NATURAL α-RE DEGREES

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In this paper we provide an explicit positive solution to Post's Problem in α-recursion theory, for many admissible ordinals α. The Sacks-Simpson Theorem (Sacks-Simpson [72]) yields a positive solution for all admissible α via an α-finite injury argument. By way of contrast, our approach makes no use of the priority method. Instead we find new ways to combine Skolem hulls with the transitive collapse lemma. Thus our proof is really very close to Gödel's proof of the G C H in L (Gödel [39]).

If λ is a limit ordinal, $\lambda < \alpha$, then $\alpha\text{-cof}(\lambda) = \alpha$-cofinality of λ is just the cofinality of λ when evaluated inside L_α; thus, $\alpha\text{-cof}(\lambda)$ = least γ s.t. there is an unbounded $f: \gamma \longrightarrow \lambda$, $f \in L_\alpha$. A set $X \subseteq \alpha$ is <u>low</u> if $X' \leq_\alpha \emptyset'$ where X' is the α-jump of X. And, X is <u>hyperregular</u> if $\langle L_\alpha, X\rangle$ is an admissible structure. We suggest consulting Simpson [74] for further clarification of the basic notions of α-recursion theory.

<u>Theorem 1</u>. Suppose $\alpha > \omega$ is admissible and $L_\alpha \models$ There is no largest cardinal. Then $S(\omega) = \{\lambda < \alpha \mid \alpha\text{-cof}(\lambda) = \omega\}$ is a low, hyperregular α-RE set whose α-degree is strictly between 0 and $0'$.

<u>Proof</u>: An <u>α-cardinal</u> is an ordinal κ such that $L_\alpha \models \kappa$ is a cardinal. If κ is an α-cardinal then we let κ^+ denote the least α-cardinal greater than κ. We show that if $\omega < \kappa$ is a <u>regular</u> α-cardinal (that is, $L_\alpha \models \kappa$ is regular) then $\langle L_{\kappa^+}, S(\omega) \cap L_{\kappa^+}\rangle$ is a Σ_1-elementary substructure of $\langle L_\alpha, S(\omega)\rangle$. Thus any $\Sigma_1\langle L_\alpha, S(\omega)\rangle$ function f with domain $\subseteq\kappa$ (and defining parameter $p \in L_{\kappa^+}$) has range contained in L_{κ^+}. This establishes the admissibility of $\langle L_\alpha, S(\omega)\rangle$. Also note that $X = \{\kappa^+ \mid \omega < \kappa$ is a regular α-cardinal$\}$ is Π_1 over L_α and thus $\leq_\alpha 0'$. Moreover if $\phi(x)$ is a Σ_1 formula defining the complete Σ_1 set C for $\langle L_\alpha, S(\omega)\rangle$ then for $\gamma \in X$:

$$C \cap L_\gamma = \{x \in L_\gamma \mid \langle L_\gamma, S(\omega) \cap L_\gamma\rangle \models \phi(x)\}$$

and therefore $C \leq_\alpha X \vee S(\omega)$, where $\vee$ denotes α-recursive join. But C has α-degree $S(\omega)'$ and $X \vee S(\omega) \leq_\alpha 0'$, so $S(\omega)$ is low.

Suppose then that $\omega < \kappa$ is a regular α-cardinal and $\langle L_\alpha, S(\omega)\rangle \models \exists y \phi(x, y, S(\omega))$ where ϕ is Δ_0 and $x \in L_{\kappa^+}$. We wish to show that $\langle L_{\kappa^+}, S(\omega) \cap L_{\kappa^+}\rangle \models \exists y\, \phi(x, y, S(\omega) \cap L_{\kappa^+})$. Choose $y \in L_\alpha$ so that $\langle L_\alpha, S(\omega)\rangle \models \phi(x, y, S(\omega))$ and let λ be a regular α-cardinal so that $y \in L_\lambda$. Also choose δ between κ and κ^+ so that $x \in L_\delta \prec L_{\kappa^+}$ and α-cof$(\delta) > \omega$. Thus any α-finite ω-sequence from L_δ belongs to L_δ: If $f:\omega \to L_\delta$ is α-finite then $\gamma = \sup(\text{Range}(f)) < \delta$ and $g \circ f \in L_\kappa$ where g is a δ-finite injection of γ into κ. Thus $f = g^{-1} \circ (g \circ f) \in L_\delta$.

Now we let $H = \Sigma_1$ Skolem hull of $L_\delta \cup \{y\}$ inside $\langle L_\lambda, S(\omega) \cap L_\lambda \rangle$. We claim that any α-finite ω-sequence from H belongs to H. For, let h be a Σ_1 Skolem function for L_λ; thus $H = h[\omega \times (L_\delta \cup \{y\})^{<\omega}]$. If $y_0, y_1, \ldots \in H$ is α-finite then we can choose α-finite sequences $\vec{x}_0, \vec{x}_1, \ldots$ (from $(L_\delta)^{<\omega}$) and $n_0, n_1, \ldots$ (from ω) so that $y_i = h(n_i, \vec{x}_i, y)$ for each i. But then $\langle (n_0, \vec{x}_0), (n_1, \vec{x}_1), \ldots \rangle \in L_\delta$ and as the Σ_1 sentence $\exists z\, \forall i (z(i) = h(n_i, \vec{x}_i, y))$ is true in L_λ, it is true in H. So $\langle y_0, y_1, \ldots \rangle \in H$.

Transitively collapse H to L_γ, $\delta \leq \gamma < \kappa^+$. Then <u>$S(\omega) \cap H$ collapses to $S(\omega) \cap L_\gamma$</u>: Call the collapsing map π. If $\beta \in S(\omega) \cap H$ then H contains an ω-sequence $\beta_0, \beta_1, \ldots$ cofinal in β. Then $\pi(\beta_0, \beta_1, \ldots)$ is cofinal in $\pi(\beta)$ so $\pi(\beta) \in S(\omega)$. Conversely if $\pi(\beta) \in S(\omega) \cap L_\gamma$ then there is an ω-sequence $\pi(\beta_0)$, $\pi(\beta_1), \ldots$ cofinal in $\pi(\beta)$; then $\langle \beta_0, \beta_1, \ldots \rangle \in H$ and $\beta_0, \beta_1, \ldots$ must be cofinal in β, else $\sup_i \beta_i < \bar{\beta} < \beta$ for some $\bar{\beta} \in H$ and $\sup_i \pi(\beta_i) < \pi(\bar{\beta}) < \pi(\beta)$ shows that $\pi(\beta_0), \pi(\beta_1), \ldots$ is not cofinal in $\pi(\beta)$.

We now have $\langle L_\lambda, S(\omega) \cap L_\lambda \rangle \models \phi(x,y,S(\omega) \cap L_\lambda) \to \langle H, S(\omega) \cap H \rangle \models \phi(x,y,S(\omega) \cap H) \to \langle L_\gamma, S(\omega) \cap L_\gamma \rangle \models \phi(x, \pi(y), S(\omega) \cap L_\gamma)$. Thus since $\gamma < \kappa^+$ we have $\langle L_{\kappa^+}, S(\omega) \cap L_{\kappa^+} \rangle \models \exists y \;\; \phi(x,y,S(\omega) \cap L_{\kappa^+})$.

It only remains to show that $S(\omega)$ is not α-recursive. Suppose that $\exists y\, \phi(x,y,p)$ is a Σ_1 formula (with parameter p, ϕ is Δ_0) and we show that this formula does not define the complement of $S(\omega)$. Choose a successor α-cardinal κ^+ so that $p \in L_\kappa$ and $y \in L_\alpha$ so that $\phi(\kappa^+, y, p)$ holds (note that $\kappa^+ \notin S(\omega)$). Choose $\delta < \alpha$ so that $\kappa^+, y \in L_\delta$ and inductively define:

$$H_0 = \Sigma_1 \text{ Skolem hull}(L_\kappa \cup \{\kappa^+, y\}) \text{ inside } L_\delta$$
$$\kappa_0 = H_0 \cap \kappa^+$$
$$H_{n+1} = \Sigma_1 \text{ Skolem hull}(L_\kappa \cup \{\kappa_n, \kappa^+, y\}) \text{ inside } L_\delta$$
$$\kappa_{n+1} = H_{n+1} \cap \kappa^+.$$

Then $H = \bigcup_n H_n$ is the Σ_1 Skolem hull of $L_{\bar{\kappa}} \cup \{\kappa^+, y\}$ inside L_δ, where $\bar{\kappa} = \sup_n \kappa_n$. If $\pi: H \,\tilde{\to}\, L_\lambda$ then $\pi(\kappa^+) = \bar{\kappa}$. Moreover $L_\lambda \models \phi(\bar{\kappa}, \pi(y), p)$. But then $\exists y \phi(\bar{\kappa}, y, p)$ is true since ϕ is Δ_0. This shows that $\exists y \phi\, (x,y,p)$ does not define the complement of $S(\omega)$ as $\bar{\kappa}$ has α-cofinality ω. —|

The preceding proof is easily modified as follows: If κ is a regular α-cardinal then let $S(\kappa) = \{\lambda < \alpha \mid \alpha\text{-cof}(\lambda) = \kappa\}$. If there is no largest α-cardinal and $\kappa < \mu$ are regular α-cardinals then $\langle L_{\mu^+}, S(\kappa) \cap L_{\mu^+}\rangle$ is a Σ_1-elementary substructure of $\langle L_\alpha, S(\kappa)\rangle$. Moreover the same proof shows that if κ_1, κ_2 are both regular α-cardinals less than a regular α-cardinal μ then $\langle L_{\mu^+}, S(\kappa_1) \cap L_{\mu^+}, S(\kappa_2) \cap L_{\mu^+}\rangle$ is a Σ_1-elementary substructure of $\langle L_\alpha, S(\kappa_1), S(\kappa_2)\rangle$. Thus $S(\kappa_1) \vee S(\kappa_2)$ is low, hyperregular and α-RE. The next result shows that $S(\kappa_1)$, $S(\kappa_2)$ represent incomparable α-degrees.

Theorem 2. Suppose there is no largest α-cardinal and κ_1, κ_2 are distinct regular α-cardinals. Then $S(\kappa_1)$ is not α-recursive in $S(\kappa_2)$.

Proof: Suppose that ϕ is Δ_0 and $\exists y \phi(x,y,p,S(\kappa_2))$ defines the complement of $S(\kappa_1)$ over $\langle L_\alpha, S(\kappa_2)\rangle$. We derive a contradiction by showing that $\langle L_\alpha, S(\kappa_2)\rangle \models \exists y \phi(x, y, p, S(\kappa_2))$, for some $x \in S(\kappa_1)$. Choose a regular α-cardinal κ so that $p, \kappa_1, \kappa_2 \in L_\kappa$ and let $y \in L_\alpha$ be so that $\phi(\kappa^+, y, p, S(\kappa_2))$. The choice of y is possible as $\kappa^+ \notin S(\kappa_1)$. If λ is a regular α-cardinal such that $y \in L_\lambda$ then inductively define a κ_1-sequence of Skolem hulls:

$$H_0 = \Sigma_1 \text{ Skolem hull of } L_\kappa \cup \{\kappa^+, y\} \text{ in } \langle L_\lambda, S(\kappa_2) \cap L_\lambda\rangle$$

$$\gamma_0 = H_0 \cap \kappa^+$$

$$H_{\delta+1} = \Sigma_1 \text{ Skolem hull of } L_\kappa \cup \{\kappa^+, y, \gamma_\delta\} \text{ in } \langle L_\lambda, S(\kappa_2) \cap L_\lambda\rangle$$

$$\gamma_{\delta+1} = H_{\delta+1} \cap \kappa^+$$

$$H_\beta = \bigcup_{\beta'<\beta} H_{\beta'},\ \gamma_\beta = \bigcup_{\beta'<\beta} \gamma_{\beta'}, \quad \beta \text{ limit.}$$

Lemma. For $\delta < \kappa_1$, any α-finite κ_2-sequence from $H_{\delta+1}$ belongs to $H_{\delta+1}$.

Proof of Lemma: First note that if $f: \kappa_2 \to H_{\delta+1}$ is $\Sigma_1\langle L_\lambda, S(\kappa_2) \cap L_\lambda\rangle$ (with parameter in $H_{\delta+1}$) then $f \in H_{\delta+1}$. For, if $f(x) = y \leftrightarrow \langle L_\lambda, S(\kappa_2) \cap L_\lambda\rangle \models \exists z \psi(x, y, z)$ where ψ is Δ_0 then $\langle H_{\delta+1}, S(\kappa_2) \cap H_{\delta+1}\rangle \models \exists b \forall x \in \kappa_2 \exists y, z \in b \psi(x,y,z)$, as this sentence is true in $\langle L_\lambda, S(\kappa_2) \cap L_\lambda\rangle$. So $f \in H_{\delta+1}$. Now it suffices to show that any α-finite $f: \kappa_2 \to H_{\delta+1}$ is $\Sigma_1\langle L_\lambda, S(\kappa_2) \cap L_\lambda\rangle$. But there is a $\Sigma_1\langle H_{\delta+1}, S(\kappa_2) \cap H_{\delta+1}\rangle$ injection $g: H_{\delta+1} \to L_\kappa$ and $g \circ f \in L_\kappa$ (since $\kappa_2 < \kappa$ and κ is a regular α-cardinal). So $f = g^{-1} \circ (g \circ f)$ is $\Sigma_1\langle H_{\delta+1}, S(\kappa_2) \cap H_{\delta+1}\rangle$ and hence $\Sigma_1\langle L_\lambda, S(\kappa_2) \cap L_\lambda\rangle$. $\dashv$

Let $H = \bigcup_{\delta<\kappa_1} H_\delta$ and $\gamma = H \cap \kappa^+$. Then $\gamma \in S(\kappa_1)$ and if $\pi: H \xrightarrow{\sim} L_{\bar{\gamma}}$ then $\pi(\kappa^+) = \gamma$. For our contradiction it suffices to show that $\pi[S(\kappa_2) \cap H] = S(\kappa_2) \cap L_{\bar{\gamma}}$ as this implies $\langle L_{\bar{\gamma}}, S(\kappa_2) \cap L_{\bar{\gamma}}\rangle \models \phi(\gamma, \pi(y), p, S(\kappa_2) \cap L_{\bar{\gamma}})$, showing that

$\exists y\phi(x,y,p,S(\kappa_2))$ does not define the complement of $S(\kappa_1)$.

If $\beta \in S(\kappa_2) \cap H$ then there is a cofinal $f:\kappa_2 \to \beta$, $f \in H$ since H is a Σ_1-elementary substructure of L_λ. Then $\pi\circ f$ is cofinal in $\pi(\beta)$ so $\pi(\beta) \in S(\kappa_2)$ Conversely, suppose $\pi(\beta) \in S(\kappa_2)$. Choose an α-finite, increasing function $f:\kappa_2 \to H$ so that $\pi\circ f$ is cofinal in $\pi(\beta)$. Then for some $\delta < \kappa_1$ and unbounded α-finite $X \subseteq \kappa_2$, $g = f \upharpoonright X$ has range contained in $H_{\delta+1}$. By the lemma $g \in H_{\delta+1}$ But g is cofinal in β as otherwise for some $\overline{\beta} \in H$, $\cup\text{Range}(g) < \overline{\beta} < \beta$ and $\cup\text{Range}(\pi\circ f) < \pi(\overline{\beta}) < \pi(\beta)$, contradicting the fact that $\pi\circ f$ is cofinal in $\pi(\beta)$.—|

Some Remarks and Questions

1) The proof of Theorem 2 is easily extended to show: If there is no largest α-cardinal and $\kappa,\kappa_1,\ldots,\kappa_n$ are distinct α-cardinals then $S(\kappa)$ is not α-recursive in $S(\kappa_1)\vee\ldots\vee S(\kappa_n)$. Moreover $S(\kappa_1)\vee\ldots\vee S(\kappa_n)$ is low and hyperregular.

2) Suppose there is no largest α-cardinal and κ_1,κ_2 are distinct α-cardinals. Then is $S(\kappa_1)$, $S(\kappa_2)$ a minimal pair (i.e., does $A \leq_\alpha S(\kappa_1)$, $A \leq_\alpha S(\kappa_2)$ imply A is α-recursive)?

3) Are there any incomplete α-RE degrees greater than all of the α-degrees of $S(\kappa)$ for regular α-cardinals κ(assuming there is no largest α-cardinal)?

4) Problem: Find natural, intermediate α-RE degrees when there is a largest α-cardinal. If $\alpha^* = \alpha$ then there is no largest α-stable; is there a way of making use of the α-stables similar to the above use of α-cardinals?

References

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