^L_{\lambda+1}$.

Lemma 1. Let $X \subseteq \omega_1^L$ be unbounded, $Y = \{S_\lambda \mid \lambda \in X\}$. Then $C^* \leq_{f\beta} Y$.

Proof. Let $L_{\hat{S}_\lambda}$ be the transitive collapse of $h[\omega \times \aleph_\lambda^L]$ for $\lambda \geq \lambda_0$. Let $p_\lambda = \langle (\aleph_{\omega_1})^{L_{\hat{S}_\lambda}}, p_0 \rangle$. If $C^* = \{x \in L_\beta \mid L_\beta \models \varphi(x, p)\}$ where φ is Σ_1, then $C^* \cap \aleph_\lambda^L = \{x < \aleph_\lambda^L \mid L_{\hat{S}_\lambda} \models \varphi(x, p_\lambda)\}$. So it suffices to show that $\{\hat{S}_\lambda \mid \lambda \in X, \lambda \geq \lambda_0\} \leq_{f\beta} Y$. But $S_\lambda = (\aleph_{\lambda+1})^{L_{\hat{S}_\lambda}}$ so $L_{\hat{S}_\lambda} \models S_\lambda$ is regular, while $(S_\lambda)^* = \aleph_\lambda^L$ so $L_{\hat{S}_\lambda+1} \models S_\lambda$ is singular. Thus $\hat{S}_\lambda$ can be found β-recursively from S_λ. ⊣

The idea of the proof is to compare the "growth rate" of A with that of the sequence $\{S_\lambda \mid \lambda \geq \lambda_0\}$. The "growth rate" of A is measured by $f_A : \omega_1 \to \aleph_{\omega_1}^L$ defined by:

$$f_A(\lambda) = \mu\gamma[A \cap \aleph_\lambda^L \text{ is definable over } L_\gamma].$$

Lemma 2. $f_A(\lambda) \leq \hat{S}_\lambda$ for all $\lambda \geq \lambda_0$.

Proof. It is enough to show that $A \cap \aleph_\lambda^L$ is definable over $L_{\hat{S}_\lambda}$. But as in the proof of Lemma 1, if $A = \{x \mid L_\beta \models \varphi(x, p)\}$, $\varphi\Sigma_1$, then:

$$A \cap \aleph_\lambda^L = \{x < \aleph_\lambda^L \mid L_{\hat{S}_\lambda} \models \varphi(x, p_\lambda)\}, \quad \lambda \geq \lambda_0. \quad ⊣$$

Now there are two cases. Either $f_A(\lambda) \geq S_\lambda$ for unboundedly many $\lambda < \omega_1^L$, or $f_A(\lambda) < S_\lambda$ for sufficiently large $\lambda < \omega_1^L$. In the first case we show that $C^* \leq_{w\beta} A$. In the second case, A is β-finite.

Lemma 3. If $f_A(\lambda) \geq S_\lambda$ for unboundedly many $\lambda < \omega_1^L$ then $C^* \leq_{w\beta} A$.

Proof. Let $X = \{\lambda \mid f_A(\lambda) \geq S_\lambda\} \subseteq \omega_1^L$. For $\lambda \in X$, $L_{f_A(\lambda)} \models S_\lambda = \aleph_{\lambda+1}$. Since X is β-finite, this shows that $Y = \{S_\lambda \mid \lambda \in X\} \leq_{w\beta} A$ and so we are done by Lemma 1. ⊣

<u>Lemma 4.</u> If $f_A(\lambda) < S_\lambda$ for sufficiently large $\lambda < \omega_1^L$, then A is β-finite.

<u>Proof.</u> Suppose $f_A(\lambda) < S_\lambda$ for $\lambda \geq \lambda_1 \geq \lambda_0$. Define:

$$g(\lambda) = \mu n [f_A(\lambda) < S_\lambda^n] , \quad \text{for } \lambda \geq \lambda_1 .$$

Then for some fixed n_0,

$$X = \{\lambda \mid f_A(\lambda) < S_\lambda^{n_0}\}$$

is <u>stationary</u> in ω_1^L (with respect to constructible closed, unbounded sets).

At this point, the proof proceeds much as in [5].

For $\lambda \in X$, let $j_\lambda : L_{S_\lambda^{n_0}} \xrightarrow{1-1} \aleph_\lambda^L$ be the $<_L$-least injection. We assume that $\lambda \in X \rightarrow \lambda$ a limit ordinal. Then

$$\lambda \in X \rightarrow j_\lambda(A \cap \aleph_\lambda^L) < \aleph_\lambda^L \rightarrow j_\lambda(A \cap \aleph_\lambda^L) < \aleph_{\lambda'}^L \text{ some } \lambda' < \lambda.$$

We therefore have a regressive function j on the stationary set X defined by:

$$j(\lambda) = \mu\lambda' [j_\lambda(A \cap \aleph_\lambda^L) < \aleph_{\lambda'}^L].$$

By Fodor's Theorem, there is an unbounded $X_0 \subseteq X$ and $\lambda_2 < \omega_1^L$ such that

$$\lambda \in X_0 \rightarrow j(\lambda) < \lambda_2 \rightarrow j_\lambda(A \cap \aleph_\lambda^L) < \aleph_{\lambda_2}^L .$$

Now the function $f: X_0 \rightarrow \aleph_{\lambda_2}^L$ defined by:

$$f(\lambda) = j_\lambda(A \cap \aleph_\lambda^L), \quad \lambda \in X_0$$

is β-finite. Moreover, the sequence $\{j_\lambda\}_{\lambda \in X}$ is definable over $L_{f(n_0)}$ and hence β-finite. But we have:

$$x \in A \leftrightarrow \exists \lambda \in X_0 [x \in j_\lambda^{-1}(f(\lambda))]$$

so A itself is β-finite. $\dashv$

§3. Extensions

The techniques used here can be used to obtain further information about the α- and β-degrees. The following results will appear in future papers:

1) Assume $V = L$ and let $\alpha = \aleph_{\omega_1}$. Then the α-degrees $\geq 0'$ are well-ordered by $\leq_\alpha$ with successor given by α-jump. The α-jump operation on α-degrees is definable in terms of $\leq_\alpha$.

2) Assume $V = L$ and let $\beta = \omega_1 \underline{\text{st}}$ p.r. closed ordinal greater than $\aleph_{\omega_1}$. Then the β-degrees are well-ordered with successor given by the $1/2$-β-jump. The only β-r.e. degrees are $\underset{\sim}{0}, \underset{\sim}{0}^{1/2}, \underset{\sim}{0}'$.

3) Let α = first stable ordinal greater than $\aleph_{\omega_1}$ and $\beta = \aleph_{\omega_1} \underline{\text{st}}$ p.r. closed ordinal greater than α. Then there are incomparable β-r.e. degrees if and only if there are incomparable α-degrees α-r.e. in and above $0'$ if and only if $0^{\#}$ exists.

4) If $0^{\dagger}$ does not exist, then the Turing Degrees and the $\beth_{\omega_1}$-degrees are not elementarily equivalent as partial-orderings. If κ is a Strongly Compact Cardinal, then the Turing Degrees and the $\beth_{\omega_1}(\kappa)$-degrees are not elementarily equivalent as partial-orderings.

References

1. Friedman, Sy D., An Introduction to β-Recursion Theory, this volume.

2. ______________, β-Recursion Theory, to appear.

3. ______________, Post's Problem without Admissibility, to appear.

4. ______________, Negative Solutions to Post's Problem, II, in preparation.

5. Prikry, Karel, On a Theorem of Silver, handwritten notes

6. Sacks, Gerald E. and Simpson, Stephen G., The α-Finite Injury Method, Annals of Mathematical Logic 4, 1972

7. Silver, Jack, On the Singular Cardinals Problem, Proceedings of the International Congress of Mathematicians in Vancouver, 1974.

8. Simpson, Stephen G., Thesis, M.I.T., 1971