$0^\#$ and Inner Models

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In this paper we examine the cardinal structure of inner models that satisfy GCH but do not contain $0^\#$. We show, assuming that $0^\#$ exists, that such models necessarily contain Mahlo cardinals of high order, but without further assumptions need not contain a cardinal κ which is κ-Mahlo. The principal tools are the Covering Theorem for L and the technique of reverse Easton iteration.

Let I denote the class of Silver indiscernibles for L and $\langle i_\alpha \mid \alpha \in \mathrm{ORD}\rangle$ its increasing enumeration. Also fix an inner model M of GCH not containing $0^\#$ and let ω_α denote the ω_α of the model $M[0^\#]$, the least inner model containing M as a submodel and $0^\#$ as an element.

Theorem 1 *Suppose that α is greater than 0. (a) $i_{\omega_1\cdot\alpha}$ is an M-cardinal, and unless α is a limit ordinal of countable $M[0^\#]$-cofinality, so is its L-cardinal successor.*
(b) If β is less than $i_{\omega_1^{L[0^\#]}\cdot\omega}$ then there is a proper inner model M of $L[0^\#]$ satisfying GCH in which the only ordinals between ω and β which are M-cardinals are those which are required to be by part (a).

It follows from (a) that for finite n, ω^M_{2n+1} is at most $i_{\omega_1\cdot(n+1)}$ and that ω^M_{2n+2} is at most the L-cardinal successor to $i_{\omega_1\cdot(n+1)}$. It follows from (b) that these bounds are optimal. The restriction in (b) on β cannot be weakened, as otherwise an increasing ω-sequence of Silver indiscernibles, and hence $0^\#$

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itself, would belong to M. In fact the supremum of the $i_{\omega_1 \cdot n}$'s must be large in M:

Theorem 2 *(a) $i_{\omega_1 \cdot \alpha}$ is inaccessible in M for limit α.*
(b) If β is less than $i_{\omega_1^{L[0^\#]} \cdot \omega \cdot \omega}$ then there is a proper inner model M of $L[0^\#]$ satisfying GCH in which the only ordinals less than β which are M-inaccessible are those which are required to be by part (a).

It follows from (a) that for finite n, the n-th M-inaccessible is at most $i_{\omega_1 \cdot \omega \cdot n}$. It follows from (b) that these bounds are optimal. As before, the restriction in (b) on β cannot be weakened, as otherwise $0^\#$ would belong to M.

We can also obtain Mahlo cardinals of high order in M. Define: κ is 0-*Mahlo* (or simply *Mahlo*) iff the set of inaccessible $\bar{\kappa} < \kappa$ is stationary in κ, κ is $\alpha + 1$-*Mahlo* iff the set of α-Mahlo $\bar{\kappa} < \kappa$ is stationary in κ, and for limit λ, κ is λ-*Mahlo* iff κ is α-Mahlo for every $\alpha < \lambda$.

Theorem 3 *(a) $i_{\omega_1 \cdot \beta}$ for β of $M[0^\#]$-cofinality at least $\omega_{\alpha+1}$ is α-Mahlo in M.*
(b) Suppose that α is not $L[0^\#]$-inaccessible. If γ is less than $i_{\omega_1^{L[0^\#]} \cdot \omega_{\alpha+1}^{L[0^\#]} \cdot \omega}$ then there is a proper inner model of $L[0^\#]$ satisfying GCH in which the only α-Mahlo cardinals less than γ are those which are required to be by part (a).

It follows from (a) that for finite $n > 0$, the n-th α-Mahlo cardinal of M is at most $i_{\omega_1 \cdot \omega_{\alpha+1} \cdot n}$. It follows from (b) that these bounds are optimal when α is not $M[0^\#]$-inaccessible . And, as before, the bound on γ in (b) cannot be improved. Part (a) of Theorem 3, when $\alpha = 0$, was proved independently by Amir Leshem.

Theorem 4 *(a) If α is inaccessible in $M[0^\#]$ then it is α-Mahlo in M.*
(b) If there are only finitely many $L[0^\#]$-inaccessibles less than β then there is a proper inner model of $L[0^\#]$ satisfying GCH in which the only cardinals α less than β which are α-Mahlo are inaccessible in $L[0^\#]$. If there are no inaccessibles in $L[0^\#]$ then there is a proper inner model M of $L[0^\#]$ satisfying GCH which contains no α which is α-Mahlo.

Thus we have reached the limit of large cardinal properties which must hold in M, in the theory ZFC$+0^\#$ exists. Of course if $0^{\#\#}$ exists, then much stronger large cardinal properties, such as n-subtlety, are witnessed in inner models of $L[0^\#]$, as these properties are witnessed in $L[0^\#]$ and are downward absolute to inner models. The next result implies that the assumption of $0^\#$ plus an $\omega+\omega$-Erdős cardinal maximizes the large cardinal properties that must hold in inner models which are generic extensions of L.

Theorem 5 *Suppose that $0^\#$ exists, there is an $\omega+\omega$-Erdős cardinal and M is an inner model generic over L. Suppose that φ is a sentence true in M. Then φ is also true in an inner model of $L[R]$ for some real R such that $R^\#$ exists.*

To see the implications of Theorem 5, suppose that P is a property downward absolute to inner models, and that P is witnessed in $L[R]$ whenever R is a real and $R^\#$ exists. An example of such a property is the existence of α-Erdős cardinals for α countable in L. Then by Theorem 5, P is also witnessed in all inner models which are class-generic extensions of L. This applies to the strongest large cardinal properties that hold in L. It follows from the last statement of Theorem 4 (b) that the hypothesis of an $\omega+\omega$-Erdős cardinal cannot be deleted in Theorem 5.

Proof of Theorem 1. (a) We show that if κ is an indiscernible of the form $i_{\omega_1\cdot\alpha}$, $\alpha>0$ then κ is a cardinal of M, and if κ is an indiscernible of uncountable cofinality in $M[0^\#]$ then its L-cardinal successor is a cardinal of M.

Lemma 6 *Suppose that κ is L-regular and has uncountable $M[0^\#]$-cofinality. Then κ is a limit of indiscernibles and κ^+ of M equals κ^+ of L.*

This is proved as follows. For each finite n consider $C_n=\{\bar\kappa<\kappa\mid$ No ordinal between $\bar\kappa$ and κ is L-definable from ordinals less than $\bar\kappa$ together with n indiscernibles $\geq\kappa\}$. For each n, C_n is CUB in κ, by the L-regularity of κ. And the intersection of the C_n's is equal to $I^*=I\cap\kappa$, since every ordinal is L-definable from finitely many indiscernibles. As κ has uncountable $L[0^\#]$-cofinality, it follows that I^* is unbounded in κ, and hence κ is a limit of indiscernibles. Now if κ^+ of L were collapsed in M, then there would be in M only κ-many constructible CUB subsets of κ. By taking diagonal intersection, there would then be in M a CUB subset C of κ which is almost contained in (i.e., contained in except for a bounded subset of κ) each constructible CUB subset of κ. But then since κ has uncountable cofinality in $M[0^\#]$ and I^* is the countable intersection of constructible CUB subsets of κ, C is almost contained in I^*. It follows that M contains an infinite set of indiscernibles, and therefore $0^\#$ belongs to M.

Lemma 7 *If κ is both L-regular and at least ω_2^M, then the cofinality of κ in M equals its cardinality in M.*

This is proved as follows. By the Covering Theorem ([2]), there is a constructible cofinal subset of κ of M-cardinality at most the maximum of ω_1^M and the M-cofinality of κ. By hypothesis, the M-cardinality of κ is greater than ω_1^M. Therefore if the M-cofinality of κ were less than the M-cardinality of κ, it would follow that κ would be singular in L, against our hypothesis.

Now we show by induction on κ, that if κ is an indiscernible of the form $i_{\omega_1 \cdot \alpha}$, $\alpha > 0$ then κ is a cardinal of M. The base case is where κ equals ω_1 of $M[0^\#]$; clearly κ is a cardinal of M since M is contained in $M[0^\#]$. The result follows immediately by induction if κ is of the form $i_{\omega_1 \cdot \lambda}$, λ limit. So we may assume that κ is of the form $i_{\omega_1 \cdot \alpha + \omega_1}$, $\alpha > 0$. Now κ is L-regular and also at least ω_2^M, since by Lemma 6 the latter is the L-cardinal successor to ω_1. Therefore by Lemma 7, the cardinality of κ in M is equal to its cofinality in M. In particular, the cardinality of κ in M is regular in M and has cofinality ω_1 in $M[0^\#]$. Now assume that κ is not a cardinal of M. Then $\gamma =$ the M-cardinality of κ is an L-regular ordinal in the interval $[i_{\omega_1 \cdot \alpha}, \kappa)$ of uncountable $M[0^\#]$-cofinality. By Lemma 6, γ is a limit of indiscernibles, but every limit of indiscernibles in this interval, with the possible exception of $i_{\omega_1 \cdot \alpha}$, has $L[0^\#]$-cofinality ω. It follows that γ equals $i_{\omega_1 \cdot \alpha}$, and that the latter has uncountable $M[0^\#]$-cofinality. But then Lemma 6 implies that the L-cardinal successor to $i_{\omega_1 \cdot \alpha}$ is an M-cardinal, and this is a contradiction. So κ is a cardinal of M.

(b) We use reverse Easton forcing. To simplify notation, assume that $V = L[0^\#]$. First consider the reverse Easton iteration with Easton supports, where $P(\leq \alpha) = P(< \alpha) * P(\alpha)$ and at an L-regular stage α, $P(\alpha)$ is the forcing with finite conditions for collapsing α to ω. Thus $P = P(< \omega_1)$

makes every ordinal less than ω_1 countable in the generic extension. We claim that there exists a P-generic. Notice that for any indiscernible $i < \omega_1$ there does exist a $P(< i)$-generic, because i^+ of L is countable. And if $j_0 < j_1 < \cdots$ is an increasing ω-sequence of indiscernibles with supremum j, G_n is $P(< j_n)$-generic over L and $G_n \subseteq G_{n+1}$ for each n, then $G = \cup\{G_n \mid n \in \omega\}$ is $P(< j)$-generic over L, since the Mahloness of j in L implies that constructible antichains in $P(< j)$ have L-cardinality less than j. It is now straightforward to build a generic for P as the union of generic $P(< j)$ for countable indiscernibles j. In the resulting generic extension M, $\omega_1^M = \omega_1$ and ω_2^M is the L-cardinal successor to ω_1.

Now repeat the same forcing construction on the interval between the L-cardinal successor to ω_1 and $i_{\omega_1+\omega_1}$, collapsing every L-regular cardinal in this interval to the L-cardinal successor to ω_1, using conditions of size $\leq \omega_1$. As this forcing is $\leq \omega_1$-distributive and for each indiscernible i between ω_1 and $i_{\omega_1+\omega_1}$, i^+ of L is the countable union of sets of L-cardinality ω_1, there exists a $P(< i)$-generic for each indiscernible i between ω_1 and $i_{\omega_1+\omega_1}$. By taking the union of such generics, we get a $P(< i_{\omega_1+\omega_1})$-generic, which ensures that in the resulting extension M, $\omega_3^M = i_{\omega_1+\omega_1}$ and ω_4^M is the L-cardinal successor to $i_{\omega_1+\omega_1}$.

Continue in this way to obtain, for any fixed finite m, a generic extension of L in which $\omega_{2n+1} = i_{\omega_1\cdot(n+1)}$ and ω_{2n+2} is the L-cardinal successor to $i_{\omega_1\cdot(n+1)}$ for each $n \leq m$. The fact that GCH holds in these models follows by standard techniques. (One cannot achieve this for all finite n simultaneously, because defining the forcing would need an infinite sequence of indiscernibles as a parameter, not available in the ground model L). $\square$

Proof of Theorem 2. (a) Let λ be of the form $i_{\omega_1\cdot\alpha}$, α limit. In the proof of Theorem 1 we showed that each $i_{\omega_1\cdot\alpha}$, $\alpha > 0$ is an M-cardinal, and therefore λ is a limit M-cardinal. As λ is regular in L, it is also regular in M by the Covering Theorem, and therefore λ is inaccessible in M.

(b) Assume that $V = L[0^\#]$. We show that there is a generic extension M of L such that $\kappa = i_{\omega_1\cdot\omega}$ is the least M-inaccessible.

First we force a CUB subset C of κ, containing the $i_{\omega_1\cdot n}$'s for $0 < n \in \omega$, whose limit points are L-singular. Consider the forcing P whose conditions are bounded closed subsets of κ with L-singular limit points. This forcing is $< \kappa$-distributive. And κ^+ of L is the countable union of constructible sets

X_n, $n \in \omega$ of L-cardinality less than κ. Moreover we can choose X_n to be L-definable from $\kappa_n = i_{\omega_1 \cdot (n+1)}$ together with n indiscernibles $\geq \kappa$. Thus we can build an ω-sequence $p_0 \geq p_1 \geq \cdots$ of conditions in P such that the union of the p_n's is P-generic, the maximum of p_n is less than κ_{n+1} and κ_n is an element of p_n. It follows that $C =$ the union of the p_n's is a CUB subset of κ containing each κ_n whose limit points are L-singular.

Now over this generic extension as ground model, force to collapse each L-regular $\beta < \kappa$ not in C to α^+ of L where α is $\omega \cup$ the maximum of $C \cap \beta$. A generic for this reverse Easton iteration can be obtained *almost* as in the proof of part (b) of Theorem 1, by successively choosing generics for the $\leq \kappa_n$-distributive part of the iteration between κ_n and κ_{n+1}. However we now need a new argument to ensure that the union of the generics below the κ_n's produces a generic for the entire forcing, since the Mahloness of κ has been destroyed. So proceed as follows: Let S_n consist of all dense constructible subsets of this forcing which are definable in L from n indiscernibles $\geq \kappa$ together with parameters less than κ_n. At stage n of the construction, extend what has already been chosen to a generic for the forcing up to κ_n, and also meet all dense sets in S_n. After ω steps the result is the generic for the entire forcing.

Now let M be the generic extension of L resulting from this last forcing. Then in M, the only limit cardinals less than κ are limit points of C and hence are L-singular. It follows that κ is the least M-inaccessible. By a similar construction, we can obtain, for any fixed finite m, a model M, generic over L, in which the first m inaccessibles are the ordinals $i_{\omega_1 \cdot \omega \cdot n}$ for $1 \leq n \leq m$. Again the fact that GCH holds in these models follows by standard techniques. □

Proof of Theorems 3, 4. (a) Note that part (a) of Theorem 3 implies part (a) of Theorem 4. We prove part (a) of Theorem 3 by induction on α. If $\alpha = 0$, we must show that $i_{\omega_1 \cdot \beta}$ is Mahlo in M when β has uncountable cofinality in $M[0^\#]$. Suppose that $C \in M$ is CUB in $i_{\omega_1 \cdot \beta}$; then C has an element of the form $i_{\omega_1 \cdot \gamma}$, γ limit, using the uncountable $M[0^\#]$-cofinality of β. By part (a) of Theorem 2, this element of C is inaccessible in M, and therefore we have established the Mahloness of $i_{\omega_1 \cdot \beta}$ when β has uncountable $M[0^\#]$-cofinality. If α is a limit ordinal, then (a) follows easily by induction. Finally, suppose that $\alpha = \alpha' + 1$ and β has $M[0^\#]$-cofinality at least $\omega_{\alpha'+2}$; we must show that $i_{\omega_1 \cdot \beta}$ is $\alpha' + 1$-Mahlo in M. If $C \in M$ is CUB in $i_{\omega_1 \cdot \beta}$ then C has an element of the form $i_{\omega_1 \cdot \gamma}$, where γ has $M[0^\#]$-cofinality at least

$\omega_{\alpha'+1}$; it follows by induction that this element of C is α'-Mahlo in M, and therefore we have shown that $i_{\omega_1 \cdot \beta}$ is $\alpha' + 1$-Mahlo in M, as desired.

(b) Assume $V = L[0^\#]$. We first consider the case $\alpha = 0$ of Theorem 3 and begin by constructing a generic extension M of L in which $\kappa = i_{\omega_1 \cdot \omega_1}$ is the least Mahlo cardinal of M. We would like to perform a reverse Easton iteration of length κ, in which the Mahloness of cardinals less than κ is destroyed; unfortunately it is not possible to obtain a generic for the natural such iteration at ordinal stages of uncountable cofinality. Our solution is to first add a generic CUB $C \subseteq \kappa$ containing no such ordinal as a limit point, ensure that all limit cardinals less than κ are limit points of C and finally kill the Mahloness of elements of Lim C by a reverse Easton iteration indexed by Lim C.

To add C, use the reverse Easton iteration P where at an L-regular stage $\alpha \leq \kappa$, $P(\alpha)$ either forces a closed unbounded subset of α using bounded closed conditions or chooses a closed bounded subset of α. Then build a P-generic G meeting the following requirements:

1. At an indiscernible i not of the form $i_{\omega_1 \cdot \alpha}$, $\alpha > 0$, $G(i)$ chooses the empty set.
2. At an indiscernible i of the form $i_{\omega_1 \cdot \alpha}$, α limit, $G(i)$ is the union of $G(i_{\omega_1 \cdot \beta})$, $\beta < \alpha$.
3. At an indiscernible i of the form $i_{\omega_1 \cdot (\lambda + n)}$, λ limit or 0, $n \in \omega$, $G(i)$ is taken to be the least condition p in the forcing $P(i)$ for adding a CUB subset of i such that: (a) p extends $G(i_{\omega_1 \cdot (\lambda+n)}) \cup \{i_{\omega_1 \cdot (\lambda+n)}\}$. (b) p meets all dense subsets of $P(i)$ which are definable in $L[G(< i)]$ from ordinals less than or equal to $i_{\omega_1 \cdot (\lambda+n)}$ together with the first $n + 1$ indiscernibles greater than or equal to i.

Notice that in Step 3 the maximum of $G(i)$ is less than the least indiscernible greater than $i_{\omega_1 \cdot (\lambda+n)}$. The result is that if $C = G(\kappa)$ then Lim $C \cap I$ consists of the indiscernibles of the form $i_{\omega_1 \cdot \alpha}$, α limit, $\alpha < \omega_1$. Notice that the Mahloness of these indiscernibles, as well as of κ, has been preserved.

Now force to collapse each $\beta < \kappa$ not in C to α^+ of L where α is the maximum of $C \cap \beta$. (This is as in the proof of part (b) of Theorem 2, except it is easier here, since the Mahloness of κ has been preserved). The result is that each limit cardinal less than κ belongs to Lim C, and again the Mahloness of elements of Lim $C \cap I$ has been preserved.

Finally we force to kill the Mahloness of elements of Lim C. This uses the reverse Easton iteration where at regular stages $\alpha < \kappa$ in Lim C, one forces a CUB subset of α consisting of singular cardinals. It is possible to inductively choose generics $G(\leq i)$ for $i \in \text{Lim } C \cap I$, using the Mahloness of i to obtain the genericity of $G(< i)$ and the countable cofinality of i to select $G(i)$. The result is that no cardinal less than κ is Mahlo, and therefore κ is the least Mahlo cardinal in the final model M. A similar argument produces a generic extension M of L (contained in $L[0^\#]$) in which $i_{\omega_1 \cdot \omega_1}, i_{\omega_1 \cdot \omega_1 \cdot 2}, \ldots, i_{\omega_1 \cdot \omega_1 \cdot n}$ are the first n Mahlo cardinals of M. The GCH can be easily verified in these models. This proves part (b) of Theorem 3 when $\alpha = 0$.

We next treat the case $\alpha = 1$ of Theorem 3. Let κ now denote ω_2. We show that there is a generic extension M of L satisfying GCH such that κ is the least 1-Mahlo in M. A similar argument will give the existence of a generic extension M of L contained in $L[0^\#]$ and satisfying GCH in which $\kappa, i_{\kappa \cdot 2}, \ldots, i_{\kappa \cdot n}$ are the first n 1-Mahlo cardinals, which will therefore establish part (b) of Theorem 3 when $\alpha = 1$.

We would like to use a reverse Easton iteration to kill the 1-Mahloness of each cardinal less than κ, by adding CUB sets of non-Mahlos. Once again, we have difficulty choosing a generic $G(i)$ at an indiscernible $i < \kappa$ of uncountable cofinality. The solution is to use a $\square_{\omega_1}$-sequence to obtain $G(i)$ as the union of generic $G(\bar{i})$, for $\bar{i} < i$ of countable cofinality. However, we must be sure that such $\bar{i}$ are not Mahlo, before they can be included in $G(i)$. Thus we must also kill the Mahloness of indiscernibles of countable cofinality.

We perform the following reverse Easton iteration of length κ: At an L-regular stage α we either add only a CUB subset $D_1(\alpha)$ of α consisting of ordinals which are not Mahlo, or both $D_1(\alpha)$ and $D_0(\alpha)$, a CUB subset of α consisting of ordinals which are not regular. Our intention is to add only $D_1(i)$ if i is an indiscernible of uncountable cofinality and both $D_0(i)$ and $D_1(i)$ if i is an indiscernible of countable cofinality. To build a generic for this forcing, we make use of a $\square_{\omega_1}$ sequence $\langle C_\alpha \mid \omega_1 < \alpha < \omega_2,\ \alpha \text{ limit}\rangle$. Thus, C_α is CUB in α of ordertype at most ω_1 and $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$ if $\bar{\alpha}$ is a limit point of C_α. Also assume that if α is a limit of indiscernibles then C_α consists only of indiscernibles. Now inductively build generics $G(\leq i)$, $i \in I \cap \kappa$ with the property that if $i > \omega_1$ and $\bar{i}$ is a limit point of C_i then $D_1(i)$ extends $D_1(\bar{i})$. Note that if C_i is undefined or does not have unboundedly many limit points then i has cofinality ω and therefore it is straightforward to build $G(i)$.

We must of course guarantee in this case that $D_1(i)$ extends $D_1(\bar{i})$ where $\bar{i}$ is the largest limit point of C_i. But as $\bar{i}$ is not Mahlo in $G(< i)$ it can be included as an element of $D_1(i)$. Also note that we obtain a generic $G(< i)$ at limit stages i (and a generic $G(\leq i)$ when C_i is unbounded in i) using the fact that for any constructible dense $D \subseteq G(\leq i)$, there is $\bar{i} < i$ such that $D \cap G(\leq \bar{i})$ is dense on $P(\leq \bar{i})$.

The final result is that κ is the least 1-Mahlo in the resulting generic extension M. M can be shown to satisfy the GCH using standard techniques.

New problems arise when $\alpha = 2$. We must construct a generic extension M of L in which $\kappa = \omega_3$ is the least 2-Mahlo. As in the previous case, we can kill the Mahloness of indiscernibles of countable cofinality. We would also like to kill the 1-Mahloness of indiscernibles of cofinality ω_1 using $\Box$; the difficulty is that the ordertype of the usual $\Box$-sequence at an ordinal of cofinality ω_1 can be greater than ω_1, which makes it impossible to cohere generics along this sequence when forcing to kill 1-Mahloness. The solution is to use a different form of $\Box$.

$\Box_\kappa^{cof\leq\lambda}$ ($\lambda \leq \kappa$ infinite cardinals):
There exists $\langle C^*_\alpha \mid \kappa < \alpha < \kappa^+, \alpha \text{ limit}, \text{cof } \alpha \leq \lambda \rangle$ such that C^*_α is CUB in α of ordertype $\leq \lambda$ and $C^*_{\bar{\alpha}} = C^*_\alpha \cap \bar{\alpha}$ if $\bar{\alpha}$ is a limit point of C^*_α.

Lemma 8 *Assume Global $\Box$: There exists $\langle C_\alpha \mid \alpha$ a singular limit ordinal$\rangle$ such that C_α is CUB in α of ordertype $< \alpha$ and $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$ if $\bar{\alpha}$ is a limit point of C_α. Then $\Box_\kappa^{cof\leq\lambda}$ holds for each $\lambda \leq \kappa$.*

This is proved as follows. For a limit ordinal $\alpha > \lambda$ of cofinality $\leq \lambda$, define: $\alpha_0 = \alpha$, $\alpha_{k+1} =$ ordertype of C_{α_k} (if α_k is greater than λ) and let $k(\alpha)$ be least so that $\alpha_{k+1} \leq \lambda$. Let $f_l : \alpha_{l+1} \to \alpha_l$ be the increasing enumeration of C_{α_l} for each $l \leq k(\alpha)$. Then define C^*_α to be the range of $f_0 f_1 \cdots f_k$ on $\{\gamma < \alpha_{k+1} \mid f_k(\gamma) > \lambda\}$ if $k(\alpha) > 0$ and $C^*_\alpha = C_\alpha$ otherwise. Note that the value of k is the same for elements of C^*_α as it is for α. It is straightforward to verify that the C^*_α's for $\kappa < \alpha < \kappa^+$ provide a $\Box_\kappa^{cof\leq\lambda}$-sequence, using the coherence properties of the given Global $\Box$-sequence and the fact that $k(\alpha) = k(\bar{\alpha})$ for $\bar{\alpha}$ in C^*_α.

As Global $\Box$ holds in $L[R]$ for each real R, it follows that $\Box_\kappa^{cof\leq\lambda}$ holds in $V = L[0^\#]$ for each pair of infinite cardinals $\lambda \leq \kappa$.

Now we present the construction of the desired model M in which $\kappa = \omega_3$ is the least 2-Mahlo, using a $\square_{\omega_2}^{cof \leq \omega_1}$-sequence. Let $\langle C^*_\alpha \mid \omega_2 < \alpha < \omega_3,\ \alpha \text{ limit, cof } \alpha \leq \omega_1\rangle$ be such a sequence, and for limit $\alpha \in (\omega_1, \omega_2)$ let C^*_α be C_α, where $\langle C_\alpha \mid \alpha \text{ limit}, \omega_1 < \alpha < \omega_2\rangle$ witnesses $\square_{\omega_1}$. Also let $\langle C_\alpha \mid \alpha \text{ limit}, \omega_2 < \alpha < \omega_3\rangle$ witness $\square_{\omega_2}$. We assume that if both C^*_α and C_α are defined then $C^*_\alpha \subseteq C_\alpha$, and if i is a limit of indiscernibles then C^*_i and C_i (when defined) consist only of indiscernibles.

At an L-regular stage α we either add only $D_2(\alpha)$, a CUB subset of α consisting of ordinals which are not 1-Mahlo, or both $D_2(\alpha)$ and $D_1(\alpha)$, a CUB subset of α consisting of ordinals which are not Mahlo, or $D_2(\alpha)$, $D_1(\alpha)$ and $D_0(\alpha)$, a CUB subset of α consisting of ordinals which are not regular. Our intention at an indiscernible stage i is to add $D_0(i)$ only if i has countable cofinality and $D_1(i)$ only if i has cofinality at most ω_1. To build a generic for this forcing, we respect the coherence properties: If $\bar{i}$ is a limit point of C^*_i then $D_1(i)$ and $D_2(i)$ extend $D_1(\bar{i})$ and $D_2(\bar{i})$, respectively; if $\bar{i}$ is a limit point of C_i then $D_2(i)$ extends $D_2(\bar{i})$. We must guarantee that when C^*_i has a largest limit point $\bar{i}$, then $\bar{i}$ can be included in $D_1(i)$ and $D_2(i)$. But $\bar{i}$ has countable cofinality and therefore is not Mahlo in $G(< i)$; it follows that $\bar{i}$ can be put into $D_1(i)$ and $D_2(i)$. We must also ensure that when C_i has a largest limit point $\bar{i}$, then $\bar{i}$ can be included in $D_2(i)$. But in this case $\bar{i}$ has cofinality $\leq \omega_1$ and therefore is not 1-Mahlo in $G(< i)$; it follows that $\bar{i}$ can be put into $D_2(i)$. As before we obtain generics at limit stages, and in the resulting model M, κ is the least 2-Mahlo cardinal.

The general case of Theorem 3 is based upon the previous one. First assume that there is no inaccessible in $V = L[0^\#]$. Define the function F on indiscernibles by $F(i) = \alpha$ where ω_α is the cofinality of i. Note that $F(i) < i$ is either 0 or a successor ordinal for every i, as we have assumed that there are no inaccessibles.

Now we describe a reverse Easton iteration designed to guarantee that i is not $F(i)$-Mahlo for each indiscernible i. At an L-regular stage α of this iteration, we choose $\beta(\alpha) \in \{-1\} \cup \alpha$ and let $P(\alpha, \beta)$ be the forcing to add a CUB subset $C(\alpha, \beta)$ of α consisting of ordinals which are not β-Mahlo, for each β in the interval $[\beta(\alpha), \alpha)$. (When β is -1, we interpret β-Mahlo to mean regular.) Then $P(\alpha)$ is the forcing obtained by taking the product of these $P(\alpha, \beta)$'s, with $< \alpha$ support. Our intention is to choose $\beta(i)$ to be $F(i) - 1$ at each indiscernible i.

We now build a generic for this iteration, with the desired choices of $F(i)$ for $i \in I$. Let $\langle C_\alpha \mid \alpha \text{ limit}\rangle$ be a Global $\Box$-sequence such that C_α has ordertype at most the cardinality of α for each α and if i is a limit of indiscernibles then C_i consists only of indiscernibles. Now as in the proof of Lemma 8, if i is an indiscernible which is singular, define $i_0 = i$, $i_{k+1} =$ ordertype C_{i_k} if i_k is singular, $k(i)$ the least k such that the ordertype of C_{i_k} is regular. (The i_k's for $k > 0$ need not be indiscernibles.) Let C_i^0 be C_i and for $k < k(i)$, let $f_k : i_{k+1} \to i_k$ be the increasing enumeration of C_{i_k} and C_i^{k+1} the image of $C_{i_{k+1}}$ under the composition $f_0 f_1 \cdots f_k$. Also for $k \le k(i)$ let $\alpha_k(i)$ be the α such that the ordertype of C_{i_k} has cardinality ω_α.

Our generics $G(\le i)$ for $P(\le i)$ are defined by induction on $i \in I$ to have the following coherence property: If $\bar{i}$ is a limit point of C_i^k then $G(i,\beta)$ extends $G(\bar{i},\beta)$ for $\alpha_k(i) \le \beta < \bar{i}$. To choose $G(i)$, we must first require that $G(i,\beta)$ extend $G(\bar{i},\beta)$ for $\bar{i}$ a limit point of C_i and $\alpha_0(i) \le \beta < \bar{i}$. For this to be possible we need to know that $\bar{i}$ is not β-Mahlo in $L[G(<i)]$; but $\bar{i}$ is not $F(\bar{i})$-Mahlo in this model, and therefore not β-Mahlo since $F(\bar{i}) \le \alpha_0(i) \le \beta$. For the same reason we may require that $G(i,\beta)$ extend $G(\bar{i},\beta)$ for $\bar{i}$ a limit point of C_i^k and $\alpha_k(i) \le \beta < \bar{i}$, for each $k \le k(i)$. Therefore the desired $G(i)$ can be chosen, when some C_i^k has a largest limit point. If no C_i^k has a largest limit point then by induction $G(i)$ can be chosen as the union of the $G(\bar{i})$, for $\bar{i}$ a limit point of C_i (or arbitrarily if C_i has ordertype ω).

So there exists a generic for this iteration. If we restrict this iteration to $\omega_{\gamma+1}$, where γ is not inaccessible and the $\beta(\alpha)$ are required to take values less than γ, then we produce a model M in which $\omega_{\gamma+1}$ is the least γ-Mahlo. By a similar argument we can arrange that $\omega_{\gamma+1}, i_{\omega_{\gamma+1}\cdot 2}, \ldots, i_{\omega_{\gamma+1}\cdot n}$ are the first n γ-Mahlos. This proves part (b) of Theorem 3. If there is no inaccessible, then we have obtained a generic extension M of L in which no cardinal κ is κ-Mahlo. And if the above iteration is restricted to the least inaccessible, then we obtain a model M such that the least κ which is κ-Mahlo in M is the least inaccessible. By a similar argument we can arrange that the first n cardinals κ which are κ-Mahlo in M are the first n inaccessibles. This proves part (b) of Theorem 4. $\Box$

Proof of Theorem 5. Suppose that M is a generic extension of L obtained by the forcing P, and M is contained in $L[0^\#]$. If φ is true in M, then φ is forced by some condition in P; we may assume that φ is forced by the weakest condition of P. By assumption, there is an $\omega + \omega$-Erdős cardinal in $L[0^\#] = L[M, 0^\#]$. It follows from the results of [1] that P has a generic relative to which a periodic subclass of I is a class of indiscernibles, and this generic can be generically coded by a real which belongs to a set-generic extension of $L[0^\#]$. Since φ is a parameter-free sentence there is such a real in $L[0^\#]$, by absoluteness. It follows that φ is true in an inner model of $L[R]$ for some real R such that $R^\#$ exists. $\Box$

References

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