MINIMAL UNIVERSES

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We work in Gödel-Bernays class theory. And we say that a structure $\langle M, A\rangle$ is a *model of* ZFC if M is a model of ZFC and obeys replacement for formulas which are permitted to mention $A \subseteq M$ as a unary predicate. An inner model M is *minimal* if there is a class A such that $\langle M, A\rangle$ is amenable yet has no transitive proper elementary submodel. M is *strongly minimal on a club* if there is a club C such that $\langle M, C\rangle$ is amenable and $\alpha \in C \longrightarrow \langle V_\alpha^M, C \cap \alpha\rangle$ is not a model of ZFC. Strong minimality on a club implies minimality, but not conversely. It is consistent for L to be strongly minimal on $C = ORD$ and if $0^\#$ exists, L is not minimal yet $L[0^\#]$ may or may not be minimal.

If $M_1 \subseteq M_2$ are inner models then M_2 is a *locally generic* extension of M_1 if every $x \in M_2$ belongs to a set-generic extension of M_1. Our main result states that if V is strongly minimal on a club and $0^\#$ exists then some inner model is both minimal and a locally generic extension of L. V can always be made strongly minimal on a club by forcing a strongly minimalizing club without adding sets (Theorem 1). Thus if $0^\#$ exists then there does exist an inner model which is both minimal and a locally generic extension of L, definable in a forcing extension of V that adds no sets. A special case is when V = the minimal model of $ZFC + 0^\#$ exists, in which case there is an inner model which is minimal and does not contain $0^\#$. This answers a question of Mack Stanley.

Theorem 1. *(Folklore) There is a class forcing to add a club C such that $\langle V, C\rangle$ is a model of ZFC and $\alpha \in C \longrightarrow \langle V_\alpha, C \cap \alpha\rangle$ is not a model of ZFC.*

Proof. Conditions are bounded closed sets p such that $\alpha \in p \longrightarrow \langle V_\alpha, p \cap \alpha\rangle$ is not a model of ZFC. Conditions are ordered by end extension. To preserve ZFC it's enough to show that if $\langle D_i | i < \lambda\rangle$ is a Σ_n definable sequence of open dense classes then the intersection of the D_i's is dense. Given a condition p, first extend if

*Research supported by NSF contract # 8903380.

necessary so that p contains an ordinal greater then λ and the parameters defining $\langle D_i | i < \lambda \rangle$ and then build a canonical Σ_n-elementary chain of models $\langle V_{\alpha_i} | i < \lambda \rangle$ and extensions p_i of p in $V_{\alpha_{i+1}} - V_{\alpha_i}$ such that p_i meets D_i. Then at limit stages $\bar{\lambda} \le \lambda$, $p_{\bar{\lambda}}$ is a condition since $\alpha_{\bar{\lambda}}$ is $V_{\alpha_{\bar{\lambda}}}$-definably singularized by $\langle \alpha_i | i < \bar{\lambda} \rangle$. $\dashv$

Theorem 2. *Suppose V is strongly minimal on a club and $0^\#$ exists. Then there is a minimal locally generic extension of L.*

Theorem 2 is proved using backwards Easton forcing, where $0^\#, C$ are used to select the appropriate (minimal) almost generic extension, C being a strongly minimalizing club. We first describe the building blocks of this backwards Easton iteration, which are designed to produce "generic stability systems".

Definition. A *stability system* p consists of a successor ordinal $|p| = \alpha(p) + 1$ and functions $f_k = f_k^p, k > 0$ such that

(a) $\mathrm{Dom}\, f_1 = \mathrm{Lim} \cap |p|$, $f_1(\alpha) \le \alpha$ for $\alpha \in \mathrm{Dom}\, f_1$, $f_1(\alpha) = \underline{\lim}\ \langle f_1(\bar{\alpha}) | \bar{\alpha} \in \mathrm{Lim} \cap \alpha \rangle$ for $\alpha \in \mathrm{Lim}^2 \cap |p|$. Define $\alpha <_1 \beta \iff \alpha < \beta$ and $\alpha < \gamma \le \beta$, $\gamma \in \mathrm{Dom}\, f_1 \longrightarrow f_1(\gamma) \ge \alpha$. Then $\alpha \in \mathrm{Dom}\, f_1 \longrightarrow f_1(\alpha) \le_1 \alpha$.

(b) $\mathrm{Dom}\, f_{k+1} = \{\alpha < |p| \,\big|\, \alpha\ a <_k\text{-limit}\}$, $f_{k+1}(\alpha) \le \alpha$ for $\alpha \in \mathrm{Dom}\, f_{k+1}$, $f_{k+1}(\alpha) = \underline{\lim}\ \langle f_{k+1}(\bar{\alpha}) | \bar{\alpha} <_k \alpha,\ \bar{\alpha} \in \mathrm{Dom}\, f_{k+1} \rangle$ for $\alpha < |p|$, $\alpha \in <_k - \lim^2$. Define $\alpha <_{k+1} \beta \longleftrightarrow \alpha <_k \beta$ and $\alpha < \gamma \le_k \beta$, $\gamma \in \mathrm{Dom}\, f_{k+1} \longrightarrow f_{k+1}(\gamma) \ge \alpha$. Then $\alpha \in \mathrm{Dom}\, f_{k+1} \longrightarrow f_{k+1}(\alpha) \le_{k+1} \alpha$.

Intuitively, $f_k(\alpha)$ represents the supremum of the ordinals which are "Σ_k stable in α", but only in a formal sense.

Definition. Suppose κ is regular, $\ell > 0$, $\gamma < \kappa$. The forcing $\mathcal{P}(\kappa, \ell, \gamma)$ consists of all stability systems p such that $\gamma \le_\ell \alpha(p) < \kappa$. Extension of conditions is defined by: $q \le p \longleftrightarrow f_k^q \supseteq f_k^p$ for all k and $\alpha(p) \le_{\ell-1}^q \alpha(q)$, where $\le_0^q = \le$. We will see that $\le$ is transitive.

Lemma 1. *For any stability system and k; $\le_k^p$ is a tree ordering and $\alpha \leqslant \beta \le_k^p \gamma$, $\alpha \le_{k+1}^p \gamma \longrightarrow \alpha \le_{k+1}^p \beta$. Also $\{\alpha | \alpha <_k^p \beta\}$ is closed in β.*

Proof. We first prove that $\le_k = \le_k^p$ is a tree ordering, by induction on k. For $k = 0$ we define $\le_0 = \le$ and the result is clear. Suppose that the result holds for k and we wish to show that $\leq_{k+1}$ is a tree ordering. Reflexivity is clear since we mean $\leq_{k+1}$ to include $=$. Antisymmetry is clear since $\alpha \leq_{k+1} \beta \longrightarrow \alpha \leq \beta$. Suppose $\alpha \leq_{k+1} \beta \leq_{k+1} \gamma$ and we want $\alpha \leq_{k+1} \gamma$. Since we have by definition $\alpha \leq_k \beta \leq_k \gamma$ by induction we know $\alpha \leq_k \gamma$. Suppose $\alpha < \delta \leq_k \gamma$, $\delta \in \operatorname{Dom} f_{k+1}$. If $\delta > \beta$ then since $\beta \leq_{k+1} \gamma$ we have $f_{k+1}(\delta) \geq \beta \geq \alpha$. If $\delta \leq \beta$ then $\delta \leq_k \beta$ since $\leq_k$ is a tree ordering and both δ and β are $\leq_k \gamma$. Since $\alpha \leq_{k+1} \beta$ we have $f_{k+1}(\delta) \geq \alpha$. So we have shown that $\leq_{k+1}$ is transitive. Now suppose $\alpha \leq \beta$ are both $\leq_{k+1} \gamma$. By induction $\alpha \leq_k \beta$. If $\alpha < \delta \leq_k \beta$, $\delta \in \operatorname{Dom} f_{k+1}$ then $\delta \leq_k \gamma$ since $\leq_k$ is transitive so $f_{k+1}(\delta) \geq \alpha$ since $\alpha \leq_{k+1} \gamma$. So $\alpha \leq_{k+1} \beta$ and we have shown that $\leq_{k+1}$ is a tree ordering.

If $\alpha \leqslant \beta \leq_k \gamma$, $\alpha \leq_{k+1} \gamma$ then $\alpha \leq_k \beta$ since $\leq_k$ is a tree ordering. If $\alpha < \delta \leq_k \beta$, $\delta \in \operatorname{Dom} f_{k+1}$ then $\delta \leq_k \gamma$ since $\leq_k$ is transitive so $f_{k+1}(\delta) \geq \alpha$ since $\alpha \leq_{k+1} \gamma$. So $\alpha \leq_{k+1} \beta$.

Finally we show that $\{\alpha | \alpha <_k \beta\}$ is closed in β, by induction on k. This is clear for $k = 0$. Suppose it holds for k and $\bar{\alpha}$ is a limit of $\{\alpha | \alpha <_{k+1} \beta\}$, $\bar{\alpha} < \beta$. Then $\bar{\alpha} <_k \beta$ by induction. Suppose $\bar{\alpha} < \gamma \leq_k \beta$, $\gamma \in \operatorname{Dom} f_{k+1}$. Then $f_{k+1}(\gamma) \geq \alpha$ for all $\alpha <_{k+1} \beta$, $\alpha < \gamma$. In particular this holds when $\alpha < \bar{\alpha}$ so $f_{k+1}(\gamma) \geq \bar{\alpha}$ since $\bar{\alpha}$ is a limit of such α. So $\bar{\alpha} <_{k+1} \beta$. $\dashv$

Lemma 2. *Let f_k, $<_k$ arise from a stability system. If $\alpha \in \operatorname{Dom} f_k$, $f_k(\alpha) < \alpha$ then $f_k(\alpha) =$ largest $\bar{\alpha} <_k \alpha$. If $\alpha \in <_k - \lim^2$, $f_{k+1}(\alpha) = \alpha$ then $\{\bar{\alpha} | \bar{\alpha} <_{k+1} \alpha\}$ is unbounded in α.*

Proof. Suppose $\alpha \in \operatorname{Dom} f_k$, $f_k(\alpha) < \alpha$. We know that $f_k(\alpha) <_k \alpha$, by definition of condition. If $f_k(\alpha) < \beta < \alpha$ then $\beta \not<_k \alpha$ since $f_k(\alpha) \not\geq \beta$. So $f_k(\alpha) =$ largest $\bar{\alpha} <_k \alpha$. Suppose $\alpha \in <_k - \operatorname{Lim}^2$, $f_{k+1}(\alpha) = \alpha$. Then $f_{k+1}(\alpha) = \alpha = \underline{\lim}\{f_{k+1}(\bar{\alpha}) | \bar{\alpha} <_k \alpha, \bar{\alpha} \in <_k - \operatorname{Lim}\}$. So $\alpha = \lim\{f_{k+1}(\bar{\alpha}) | \bar{\alpha} <_k \alpha, \bar{\alpha} \in <_k - \operatorname{Lim}\}$. For any $\alpha_0 < \alpha$ there must be $\bar{\alpha}_0 <_k \alpha$, $\bar{\alpha}_0 \in <_k - \lim$ such that $f_{k+1}(\bar{\alpha}) \geq f_{k+1}(\bar{\alpha}_0) \geq \alpha_0$ for all $\bar{\alpha} <_k \alpha, \bar{\alpha} \in <_k - \lim$, $\bar{\alpha} \geq \bar{\alpha}_0$. (For, we need only first choose $\bar{\alpha}_0'$ to guarantee $f_{k+1}(\bar{\alpha}) \geq \alpha_0$ for all $\bar{\alpha}$ beyond $\bar{\alpha}_0'$ and then minimize $f_{k+1}(\bar{\alpha}_0')$ to get $\bar{\alpha}_0$.) Then $f_{k+1}(\bar{\alpha}_0) <_k \alpha$ since either $f_{k+1}(\bar{\alpha}_0) <_{k+1} \bar{\alpha}_0 <_k \alpha$ or $f_{k+1}(\bar{\alpha}_0) = \bar{\alpha}_0 <_k \alpha$. Suppose $f_{k+1}(\bar{\alpha}_0) < \beta \leq_k \alpha$, $\beta \in \operatorname{Dom} f_{k+1}$. If $\beta \leq \bar{\alpha}_0$ then $f_{k+1}(\beta) \geq f_{k+1}(\bar{\alpha}_0)$ since $f_{k+1}(\bar{\alpha}_0) <_{k+1} \bar{\alpha}_0$. If $\bar{\alpha}_0 \leq \beta$ then $\bar{\alpha}_0 \leq_k \beta$ and $f_{k+1}(\beta) \geq f_{k+1}(\bar{\alpha}_0)$ by choice of $\bar{\alpha}_0$. So $f_{k+1}(\bar{\alpha}_0) <_{k+1} \alpha$ and $f_{k+1}(\bar{\alpha}_0) \geq \alpha_0$. So $\{\bar{\alpha} | \bar{\alpha} <_{k+1} \alpha\}$ is unbdd in α. $\dashv$

Lemma 3. *Suppose $r \leq q \leq p$ in $\mathcal{P}(\kappa, \ell, \gamma)$. Then $r \leq p$.*

Proof. We need to check that $\alpha(p) \leq^r_{\ell-1} \alpha(r)$. But $\alpha(p) \leq^q_{\ell-1} \alpha(q)$ and so $\alpha(p) \leq^r_{\ell-1} \alpha(q)$, and $\alpha(q) \leq^r_{\ell-1} \alpha(r)$. So the result follows from Lemma 1. $\dashv$

Lemma 4. *Suppose $p \in \mathcal{P}(\kappa, \ell, \gamma)$ and $\alpha(p) \leq \alpha < \kappa$. Then there exists $q \leq p$, $\alpha(q) = \alpha$.*

Proof. For limit $\lambda \in (\alpha(p), \alpha]$ define $f^q_k(\lambda) = \lambda$. It is routine to verify that the resulting q is a condition and extends p. $\dashv$

Lemma 5. *Suppose $p_0 \geq p_1 \geq \cdots$ is a sequence of conditions in $\mathcal{P}(\kappa, \ell, \gamma)$ of length $< \kappa$. Then there is $p \leq$ each p_i, $\alpha(p) = \bigcup_i \alpha(p_i)$.*

Proof. Assume that the p_i's are distinct. Let $\alpha = \bigcup_i \alpha(p_i)$. We must define $f^p_k(\alpha)$. We do so by induction on $k > 0$. If $\alpha \notin \mathrm{Lim}$, $f^p_1(\alpha)$ is undefined. If $\alpha \in \mathrm{Lim}^2$, let $f^p_1(\alpha) = \underline{\lim}\ \langle f^{p_i}_1(\bar{\alpha}) | \bar{\alpha} \leq \alpha(p_i),\ \bar{\alpha} \text{ limit}\rangle$. If $\alpha \in \mathrm{Lim} - \mathrm{Lim}^2$ then let $f^p_1(\alpha) = \alpha$. Assuming $f^p_k(\alpha)$ is defined (and $f^p_k \restriction \alpha = \bigcup_i f^{p_i}_k$) it makes sense to ask if $\alpha \in <^p_k -\lim$. If not then $f^p_{k+1}(\alpha)$ is undefined. If $\alpha \in <^p_k -\lim^2$ then set $f^p_{k+1}(\alpha) = \underline{\lim}\ \langle f^p_{k+1}(\bar{\alpha}) | \bar{\alpha} <^p_k \alpha, \bar{\alpha} \text{ a } <^p_k -\text{limit}\rangle$. If $\alpha \in <^p_k -\lim - <^p_k -\lim^2$ then set $f^p_{k+1}(\alpha) = \alpha$.

Now we show that $\alpha(p_i) <^p_{\ell-1} \alpha$, defined in terms of the above f^p_k's. Suppose $\alpha(p_i) <^p_k \alpha$ for all i, where $k < \ell - 1$ and we want $\alpha(p_i) <^p_{k+1} \alpha$. Suppose $\alpha(p_i) < \beta \leq^p_k \alpha$, $\beta \in \mathrm{Dom}\, f^p_{k+1}$. If $\beta < \alpha$ then we can choose j so that $\beta \leq^{p_j}_k \alpha(p_j)$ and then $f^{p_j}_{k+1}(\beta) \geq \alpha(p_i)$ since $p_j \leq p_i$. If $\beta = \alpha$ then $f^p_{k+1}(\beta) < \alpha(p_i)$ can only result if $f^p_{k+1}(\bar{\alpha}) < \alpha(p_i)$ for some $\bar{\alpha} <^p_k \alpha$, $\bar{\alpha} > \alpha(p)$ but then $\bar{\alpha} <^{p_j}_k \alpha(p_j)$ for large j, contradicting $p_j \leq p_i$. So $\alpha(p_i) \leq^p_{\ell-1} \alpha$ for all i.

Now it is easy to verify that p is a condition extending each p_i, since any violation created by $\alpha = \alpha(p)$ would imply a violation at some $\alpha(p_i) <^p_{\ell-1} \alpha(p)$. $\dashv$

Now we describe the backwards Easton iteration used to create our minimal inner model. $\mathcal{P}$ is the iteration with Easton supports over L where $\mathcal{P}_0 =$ the trivial forcing, $\mathcal{P}_\lambda =$ inverse limit at singular λ, direct limit at regular λ, $\mathcal{P}_{\kappa+1} = \mathcal{P}_\kappa * \dot{\mathbb{Q}}_\kappa$ where $\dot{\mathbb{Q}}_\kappa$ is a term for the trivial forcing unless κ is regular. For regular κ, $\dot{\mathbb{Q}}_\kappa$ is a term for the following forcing in $L[G_\kappa]$, G_κ denoting the $\mathcal{P}_\kappa$-generic: choose a pair

(ℓ_κ, γ_k) with $\ell_\kappa > 0$, $\gamma_\kappa < \kappa$ and apply the forcing $\mathcal{P}(\kappa, \ell_\kappa, \gamma_\kappa)$. Now $\mathcal{P}_\kappa \Vdash \dot{\mathbb{Q}}_\kappa$ is $< \kappa$-closed and has cardinality κ, so $\mathcal{P}$ preserves cofinalities.

Our goal is to build $G = \langle G_\alpha | \alpha \in \text{ ORD } \rangle$ so that G_α is $\mathcal{P}_\alpha$-generic over L and to select ordinals $\alpha_i \in [i, i^*)$, $i < i^*$ adjacent Silver indiscernibles such that (writing $G_{\alpha+1} = G_\alpha * g_\alpha$) :

1. $i < j$ in $I =$ Silver indiscernibles, $p \in g_i$, $q \in g_j$, $\alpha(q) \geq i \longrightarrow f_k^p \subseteq f_k^q$ for all k. let $f_k = \bigcup\{f_k^p | p \in g_i$ for some $i \in I\}$.

2. For $i \in I, \ell_i =$ least ℓ such that the $\langle 0^\#, C\rangle, i - \Sigma_\ell$ stables in C are bounded in i, where $C =$ the given strongly minimalizing club for V. (α is $B, \beta - \Sigma_\ell$ stable if $\langle L_\alpha[B], B \cap \alpha\rangle$ is a Σ_ℓ-elementary submodel of $\langle L_\beta[B], B \cap \beta\rangle$). Also $\gamma_i = \alpha_j$ where $j = \bigcup\{\langle 0^\#, C\rangle, i - \Sigma_{\ell_i}$ stables in $C\} \geq 0$. (By convention, $\alpha_0 = 0$.)

3. For $i \in I$, $f_k(i) = i$ if $k < \ell_i$ and $f_{\ell_i}(i) = \gamma_i$.

4. For $i \in I, i \leq_{\ell_i} \alpha_i$ (where $\leq_k$ is defined from the f_k's) and $\alpha_i \in \text{Dom } f_{\ell_i+1}, f_{\ell_i+1}(\alpha_i) = \alpha_j$ where $j = \bigcup\{\langle 0^\#, C\rangle, i - \Sigma_{\ell_i+1}$ stables in $C\}$.

Suppose that the f_k's have been constructed to obey 1–4 above and we now prove Theorem 2. The desired minimal, locally generic extension of L is $L[\langle G_\alpha | \alpha \in \text{ORD}\rangle]$, witnessed by the amenable class $\langle f_k | k \in \omega\rangle$. The reason for minimality is roughly as follows: there are unboundedly many $\alpha <_k \infty$ (defined from the f_k's) yet no ordinal α is $<_k \infty$ for all k simultaneously. More precisely:

Lemma 6. *let $i < j$ be indiscernibles, $i \langle 0^\#, C\rangle, j - \Sigma_k$ stable, $i \in C$ and (j a limit of $\langle 0^\#, C\rangle, j - \Sigma_{k-2}$ stables or $k \leq 2$). Then $i \leq_k \alpha_i <_k j$, where $\leq_k$ is defined from the f_k's.*

Proof. By induction on k and for fixed k by induction on j. By property 4, $i \leq_{\ell_i} \alpha_i$ and clearly $\ell_i \geq k$ since i is a limit of $\langle 0^\#, C\rangle, i - \Sigma_{k-1}$ stables ($k > 1$) and so $\ell_i \geq k$ follows by property 2. So we only need to check that $\alpha_i <_k j$.

Suppose $k = 1$. If $\alpha_i \not<_1 j$ then choose $\alpha_i < \beta \leq j$ so that $f_1(\beta) < \alpha_i$. Then $\beta < j$ as properties 2, 3 imply that $f_1(j) \geq \alpha_i$, since i is $\langle 0^\#, C\rangle, j - \Sigma_1$ stable and belongs to C. There are no indiscernibles between β and j as otherwise we can apply induction on j. So $\bar{j} \leq \beta < j$ where $\bar{j} = I$-predecessor to j. In fact $\bar{j} < \beta < j$ since otherwise $\alpha_i < \beta = \bar{j}$ and again 2, 3 imply that $f_1(\bar{j}) \geq \alpha_i$. If $f_1(j) < \beta$ then since $f_1(j) <_1 j$ we have $f_1(j) \leq f_1(\beta)$ and hence $f_1(\beta) \geq f_1(j) \geq \alpha_i$, contrary to assumption. So $f_1(j) \geq \beta > \bar{j}$ and by 3, $f_1(j) = \alpha_{\bar{j}}, \bar{j}$ is $\langle 0^\#, C\rangle, j - \Sigma_1$ stable and belongs to C. And $\bar{j} < \beta \leq \alpha_{\bar{j}}$. But $\bar{j} \leq_1 \alpha_{\bar{j}}$ so $f_1(\beta) \geq \bar{j}$ and $\bar{j} > \alpha_i$ since $\beta \leq \alpha_{\bar{j}}$ and $\beta > \alpha_i$. So $f_1(\beta) > \alpha_i$, contradicting our assumption.

Suppose the lemma holds for k and we prove it for $k+1$. If $\alpha_i \not<_{k+1} j$ then choose $\alpha_i < \beta \leq_k j$ so that $\beta \in \operatorname{Dom} f_{k+1}$, $f_{k+1}(\beta) < \alpha_i$. By 2, 3 we have $\beta <_k j$. By induction on j there can be no $\bar{j}$ such that $\beta \leq \bar{j}$ and $\bar{j}$ is $\langle 0^\#, C\rangle, j - \Sigma_k$ stable and in C, as otherwise induction on k implies $\bar{j} \leq_k \alpha_{\bar{j}} <_k j$ so $\bar{j} <_k j$ and $\beta \leq_k \bar{j}$ by Lemma 1. By 2, 3 $f_k(j)$ is defined and equal to $\alpha_{\bar{j}}$ where $\bar{j} = \bigcup\{\langle 0^\#, C\rangle, j - \Sigma_k$ stables in $C\}$ and since $\beta <_k j$ we have $\beta \leq \alpha_{\bar{j}}$. And $\bar{j} < \beta \leq \alpha_{\bar{j}}$ since there is no $\bar{j}$ such that $\beta \leq \bar{j}$ and $\bar{j}$ is $\langle 0^\#, C\rangle, j - \Sigma_k$ stable, $\bar{j} \in C$. Now $\alpha_{\bar{j}} = f_k(j) <_k j$ so since $\beta <_k j$ we have $\beta \leq_k \alpha_{\bar{j}}$ by Lemma 1. Now let $\ell = \ell_{\bar{j}}$. Clearly $\ell \geq k$ since $\bar{j}$ is $\langle 0^\#, C\rangle, j - \Sigma_k$ stable, $\bar{j} \in C$ and hence $\bar{j} = \bigcup\{\langle 0^\#, C\rangle, \bar{j} - \Sigma_{k-1}$ stables in $C\}$. But if $\ell > k$ then by 4, $\bar{j} <_{k+1} \alpha_{\bar{j}}$ and this contradicts $f_{k+1}(\beta) < \alpha_i < \bar{j}$. So $\ell = k$, $\bar{j} <_k \alpha_{\bar{j}}$ and by 4, $f_{k+1}(\alpha_{\bar{j}})$ is defined and equal to $\alpha_{\bar{\bar{j}}}$ where $\bar{\bar{j}} = \bigcup\{\langle 0^\#, C\rangle, \bar{j} - \Sigma_{k+1}$ stables in $C\}$. But i is $\langle 0^\#, C\rangle, j - \Sigma_{k+1}$ stable, $i \in C$ and $\bar{j}$ is $\langle 0^\#, C\rangle, j - \Sigma_k$ stable, $\bar{j} \in C$ and $i < \bar{j}$ so i is $\langle 0^\#, C\rangle, \bar{j} - \Sigma_{k+1}$ stable and we get $i \leq \bar{\bar{j}}$. Thus $\alpha_i \leq f_{k+1}(\alpha_{\bar{j}}) < \beta$. This contradicts $\beta \leq_k \alpha_{\bar{j}}, f_{k+1}(\beta) < \alpha_i$ since $f_{k+1}(\alpha_{\bar{j}}) <_{k+1} \alpha_{\bar{j}}$. ⊣

Corollary 7. *Define $\alpha <_k \infty \longleftrightarrow \alpha <_k \beta$ for cofinally many $\beta <_{k-1} \infty$ (where $\beta <_0 \infty$ is vacuous). Then for each k there are cofinally many $\alpha <_k \infty$.*

Proof. By Lemma 6 if i is $\langle 0^\#, C\rangle - \Sigma_k$ stable and belongs to C then $i <_k \infty$ (by induction on k). The class of all such i is cofinal in the ordinals. ⊣

Corollary 8. *No α is $<_k \infty$ for all k.*

Proof. Choose k large enough so that $i =$ the least $\langle 0^\#, C\rangle - \Sigma_k$ stable in C is larger than α. Then $i <_k \infty$ is $f_k(i) = \alpha_0 = 0$. So $\alpha \not<_k \infty$. ⊣

Thus we have minimality, since if $\langle L[\langle G_\alpha | \alpha \in \mathrm{ORD}\rangle], \langle f_k | k \in \omega\rangle\rangle$ had a transitive elementary submodel, by Corollary 7 its height α would be $<_k \infty$ for each k, in contradiction to Corollary 8.

It remains to construct the G_α's so as to obey 1–4.

The Construction. We build $G_{i+1} = G_i * g_i$ by induction on $i \in I$. When defining G_{i^*+1} we also specify $\alpha_i \in [i, i^*)$, where i^* denotes the I-successor to i.

$\mathbf{G_{i_0+1}, i_0 = \min I}$ Choose G_{i_0} to be the $L[0^\#]$-least generic for $\mathcal{P}_{i_0}$, using the countability of i_0. Set $\ell_{i_0} = 1, \gamma_{i_0} = 0 = \alpha_0$ and choose g_{i_0} to be the $L[0^\#]$-least generic for $\mathcal{P}(i_0, 1, 0)$ as defined in $L[G_{i_0}]$.

$\mathbf{G_{i^*+1}, i \in I}$ First choose G_{i^*} to be the $L[0^\#, C]$-least generic for $\mathcal{P}_{i^*}$ extending G_{i+1}, using the $\leq i$-closure of $\mathcal{P}_{i+1,i^*}$ in $L[G_{i+1}]$, where $\mathcal{P}_{i^*} = \mathcal{P}_{i+1} * \mathcal{P}_{i+1,i^*}$. (Note that the dense sets in $\mathcal{P}_{i+1,i^*}$ can be grouped into countably many collections of size i, enabling an easy construction of a generic.) The key step involves the choice of g_{i^*}.

Choose $n \geq 0$ so that the ordertype of the $\langle 0^\#, C\rangle, i - \Sigma_\ell$ stables in C is $\lambda + n$, λ limit or 0, where $\ell = \ell_i =$ least ℓ such that the $\langle 0^\#, C\rangle, i - \Sigma_\ell$ stables in C are bounded in i. Let $\overline{Q}_{i^*}$ be the forcing $\mathcal{P}(i^*, \ell + 1, \alpha_j)$ as defined in $L[G_{i^*}]$, where $j = \bigcup\{\langle 0^\#, C\rangle, i - \Sigma_{\ell+1}$ stables in $C\} \geq 0$. Let $p_0 \in \overline{Q}_{i^*}$ be the "condition" defined by $\alpha(p_0) = i$, $f_k^{p_0} \restriction i = \bigcup\{f_k^p | p \in g_i\}$, $f_k^{p_0}(i) = i$ if $k < \ell$, $f_\ell^{p_0}(i) = \gamma_i = \alpha_{j'}$ where $j' = \bigcup\{\langle 0^\#, C\rangle, i - \Sigma_\ell$ stables in $C\} \geq 0$. *We will verify* later that p_0 is indeed a condition in $\overline{Q}_{i^*}$. Choose $p_1 \leq p_0$ in $\overline{Q}_{i^*}$ to meet all dense Δ in $L[G_{i^*}]$ definable in $L[G_{i^*}]$ from G_{i^*} and parameters in $(i+1) \cup \{j_1 \cdots j_n\}$ where $j_1 \cdots j_n$ are the first n indiscernibles $\geq i^*$. Also arrange that $\alpha(p_1)$ is a $<_\ell^{p_1}$ $-$limit, $f_{\ell+1}^{p_1}(\alpha(p_1)) = \alpha_j$. Now set $\alpha_i = \alpha(p_1)$ and $\ell_{i^*} = 1$, $\gamma_{i^*} = \alpha_i$. Choose g_{i^*} to be generic for $Q_{i^*} = \mathcal{P}(i^*, 1, \alpha_i)$ over $L[G_{i^*}]$, extending the condition $p_1 \in Q_{i^*}$.

$\mathbf{G_{i+1}, i \in \mathrm{Lim}\, I}$ $G_i = \bigcup\{G_j | j \in I \cup i\}$. Let $Q_i = \mathcal{P}(i, \ell_i, \gamma_i)$ in $L[G_i]$ where ℓ_i is the least ℓ such that the $\langle 0^\#, C\rangle, i - \Sigma_\ell$ stables in C are bounded in i and $\gamma_i = \alpha_j$ where $j = \bigcup\{\langle 0^\#, C\rangle, i - \Sigma_{\ell_i}$ stables in $C\}$. Let $f_k \restriction i = \bigcup\{f_k^p | p \in g_j$ for some $j \in I \cap i\}$. Then $g_i = Q_i$-generic determined by $\langle f_k \restriction i | k \in \omega\rangle$. *We will verify* later that $\langle f_k \restriction i | k \in \omega\rangle$ does indeed determine a Q_i-generic over $L[G_i]$.

Lemma 9. *Assume that the verifications claimed in the construction can be carried out. Then 1–4 hold.*

Proof. Everything is clear, with the possible exception of the first statement in 4: $i \leq_{\ell_i} \alpha_i$. But note that in the construction of G_{i^*+1}, $i = \alpha(p_0)$, $\alpha_i = \alpha(p_i)$ where

$p_1 \le p_0$ in $\mathcal{P}(i^*, \ell_i + 1, \alpha_j)$ for some j, so we're done by the definition of extension of conditions. $\dashv$

Lemma 10. *By induction on $i \in I$:*

(a) g_i is well-defined and Q_i-generic, where $Q_i = \mathcal{P}(i, \ell_i, \gamma_i)$ as interpreted by G_i.

(b) Define p_0 by: $\alpha(p_0) = i$, $f_k^{p_0} \restriction i = \bigcup\{f_k^p | p \in g_i\}$, $f_k^{p_0}(i) = i$ if $k < \ell_i$, $f_{\ell_i}^{p_0}(i) = \gamma_i$. Then p_0 is a stability system.

(c) Lemma 6 holds for indiscernibles $\le i$.

(d) If p_0 is defined as in (b) then $p_0 \in \overline{Q_{i^*}}$, as defined in the construction.

Proof. (a) This follows by induction unless $\ell_i > 1$ and there is a final segment $i_0 < i_1 < \cdots$ of the $\langle 0^\#, C\rangle, i - \Sigma_{\ell_i - 1}$ stables C of ordertype ω. We may also assume that i_0 is big enough so that $j = \bigcup\{\langle 0^\#, C\rangle, i - \Sigma_{\ell_i}$ stables in $C\} = \bigcup\{\langle 0^\#, C\rangle, i_n - \Sigma_{\ell_i}$ stables in $C\}$ for all n. Note that $\ell_{i_n} = \ell_i - 1$ and the ordertype of the $\langle 0^\#, C\rangle, i_n - \Sigma_{\ell_{i_n}}$ stables in $C = \lambda + n'$ with $n \le n' < \omega$, λ limit or 0. By construction, p_1^n meets Δ in $L[G_{i_n^*}]$ defined from $G_{i_n^*}$ and parameters in $(i_n + 1) \bigcup$ (least n indiscernibles $\ge i_n^*$) where Δ is dense on $\mathcal{P}(i_n^*, \ell_i, \gamma_i)$ and p_1^n is the condition in $g_{i_n^*}$ with $\alpha(p_1^n) = \alpha_{i_n}$. By an inductive use of (c), $i_n <_{\ell_i - 1}^{p_0} \alpha_{i_n} <_{\ell_i - 1}^{p_0} i_m$ for all $m > n$, where p_0 is defined as in (b). So $\alpha_{i_0} <_{\ell_i - 1}^{p_0} \alpha_{i_1} <_{\ell_i - 1}^{p_0} \cdots$ and $p_1^0 \ge p_1^1 \ge p_1^2 \ge \cdots$ in $\mathcal{P}(i, \ell_i, \gamma_i)$ determine the generic g_i containing the p_1^n's.

(b) The genericity of g_i established in (a) implies that i is a $<_{\ell_i - 1}^{p_0} - \lim^2$ and $f_k^{p_0}(i)$ is determined correctly by $f_k^{p_0}(\gamma)$, $\gamma <_{k-1}^{p_0} i$, γ a $<_{k-1}^{p_0} - \lim$, for $k \le \ell_i$. So p_0 obeys the requirements for a stability system.

(c) The proof of Lemma 6 for indiscernibles $\le i$ only used the facts that p_0 is a stability system and 1–4 hold $\le i$. So we are done by Lemma 9 through i.

(d) We must verify that $\alpha_j <_{\ell+1}^{p_0} i$ in the definition of g_{i^*}. This follows from (c). $\dashv$