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MINIMAL CODING

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The purpose of the present paper is to establish the following strengthening of Jensen's Coding Theorem.

Theorem. *Suppose $V = L[A]$ where $A \subseteq$ ORD and GCH holds. Then there is a cofinality and cardinal preserving forcing for producing a real R such that $V[R] = L[R]$ and A is $L[R]$-definable from R. Moreover we can require R to be* ***minimal*** *over V: if $x \subseteq$ ORD belongs to $V[R]$, then either $x \in V$ or $R \in V[x]$.*

The first part of the Theorem is Jensen's Coding Theorem (see Beller–Jensen–Welch [1]). Thus our goal is to establish the Coding Theorem using branching conditions, which are suitable for showing minimality. Minimal reals were first constructed by Sacks [6] using forcing with perfect trees. Our forcing conditions are obtained by replacing the building blocks R^s of Jensen's forcing by forcings constructed out of perfect trees. A peculiarity is that our notion of 'path' p through such a tree of height κ does not require that $p \upharpoonright \gamma$ belong to V for $\gamma < \kappa$. This is necessary as it is easily shown that if R is minimal over V, $p: \kappa \to 2$ belongs to $V[R] - V$ and $\kappa > \aleph_0$, then $p \upharpoonright \gamma \notin V$ for some $\gamma < \kappa$.

Our Theorem has some corollaries concerning reals which are minimal over L.

Corollary. *There is an L-definable forcing for producing a real R which is minimal over L but not set-generic over L.*

Proof. Let Q be the forcing that adds a Cohen subclass of ORD using constructible conditions $p: \gamma \to 2$, $\gamma \in$ ORD. Consider $Q * P$ where P arises from the Theorem applied to $\langle L, A\rangle$, $A =$ the generic class added by Q. Note that A is not L-definable. We now show that if R denotes the generic real added by $Q * P$, then R is not set-generic over L: Suppose $\phi(x, R)$ is a formula such that $p \subseteq A$ iff $L[R] \models \phi(p, R)$ and that $L[R] = L[G]$ where G is generic for a set of conditions $\mathcal{R} \in L$. By the Truth Lemma $p \subseteq A$ iff $\exists r \in G\ r \Vdash_{\mathcal{R}} \phi(p, R)$ and thus for some fixed $r_0 \in G$, $r_0 \Vdash_{\mathcal{R}} \phi(p, R)$ for *class-many* p. Thus $p \subseteq A \leftrightarrow \exists p'\ r_0 \Vdash_{\mathcal{R}} \phi(p', R)$ and $p \subseteq p'$ and we have contradicted the fact that A is not L-definable. □

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It was shown by Jensen [5] that if $0^{\#}$ exists, then there exists a real $R \in L[0^{\#}]$ which is not set-generic over L. By imitating that construction we can obtain the following.

Corollary (to proof of Theorem). *If $0^{\#}$ exists, then there is a real R which is minimal over L but not set-generic over L.*

Our proof will be presented in stages. Initially we shall assume that $A \subseteq \omega_1$. then we consider the cases where ω_1 is replaced by ω_2 and $\omega_{\omega+1}$ before turning to the general case, where $A \subseteq \mathrm{ORD}$ is a class.

Some Notation. (a) We use $*$ to denote concatenation of sequences. Thus $\sigma * \tau(i) = \sigma(i)$ for $i < \mathrm{length}(\sigma)$ and $= \tau(i - \mathrm{length}(\sigma))$ for $\mathrm{length}(\sigma) \leqslant i < \mathrm{length}(\sigma) + \mathrm{length}(\tau)$. Also, $\sigma * 0$, $\sigma * 1$ denote $\sigma * \langle 0 \rangle$, $\sigma * \langle 1 \rangle$.

(b) We confuse sets with their characteristic functions. Thus $B(i) = 1$ if $i \in B$, $= 0$ if $i \notin B$.

(c) For $R \subseteq \omega$, $(R)_n$ denotes $\{m \mid 2^n 3^m \in R\}$.

1. Minimally coding a subset of ω_1

Assume that $A \subseteq \omega_1$. In this section, we shall establish the theorem in this special case. (This was done independently by Groszek under the extra hypothesis that $2^{\omega} \subseteq L$.)

For this purpose we use a technique called 'canonical coding': To each real R we will define a canonical sequence of perfect trees T^R_α for an initial segment of ordinals α. R will be a path through T^R_α whenever it is defined, and in fact T^R_α will be defined for all $\alpha < \omega_1$ for the desired real R. Then we code A by: $\alpha \in A \leftrightarrow R$ 'goes right' at large enough even levels of T^R_α (this is defined precisely below).

This strategy does lead to success if we make an extra assumption about A. We say that A is *efficient* if $\beta < \omega_1 \rightarrow \beta$ is countable in $L[A \cap \beta]$. Our assumptions do not imply that an efficient A exists. If A is efficient, then we can use conditions which are perfect trees T such that for some α (called the rank of T), $R \in [T] = \{\text{paths through } T\} \rightarrow T = T^R_\alpha$ and ($\beta < \alpha \rightarrow \beta \in A$ iff R 'goes right' at large enough even levels of T^R_β). The efficiency of A is used to show that for $\alpha' > \alpha$, T can be extended to a condition T' of rank α'; we need that α' is countable in $L[A \cap \alpha']$ to inductively fuse countably many extensions of T to conditions of ranks unbounded in α' (for limit α').

If we cannot assume that A is efficient, then the coding must be modified. The idea now is to arrange that our generic real code not A but instead an efficient $B_R \subseteq \omega_1$ such that $A = \mathrm{even}(B_R) = \{\alpha \mid 2\alpha \in B_R\}$. Thus a condition should be a perfect tree T on ω such that if R is a branch through T, then R codes some $B_R \cap |T|$ which is efficient through $|T|$ and such that $\{\alpha \mid 2\alpha \in B_R \cap |T|\}$ is an initial segment of A.

Note that our forcing is very 'thin' in the sense that if R is a branch through some condition T, $\text{rank}(T) = \alpha$, then $T = T^R_\alpha$ is uniquely determined. The way in which this is done is to use the $L[B_R \cap \alpha]$-least counting of α to construct T^R_α as a fusion of finite unions of 'canonical' conditions T^S_β, $\beta < \alpha$. We also use a $\Diamond$-like construction to anticipate fusion sequences that must be considered to show that our forcing preserves ω_1 and produces a minimal real.

We are ready to begin the definition of the forcing R^A for minimally coding A. *Perfect trees* on ω are defined to be collections T of finite functions $s : |s| \to 2$ with the properties that

$$s \in T \to \exists t_1, t_2 \in T\ (s \subseteq t_1 \cap t_2,\ t_1(n) \neq t_2(n) \text{ for some } n)$$

and

$$s \in T,\ t \subseteq s \to t \in T.$$

A real $R \subseteq \omega$ is a *branch* through T if

$$R \in [T] = \{S \mid c_S \upharpoonright n \in T \text{ for all } n\}$$

where c_S = the characteristic function of S. If T is a perfect tree and $s \in T$, then $(T)_s$ is the perfect tree $\{t \in T \mid t \subseteq s \text{ or } s \subseteq t\}$ and $\|s\|$ denotes the cardinality of $\{t \mid t \subseteq s \text{ and } t * 0,\ t * 1 \in T\}$.

We define R^A_α, the collection of conditions of rank $\leqslant \alpha$, by induction on $\alpha < \omega_1$. We then define the desired forcing R^A as the union of the R^A_α, $\alpha < \omega_1$, and for $T \in R^A$ we write $|T| = \alpha$ if $T \in R^A_\alpha - R^A_{<\alpha}$. The notion of extension for R^A is defined to be inclusion: T_1 is stronger than T_2 if $T_1 \subseteq T_2$ (in which case we write $T_1 \leqslant T_2$). Also define $T_1 \leqslant_k T_2$ if $T_1 \leqslant T_2$ and $s \in T_2$, $\|s\| \leqslant k \to s \in T_1$.

Simultaneously with the R^A_α we shall define for certain reals R a canonical condition $T^R_\alpha \in R^A_\alpha$. The condition T^R_α is defined whenever $R \in [T]$ for some $T \in R^A_\alpha - R^A_{<\alpha}$. We say that T is *canonical if* $T = T^R_\alpha$ for all $R \in [T]$.

We also want certain closure properties for the R^A_α, $\alpha < \omega_1$. In fact, R^A_α is obtained by closing $R^A_{<\alpha} \cup \tilde{R}^A_\alpha$ under the operations (*), (**) where $R^A_{<\alpha} = \bigcup \{R^A_\beta \mid \beta < \alpha\}$ and $\tilde{R}^A_\alpha$ = the canonical conditions in $R^A_\alpha - R^A_{<\alpha}$. (*), (**) are defined by:

$$(*) \quad T \in R^A_\alpha,\quad s \in T \to (T)_s \in R^A_\alpha,$$

$$(**) \quad T_1, \ldots, T_n \in R^A_\alpha \to T_1 \cup \cdots \cup T_n \in R^A_\alpha.$$

We will also need to make use of a $\Diamond(E)$-sequence where $E \subseteq E_0 = \{\alpha < \omega_1 \mid \alpha$ is countable in $L[A \cap \alpha]\}$ is defined below.

Lemma 1.1. *E_0 is stationary.*

Proof. If C is closed unbounded, let M be the least elementary submodel of $L_{\omega_2}[A]$ containing A and C as elements. If $\alpha = M \cap \omega_1$, then $\alpha \in C$ and $\alpha \in E_0$ since there is a counting of α in $L_{\delta+2}[A \cap \alpha]$ where $L_\delta[A \cap \alpha] \simeq M$. $\square$

A ordinal α is *A-locally countable* if $\beta < \alpha \rightarrow \beta$ is countable in $L_\alpha[A]$. Now let $E = E_0 \cap \{\alpha \mid \alpha$ is p.r. closed and A-locally countable$\}$. Then E is stationary and we let $\langle S_\alpha \mid \alpha \in E\rangle$ be the canonical $\Diamond(E)$-sequence from [2]. Thus $S_\alpha \subseteq \alpha$ for $\alpha \in E$ and if $X \subseteq \omega_1$, then $X \cap \alpha = S_\alpha$ for stationary-many $\alpha \in E$. We also have good definability properties for $\langle S_\alpha \mid \alpha \in E\rangle$: If $\alpha \in E$, then $\langle S_\beta \mid \beta \in E \cap \alpha\rangle$ is definable over $L_\alpha[A]$ and $S_\alpha \in L_\gamma[A \cap \alpha]$ where γ is the least p.r. closed ordinal greater than α which is $A \cap \alpha$-locally countable.

We can now begin the inductive definition of the R^A_α, $\alpha < \omega_1$.

Case 1: $\alpha = 0$. If $s_1, \ldots, s_n$ are finite strings, then $(2^{<\omega})_{s_1} \cup \cdots \cup (2^{<\omega})_{s_n}$ belongs to R^A_0 and these are the only elements of R^A_0. For any real R, $T^R_0 = 2^{<\omega}$, so $\tilde{R}^A_0 = \{2^{<\omega}\}$.

Case 2: $\alpha = \beta + 1$. Let $T \in \tilde{R}^A_\beta$. We describe the extensions of T in $\tilde{R}^A_\alpha$. Let $\text{Split}(T) = \{a \in T \mid a * 0, a * 1 \in T\}$ and let $f: 2^{<\omega} \rightarrow \text{Split}(T)$ be bijective so that $f(a * 0) \supseteq f(a) * 0$ for all a (f is unique). Define T_0, T_1 to be $T_j = \{a \mid a \subseteq f(b)$ for some $b \in 2^{<\omega}$, $b(2i) = j$ for $2i < |b|\}$. We now define T^k_i for $k \in \omega$, $i < 2^{2^k}$, by induction: To define T^k_i, $i < 2^{2^k}$ first list all subsets x of $\{a \in 2^{<\omega} \mid \text{length}(a) = k\}$ as $x_0, x_1, \ldots$ for each x_i choose subtrees $T(i) \leqslant_k T$ so that no $T(i)$ shares a path with any $T^{k'}_j$, $k' < k$, nor with any other $T(i')$, and so that

$$a \in x_i \rightarrow (T(i))_{f(a)} \subseteq T_1, \qquad a \in 2^k - x_i \rightarrow (T(i))_{f(a)} \subseteq T_0.$$

Then add all the $T(i)$ to $\tilde{R}^A_\alpha$ if β is not even and if $\beta = 2\gamma$, then add $T(i)$ to $\tilde{R}^A_\alpha$ iff ($x_i = 2^k$, $\gamma \in A$ or $x_i = \emptyset$, $\gamma \notin A$). Set $T^k_i = T(i)$. To obtain $\tilde{R}^A_\alpha$ close $\tilde{R}^A_{<\alpha} \cup \tilde{R}^A_\alpha$ under $(*)$, $(**)$.

Case 3: α *limit*, $\alpha \notin E$. Our goal here is to extend each canonical $T \in R^A_{<\alpha} = \bigcup \{R^A_\beta \mid \beta < \alpha\}$ to a canonical condition in $R^A_\alpha - R^A_{<\alpha}$. (It then follows easily that every condition in $R^A_{<\alpha}$ can be extended to a condition in $R^A_\alpha - R^A_{<\alpha}$.) Thus it is important to arrange that if T^* is added to R^A_α, then T^* can be recovered canonically from each path $R \in [T^*]$ (so that T^* will be canonical). We achieve this by insisting that a final segment of $B_R \cap \alpha$ is constant for $R \in [T^*]$ and that $B_R \cap \alpha$ codes the construction of T^*, where $B_R \cap \alpha$ is the predicate coded by R: $\beta \in B_R \cap \alpha \leftrightarrow \beta < \alpha$ and R goes right at large enough even levels of T^R_β, where R *goes right at level i on T* if $a \in T$, $a \subseteq R$, $\|a\| = i \rightarrow a * 1 \subseteq R$.

Given a real R we first define what it means for $B_R \cap \alpha$ to code an ω-sequence $\alpha^R_0 < \alpha^R_1 < \cdots$ cofinal in α. Define $R^* = \{n \mid 2(\lambda + 2^n 3^i) + 1 \in B_R$ for unboundedly many $2(\lambda + 2^n 3^i) + 1 < \alpha$, λ limit or $0\}$. Note that R^* depends only on a final segment of $B_R \cap \{2\beta + 1 \mid \beta < \alpha\}$. If α is countable in $L[R^*]$ and $0 \notin R^*$ say that $B_R \cap \alpha$ codes the ω-sequence $\alpha^R_0 < \alpha^R_1 < \cdots$ defined by: $\langle \alpha^R_i \mid i < \omega\rangle$ is the $L[R^*]$-least increasing ω-sequence cofinal in α such that α a limit of limit ordinals $\rightarrow \alpha^R_i$ a limit ordinal for all i. This extra condition on the α^R_i's will be useful in the proof of extendibility.

Now pick an ordinal $\hat{\beta} < \alpha$ and an integer $\hat{k} \in \omega$. Let n be least so that $\hat{\beta} < \alpha_n^R$ and define $T_0 \geq T_1 \geq \cdots$ as follows: $T_0 = L[A \cap \alpha, R^*]$-least canonical $T \leq T_{\hat{\beta}}^R$ in $R_{\alpha_n}^A$ such that $T \leq_{\hat{k}} T_{\hat{\beta}}^R$ and $S \in [T] \to B_S$ and B_R agree on $[\hat{\beta}, \alpha_n^R)$. If T_k is defined, then $T_{k+1} = L[A \cap \alpha, R^*]$-least canonical $T \leq T_k$ in $R_{\alpha_{n+k+1}^R}^A$ such that $T \leq_{\hat{k}+k+1} T_k$ and $S \in [T] \to B_S$ and B_R agree on $[\alpha_{n+k}^R, \alpha_{n+k+1}^R)$. Let $T^R(\hat{\beta}, \hat{k}) = \bigcap \{T_k \mid k < \omega\}$.

Now include in $\tilde{R}_\alpha^A$ all perfect trees T such that $T \in L_\beta[A]$ where β is least so that $\beta > \alpha$ and β is both p.r. closed and A-locally countable, and for some $\hat{\beta} < \alpha$, $\hat{k} \in \omega$ and all $R \in [T]$, $T = T^R(\hat{\beta}, \hat{k})$ as built above. If $R \in [T]$ for such a T, then $T_\alpha^R = T$. Note that T_α^R is uniquely determined as $\bigcap \{T_{\alpha_n}^R \mid n_0 \leq n \in \omega\}$ for sufficiently large n_0. Then R_α^A is obtained from $R_{<\alpha}^A \cup \tilde{R}_\alpha^A$ by closing under $(*)$, $(**)$.

Case 4: α limit, $\alpha \in E$. We treat this case as in Case 3 but with one change. First see if S_α, obtained from the $\Diamond(E)$-sequence $\langle S_\alpha \mid \alpha \in E \rangle$, is of the form $\bigcup \{\{i\} \times D_i \mid i \in \omega\} \cup \{T_0\}$ where $T_0 \in R_{<\alpha}^A$, D_i is open dense on $R_{<\alpha}^A$ and in fact $|T| < \alpha \to \exists T' \leq_i T$, $|T'| < \alpha$ such that $T' \in D_i$. (We identify $L_\alpha[A]$ with α so that the previous set can be viewed as a subset of α.) If not, proceed exactly as in Case 3. If so, proceed as in Case 3 except add to R_α^A an additional condition T, described as follows: Let $\alpha_0 < \alpha_1 < \cdots$ be the $L[A \cap \alpha]$-least ω-sequence cofinal in α and inductively define T_{k+1} to be the $L[A \cap \alpha]$-least $T \leq_k T_k$ such that $|T| \geq \alpha_k$, $T \in D_k$ and for some limit $\lambda \geq \alpha_k$, $2(\lambda + 3^n) + 1 \in B_R$ for all $R \in [T]$, for all $n \leq k$. The purpose of the last clause is to guarantee that if $T = \bigcap \{T_k \mid k < \omega\}$, then $R \in [T] \to 0 \in R^*$ (R^* as defined in Case 3). Now add T to $\tilde{R}_\alpha^A$ and obtain R_α^A by closing $R_{<\alpha}^A \cup \tilde{R}_\alpha^A$ under $(*)$, $(**)$. For $R \in [T]$, $T_\alpha^R = T$. This completes the construction of the R_α^A, $\alpha < \omega_1$ and hence that of $R^A = \bigcup \{R_\alpha^A \mid \alpha < \omega_1\}$.

Remark. We claim that the tree T_α^R (and hence $A \cap \alpha$) can be recovered uniformly from R in $L[R]$, for all R such that $R \in [T]$ for some $T \in R_\alpha^A - R_{<\alpha}^A$. Of course T_0^R is just $2^{<\omega}$. If $\alpha = \beta + 1$, then given T_β^R it is easy to compute $T_{\beta+1}^R = T_\alpha^R$.

If α is a limit ordinal, then first test $\alpha \in E$ using $A \cap \alpha$. If $\alpha \notin E$, then T_α^R is easily determined from $B_R \cap \alpha$, as described in Case 3. If $\alpha \in E$, then compute R^* from $B_R \cap \alpha$ as in Case 3 and see if $0 \in R^*$. If not, then T_α^R is computable from R as in the case $\alpha \notin E$. If so, then T_α^R can be computed if we know S_α, $B_R \cap \alpha$ and $A \cap \alpha$. But all three of these can be determined from $B_R \cap \alpha$ in $L[B_R \cap \alpha]$ due to the facts that $S_\alpha \in L[A \cap \alpha]$, $A \cap \alpha =$ even part$(B_R \cap \alpha)$. As $B_R \cap \alpha$ can be computed in $L[R]$ we are done.

The main things to show about R^A are Extendibility and Fusion.

Lemma 1.2 (Extendibility). *Suppose $T \in R^A$ and $|T| \leq \alpha < \omega_1$. Then for any $l \in \omega$ there exists $T' \leq_l T$, $|T'| = \alpha$.*

Proof. By induction on α. Obviously we can assume that $\alpha \neq 0$. We can also assume that T is canonical, as otherwise write $T = (T_0)_{s_0} \cup \cdots \cup (T_n)_{s_n}$ where $s_i \in T_i$ and T_i is canonical; then choose $l' = \max(l, \|s_0\|, \ldots, \|s_n\|)$ and let $T_i' \leqslant_{l'} T_i$ for each i, $|T_i'| = \alpha$. Then $T' = (T_0')_{s_0} \cup \cdots \cup (T_n')_{s_n}$ is as desired.

To successfully carry out our induction we must prove somewhat more than what is stated in the lemma. We inductively define the notion "$b \subseteq [|T|, \alpha)$ is *T-special at* α" and prove by induction on α that T canonical, $|T| \leqslant \alpha < \omega_1$, $b \subseteq [|T|, \alpha)$ T-special at $\alpha \rightarrow$ there exists a canonical $T' \leqslant_l T$, $|T'| = \alpha$ such that $R \in [T'] \rightarrow B_R \cap [|T|, \alpha) = b$. (We shall also have to prove the existence of such b.)

If $\alpha = |T|$, then ϕ is T-special at α.

Suppose $\alpha = \beta + 1$. Then $b \subseteq [|T|, \alpha)$ is T-special at α iff b-$\{\beta\}$ is T-special at β and $\beta = 2\gamma \rightarrow (\beta \in b$ iff $\gamma \in A)$. By induction we can extend T to a canonical $T^* \leqslant_l T$ such that $|T^*| = \beta$ and $R \in [T^*] \rightarrow B_R \cap [|T|, \beta) = b \cap \beta$. So we can assume that $|T| = \beta$ and we must show that

(a) β odd implies there are canonical T_0', $T_1' \in \tilde{R}^A_{\beta+1}$ such that $T_i' \leqslant_l T$, $R \in [T_0']$ ($R \in [T_1']$, respectively)$\rightarrow R$ goes right at large enough even levels of T (R goes left at infinitely many even levels of T, respectively), and

(b) $\beta = 2\gamma$ implies that if $\gamma \in A$ (if $\gamma \notin A$, respectively), then there exists $T_1' \leqslant_l T$ ($T_0' \leqslant_l T$, respectively) in $\tilde{R}^A_{\beta+1}$ such that $R \in [T_1'] \rightarrow R$ goes right at large enough even levels of T ($R \in [T_0'] \rightarrow R$ goes left at infinitely many even levels of T, respectively). But an inspection of the construction of Case 2 reveals that conditions T_0, $T_1 \leqslant_l T$ were added to $\tilde{R}^A_{\beta+1}$ so as to meet the above requirements on T_0', T_1'.

Suppose 'T-special at β' is defined for all $\beta < \alpha$ where α is a limit ordinal $<\omega_1$ and for $\beta < \alpha$, $\beta > |T|$ and $b \subseteq [|T|, \beta)$ T-special at β, T canonical$\rightarrow$there is a canonical $T' \leqslant_l T$, $|T'| = \beta$ and $R \in [T'] \rightarrow B_R \cap [|T|, \beta) = b$. Define R^* as in Case 3 with B_R replaced by $b \subseteq [|T|, \alpha)$. We require for b to be T-special at α that α is countable in $L[R^*]$ and $0 \notin R^*$. We also require that $b \cap \alpha_n$ is T-special at α_n where n is least so that $\bar{\beta} = |T| < \alpha_n$ and $\alpha_0 < \alpha_1 < \cdots$ is defined as was $\alpha_0^R < \alpha_1^R < \cdots$ in Case 3. Let $T_0 = L[A \cap \alpha, R^*]$-least canonical $\bar{T} \leqslant_l T$ in $R^A_{\alpha_n}$ such that $R \in [\bar{T}] \rightarrow B_R \cap [|T|, \alpha_n) = b \cap \alpha_n$. Then we also require that $b \cap [\alpha_n, \alpha_{n+1})$ is T_0-special at α_{n+1}. Continue in this way to define $\langle T_k \mid k \in \omega \rangle$ and require that $b \cap [\alpha_{n+k}, \alpha_{n+k+1})$ is T_k-special at α_{n+k+1}. Finally we insist that $b \in L_\beta[A]$ where $\beta > \alpha$ is least so that β is p.r. closed and A-locally countable. This completes the definition of 'T-special at α'.

We must show that T can be $\leqslant_l$-extended to a canonical $T' \in R^A_\alpha$ such that $R \in [T'] \rightarrow B_R \cap [|T|, \alpha) = b$, whenever b is T-special at α. But define $T_0 \geqslant T_1 \geqslant \cdots$ as in the preceding paragraph using b and let $T' = \bigcap \{T_k \mid k < \omega\}$. If $R \in [T']$, then $B_R \cap [|T|, \alpha) = b$ by construction. And clearly $T^R = T'$ where T^R is defined as in Case 3; $T' \in L_\beta[A]$ when $\beta > \alpha$ is p.r. closed and $L_\beta[A]$ is locally countable, since $b \in L_\beta[A]$ for such β. Thus by definition $T' \in R^A_\alpha$ and T' is canonical.

Finally it remains to show that if $|T| < \alpha$, then there exists $b \subseteq [|T|, \alpha)$ which is T-special at α. This is shown by induction on α. We can assume that α is a limit ordinal. The first step is to define $b \cap \{2\gamma \mid \gamma \in \mathrm{ORD}\}$ so that if $2\gamma \in [|T|, \alpha)$, then $\gamma \in A \leftrightarrow 2\gamma \in b$. Next define $b \cap \{2\gamma + 1 \mid \gamma \in \mathrm{ORD}\}$ on an ω-sequence so that $0 \notin R^* = \{n \mid 2(\lambda + 2^n 3^i) + 1 \in b$ for unboundedly many $2(\lambda + 2^n 3^i) + 1 < \alpha$, λ limit or $0\}$ codes α and $R^* \in L_\delta[A]$ when $\delta > \alpha$ and δ is p.r. closed, A-locally countable. We have now determined the sequence $\alpha_0 < \alpha_1 < \cdots$ cofinal in α as we determined $\alpha_0^R < \alpha_1^R \cdots$ in Case 3.

Now let n be least so that $|T| < \alpha_n$. Consider $b \cap \alpha_n$. We want to arrange that it is T-special at α_n.

As we can assume that α is a limit of limit ordinals (otherwise the construction of b is easy by induction and the definition of T-special at successor ordinals) we know that $\alpha_n < \alpha_{n+1} < \cdots$ are limit ordinals; also note that we have only committed ourselves thus far on finitely many ordinals from $\{2\gamma + 1 \mid \gamma < \alpha_m\}$ for each $m \geqslant n$. Now fill in $b \cap \alpha_n$ so as to be T-special at α_n. We assume here inductively that for $\beta < \alpha$ and $|T| < \beta$ there exists a T-special set $b_\beta \subseteq [|T|, \beta)$ which agrees with a given finite assignment on $\{2\gamma + 1 \in [|T|, \beta) \mid \gamma \in \mathrm{ORD}\}$. (It is clear that this inductive hypothesis is being maintained at α.) Next define $T_0 \in R^A_{\alpha_n}$ to be $L[A \cap \alpha, R^*]$-least so that T_0 is canonical, $T_0 \leqslant_l T$ and $S \in [T_0] \rightarrow B_S$ and $b \cap \alpha_n$ agree on $[|T|, \alpha_n)$. The next step is to fill in $b \cap [\alpha_n, \alpha_{n+1})$ so as to be T_0-special at α_{n+1}. Then define $T_1 \leqslant_{l+1} T_0$ as before and make $b \cap [\alpha_{n+1}, \alpha_{n+2})$ T_1-special at α_{n+2}. After ω steps one obtains the desired b. (One should also avoid including any new ordinals in $b \cap [\alpha_m, \alpha_{m+1})$ of the form $2(\lambda + 2^{\bar{m}} 3^i) + 1$ for $\bar{m} \leqslant m$, so as to not alter the definition of R^*.) □

In order to state the Fusion Lemma we make a definition: A set $D \subseteq R^A$ is *n-dense below* T if $T' \leqslant T \rightarrow \exists T'' \leqslant_n T'$, $T'' \in D$.

Lemma 1.3 (Fusion). *Suppose $T \in R^A$ and for each n, $D_n \subseteq R^A$ is n-dense below T and open. Then there exists $T' \leqslant T$ such that $T' \in D_n$ for all n.*

Proof. As we have insisted that $T \in R^A_\alpha \rightarrow T \in L_\beta[A]$ where $\beta > \alpha$ is least so that $L_\beta[A]$ is locally countable and admissible, $\Diamond(E)$ implies that we can obtain $\alpha \in E$ such that S_α is $\bigcup \{\{i\} \times (D_i \cap L_\alpha[A]) \mid i \in \omega\} \cup \{T\}$ and $R^A \cap L_\alpha[A] = R^A_{<\alpha}$. Then by the construction of Case 4 we added $T' \leqslant T$, $T' \in R^A_\alpha$ such that $T' \in D_k$ for all k. (Note that the technical restriction on the T_k's in Case 4 for guaranteeing $R^* = \omega$ for $R \in [T]$ can be satisfied, using the proof of the Extendibility Lemma.) □

Using these lemmas we can complete the proof of Minimal Coding when $V = L[A]$, $A \subseteq \omega_1$. Define $R_G \subseteq \omega$ for R^A-generic G by the equation $\{R_G\} = \bigcap \{[T] \mid T \in G\}$. By Lévy absoluteness we know that if $R_G \in [T]$, then $\alpha \in A \leftrightarrow$ R_G goes right at unboundedly many even levels of $T^{R_G}_{2\alpha}$ for all $\alpha < |T|$ and thus by

Extendibility this holds for all $\alpha<\omega_1$. We can now define $A\cap\alpha$, $T_\alpha^{R_G}$ by a simultaneous induction inside $L[R_G]$ and hence $A\in L[R_G]$.

Fusion can be used to show that ω_1 is preserved by R_G and that R_G is minimal. For the former suppose $T\Vdash f:\omega\to\omega_1$ and let $D_n=\{T'\leqslant T \mid s\in T',\ \|s\|=n\to (T')_s\Vdash f(n)=\alpha$ for some $\alpha\}$. Clearly D_n is open and n-dense below T for each n so by Fusion there exists $T'\leqslant T$, $T'\in D_n$ for all n. But then $T'\Vdash f$ is bounded. For the latter suppose $T\Vdash x\subseteq \mathrm{ORD}$, $x\notin L[A]$ and let $D_n=\{T'\leqslant T \mid s_1,s_2\in T',\ \|s_1\|=\|s_2\|=n,\ s_1\neq s_2\to$ for some α, $((T')_{s_1}\Vdash\alpha\in x,\ (T')_{s_2}\Vdash\alpha\notin x)$ or $((T')_{s_1}\Vdash \alpha\notin x$ and $(T')_{s_2}\Vdash\alpha\in x)\}$. Then D_n is open and n-dense below T for each n so there exists $T'\leqslant T$, $T'\in D_n$ for each n. But then $T'\Vdash R_G\in L[x]$.

As cardinals $>\omega_1$ are clearly preserved, this completes the proof of our Theorem when $V=L[A]$, $A\subseteq\omega_1$.

2. Minimally coding a subset of ω_2

Assume that $A\subseteq\omega_2$, $V=L[A]$ and $2^\omega\subseteq L_{\omega_1}[A]$. In this section we show how to minimally code A by a real. This construction reveals the main ideas in the proof of the full theorem.

The basic approach is analogous to Jensen's in that we will code A by a subset x of ω_1, which in turn is coded by a real. Coding A by x could be accomplished using a forcing analogous to the forcing of Section 1; however, the need to make R minimal requires us to mix this with the forcing for coding x into R.

It is now clear that x cannot result from a forcing with ω_1-trees in the usual sense, for such a forcing would produce an amenable $y\subseteq\omega_1$; i.e., a y with the property that $y\cap\beta\in V$ for all $\beta<\omega_1$. Instead we must simultaneously define the 'generalized trees' T for coding A into x, a 'path through T' and the canonical trees $t\in\bar{R}_\alpha^T$ of 'rank α' for coding $A\cap\alpha$ and a '$T\restriction\alpha$-path' $x^R\restriction\alpha$ into R. Then T assigns to each condition $t\in\bar{R}_\alpha^T$ two 'terms' for strings σ_0, σ_1. The idea is that if R is generic, then R canonically recovers $t\in\bar{R}_\alpha^T$ such that $R\in[t]$ and then R codes either $(x^R\restriction\alpha)*\sigma_0(R)$ or $(x^R\restriction\alpha)*\sigma_1(R)$. So in a sense T dictates possible ways of extending a branch through $T\restriction\alpha$, but where that branch may possibly fail to belong to V; the condition t in $\bar{R}_\alpha^T$ completely describes a branch through $T\restriction\alpha$ for each $R\in[t]$.

The trees that comprise R^T come from the 'universal' collection of trees R^*, used to code branches through any of the various 'generalized trees' T. The inductive construction of $R^*=\bigcup\{R_\alpha^*\mid\alpha<\omega_1\}$ is similar in outline to the construction of Section 1, but there are several major differences. We must abandon the idea of using a 'thin' set of conditions, in the sense that we now have that R_α^* is uncountable for $\alpha\geqslant\omega$. We also handle fusions quite differently, due to the lack of $\Diamond$. Instead we make a more explicit guess at the collapse of an elementary submodel that could arise in a fusion argument. The Recursion Theorem is necessary both for guessing at the collapse of the final forcing $\mathcal{P}^A$ and for coding an index for a fusion sequence into each path through the tree resulting from that fusion.

The inductive construction of $R^A = \bigcup \{R^A_\alpha \mid \alpha < \omega_2\}$ is also close in outline to that of Section 1, the major differences arising from the necessity of dealing with ordinals of uncountable cofinality. For this purpose we use $\Box$. In addition we must anticipate fusions as outlined in the preceding paragraph. There are two types of fusion here, the usual kind as well as those obtained by successively thinning the extensions of nodes on a fixed level of a given generalized tree.

Now for the construction of R^*. Let i_0 be an index for both the desired forcing $\mathcal{P}^A$ as a $\Sigma_1\langle L_{\omega_2}[A], A\rangle$-set with parameter ω_1, together with a description of how A is coded by a $\mathcal{P}^A$-generic real. The 'canonical' trees in $R^*_\alpha - R^*_{<\alpha}$ form $\tilde{R}^*_\alpha$, which yields R^*_α when added to $R^*_{<\alpha} = \bigcup \{R^*_\beta \mid \beta < \alpha\}$ and closed under the operations:

$(*)$ $t \in R^*_\alpha$, $\quad a \in t \to (t)_a \in R^*_\alpha$, and

$(**)$ $t_0, \ldots, t_n \in R^*_\alpha = \cup \{R^*_\beta \mid \beta \leq \alpha\} \to t_0 \cup \cdots \cup t_n \in R^*_\alpha$.

In Case 3 we will refer to 'acceptable terms'. This notion will be defined at the end of the construction.

Case 1: $\alpha = 0$. R^*_0 consists of all trees of the form $(2^{<\omega})_{a_1} \cup \cdots \cup (2^{<\omega})_{a_n}$ where $a_1 \cdots a_n$ are finite strings. For any real R, $t^R_0 = 2^{<\omega}$ so $\tilde{R}^*_0 = \{2^{<\omega}\}$.

Case 2: $\alpha = \beta + 1$. Let $t \in \tilde{R}^*_\beta$. We describe the extensions of t in $\tilde{R}^*_\alpha$. Let $\mathrm{Split}(t) = \{a \in t \mid a * 0, a * 1 \in t\}$ and let $f: 2^{<\omega} \to \mathrm{Split}(t)$ be bijective so that $f(a * 0) \supseteq f(a) * 0$ for all a (f is unique). Define t_0, t_1 to be $t_j = \{a \mid a \subseteq f(b)$ for some $b \in 2^{<\omega}$, $b(2i) = j$ for $2i < |b|\}$. Suppose the $t^{k'}_i$ have been defined for all $0 \leq k' < k$, $i < 2^{2^{k'}}$. To define the t^k_i, $i < 2^{2^k}$, first list all subsets x of $\{a \in 2^{<\omega} \mid \mathrm{length}(a) = k\}$ as $x_0, x_1, \ldots$ and for each x_i choose subtrees $t(i) \leq_k t$ so that no $t(i)$ shares a path with any $t^{k'}_j$, $k' < k$, $j < 2^{2^{k'}}$, nor with any other $t(i')$, and so that $a \in x_i \to (t(i))_{f(a)} \subseteq t_1$, $a \in 2^k - x_i \to (t(i))_{f(a)} \subseteq t_0$. Then add all the $t(i)$ to $\tilde{R}^*_\alpha$ if β is not even and if $\beta = 2\gamma$, then add $t(i)$ to $\tilde{R}^*_\alpha$ iff ($x_i = 2^k$, $\gamma \in A$ or $x_i = \emptyset$, $\gamma \notin A$). Set $t^k_i = t(i)$.

To obtain R^*_α close $R^*_{<\alpha} \cup \tilde{R}^*_\alpha$ under $(*)$, $(**)$.

Case 3: α limit, $L_\alpha[A]$ is not locally countable. In this case we are not concerned with fusions, only with guaranteeing extendibility to level α.

Given a real R define $R^* = \{n \mid 4(\lambda + 2^n 3^i) + 3 \in B^R$ for unboundedly many such ordinals $< \alpha$, λ limit or $0\}$ where B^R is defined as follows. If t is an ω-tree, $R \in [t]$, then we say that R *goes right at level i on t* if $a \in t$, $\|a\| = i$, $a \subseteq R \to a * 1 \subseteq R$; otherwise R goes left at level i on t. Our new coding is given by: $\delta \in B^R$ iff R goes right at sufficiently large even levels of t^R_δ. (We are reserving the ordinals $4\beta + 1 < \alpha$ for coding an $x \subseteq \omega_1$ which can be used to code $A \cap \omega_2$.) If α is countable in $L[R^*]$ and $0 \notin R^*$, we define $\alpha^R_0 < \alpha^R_1 < \cdots$ to be the

$L[R^*]$-least ω-sequence cofinal in α such that α a limit of limit ordinals$\rightarrow$each α_i^R is a limit ordinal.

Now pick $\hat{\beta}<\alpha$, $\hat{k}\in\omega$, an acceptable term $\hat{\sigma}$ and an ordinal $\hat{\alpha}\leqslant\hat{\beta}$, $\hat{\alpha}$ limit or 0. Let α_k denote α_k^R. Choose n to be least so that $\hat{\beta}<\alpha_n$ and define $t_0\geqslant t_1\geqslant\cdots$ as follows: $t_0=L[A\cap\alpha, R^*]$-least canonical $t\leqslant_{\hat{k}} t_{\hat{\beta}}^R$ in $\bar{R}^*_{\alpha_n}$ such that $S\in[t]\rightarrow B^S$, B^R agree on $[\hat{\beta}, \alpha_n)-\{4\gamma+1 \mid \gamma\in\mathrm{ORD}\}$ and $B^S(\hat{\alpha}+4\gamma+1)=\hat{\sigma}(s)(\gamma)$ for $4\gamma+1<\min(|\hat{\sigma}|, \alpha_n-\hat{\alpha})$ (if it exists). If t_k is defined, then $t_{k+1}=L[A\cap\alpha, R^*]$-least canonical $t\leqslant_{k+\hat{k}+1} t_k$ in $\bar{R}^*_{\alpha_{n+k+1}}$ such that $S\in[t]\rightarrow B^S$, B^R agree on $[\alpha_{n+k}, \alpha_{n+k+1})-\{4\gamma+1 \mid \gamma\in\mathrm{ORD}\}$ and $B^S(\hat{\alpha}+4\gamma+1)=\hat{\sigma}(S)(\gamma)$ for $4\gamma+1<\min(|\hat{\sigma}|, \alpha_{n+k+1}-\hat{\alpha})$. Let $t^R(\hat{\beta}, \hat{k}, \hat{\sigma}, \hat{\alpha})=\bigcap\{t_k \mid k\in\omega\}$ when all the t_k exist.

Include in $\bar{R}^*_\alpha$ all perfect trees t such that for some $\hat{\beta}$, $\hat{k}$, $\hat{\sigma}$, $\hat{\alpha}$ and all $R\in[t]$, $t=t^R(\hat{\beta}, \hat{k}, \hat{\sigma}, \hat{\alpha})$ as built above. Then R^*_α is obtained from $R^*_{<\alpha}\cup R^*_\alpha$ by closing under the operations $(*)$ and $(**)$.

Case 4: α limit, $L_\alpha[A]$ is locally countable. Two kinds of conditions are added to $\bar{R}^*_\alpha$ in this case. First add conditions exactly as in Case 3. We now describe the conditions of the second kind, needed to anticipate fusions.

Given a real R we first define what it means for R to code a *fusion index* $\hat{i}\in\omega$. In order for this to be defined the following conditions must be met. Let $\eta\geqslant\alpha$ be least so that α is not regular in $J_{\eta+1}[R]$; we must have that η exists and $k\geqslant 2$ where $k=$ least k such that α is $\Sigma_k(J_\eta[R])$-projectible. Now using the index i_0 fixed at the start, we must be able to decode from R a predicate $\bar{A}\subseteq(\omega_2)^{J_\eta[R]}$ so that R is $\mathcal{P}^{\bar{A}}$-generic over $J_\eta[\bar{A}]$. (Here we use $\mathcal{P}^{\bar{A}}$ to denote the forcing for coding $\bar{A}$ into a real over $J_\eta[\bar{A}]$. Also we only require genericity for predense $D\in J_\eta[\bar{A}]$.) We also require that Σ_{k-1}-projectum$(J_\eta[\bar{A}])=(\omega_2)^{J_\eta[\bar{A}]}=(\omega_2)^{J_\eta[R]}$ and so we can form the Σ_{k-2}-Master code structure $\mathcal{A}$ for $J_\eta[\bar{A}]$. Thus $X\subseteq\mathcal{A}\cap\mathrm{ORD}$ is $\Sigma_1(\mathcal{A})$ iff X is $\Sigma_{k-1}(J_\eta[\bar{A}])$. Now let p be the least parameter for Σ_1 projecting $\mathcal{A}$ into $(\omega_2)^{J_\eta[\bar{A}]}$ and let $\beta_0<\beta_1<\cdots$ be the first ω ordinals $\beta<(\omega_2)^{J_\eta[\bar{A}]}$ such that $\beta\notin H'_\beta=\Sigma_1$-Skolem hull of $\alpha\cup\beta\cup\{p\}$ in $\mathcal{A}$. We require that $\bigcup\{\beta_i \mid i<\omega\}=(\omega_2)^{J_\eta[\bar{A}]}$. Let $H^q_\beta=\Sigma_1$-Skolem hull of $\beta\cup\{q,p\}$ in $\mathcal{A}$ and define $\alpha_0<\alpha_1<\cdots$ by $\alpha_0=H^0_0\cap\omega_1$, $\alpha_{n+1}=H^{\beta_n}_{\alpha_n+1}\cap\omega_1$. We assume that $\bigcup\{\alpha_i \mid i<\omega\}=\alpha$. Let $\mathcal{A}'=\Sigma_{k-1}$-Master Code structure for $J_\eta[\bar{A}]$ and let $h:(\omega_2)^{J_\eta[\bar{A}]}\rightarrow\omega$ be a canonical $\Sigma_1(\mathcal{A}')$-injection.

For R to code a fusion index $\hat{i}$ when the conditions of the previous paragraph are met consider the sequence $i_0, i_1, \ldots$ defined by $i_k=$ least $i<\omega$ such that $\alpha_{2^k3^{i+1}}\in B^R$; we require that $0\in R^*$ and that the i_k are defined and equal to $\hat{i}$ for sufficiently large k. Then the key requirement is that the $\Sigma_1(\mathcal{A}')$-set defined by the Σ_1-formula with index $h^{-1}(\hat{i})$ is a fusion sequence $t_0\geqslant t_1\geqslant_1 t_2\geqslant_2\cdots$ of (not necessarily canonical) conditions in $R^*_{<\alpha}$.

This completes the definition of "R codes the fusion index $\hat{i}$". If the above conditions are met then we set $t^R=\bigcap\{t_i \mid i<\omega\}$.

We want to add the results of such fusions to $\bar{R}^*_\alpha$ but must be careful to arrange that an index for each such fusion t^R can be canonically recovered from R. We will be able to show with help from the recursion theorem that the addition of these fusions does suffice to establish the fusion lemma for $\mathcal{P}^A$.

We can now describe the second type of condition that must be added to $\tilde{R}^*_\alpha$. Add t if for all $R \in [t]$ all of the above conditions hold and $t = t^R$, where t^R is defined as above. Finally obtain R^*_α by closing $R^*_\alpha \cup \tilde{R}^*_{<\alpha}$ under the $(*)$, $(**)$ operations.

This completes the construction of $R^* = \bigcup \{R^*_\alpha \mid \alpha < \omega_1\}$. For $t \in R^*$ let $\alpha(t) =$ unique α such that $t \in R^*_\alpha - R^*_{<\alpha}$.

As in Section 1 we can check that $t \in R^* \to t = (t_0)_{a_0} \cup \cdots \cup (t_n)_{a_n}$ where the t_i's are canonical and $a_i \in t_i$. Also a real can belong to at most one canonical $t \in R^*_\alpha$, so t^R_α is well-defined (for any $\alpha < \omega_1$).

We now clarify the above construction by discussing acceptability.

Acceptable terms

A *term* $\sigma = \sigma(\boldsymbol{R})$ is an $L[\boldsymbol{R}, R_0]$-name for a string in $2^{<\omega_1}$, where $\boldsymbol{R}$ denotes a real and R_0 is a real parameter. We also assume that there is a fixed ordinal $|\sigma| < \omega_1^{L[R_0]}$ such that $\text{length}(\sigma(R)) = |\sigma|$ for all reals R. The class of *acceptable terms* is defined inductively as follows:

(a) Any constant term $\sigma(R) = S_0$ (S_0 a fixed element of $2^{<\omega_1}$) is acceptable, provided $\text{length}(S_0)$ is a limit ordinal.

(b) If σ_1, σ_2 are acceptable, $|\sigma_1| = |\sigma_2|$ and $a \in 2^{<\omega}$, then σ is acceptable where $\sigma(R) = \sigma_1(R)$ if $a \subseteq R$, $= \sigma_2(R)$ if $a \nsubseteq R$.

(c) If σ_1 is acceptable and σ_2 is acceptable, then $\sigma_1 * \sigma_2$ is acceptable where $\sigma_1 * \sigma_2(R) = \sigma_1(R) * \sigma_2(R)$ and $*$ denotes concatenation.

(d) If $\sigma_1, \sigma_2, \ldots$ are acceptable and $\sigma_n \subseteq \sigma_{n+1}$ for all n (i.e., $\sigma_n(R) \subseteq \sigma_{n+1}(R)$ for all R), then σ is acceptable where $\sigma(R) = \bigcup \{\sigma_n(R) \mid n < \omega\}$.

We can establish extendibility for R^* like we did for R^A in Section 1, using the notion 't-special at α' for $b \subseteq [|t|, \alpha)$. We will need, however, a version of extendibility that is stronger than what is established there, in order to facilitate our study of the forcings R^T, T a generalized tree.

First we need to define the notion of a *type* 1 *extension* $t \geqslant t^*$ in $\tilde{R}^*$. These are extensions which arise by applying any of the cases in the construction of R^* with the exception of the second half of Case 4 (where fusions were added). For $t \in R^*$ recall that $\alpha(t)$ denotes the unique α such that $t \in R^*_\alpha - R^*_{<\alpha}$. We define "$t \geqslant t^*$ is type 1" by induction on $\alpha(t^*)$ as follows. The trivial extension $t = t^*$ is type 1. If $\alpha(t^*) = \beta + 1$, then $t \geqslant t^*$ is type 1 if there is a sequence $t \geqslant t' \geqslant t^*$ where $t \geqslant t'$ is type 1 and $\alpha(t') = \beta$. Finally, if $\alpha(t^*) > \alpha(t)$ is a limit ordinal, then $t \geqslant t^*$ is type 1 if $t' \geqslant t \geqslant t^*$ where t^* arises from t' as in Case 3 or as in the first part of Case 4 (the nonfusion case), but also where the conditions t_k arising there have the property that $t = t^R_\beta \geqslant t_0 \geqslant t_1 \geqslant \cdots$ are all type 1 extensions. An examination of the proof of Lemma 1.2 yields the following.

Lemma 2.1. *Suppose* $t \in \tilde{R}^*$, $k \in \omega$, σ *is acceptable and* $\hat{\alpha} \leqslant \alpha(t)$ *is* 0 *or a limit* *ordinal. Also suppose that* $R \in [t] \rightarrow B^R(\hat{\alpha}+4\gamma+1) = \sigma(R)(\gamma)$ *for* $4\gamma+1 < \min(|\sigma|, \alpha(t)-\hat{\alpha})$. *Then if* $\alpha > \alpha(t)$, *there exists* $t^* = \tilde{R}^*_\alpha$ *such that* $t^* \leqslant_k t$, $t \geqslant t^*$ *is a type* 1 *extension and* $R \in [t^*] \rightarrow B^R(\hat{\alpha}+4\gamma+1) = \sigma(R)(\gamma)$ *for* $4\gamma+1 < \min(|\sigma|, \alpha-\hat{\alpha})$.

Proof. We follow the outline of the proof of Lemma 1.2. First suppose that α is a successor ordinal. Then the result is clear by induction unless σ falls under (b) in the inductive definition of acceptable term. In that case the following Sublemma shows that Case 2 of the construction of R^* was designed so as to allow the desired extendibility.

Sublemma. *If* σ *is an acceptable term, then for each* $\alpha < |\sigma|$ *the function* $R \mapsto \sigma(R)(\alpha)$ *is a* ***continuous*** *function from* 2^ω *to* 2 *and hence* $\{R \mid \sigma(R)(\alpha) = 0\}$ *can be written* $[(2^{<\omega})_{a_0} \cup \cdots \cup (2^{<\omega})_{a_n}]$ *for some* $a_0, \ldots, a_n \in 2^{<\omega}$.

Proof of Sublemma. Clear, by induction on the formation of σ. □

Finally, if α is a limit ordinal, then the existence of t^* follows as in Lemma 1.2 using a 't-special at α' set $b \subseteq [\alpha(t), \alpha)$ to obtain t^* so that $R \in [t^*] \rightarrow B^R$, b agree on $[\alpha(t), \alpha) - \{4\gamma+1 \mid \gamma \in \mathrm{ORD}\}$, $B^R(\hat{\alpha}+4\gamma+1) = \sigma(R)(\gamma)$ for $4\gamma+1 < \min(|\sigma|, \alpha-\hat{\alpha})$. Also as no reference is needed to Case 4 of the construction of R^*, all extensions as above are in fact type 1. □

Also the following can be checked by a simple induction.

Lemma 2.2. *Suppose* $t \geqslant t^*$ *is a type* 1 *extension in* $\tilde{R}^*$. *then for each* $\alpha \in [\alpha(t), \alpha(t^*)]$ *there is a unique* $t' \in \tilde{R}^*_\alpha$ *such that* $t \geqslant t' \geqslant t^*$.

The preceding lemma is needed to define an important equivalence relation on elements of $\tilde{R}^*$.

The equivalence relation ~. If $t_1, t_2 \in \tilde{R}^*$, then $t_1 \sim t_2$ provided $\alpha(t_1) = \alpha(t_2)$ and either $t_1 = t_2$ or there are $\bar{t}_1, \bar{t}_2 \in \tilde{R}^*$ such that:

(a) $\bar{t}_1 \geqslant t_1$, $\bar{t}_2 \geqslant t_2$ are type 1 extensions,
(b) $\bar{t}_1 \sim \bar{t}_2$,
(c) $R, S \in [t_1] \cup [t_2] \rightarrow B^R$, B^S agree on $[\alpha(\bar{t}_1), \alpha(t_1)) - \{4\gamma+1 \mid \gamma \in \mathrm{ORD}\}$.

Note. The previous is an inductive definition.

Lemma 2.3. ~ *is an equivalence relation.*

Proof. If $\bar{t}_1 \geqslant t_1$, $\bar{t}_2 \geqslant t_2$ are type 1 extensions, $\bar{t}'_2 \geqslant t_2$, $\bar{t}_3 \geqslant t_3$ are type 1 extensions and $\bar{t}_1 \sim \bar{t}_2$, $\bar{t}'_2 \sim \bar{t}_3$, then by Lemma 2.2 we must have $\bar{t}_2 \geqslant \bar{t}'_2$ or $\bar{t}'_2 \geqslant \bar{t}_2$. Without loss of generality assume the former. Choose $\bar{t}_1'$ so that $\alpha(\bar{t}_1') = \alpha(\bar{t}_2')$ and $\bar{t}_1 \geq \bar{t}_1' \geq t_1$, again by Lemma 2.2. Then $\bar{t}_2' \sim \bar{t}_1'$, as witnessed by $\bar{t}_1$, $\bar{t}_2$. So $\bar{t}_1' \sim \bar{t}_3$ witnesses $t_1 \sim t_3$. □

The relation $\sim$ is needed to give the proper definition of generalized tree. T is a *generalized tree* if $T = \langle (f_i, g_i) \mid i < \omega_1 \rangle$ where:

(a) $f_i, g_i : \tilde{R}_i^T \to$ Acceptable Terms, where $\tilde{R}_i^T =$ canonical elements of $R_i^T - R_{<i}^T$.

(b) $|f_i(t)| = |g_i(t)|$ is a limit ordinal for all $t \in \tilde{R}_i^T$ and all i. In addition $f_0(2^{<\omega}) = g_0(2^{<\omega})$ and for $i > 0$, $t \in \tilde{R}_i^T : f_i(t)(R)(0) = 0$, $g_i(t)(R)(0) = 1$, for all reals R.

(c) If $t_1 \sim t_2$ belong to $\tilde{R}_i^T$, then $f_i(t_1) = f_i(t_2)$, $g_i(t_1) = g_i(t_2)$.

To complete the above definition we must define R_i^T, $\tilde{R}_i^T$. We define R_i^T, $\tilde{R}_i^T$ by induction on i. If R is a real and $R \in [t]$ for some $t \in \tilde{R}_\alpha^*$, α limit, then define $B^R \cap \alpha$ as we did earlier and set $x^R \cap \alpha = \{\gamma \mid 4\gamma + 1 \in B^R \cap \alpha\}$. Now $\tilde{R}_0^T = \{2^{<\omega}\}$ and R_0^T is the $(*)$, $(**)$ closure of $\tilde{R}_0^T$. If $\tilde{R}_i^T$ is defined, then $t' \in \tilde{R}_{i+1}^T$ if for some $t \in \tilde{R}_i^T$, $t \geq t' \in \tilde{R}^*_{\alpha(t)+\eta}$ where $\eta = |f_i(t)|$ and moreover we have that either $R \in [t'] \to x^R(\alpha(t) + \eta') = f_i(t)(R)(\eta')$ for $\eta' < \eta$ or $R \in [t'] \to x^R(\alpha(t) + \eta') = g_i(t)(R)(\eta')$ for $\eta' < \eta$. (Note that η is a limit ordinal so $x^R \cap (\alpha(t) + \eta)$ is defined for $R \in [t']$.) To obtain R_{i+1}^T, close $\tilde{R}_{i+1}^T \cup R_i^T$ under the operations $(*)$, $(**)$. Finally to define $\tilde{R}_\lambda^T$ for a limit ordinal λ take all $t \in \tilde{R}^*$ which can be written $t = \bigcap \{t_i \mid i < \omega\}$ where $t_0 > t_1 > \cdots$, $\alpha(t) = \bigcup \{\alpha(t_i) \mid i < \omega\}$, $t_i \in R_{<\lambda}^T = \bigcup \{R_i^T \mid i < \lambda\}$ and $\lambda = \bigcup \{\alpha \mid t_i \in R_\alpha^T - R_{<\alpha}^T$ for some $i\}$. And R_λ^T is the $(*)$, $(**)$ closure of $\tilde{R}_\lambda^T \cup R_{<\lambda}^T$. For $t \in R^T$ let $|t|$ denote the unique i such that $t \in R_i^T - R_{<i}^T$.

Finally set $R^T = \bigcup \{R_i^T \mid i < \omega_1\}$.

Lemma 2.4 (Extendibility for R^T). *Suppose $t \in R_i^T$, $k \in \omega$ and $i \leq j < \omega_1$. Then there exists $t' \leq_k t$ such that $|t'| = j$.*

Proof. As in the proof of Lemma 1.2 it suffices to consider $t \in \tilde{R}_i^T$. We show that there is a type 1 extension $t' \leq_k t$ with $t' \in \tilde{R}_j^T$, by induction on j.

If $j = i$, there is nothing to show. If $j > i$ is a successor ordinal, then the result is clear by induction and Lemma 2.1.

If $j > i$ is a limit ordinal, then choose a cofinal ω-sequence $i < j_0 < j_1 < \cdots$ below j and select a corresponding sequence $t \geq_k t_0 \geq_k t_1 \geq \cdots$ of type 1 extensions so that $t_n \in \tilde{R}_{j_n}^T$. Let $b_n^3 = \{4\gamma + 3 \in B^R \mid \gamma \in [\alpha(t), \alpha(t_n))\}$ for $R \in [t_n]$ and $R^*(n) = \{m \mid 4(\lambda + 2^m 3^i) + 3 \in b_n^3$ for unboundedly many such ordinals $< \alpha(t_n)$, λ limit or $0\}$. We can also arrange the choice of t_n's so that for $n \leq n'$, $4(\lambda + 2^m 3^i) + 3 \in [\alpha(t_n), \alpha(t_{n'}))$, we have $(m \in (R^*(n))_n \leftrightarrow 4(\lambda + 2^m 3^i) + 3 \in b_{n'}^3)$ and $(R^*(n))_n$ codes the ordinal $\alpha(t_n)$. The net effect is that $\alpha = \bigcup \{\alpha(t_n) \mid n \in \omega\}$ is countable in $L[R^*(\omega)]$ where $R^*(\omega)$ is defined from $b_\omega^3 = \bigcup \{b_n^3 \mid n \in \omega\}$, α as was $R^*(n)$ from b_n^3, $\alpha(t_n)$. Also choose acceptable terms σ_n so that $R \in [t_n] \to B^R(\alpha(t) + 4\gamma + 1) = \sigma_n(R)(\gamma)$ for $4\gamma + 1 < |\sigma_n| = \alpha(t_n) - \alpha(t)$.

Now we see that $b = b^3_\omega \bigcup \{2\gamma < \alpha \mid \gamma \in A\}$ is t-special at α. The latter implies that we can choose a type 1 extension $t' \leqslant_k t$, $\alpha(t') = \alpha$ so that $R \in [t'] \rightarrow B^R(\alpha(t) + 4\gamma + 1) = \sigma_\omega(R)(\gamma)$ for $\gamma < \alpha - \alpha(t)$, $\sigma_\omega = \bigcup \{\sigma_n \mid n \in \omega\}$ and B^R, b agree on $[\alpha(t), \alpha) - \{4\gamma + 1 \mid \gamma \in \text{ORD}\}$.

We claim that $t' \in \tilde{R}^T_j$. To see this, by Lemma 2.2 define t'_n so that $t \geqslant t'_n \geqslant t'$, $\alpha(t'_n) = \alpha(t_n)$. Then it suffices to show that $t'_n \in \tilde{R}^T_{j_n}$ for each n. As $t'_n \sim t_n$ our proof reduces to the following lemma. □

Lemma 2.5. *Suppose $t \geqslant t_1$, $t \geqslant t_2$ are type 1 extensions, $t_1 \sim t_2$ and $t, t_1 \in \tilde{R}^T$. Then $t_2 \in \tilde{R}^T_{|t_1|}$ where $|t_1|$ = the unique j such that $t_1 \in \tilde{R}^T_j$.*

Proof. By induction on $|t_1|$.

If $|t_1| = |t|$, then $t = t_1 = t_2$ so there is nothing to show.

If $|t_1| = j + 1 > |t|$, then choose $\bar{t}_1$, t_2 so that $t \geqslant \bar{t}_1 \geqslant t_1$, $t \geqslant \bar{t}_2 \geqslant t_2$ and $\alpha(\bar{t}_1) = \alpha(\bar{t}_2)$ where $\bar{t}_1 \in \tilde{R}^T_j$. Then as $\bar{t}_1 \sim \bar{t}_2$ we have that $\bar{t}_2 \in \tilde{R}^T_j$. As $R \in [t_2]$, $S \in [t_1] \rightarrow B^R$, B^S agree on $[\alpha(\bar{t}_1), \alpha(t_1)) - \{4\gamma + 1 \mid \gamma \in \text{ORD}\}$ we see that $R \in [t_2] \rightarrow B^R(\alpha(\bar{t}_2) + 4\gamma + 1) = \sigma(R)(\gamma)$ for $\gamma < \alpha(t_2) - \alpha(\bar{t}_2)$ where $\sigma = (f_j(\bar{t}_1)$ or $g_j(\bar{t}_1)) = (f_j(\bar{t}_2)$ or $g_j(\bar{t}_2))$ since $\bar{t}_1 \sim \bar{t}_2$. So $t_2 \in \tilde{R}^T_{j+1}$. If $|t_1| = \lambda$ is a limit ordinal $> |t|$, then the result is clear by induction and the definition of $\tilde{R}^T_\lambda$. □

A useful fact is the 'countable closure' of the collection of generalized trees. If T_1, T_2 are generalized trees, then we define $T_1 \leqslant T_2$ if $R^{T_1} \subseteq R^{T_2}$.

Lemma 2.6. *Suppose $T_0 \geqslant T_1 \geqslant \cdots$ are generalized trees. Then there is a generalized tree T so that T = greatest lower bound of $\langle T_i \mid i < \omega \rangle$.*

Proof. If $T \restriction \gamma = \langle (f_i, g_i) \mid i < \gamma \rangle$ has been defined and hence so has $\tilde{R}^T_\gamma$, define $f_\gamma(t) = \bigcup \{f^n_{\gamma_n}(t) \mid n \in \omega\}$, $g_\gamma(t) = \bigcup \{g^n_{\gamma_n}(t) \mid n \in \omega\}$, where $t \in \tilde{R}^{T_n}_{\gamma_n}$ and $T_n = \langle (f^n_i, g^n_i) \mid i < \omega_1 \rangle$. If $T' \leqslant T_n$ for all n, then $f'_{\gamma'}(t') \supseteq f_\gamma(t')$ follows for any $t' \in \tilde{R}^{T'}_{\gamma'}$ where $t' \in \tilde{R}^T_\gamma$; using this it is clear that $T' \leqslant T$. □

Coding A into $x \subseteq \omega_1$: the forcing R^A

We turn now to a dicusssion of how generalized trees can be used to code A into a subset of ω_1. Analogously to the construction of R^* we will inductively define collections R^A_α of generalized trees for $\alpha < \omega_2$, as well as $\tilde{R}^A_\alpha$ = the canonical elements of $R^A_\alpha - R^A_{<\alpha}$. If $x : \omega_1 \rightarrow \tilde{R}^*$ is a 'path through T' for some $T \in R^A_\alpha - R^A_{<\alpha}$, then x canonically defines the unique $T^x_\alpha \in \tilde{R}^A_\alpha$ such that x is a 'path through T^x_α.'

Define $[T \restriction \gamma]$ = the collection of paths through $T \restriction \gamma$ as follows. Write $T = \langle (f_i, g_i) \mid i < \omega_1 \rangle$ and $T \restriction \gamma = \langle (f_i, g_i) \mid i < \gamma \rangle$. If $0 \leqslant \gamma < \omega_1$, then x is *a path through* $T \restriction \gamma$ if there exists $t \in \tilde{R}^T_\gamma$ and $R \in [t]$ such that $\text{Dom}(x) = \alpha(t)$ and $x(i) = t^R_i$ for $i < \alpha(t)$. And $x : \omega_1 \rightarrow \tilde{R}^*$ is a *path through* T, $x \in [T]$, if $x \restriction \alpha \in$

$\bigcup\{[T \restriction \gamma] \mid \gamma < \omega_1\}$ for unboundedly many $\alpha < \omega_1$. Note that $T_1 \leq T_2 \rightarrow [T_1] \subseteq [T_2]$.

Now suppose T^x_α is defined for all $\alpha < \omega_2$. Fix $\alpha < \omega_2$ and let $T = T^x_\alpha$. For each $\beta < \omega_1$ let t^x_β be the unique element of Range$(x) \cap \tilde{R}^T_\beta$. We say that x *goes left at* β *on* T^x_α if $R \in [x(\alpha(t^x_\beta)+2)] \rightarrow \alpha(t^x_\beta)+1 \notin B^R$, x *goes right at* β *on* T^x_α otherwise. Note that in the latter case, $\alpha(t^x_\beta)+1 \in B^R$ for all $R \in [x(\alpha(t^x_\beta)+2)]$, as in general: $t \in \tilde{R}^*$, $\alpha(t) = \beta + 1 \rightarrow B^R$, B^S agree at β for R, $S \in [t]$.

Then x codes $B^x \subseteq \omega_2$ defined by: $\alpha \in B^x$ iff x goes right at $\beta + 1$ on T^x_α for sufficiently large $\beta < \omega_1$. We code A into $x \subseteq \omega_1$ by requiring that $A =$ even$(B^x) = \{\gamma \mid 2\gamma \in B^x\}$.

We should comment on the fact that some of our definitions will appear to not take place in V, due to the need to refer to paths $R \in [t]$, $x \in [T]$ which may not belong to V. If t is an ω-tree and $\phi(R)$ is a formula of countable rank, then $\forall R \in [t]\ \phi(R)$ holds in V iff it holds in all extensions of V, by Lévy–Shoenfield absoluteness. The analogous property for generalized trees T is false. However, we can talk about 'truth in all extensions of V' using the forcing $\Vdash_{\overline{\omega_1}}$ for collapsing ω_1 to ω with finite conditions. Thus if $\phi(x)$ has rank $< \omega_2$ and T is a generalized tree, then when we say '$\phi(x)$ for all paths x through T' we actually mean $\Vdash_{\overline{\omega_1}} \phi(x)$ for all paths x through T.

We are almost ready to begin the inductive definition of R^A_α, $\alpha < \omega_2$. We shall need a form of $\Box$: let $\langle C^y_\alpha \mid \alpha$ limit, $\omega_1 \leq \alpha < (\omega_2)^{L[y]}$, $y \subseteq \alpha\rangle$ be a sequence so that C^y_α is a closed subset of α of ordertype $\leq \omega_1$, $\beta \in C^y_\alpha \rightarrow C^y_\beta \cap \beta = C^y_\alpha \cap \beta$ and C^y_α is uniformly definable as an element of $L_{\beta(\alpha)}[y]$ where $\beta(\alpha) =$ least β such that $L_\beta[y] \models \text{card}(\alpha) = \omega_1$, $\bigcup C^y_\alpha < \alpha \rightarrow \text{cof}(\alpha) = \omega$ in $L_{\beta(\alpha)}[y]$.

And we must introduce the operation $(***)$ that generates R^A_α from $R^A_{<\alpha} \cup \tilde{R}^A_\alpha$. First define $T(t)$, for T a generalized tree and $t \in \tilde{R}^T$ by: $\alpha(t) \leq \alpha(t')$, $[t] \cap [t'] \neq \emptyset \rightarrow t' \in T(t)$; $t_1 \sim t_0$, $t_0 \in T(t) \rightarrow t_1 \in T(t)$. Note that $t' \in \tilde{R}^T \rightarrow t' \in T(t)$ for at most countably many t, as an induction on $\alpha(t')$ shows that t' shares a path with at most countably many $t \in \tilde{R}^T \cap \tilde{R}^*_\alpha$ for each $\alpha \leq \alpha(t')$.

Now suppose T_0, $T_1 \leq T$ are generalized trees, $t \in \tilde{R}^{T_0} \cap \tilde{R}^{T_1}$ and $a \in 2^{<\omega}$. Define $T^* = T(a, t, T_0, T_1) = \langle (f^*_i, g^*_i) \mid i < \omega_1\rangle$ as follows. Suppose (f^*_i, g^*_i) is defined for $i < \gamma$ so we have defined $\tilde{R}^{T^*}_\gamma$. Pick $t^* \in T(t)$, $t^* \in \tilde{R}^{T^*}_\gamma$ and canonically choose type 1 extensions t_0, t_1 of t^* in $\tilde{R}^{T_0}$, $\tilde{R}^{T_1}$ so that $\alpha(t_0) = \alpha(t_1)$, with corresponding acceptable terms σ_0, σ_1 so that $\sigma_0(R)(0) = 0$, $\sigma_1(R')(0) = 0$ (for $R \in [t_0]$, $R' \in [t_1]$). (If $t^* \notin \tilde{R}^{T_0} \cap \tilde{R}^{T_1}$ or $t^* \notin T(t)$, then ignore the previous.) Then $f^*_\gamma(t^*)$ is defined to be σ where $\sigma(R) = \sigma_0(R)$ if $a \subseteq R$, $= \sigma_1(R)$ if $a \nsubseteq R$. Define g^*_γ similarly, but with $\sigma_0(R)(0) = 1$, $\sigma_1(R')(0) = 1$. (If $t^* \in T(t)$, $t^* \notin \tilde{R}^{T_i}$, then $f^*_\gamma(t^*) = f^{1-i}_{\gamma_{1-i}}(t^*)$ where $t^* \in \tilde{R}^{T_{1-i}}_{\gamma_{1-i}}$ and $T_i = \langle (f^i_j, g^i_j) \mid j < \omega_1\rangle$. If $t^* \notin T(t)$, then $(f^*_\gamma(t^*), g^*_\gamma(t^*)) = (f_{\bar{\gamma}}(t^*), g_{\bar{\gamma}}(t^*))$ where $t^* \in R^T_{\bar{\gamma}}$ and $T = \langle (f_i, g_i) \mid i < \omega_1\rangle$.)

$(***)$ is the operation that produces T^* from a, t, T_0, T_1, T.

Now for the construction of the $\tilde{R}^A_\alpha, \alpha < \omega_2$.

Case 1: $\alpha \leq \omega_1$. R^A_α contains only one generalized tree T defined by $T_\emptyset =$

$\langle (f_i, g_i) \mid i < \omega_1 \rangle$ where $f_i(t)(R) = \langle 0, 0, \ldots \rangle$, $g_i(t)(R) = \langle 1, 0, \ldots \rangle$ for $i > 0$ and $t \in \tilde{R}_i^{T_0}$, $f_0(2^{<\omega}) = g_0(2^{<\omega}) = \emptyset$.

Case 2: $\alpha = \beta + 1 > \omega_1$. Let $T \in \tilde{R}^A_\beta$. We shall describe the extensions of T in $\tilde{R}^A_\alpha$. We first define two generalized trees $T_0, T_1 \leqslant T$ so that $x \in [T_0] \rightarrow x$ goes left at $i+1$ on T for all $i < \omega_1$, $x \in [T_1] \rightarrow x$ goes right at $i+1$ on T for all $i < \omega_1$. Write $T = \langle (f_i, g_i) \mid i < \omega_1 \rangle$ and suppose $T_0 \restriction \gamma = \langle (f_i^0, g_i^0) \mid i < \gamma \rangle$ is defined so that $\tilde{R}^{T_0}_\gamma \subseteq \tilde{R}^T_{\omega \cdot \gamma}$. To define f^0_γ, for each $t \in \tilde{R}^{T_0}_\gamma$ choose a sequence $t = t_0 \geqslant t_1 \geqslant t_2 \geqslant \cdots$ of type 1 extensions with the following properties: $t_n \in \tilde{R}^T_{\omega \cdot \gamma + n}$, $R \in [t_{n+1}] \rightarrow x^R(\alpha(t_n) + \delta) = f_{\omega \cdot \gamma + n}(t_n)(R)(\delta)$ for $\delta < |f_{\omega \cdot \gamma + n}(t_n)|$, for $n \geqslant 0$. Then $f^0_\gamma(t)$ is characterized by

$$|f^0_\gamma(t)| = \sum_n |f_{\omega \cdot \gamma + n}(t_n)|,$$

$$f^0_\gamma(t)(R)(\alpha(t_n) + \delta) = f_{\omega \cdot \gamma + n}(t_n)(R)(\delta) \quad \text{for } \delta < |f_{\omega \cdot \gamma + n}(t_n)| \text{ and } n \geqslant 0.$$

To define g^0_γ, for each $t \in \tilde{R}^{T_0}_\gamma$ choose a sequence $t = t_0 \geqslant t_1 \geqslant \cdots$ of type 1 extensions as above, except that '$n \geqslant 0$' should be '$n > 0$' and $R \in [t_1] \rightarrow x^R(\alpha(t) + \delta) = g_{\omega \cdot \gamma}(t)(R)(\delta)$ for $\delta < |f_{\omega \cdot \gamma}(t)|$, $g^0_\gamma(t)(R)(\alpha(t) + \delta) = g_{\omega \cdot \gamma}(t)(R)(\delta)$ for $\delta < |f_{\omega \cdot \gamma}(t)|$. The existence of the t_n's as well as the fact that the definitions of f^0_γ, g^0_γ do not depend on the choice of t_n's follow from Lemma 2.4 and the definition of generalized tree.

Let $T_0 = \langle (f^0_i, g^0_i) \mid i < \omega_1 \rangle$ and define T_1 analogously, with the roles of f and g switched. Thus T_0, T_1 have the property stated earlier (paths through T_0 go left, paths through T_1 go right on T at successor ordinals). If β is not even, we put both T_0 and T_1 into $\tilde{R}^A_\alpha$. If $\beta = 2\gamma$, we put T_0 into $\tilde{R}^A_\alpha$ iff $\gamma \notin A$, T_1 into $\tilde{R}^A_\alpha$ iff $\gamma \in A$. Now repeat the above procedure as in Section 1, defining $T^k_0, T^k_1 \leqslant_k T$ by induction on $k < \omega_1$ ($T' \leqslant_k T$ if $T' \leqslant T$ and $\tilde{R}^{T'}_i = \tilde{R}^T_i$ for $i \leqslant k$. Equivalently: $T' \restriction k+1 = T \restriction k+1$, $T' \leqslant T$): $T^0_0 = T_0$, $T^0_1 = T_1$ are already defined. To obtain T^k_0, T^k_1 first choose $T' \leqslant_k T$ so that $[T']$ is disjoint from $\bigcup \{[T^{k'}_i] \mid k' < k, i = 0$ or $1\}$. This is easily accomplished by arranging that $x \in [T'] \rightarrow x$ goes left at $i+1$, x goes right at $i+2$ for some large $i < \omega_1$. Then apply the above procedure to T' instead of T, but this time only modifying f'_i, g'_i (where $T' = \langle (f'_i, g'_i) \mid i < \omega \rangle$) for $i > k$. We thereby obtain T^k_0, T^k_1 and $x \in [T^k_0] \rightarrow x$ goes left at $i+1$ for $k \leqslant i < \omega_1$, $x \in [T^k_i] \rightarrow x$ goes right at $i+1$ for $k \leqslant i < \omega_1$.

If β is not even, we put T^k_0, T^k_1 into R^A_α for all k. If $\beta = 2\gamma$, then we put T^k_0 into R^A_α iff $\gamma \notin A$, T^k_1 into R^A_α iff $\gamma \in A$. Then R^A_α is obtained from $R^A_{<\alpha}$ by including all T^k_i as above, $i = 0$ or 1, $k < \omega_1$ for all trees $T \in \tilde{R}^A_\beta$, and closing under the $(***)$ operation described just before the start of the construction.

*Case 3: α limit, $\alpha > \omega_1$ but α is **not** of the form $\omega_1 \cdot \lambda$, λ limit.* Write $\alpha = \beta + \delta$ where $0 < \delta \leqslant \omega_1$ and ω_1 divides β. Our goal is to extend each $T \in \tilde{R}^A_{<\alpha} = \bigcup \{\tilde{R}^A_{\alpha'} \mid \alpha' < \alpha\}$ to a condition $T' \in \tilde{R}^A_\alpha$, arranging that T' is canonically recoverable from each $x \in [T']$. We are only concerned with T such that $|T| \in [\beta, \alpha)$.

Pick $\delta_0 < \delta$ and $\delta_1 < \omega_1$. For any x define a sequence $\langle T_\gamma \mid \delta_0 \leq \gamma \leq \delta \rangle$ as follows. $T_{\delta_0} = T^x_{\beta+\delta_0}$. If T_γ is defined as an element of $\tilde{R}^A_{\beta+\gamma}$, then let $T_{\gamma+1}$ be the $L[T_{\delta_0}, A \cap \alpha]$-least $\hat{T} \leq_{\delta_1+\gamma} T_\gamma$ in $\tilde{R}^A_{\beta+\gamma+1}$ so that $y \in [\hat{T}] \rightarrow B^y$, B^x agree on the ordinal $\beta + \gamma$. If T_γ is defined for all $\gamma < \lambda$ where $\lambda \leq \delta$ is a limit ordinal, then $T_\lambda = \bigcap \{T_\gamma \mid \gamma < \lambda\}$ is the generalized tree characterized by $[T_\lambda] = \bigcap \{[T_\delta] \mid \gamma < \lambda\}$. (See Lemma 2.6.) If the above inductive definition breaks down somewhere, then we say that $T^x(\delta_0, \delta_1)$ is undefined. Otherwise $T^x(\delta_0, \delta_1)$ is defined to be T_δ.

We include in $\tilde{R}^A_\alpha$ all trees T such that for some $\delta_0 < \delta$, $\delta_1 < \omega_1 : T^x(\delta_0, \delta_1) = T$ for all $x \in [T]$. Then R^A_α is obtained by taking the $(***)$-closure of $\tilde{R}^A_\alpha \cup R^A_{<\alpha}$.

Case 4: $\alpha = \omega_1 \cdot \lambda$, λ limit and $L_\alpha[A] \nvDash \omega_1$ is the largest cardinal. In this case we are not concerned with fusions, only with guaranteeing extendibility using the form of $\square$ mentioned above.

Given $x \in [T]$, $T \in \tilde{R}^A_{<\alpha}$ we first define $x^* = \{\delta < \alpha \mid 4 \cdot \langle \delta, \delta' \rangle + 3 \in B^x$ (where $\delta' < \alpha$) for unboundedly many such ordinals $<\alpha\}$ where $\langle \cdot, \cdot \rangle$ is an α-recursive pairing on $\alpha \times \alpha$. Then $\alpha'_0 < \alpha'_1 < \cdots$ is defined if $0 \notin x^*$ and $\alpha < \omega_2^{L[x^*]}$. Then $C^{x^*}_\alpha$ is defined as a closed subset of α. If $C^{x^*}_\alpha$ is unbounded in α, then let $\alpha'_0 < \alpha'_1 < \cdots$ be the increasing enumeration of $C^{x^*}_\alpha$. Otherwise let $\alpha'_0 < \alpha'_1 < \cdots$ be the increasing enumeration of $C^{x^*}_\alpha$ followed by the $L[x^*]$-least ω-sequence $\beta_0 < \beta_1 < \cdots$ cofinal in α such that $\bigcup C^{x^*}_\alpha < \beta_0$ and each β_i is divisible by ω_1. Let γ_0 be the order-type of the sequence of ordinals α'_i.

Now pick $\hat{\beta} < \alpha$ and $\hat{\delta} < \omega_1$. Set $\hat{T} = T^x_{\hat{\beta}}$ and define $\alpha_0 < \alpha_1 < \cdots$ to be the final segment of $\alpha'_0 < \alpha'_1 < \cdots$ defined by $\alpha_0 =$ least α'_i greater than $\hat{\beta}$. Define a sequence $\langle T_\gamma \mid 0 \leq \gamma \leq \gamma_0 \rangle$ as follows, where $\gamma_0 =$ ordertype of the α_i's.

$T_0 = L[A \cap \alpha, x^*]$-least $T' \leq_{\hat{\delta}} \hat{T}$ in $\tilde{R}^A_{\alpha_0}$ so that $y \in [T'] \rightarrow B^y$ and B^x agree on $[\hat{\beta}, \alpha_0)$. If T_γ is defined as an element of $\tilde{R}^A_{\alpha_\gamma}$, then let $T_{\gamma+1}$ be the $L[A \cap \alpha, x^*]$-least $T' \leq_{\hat{\delta}+\gamma} T_\gamma$ in $\tilde{R}^A_{\alpha_{\gamma+1}}$ so that $y \in [T'] \rightarrow B^y$, B^x agree on $[\alpha_\gamma, \alpha_{\gamma+1})$. If T_γ is defined for all $\gamma < \lambda \leq \gamma_0$ where λ is a limit ordinal, then $T_\lambda = \bigcap \{T_\gamma \mid \gamma < \lambda\}$ is the generalized tree characterized by $[T] = \bigcap \{[T_\gamma] \mid \gamma < \lambda\}$. If the above inductive definition breaks down, then $T^x(\hat{\beta}, \hat{\delta})$ is undefined. Otherwise we set $T^x(\hat{\beta}, \hat{\gamma}) = T_{\gamma_0}$.

We include in $\tilde{R}^A_\alpha$ all trees T such that for some $\hat{\beta} < \alpha$, $\hat{\delta} < \omega_1$, $T^x(\hat{\beta}, \hat{\delta}) = T$ for all $x \in [T]$. Obtain R^A_α by closing $R^A_{<\alpha} \cup \tilde{R}^A_\alpha$ under $(***)$.

Case 5: $\alpha = \omega_1 \cdot \lambda$, λ limit and $L_\alpha[A] \vDash \omega_1$ is the largest cardinal. Two kinds of conditions are added to $\tilde{R}^A_\alpha$ in this case. First add conditions exactly as in Case 4. We now describe the conditions of the second kind, needed to anticipate fusions.

Given $x \subseteq \omega_1$ we first define what it means for $B^x \cap \alpha$ to code a *fusion index* $(\hat{j}, \hat{\delta})$. Let (η, k) be least so that α is $\Sigma_k(J_\eta[A \cap \alpha])$-projectible; we suppose that (η, k) exists and $k \geq 2$. Let $\mathscr{A} = \Sigma_{k-2}$-Master Code structure for $J_\eta[A \cap \alpha]$ and we suppose that Σ_1-projectum$(\mathscr{A}) = \alpha$, $\Sigma_2^{\mathscr{A}}$-cofinality$(\alpha) = \gamma_0 \leq \omega_1$. Actually we suppose that the increasing enumeration of C is a $\Sigma_2(\mathscr{A})$-continuous sequence $\alpha_0 < \alpha_1 < \cdots$ cofinal in α of ordertype $\leq \omega_1$ where C consists of all $\alpha' < \alpha$ such that $\alpha' \notin H_{\alpha'} = \Sigma_1$-Skolem hull of $\alpha' \cup \{\omega_1, p\}$ in $\mathscr{A}$, p = least p such that $\mathscr{A}$ is Σ_1-projectible to α with parameter p. Also let $f : \mathscr{A}' \to \omega_1$ be a canonical $\Sigma_1(\mathscr{A}')$-injection, where $\mathscr{A}' = \Sigma_{k-1}$-Master Code structure for $J_\eta[A \cap \alpha]$.

Now consider the sequence $\delta_0 < \delta_1 < \cdots$ defined by δ_i = least $\delta < \omega_i$ such that $4(\alpha_i + \delta) + 3 \in B^x$; we have a fusion index if δ_i eventually equals $\langle 0, \langle \hat{j}, \hat{\delta} \rangle \rangle$. (Note that this implies $0 \in x^*$ and therefore these conditions are distinguished from those of Case 4.) The idea is that $(\hat{j}, \hat{\delta})$ codes a fusion $T_0 \geqslant_{\hat{\delta},1} T_1 \geqslant_{\hat{\delta},2} T_2 \geqslant_{\hat{\delta},3} \cdots$ of length $\gamma_0 = \text{ordertype}\{\alpha_0 < \alpha_1 < \cdots\}$. For $l, l' < \omega_1$ we write $T \leqslant_{l,l'} T'$ if $T \leqslant_l T'$ and in addition $t \in \tilde{R}^{T'}_{l'} \to t \in \tilde{R}^{T}_{l'}$ or $t \in T'(t_i)$ for some i, $\omega \cdot i \geqslant l'$, where $\langle t_j \mid j < \omega_1 \rangle$ is a canonical enumeration of $\tilde{R}^*$. We assume that $t_i \sim t_j$, $i \leqslant j \to j - i$ is finite. (Note that there exists a generalized tree which is a greatest lower bound to this type of fusion sequence as $t \in T(t_i)$ for at most countably many i.) We add the results of such fusions to $\tilde{R}^A_\alpha$ but are careful to arrange that an index for each such resulting T can be canonically recovered from every $x \in [T]$. We can show with the aid of the recursion theorem that the addition of these fusions suffices to establish the Fusion Lemma for $\mathscr{P}^A$.

We can now describe the second type of condition that must be added to $\tilde{R}^A_\alpha$. For any $\hat{j} < \omega_1$ let $S(\hat{j})$ = the $\Sigma_1(\mathscr{A}')$-set with defining parameter $f^{-1}(\hat{j})$. Add T to $\tilde{R}^A_\alpha$ if T is the result of a fusion $T_0 \geqslant_{\hat{\delta},1} T_1 \geqslant_{\hat{\delta},2} T_2 \geqslant_{\hat{\delta},3} \cdots$ of length γ_0 where $|T_{i+1}| \geqslant \alpha_i$, $x \in [T] \to B^x \cap \alpha$ codes a fusion index $(\hat{j}, \hat{\delta})$ and $S(\hat{j}) = \langle T_i \mid i < \gamma_0 \rangle$. We do not require that the trees T_i are canonical. Finally obtain R^A_α by closing $R^A_{<\alpha} \cup \tilde{R}^A_\alpha$ under the operation $(***)$.

This completes the construction of $R^A = \bigcup \{R^A_\alpha \mid \alpha < \omega_2\}$. As in Section 1 we can show that if $x \in [T]$ for some $T \in \tilde{R}^A_\alpha$, then $T = T^x_\alpha$ is uniquely determined and can be defined uniformly from x. Also any $T \in R^A_\alpha$ can be recovered from elements of $R^A_{<\alpha} \cup \tilde{R}^A_\alpha$ via finitely many applications of the operation $(***)$. The main thing that we wish to show now is extendibility for R^A. For $T \in R^A$, $|T|$ denotes the unique α such that $T \in R^A_\alpha - R^A_{<\alpha}$.

Lemma 2.7 (Extendibility for $R^A, A \subseteq \omega_2$). *Suppose $T \in R^A$ and $|T| \leqslant \alpha < \omega_2$. Then for any $l \in \omega_1$ there exists $T' \leqslant_l T$, $|T'| = \alpha$.*

Proof. This is very much like the proof of Lemma 1.2, the major difference being the use of $\square$ to handle cases where α has uncountable cofinality. We can assume that T is canonical; for example if $T = T_0(a, t, T_1, T_2)$ where T_0, T_1, T_2 are canonical, then $T \geqslant_l T' \in R^A$ where $T' = T'_0(a, t, T'_1, T'_2)$ and $T'_i \geqslant_{l'} T_i$, $k' = \max(l, |t|)$.

We define the notion "$b \subseteq [|T|, \alpha)$ is T-special at α" as follows. For any ordinal $\beta \in (|T|, \alpha]$ divisible by ω_1 define $\bar{b}_\beta = \{\delta < \beta \mid 4 \cdot \langle \delta, \delta' \rangle + 3 \in b$ for unboundedly many such ordinals $< \beta\}$. We require that $0 \notin \bar{b}_\beta$ and $\beta < (\omega_2)^{L[\bar{b}_\beta]}$ for all such β, that $2\gamma \in b$ iff $\gamma \in A$ for $2\gamma \in [|T|, \alpha)$ and that whenever $\delta \leqslant \delta_0 \leqslant \delta_1$, $|T| \leqslant 4 \cdot \langle \delta, \delta_0 \rangle + 3 \leqslant 4 \cdot \langle \delta, \delta_1 \rangle + 3 < \alpha$ then $4 \cdot \langle \delta, \delta_0 \rangle + 3 \in b$ iff

$4 \cdot \langle \delta, \delta_1 \rangle + 3 \in b$. The net effect is that if $\beta_0, \beta_1 \in (|T|, \alpha)$ are divisible by ω_1, $\beta_0 < \beta_1$, then $\tilde{b}_{\beta_0} = \tilde{b}_{\beta_1} \cap \beta_0$. Clearly for any $T \in R^A$ and $\alpha > |T|$ there exists $b \subseteq [|T|, \alpha)$ which is T-special at α. (We assume for this purpose that $\gamma < \omega_1 \to \langle \delta, \gamma \rangle < \delta + \omega_1$.) In that case we show that there exists a canonical $T' \leqslant_l T$, $|T'| = \alpha$ such that $x \in [T'] \to B^x \cap [|T|, \alpha) = b$.

If $\alpha = \omega_1$, then there is nothing to show. ϕ is T-special at ω_1.

Suppose $\alpha = \beta + 1$. Then $b \subseteq [|T|, \alpha)$ is T-special at α iff b-$\{\beta\}$ is T-special at β and $\beta = 2\gamma \to (\beta \in b \leftrightarrow \gamma \in A)$. By induction we can extend T to a canonical $T^* \leqslant_l T$ such that $|T^*| = \beta$ and $x \in [T^*] \to B^x \cap [|T|, \beta) = b \cap \beta$. So we can assume that $|T| = \beta$ in which case, Case 2 of the construction of R^A makes it clear that the desired T' exists.

Suppose $\alpha = \omega_1 \cdot \gamma + \delta$ where $0 < \delta \leqslant \omega_1$ is a limit ordinal. To see that T can be l-extended to $T' \in \tilde{R}^A_\alpha$ so that $x \in [T'] \to B^x \cap [|T|, \alpha) = b$, first choose $\hat{T} \leqslant_l T$ in $\tilde{R}^A_{\omega_1 \cdot \gamma}$ such that $x \in [\hat{T}] \to B^x \cap [|T|, \omega_1 \cdot \gamma) = b \cap \omega_1 \cdot \gamma$ if $|T| < \omega_1 \cdot \gamma$; $\hat{T} = T$ otherwise. Then successively choose $\hat{T} = T_0 \geqslant_l T_1 \geqslant_{l+1} \cdots$ such that $T_{\bar{\gamma}+1}$ is the $L[T_0, A \cap \alpha]$-least $T^* \leqslant_{l+\bar{\gamma}} T_{\bar{\gamma}}$ in $\tilde{R}^A_{\omega_1\gamma+\bar{\gamma}+1}$ so that $x \in [T^*] \to B^x$ agrees with b at the ordinal $\omega_1\gamma + \bar{\gamma}$. Then $T_{\bar{\gamma}} = \text{glb}\{T_{\bar{\gamma}'} \mid \bar{\gamma}' < \bar{\gamma}\}$ belongs to $\tilde{R}^A_{\omega_1\gamma+\bar{\gamma}}$ for limit $\bar{\gamma}$ by the definition of R^A_α in Case 3.

Suppose $\alpha = \omega_1 \cdot \lambda$, λ limit. Define x^* as we did in Case 4 but with B^x replaced by b. Then note that $0 \notin x^*$, $\alpha < (\omega_2)^{L[x^*]}$. Define $\alpha_0 < \alpha_1 < \cdots$ as in Case 4, where $\hat{\beta} = |T|$. Let $\gamma_0 = \text{ordertype}\{\alpha_0 < \alpha_1 < \cdots\}$.

We note that for each $i < \gamma_0$, x^* has the same value if in its definition B^x, α are replaced by $b \cap \alpha_i$, α_i. Now define $\langle T_\gamma \mid 0 \leqslant \gamma \leqslant \gamma_0 \rangle$ exactly as in Case 4 but with $\hat{T}$, $\hat{\delta}$, B^x replaced by T, l, b. We have that $b \cap [\alpha_\gamma, \alpha_{\gamma+1})$ is T_γ-special at α_{l+1}. Now notice that for limit γ, T_γ does in fact belong to $\tilde{R}^A_{\alpha_\gamma}$ as $C^{x^*}_\alpha \cap \alpha_\gamma = C^{x^*}_{\alpha_\gamma}$, and x^* has the same definition 'at α_γ' as it does 'at α'. Thus we have proved that there exists $T' \leqslant_l T$ in $\tilde{R}^A_\alpha$ such that $x \in [T'] \to B^x \cap [|T|, \alpha) = b$, for such b. □

The forcing $\mathcal{P}^A$

Finally we can describe the desired forcing for minimally coding A by a real. A condition in $\mathcal{P}^A$ is a pair (t, T) where $t \in R^T$ and $T \in R^A$. We write $(t, T) \leqslant (\bar{t}, \bar{T})$ if $t \leqslant \bar{t}$ in $R^{\bar{T}}$ and $T \leqslant \bar{T}$ in R^A. We clearly have extendibility for $\mathcal{P}^A$ in the form $(t, T) \in \mathcal{P}^A$, $\alpha < \omega_1$, $\beta < \omega_2 \to \exists(\bar{t}, \bar{T}) \leqslant (t, T)$ such that $|\bar{t}| \geqslant \alpha$, $|T| \geqslant \beta$. Our main goal now is to establish enough fusion for $\mathcal{P}^A$ so that we may show that $\mathcal{P}^A$ preserves cardinals and produces a minimal real. It is clear from extendibility that a $\mathcal{P}^A$-generic real codes A.

We begin with fusion for R^A. Fix a canonical enumeration $\langle t_j \mid j < \omega_1 \rangle$ of R^*, as in Case 5 of the construction of R^A. Recall that $T' \leqslant_{l,l'} T$ means that $T' \leqslant_l T$ and $t \in \tilde{R}^T_{l'} \to t \in \tilde{R}^{T'}_{l'}$ unless $t \in T(t_i)$ for some i, $\omega \cdot i \geqslant l'$. A subset D of R^A is *open* if $T \in D$, $T' \leqslant T \to T' \in D$ and is *l, l'-dense below T*, $T \in R^A$ if $T' \leqslant_l T \to \exists T'' \leqslant_{l,l'} T'$ such that $T'' \in D$.

Lemma 2.8 (Fusion for R^A). *Suppose $T \in R^A$, $l < \omega_1$ and $D_{l,l'}$ is l, l'-dense below*

T and open for every $l' < \omega_1$. Then there exists $T' \leqslant T$ such that $T' \in D_{l,l'}$ for every $l' < \omega_1$.

Proof. It is here that we show that the fusions added in Case 5 of the construction suffice. Suppose the lemma fails and let $\beta < \omega_3$ be least so that there is a least counterexample $T, l < \omega_1$, $\langle D_{l,l'} \mid l' < \omega_1 \rangle$ definable over $J_\beta[A]$. Let $k \geqslant 2$ be chosen so that this counterexample is $\Sigma_{k-1}(J_\beta[A])$ with parameter ω_1 and let $\mathscr{B} = \Sigma_{k-2}$-Master Code structure for $J_\beta[A]$. Note that Σ_1-projectum(β) equals ω_2 and let p be the least parameter such that $\mathscr{B}$ is Σ_1-projectible to ω_2 with parameter p. Let $\alpha_0 < \alpha_1 < \cdots$ be the first ω_1 ordinals $\alpha' < \omega_2$ such that $\alpha' \notin H_{\alpha'} = \Sigma_1$-Skolem hull of $\alpha' \cup \omega_1 \cup \{p\}$ in $\mathscr{B}$ and let $\alpha = \bigcup \{\alpha_i \mid i < \omega_1\}$, $\mathscr{A} =$ transitive collapse of H_α. Then $\mathscr{A} = \Sigma_{k-2}$-Master Code structure for $J_\eta[A \cap \alpha]$ for some η.

Now notice that we are exactly in the situation of Case 5 of the construction of R^A. Let $\mathscr{A}' = \Sigma_{k-1}$-Master Code structure for $J_\eta[A \cap \alpha]$; note that $\mathscr{A}' \cap \mathrm{ORD} = \alpha$, Σ_1-projectum($\mathscr{A}'$) $= \omega_1$. Now pick any $\hat{j} < \omega_1$. Form the sequence $T_0 \geqslant_{l,1} T_1 \geqslant_{l,2} T_2 \geqslant_{l,3} \cdots$ as follows: Let $T_0 = T$. If T_i has been chosen then let T_{i+1} be $L[A]$-least so that $T_{i+1} \leqslant_{l,i} T_i$, $|T_{i+1}| \geqslant \alpha_i$, $T_{i+1} \in D_{l,i}$ and $\delta_i = \langle 0, \langle \hat{j}, l \rangle \rangle$ where δ_i is least so that $4(\alpha_i + \omega + \delta_i) + 3 \in B^x$, for all $x \in [T_{i+1}]$. For limit λ let $T_\lambda = \mathrm{glb}\{T_i \mid i < \lambda\}$. We assumed that $\langle D_{l,l'} \mid l' < \omega_1 \rangle$ is Σ_{k-1}-definable over $L_\beta[A]$ with parameter ω_1 and therefore $|T_{i+1}| < \alpha_{i+1}$.

We claim that $\hat{j}$ can be chosen so that the above sequence $\langle T_i \mid i < \omega_1 \rangle$ is well-defined. Indeed let $h(\hat{j})$ be a $\Sigma_1(\mathscr{A}')$-index for $\langle T_i \mid i < \omega_1 \rangle$. By the recursion theorem we may choose $\hat{j}$ so that $\hat{j}$, $h(\hat{j})$ define the same sequence $\langle T_i \mid i < \omega_1 \rangle$. Now clearly if T_i is defined then so is T_{i+1} as $D_{l,i}$ is l,i-dense and we have the freedom to $l+1+i$-extend T_i to T'_{i+1} such that $x \in [T'_{i+1}] \to \langle 0, \langle j, l \rangle \rangle$ is the least δ so that $4(\alpha_i + \omega + \delta) + 3 \in B^x$ before l,i-extending T'_{i+1} to the desired T_{i+1}. But notice that $T_\lambda \in R^A_{\alpha_\lambda}$ for limit λ, precisely because of the definition of $R^A_{\alpha_\lambda}$ in Case 5 of the construction of R^A and because $\hat{j}$, $h(\hat{j})$ define $\langle T_i \mid i < \lambda \rangle$ 'at α_λ' just as they define $\langle T_i \mid i < \omega_1 \rangle$ 'at α'. Finally, let $T' = \mathrm{glb}\langle T_i \mid i < \omega_1 \rangle$ and we obtain a contradiction to the fact that T, l, $\langle D_{l,l'} \mid l' < \omega_1 \rangle$ is a counterexample. This proves the lemma. □

Corollary 2.9 (Horizontal Fusion for R^A). *Suppose $T \in R^A$, $\alpha < \omega_1$ and for each $t \in \tilde{R}^T_\alpha$, D_t is open and t-dense below T; i.e., $T' \leqslant_\alpha T \to \exists T'' \leqslant_\alpha T'$ ($T'' \in D_t$ and $t' \in \tilde{R}^{T'} - \tilde{R}^{T''} \to t' \in T'(t)$). Then there exists $T' \leqslant_\alpha T$ such that $T' \in D_t$ for all $t \in \tilde{R}^T_\alpha$.*

Proof. Let $D_{\alpha,l'} = \{T' \leqslant_\alpha T \mid T' \in D_{t_{l'}}\}$ and apply Lemma 2.8. □

Corollary 2.10 (Vertical Fusion for R^A). *Suppose $T \in R^A$ and for each $\alpha < \omega_1$, D_α is open and α-dense below T; i.e., $T' \leqslant T \to \exists T'' \leqslant_\alpha T'$ such that $T'' \in D_\alpha$. Then there exists $T' \leqslant T$ such that $T' \in D_\alpha$ for all α.*

Proof. Let $D_{0,l'} = D_{l'}$ and apply Lemma 2.8. $\square$

Corollary 2.11 (Countable Distributivity for R^A). *Suppose $T \in R^A$, $\alpha < \omega_1$ and D_i is α-dense below T and open for each $i \in \omega$. Then there exists $T' \leqslant_\alpha T$ such that $T' \in D_i$ for each i.*

Proof. Let $D_{\alpha,l'} = D_{l'}$ for $l' < \omega$, $= R^A$ for $l' \geqslant \omega$. Then apply Lemma 2.8. $\square$

We now apply these results to the study of *density reduction* for $\mathcal{P}^A$. A set $D \subseteq \mathcal{P}^A$ is *local* if $(t', T') \in D$, $(t', T'') \in \mathcal{P}^A$, $T'(t') = T''(t') \rightarrow (t', T'') \in D$. If $T \in R^A$ and $D \subseteq \mathcal{P}^A$, then $D^T = \{t \in R^T \mid (t, T) \in D\}$. If $D \subseteq \mathcal{P}^A$ is dense below (t, T), then $T' \leqslant T$ *reduces* D if $t \in R^{T'}$ and $D^{T'}$ is dense below t on $R^{T'}$.

Lemma 2.12 (Countable Density Reduction). *Suppose $(t, T) \in \mathcal{P}^A$, $\alpha < \omega_1$ and D_i is local and dense below (t, T) for each $i < \omega$. Then there exists $T' \leqslant_\alpha T$ in R^A such that $(t, T') \in \mathcal{P}^A$ and T' reduces D_i for each $i < \omega$.*

Proof. It suffices to consider just one $D_i = D$, by Corollary 2.11. Fix $\beta < \omega_1$, β greater than α and $|t|$. For any t' in $\tilde{R}^T_\beta \cap T(t)$ the set $D_{t'} = \{T' \leqslant_\beta T \mid$ for some $t'' \leqslant t'$, (t'', T') extends an element of $D\}$ is t'-dense below T as D is local and dense below (t, T) (we also use closure under $(***)$). So by Horizontal Fusion for R^A there exists $T' \leqslant_\beta T$ such that $D^{T'}$ is dense below t on $R^{T'}_{\leqslant\beta}$. Now apply Vertical Fusion to the (β-dense below T) set of such T' for $\beta < \omega_1$, $\alpha \cup |t| < \beta$ to obtain the desired $T' \leqslant_\alpha T$. $\square$

Corollary 2.13 (Density Reduction). *Suppose $(t, T) \in \mathcal{P}^A$, $\alpha < \omega_1$, and D_i is local, dense below (t, T) for each $i < \omega_1$. Then there exists $T' \leqslant_\alpha T$ in R^A such that $(t, T') \in \mathcal{P}^A$ and T' reduces D_i for each $i < \omega_1$.*

Proof. For each $\beta > \alpha$ it is β-dense for T' to reduce D_i for $i < \beta$. Now apply Vertical Fusion for R^A to obtain $T' \leqslant_\alpha T$ reducing all of the D_i. $\square$

We now consider fusion for $\mathcal{P}^A$. A subset D of $\mathcal{P}^A$ is *n-dense below* (t, T) if whenever $(t', T') \leqslant (t, T)$ there exists $(t'', T'') \leqslant (t', T')$ such that $(t'', T'') \in D$ and $t'' \leqslant_n t'$.

Lemma 2.14 (Fusion for $\mathcal{P}^A$). *Suppose $(t, T) \in \mathcal{P}^A$ and D_i is open and i-dense below (t, T) for each $i \in \omega$. Then there exists $(t', T') \leqslant (t, T)$ such that $(t', T') \in D_i$ for each i.*

Proof. Suppose the lemma fails and as in the proof of Lemma 2.8 let $\beta < \omega_3$ be least so that there is a counterexample (t, T), $\langle D_i \mid i < \omega \rangle$ definable over $J_\beta[A]$. We choose the least such counterexample and let $k \geqslant 2$ be chosen so that this counterexample is $\Sigma_{k-1}(J_\beta[A])$ with parameter ω_1. Set $\mathscr{B} = \Sigma_{k-2}$-Master Code structure for $J_\beta[A]$. Then Σ_1-projectum$(\mathscr{B}) = \omega_2$ and we let p be the least parameter such that $\mathscr{B}$ is Σ_1-projectible to ω_2 with parameter p. Let $\hat{\beta}_0 < \hat{\beta}_1 < \cdots$ be the first ω ordinals $\beta < \omega_2$ such that $\beta \notin H'_\beta = \Sigma_1$-Skolem hull of $\omega_1 \cup \beta \cup \{p\}$ in $\mathscr{B}$ and define $\hat{\beta} = \bigcup \{\hat{\beta}_i \mid i < \omega\}$. Let $H^q_\alpha = \Sigma_1$-Skolem hull of $\alpha \cup \{q, p\}$ in $\mathscr{B}$ and define $\alpha_0 < \alpha_1 < \cdots$ by $\alpha_0 = H^0_0 \cap \omega_1$, $\alpha_{n+1} = H^{\hat{\beta}_n}_{\alpha_n+1} \cap \omega_1$. We set $\alpha = \bigcup \{\alpha_i \mid i < \omega\}$ and let $\mathscr{A}$ = transitive collapse of $\bigcup \{H^{\hat{\beta}_n}_{\alpha_n+1} \mid n \in \omega\}$. Then $\mathscr{A}$ is the Σ_{k-2}-Master Code structure for $J_\eta[\bar{A}]$ for some η and $\bar{A}$ such that $\alpha = (\omega_1)^{J_\eta[\bar{A}]}$, $\bar{A} \cap \alpha = A \cap \alpha$.

We are in the situation of Case 4 of the construction of R^*. Let $\mathscr{A}' = \Sigma_{k-1}$-Master Code structure for $J_\eta[\bar{A}]$ and pick any pair $(\hat{\imath}, \hat{\jmath}) \in \omega \times \omega_1$. Form the sequence $(t_0, T_0) \geqslant (t_1, T_1) \geqslant \cdots$ as follows. Let $(t_0, T_0) = (t, T)$. If (t_i, T_i) has been chosen, then let (t_{i+1}, T_{i+1}) be $L[A]$-least so that $t_{i+1} \leqslant_i t_i$, $(t_{i+1}, T_{i+1}) \in D_i$, $|T_{i+1}| \geqslant \hat{\beta}_i$, $\alpha(t_{i+1}) \geqslant \alpha_i$, $\delta_i = \langle 0, \langle \hat{\jmath}, 0\rangle\rangle$ where δ_i is least so that $4(\hat{\beta}_i + \delta_i) + 3 \in B^x$ (for all $x \in [T_{i+1}]$) and, if k, j are such that $i = 2^k 3^j$, so that $\alpha_{2^k 3^j} + 1 \in B^R$ (for all $R \in [t_{i+1}]$) iff $j = \hat{\imath}$.

In addition we require that T_{i+1} reduces D' whenever $D' \in H^{\hat{\beta}_i}_{\alpha_i+1}$ is local and dense below (t_i, T_i) on $\mathscr{P}^A$ and that $s \in t_{i+1}$, $\|s\| = i \rightarrow (t_{i+1})_s$ belongs to the first i open dense sets d' on R^{T_j} belonging to $H^{\hat{\beta}_j}_{\alpha_j+1}$, in a canonical ω-listing of $H^{\hat{\beta}_j}_{\alpha_j+1}$ (for each $j \leqslant i$). (It is clear that $\langle (t_i, T_i) \mid i < \omega\rangle$ is well-defined using the i-density of D_i and the definitions of α_i, $\hat{\beta}_i$.)

We claim that $(\hat{\imath}, \hat{\jmath})$ can be chosen so that $(t', T') = (\bigcap \{t_i \mid i < \omega\}, \text{glb}\{T_i \mid i < \omega\})$ belongs to $\mathscr{P}^A$.

Notice that the sequence $\langle (t_i, T_i) \mid i < \omega\rangle$ is $\Sigma_1(\mathscr{B}')$ where $\mathscr{B}' = \Sigma_{k-1}$-Master Code structure for $J_\beta[A]$. Therefore by the recursion theorem we can assume that $\hat{\jmath}$ is chosen so that $f^{-1}(\hat{\jmath})$ is an index $<\hat{\beta}$ for $\langle T_i \mid i < \omega\rangle$ as a $\Sigma_1(\mathscr{B}')$-sequence where $f: \mathscr{B}' \rightarrow \omega_1$ is a canonical $\Sigma_1(\mathscr{B}')$-injection, and $\hat{h}^{-1}(\hat{\imath})$ is an index $<\hat{\beta}$ for $\langle t_i \mid i < \omega\rangle$ as a $\Sigma_1(\mathscr{B}')$-sequence where $\hat{h}: \bigcup \{H^{\hat{\beta}_n}_{\alpha_n+1} \mid n \in \omega\} \cap \omega_2 \rightarrow \omega$ is a canonical $\Sigma_1(\mathscr{B}')$-injection. But then $h^{-1}(\hat{\imath})$ is an index $<(\omega_2)^{J_\eta[\bar{A}]}$ for $\langle t_i \mid i < \omega\rangle$ as a $\Sigma_1(\mathscr{A}')$-sequence where $h: (\omega_2)^{J_\eta[\bar{A}]} \rightarrow \omega$ is a canonical $\Sigma_1(\mathscr{A}')$-injection. Thus we see that the conditions for $T \in R^{\bar{A}}_\beta$, $t \in R^T_\alpha$ are met with the possible exception of the requirement that $R \in [t] \rightarrow R$ decodes $\bar{A} \subseteq (\omega_2)^{J_\eta[R]}$ and R is $\mathscr{P}^{\bar{A}}$-generic over $J_\eta[\bar{A}]$, $(\omega_2)^{J_\eta[R]} = (\omega_2)^{J_\eta[\bar{A}]}$. (The requirement $T_0 \geqslant_{0,1} T_1 \geqslant_{0,2} T_2 \geqslant \cdots$ is equivalent to $T_0 \geqslant T_1 \geqslant T_2 \geqslant \cdots$.) Inside $J_\eta[R]$ decode from R as A is decoded from a $\mathscr{P}^A$-generic real. It is easily seen that $\bar{A}$ is the resulting subset of $(\omega_2)^{J_\eta[\bar{A}]}$. If $\bar{D} \in L_\eta[\bar{A}]$ is predense (i.e., $\bar{D}^* = \{p \in \mathscr{P}^{\bar{A}} \mid p \leqslant \text{some } q \in \bar{D}\}$ is dense), then for some i, T_{i+1} reduces D^* where $D = \pi^{-1}(\bar{D})$ and $\pi: \bigcup \{H^{\hat{\beta}_n}_{\alpha_{n+1}} \mid n \in \omega\} \simeq J_\eta[\bar{A}]$. But then by construction for some j, $(t_{j+1})_s$ meets $(D^*)^{T_{i+1}}$ for each $s \in t_{j+1}$, $\|s\| = j$ and so $((t_{j+1})_s, T_{i+1}) \geqslant ((t)_s, T)$ meets $\bar{D}^*$ where s is an initial segment of c_R. This proves the genericity of R. The fact that $(\omega_2)^{J_\eta[\bar{A}]} = (\omega_2)^{J_\eta[R]}$ follows from this and the fact that, by the leastness of our counterexample, the fusion lemmas all hold for $\mathscr{P}^{\bar{A}}$. Cardinal preservation is a consequence of fusion, as lemmas below demonstrate. This proves that $(t, T) \in \mathscr{P}^A$, contradicting our choice of counterexample. $\square$

We are now ready to establish cardinal preservation and minimality for $\mathcal{P}^A$.

Corollary 2.15. *$\mathcal{P}^A$ preserves cardinals and cofinalities.*

Proof. Suppose $(t, T) \Vdash f: \omega_1 \to \mathrm{ORD}$. For each $i < \omega_1$ let $D_i = \{(t', T') \leq (t, T) \mid \text{for some } \alpha, (t', T') \Vdash f(i) = \alpha\}$. then D_i is open dense below (t, T) and local.

So by Density Reduction there exists $(t, T') \leq (t, T)$ such that T' reduces each D_i. But then $(t, T') \Vdash \mathrm{Range}(f) \subseteq \bigcup \{E_i \mid i < \omega_1\}$, where $E_i = \{\alpha \mid \text{for some } t', (t', T') \Vdash f(i) = \alpha\}$. So $(t, T') \Vdash \mathrm{Range}(f)$ is contained in some $x \in V$, $\mathrm{card}^V(x) = \omega_1$.

Now suppose $(t, T) \Vdash f: \omega \to \mathrm{ORD}$. Then for each $i \in \omega$, $D_i = \{(t', T') \leq (t, T) \mid \text{for some finite } y, (t', T') \Vdash f(i) \in y\}$ is open and i-dense below (t, T); this uses closure under $(*)$, $(**)$. So by Fusion there exists $(t', T') \leq (t, T)$ such that $(t', T') \in D_i$ for all i. So $(t', T') \Vdash \mathrm{Range}(f) \subseteq x$, for some $x \in V$, $\mathrm{card}^V(x) = \omega$. □

Corollary 2.16 (Minimality). *If R is $\mathcal{P}^A$-generic, then R is minimal over V.*

Proof. Suppose $(t, T) \Vdash \boldsymbol{x} \subseteq \mathrm{ORD}$, $\boldsymbol{x} \notin V$, and let R_G denote the generic real.

Claim. *For all $(t', T') \leq (t, T)$ there exist (t_0', T''), $(t_1', T'') \leq (t', T')$ so that for some α, $(t_0', T'') \Vdash \alpha \in \boldsymbol{x}$, $(t_1', T'') \Vdash \alpha \notin \boldsymbol{x}$.*

Given the Claim, let $D_i = \{(t', T') \leq (t, T) \mid s_0 \neq s_1 \text{ in } t', \|s_0\| = \|s_1\| = i \to \text{for some } \alpha, ((t')_{s_0}, T') \Vdash \alpha \,\mathrm{e}\, \boldsymbol{x} \text{ iff } ((t')_{s_1}, T') \Vdash \alpha \,\bar{\mathrm{e}}\, \boldsymbol{x}\}$. D_i is i-dense below (t, T) for each i, where e varies over $\{\in, \notin\}$, $\{\bar{\mathrm{e}}\} = \{\in, \notin\} - \{\mathrm{e}\}$. So by Fusion for $\mathcal{P}^A$ there exists $(t', T') \leq (t, T)$ in $\bigcap \{D_i \mid i \in \omega\}$. But then clearly $(t', T') \Vdash R_G \in V[\boldsymbol{x}]$.

Proof of Claim. Choose (t_0', T_0''), $(t_1', T_1'') \leq (t', T')$ so that for some α, $(t_0', T_0'') \Vdash \alpha \in \boldsymbol{x}$, $(t_1', T_1'') \Vdash \alpha \notin \boldsymbol{x}$. We assume that $t_0' = (t_0)_{s_0}$, $t_1' = (t_1)_{s_1}$ where $t_0 \neq t_1$ are elements of some $\tilde{R}_i^{T'}$ and that s_0, s_1 are incompatible. Now let $T'' = T'(s_0, 2^{<\omega}, T_0'', T_1'')$. Clearly $(t_0', T'') \Vdash \alpha \in \boldsymbol{x}$, $(t_1', T'') \Vdash \alpha \notin \boldsymbol{x}$ so we are done. □

3. Minimally coding a subset of $\omega_{\omega+1}$

It is fairly straightforward to generalize the technique of Section 2 to obtain a minimal coding for a given subset of ω_{n+1}, n finite. The notion of generalized tree becomes generalized ω_n-tree, consisting of an ω_n-sequence $T = \langle (f_i, g_i) \mid i < \omega_n \rangle$ where $f_i, g_i: \tilde{R}_i^T \to$ 'acceptable n-terms' and $\tilde{R}_i^T$ is a collection of ω_{n-1} trees. We use $\square_{\omega_n}$ to build $R^A \subseteq \{\text{generalized } \omega_n\text{-trees}\}$, where we use collapses of elementary submodels of $L_\beta[A]$, $\beta < \omega_{n+2}$ to anticipate fusions of length $\leq \omega_n$. The major difference is that for $i < n$ we must also consider such models to anticipate fusions of ω_i-trees, these fusions of length $\leq \omega_i$. So even the definition of $R^* \subseteq \{\omega\text{-trees}\}$ must be modified when carrying out this generalization.

Rather than discuss the minimal coding of subsets of ω_{n+1} for finite n, we proceed directly to a discussion of the case of subsets of $\omega_{\omega+1}$. Of course the above ideas need to be considered in this case as well; in addition we must now discuss the way in which we code at the limit cardinal ω_ω. This coding is basically like Jensen's, where a 'scale' of functions through ω_ω is used to effect an almost disjoint coding of A into ω_ω. However as in [3] we greatly modify Jensen's proof of extendibility to cope with the fact that our forcings are so 'thin' (this alternate approach also eliminates a split into cases according to whether or not $0^\#$ belongs to V). As before we have two types of extensions at limit levels: extensions of type 1 needed for extendibility, and of type 2 to anticipate fusions. In this case the fusions have any possible length ω_n, $0 \leq n < \omega$. Conditions are of the form $\langle T_n \mid 0 \leq n < \omega\rangle$ where T_n is a generalized ω_n-tree. We require that the first n components of the conditions in an ω_n-long fusion sequence remain constant. Guaranteeing that we have a condition at limit stages is like the main distributivity argument for Jensen Coding; we must take advantage of built-in 'predensity reduction' as in [3] to get genericity over the collapse. The remaining details of the construction are natural generalizations of the corresponding ones in Section 2.

We begin with the definition of R^0, the appropriate generalization of the R^* of Section 2 to the present context. Then R^1 can be defined from R^0 much as was R^A from R^* in Section 2. Continuing in this way we will have $R^2, R^3, \ldots$ and we then can finally discuss $\mathcal{P}^A$, whose conditions are sequences of elements of the different R^n.

We assume of course that $V = L[A]$, $A \subseteq \omega_{\omega+1}$ and in addition that $2^{\omega_n} \subseteq L_{\omega_{n+1}}[A]$ for $0 \leq n \leq \omega$. As in Section 2 we will use the Recursion Theorem (in two different ways). Fix i_0, an index both for the desired forcing $\mathcal{P}^A$ and for a description of how B^R (where $A = \text{even}(B^R)$, $B^R \subseteq \omega_{\omega+1}$) is coded by a $\mathcal{P}^A$-generic real R, as $\Sigma_1\langle L_{\omega_{\omega+1}}[A], A\rangle$-sets with parameter $\langle \omega_n \mid n \in \omega\rangle \in L_{\omega_{\omega+1}}[A]$. The forcing R^0 is the union $\bigcup\{R^0_\alpha \mid \alpha < \omega_1\}$; R^0_α is obtained from $\bar{R}^0_\alpha =$ the canonical elements of $R^0_\alpha - R^0_{<\alpha}$ by closing $R^0_{<\alpha} \cup \bar{R}^0_\alpha$ under: $(*)$ $t \in R^0_\alpha$, $a \in t \rightarrow (t)_a \in R^0_\alpha$ and $(**)$ $t_0, \ldots, t_n \in R^0_\alpha = \bigcup\{R^0_\beta \mid \beta \leq \alpha\} \rightarrow t_0 \cup \cdots \cup t_n \in R^0_\alpha$. We define $\bar{R}^0_\alpha$ by induction on α.

Case 1: $\alpha = 0$. R^0_0 consists of all trees of the form $(2^{<\omega})_{a_1} \cup \cdots \cup (2^{<\omega})_{a_n}$ where $a_1, \ldots, a_n$ are finite strings. For any real R, $t^R_0 = 2^{<\omega}$ so $\bar{R}^0_0 = \{2^{<\omega}\}$.

Case 2: $\alpha = \beta + 1$. Define R^0_α from R^0_β exactly as we defined R^*_α from R^*_β in Section 2.

Case 3: α limit, but α is not a limit of limit ordinals. Write $\alpha = \beta + \omega$ where β is 0 or a limit ordinal. Given a real R, B^R is defined by: $\gamma \in B^R$ iff R goes right at all sufficiently large even levels of the tree t^R_γ. Choose ordinals $\hat{\alpha} \leqslant \hat{\beta} < \alpha$ ($\hat{\alpha}$ limit or 0), an acceptable term $\hat{\sigma}$ and an integer $\hat{k}$. Also assume that $\beta \leqslant \hat{\beta}$.

Let $t_0 \leqslant_{\hat{k}} t^R_{\hat{\beta}}$ be canonical and least in $L[A \cap \alpha, t^R_{\hat{\beta}}]$ such that $S \in [t_0] \to B^S$, B^R agree on $[\hat{\beta}, \alpha(t_0)) - \{4\gamma + 1 \mid \gamma \in \mathrm{ORD}\}$ and $B^S(\hat{\alpha} + 4\gamma + 1) = \hat{\sigma}(S)(\gamma)$ for $4\gamma + 1 < \min(|\hat{\sigma}|, \alpha(t_0) - \hat{\alpha})$, where $\alpha(t_0) = \max(\hat{\beta}, \beta)$. If t_n is defined, then let $t_{n+1} \leqslant_{\hat{k}+n+1} t_n$ be canonical and least in $L[A \cap \alpha, R^*]$ such that $S \in [t_{n+1}] \to B^S$, B^R agree at $\alpha(t_0) + n$ (if $\alpha(t_0) + n$ is not of the form $\hat{\alpha} + 4\gamma + 1$) or $B^S(\hat{\alpha} + 4\gamma + 1) = \hat{\sigma}(S)(\gamma)$ (if $\alpha(t_0) + n = \hat{\alpha} + 4\gamma + 1$, $\gamma < |\hat{\sigma}|$). Define $t^R(\hat{\beta}, k, \hat{\sigma}, \hat{\alpha}) = \bigcap \{t_n \mid n \in \omega\}$ if all the t_n are defined.

We include in $\tilde{R}^0_\alpha$ all trees t such that for some $(\hat{\beta}, \hat{k}, \hat{\sigma}, \hat{\alpha})$ as above, $t^R(\hat{\beta}, \hat{k}, \hat{\sigma}, \hat{\alpha}) = t$ for all $R \in [t]$. Then R^0_α is the $(*)$,$(**)$-closure of $R^0_{<\alpha} \cup \tilde{R}^0_\alpha$.

Case 4: α is a limit of limit ordinals and $L_\alpha[A]$ is not locally countable. In this case we are concerned with extendibility, not fusion. Given a real R let $R^* = \{n \mid \lambda + 4n + 3 \in B^R$ for unboundedly many such ordinals $<\alpha$, λ limit$\}$. Then $\alpha'_0 < \alpha'_1 < \cdots$ is defined if $0 \notin R^*$ and R^* codes an ordinal $\geqslant \alpha$. Set $\alpha'_0 < \alpha'_1 < \cdots$ equal to the $L[R^*]$-least ω-sequence cofinal in α so that each α'_i is a limit ordinal. Pick $\hat{\sigma}$, $\hat{t}$, $\hat{\alpha}$, $\hat{\beta}$, $\hat{n}$ consisting of $\hat{\alpha} \leqslant \hat{\beta} < \alpha$ ($\hat{\alpha}$ limit or 0), an acceptable term $\hat{\sigma}$ and the integer $\hat{n}$. Also set $\hat{t} = t^R_{\hat{\beta}}$ and let $\alpha_0 < \alpha_1 < \cdots$ be the final segment of $\alpha'_0 < \alpha'_1 < \cdots$ defined by $\alpha_0 =$ least α'_i greater than $\hat{\beta}$.

Now define $\langle t_n \mid 0 \leqslant n < \omega \rangle$ as follows: $t_0 = L[A \cap \alpha, R^*]$-least $t' \leqslant_{\hat{n}} \hat{t}$ in $\tilde{R}^0_{\alpha_0}$ such that $S \in [t'] \to B^S$, B^R agree on $[\hat{\beta}, \alpha_0) - \{4\gamma + 1 \mid \gamma \in \mathrm{ORD}\}$ and $B^S(\hat{\alpha} + 4\gamma + 1) = \hat{\sigma}(S)(\gamma)$ for $4\gamma + 1 < \min(|\hat{\sigma}|, \alpha_0 - \hat{\alpha})$. If t_n is defined, then let t_{n+1} be the least $t' \leqslant_{\hat{n}+n} t_n$ in $\tilde{R}^0_{\alpha_{n+1}}$ so that $S \in [t'] \to B^S$, B^R agree on $[\alpha_n, \alpha_{n+1}) - \{4\gamma + 1 \mid \gamma \in \mathrm{ORD}\}$ and $B^S(\hat{\alpha} + 4\gamma + 1) = \hat{\sigma}(S)(\gamma)$ for $4\gamma + 1 < \min(|\hat{\sigma}|, \alpha_{n+1} - \hat{\alpha})$. Set $t^R(\hat{\sigma}, \hat{\alpha}, \hat{\beta}, \hat{n}) = \bigcap \{t_n \mid 0 \leqslant n \leqslant \omega\}$.

Include in $\tilde{R}^0_\alpha$ all trees t such that for some $(\hat{\sigma}, \hat{\alpha}, \hat{\beta}, \hat{n})$ as above $t^R(\hat{\alpha}, \hat{\alpha}, \hat{\beta}, \hat{n}) = t$ for all $R \in [t]$. Obtain R^0_α by closing $R^0_{<\alpha} \cup \tilde{R}^0_\alpha$ under $(*)$, $(**)$.

Case 5: α is a limit of limit ordinals, $L_\alpha[A]$ is locally countable. First add conditions to $\tilde{R}^0_\alpha$ as in Case 4. We now describe the other conditions to be added, which are needed to anticipate fusions.

Given a real R we define what it means for R to code a *type* A *fusion*. For this to be defined the following conditions must be met. Let $\eta \geqslant \alpha$ be least so that α is not regular in $J_{\eta+1}[R]$; we assume that η exists and that R codes a predicate $\bar{B}^R \subseteq (\omega_{\omega+1})^{J_\eta[R]}$ via the index i_0 for decoding $B^R \subseteq \omega_{\omega+1}$ from $\mathcal{P}^A$-generic reals R, using the parameter $\langle \bar{\omega}_n \mid n \in \omega \rangle$, $\bar{\omega}_n = (\omega_n)^{J_\eta[R]}$. Let $\bar{\beta} = (\omega_{\omega+1})^{J_\eta[R]} \leqslant \eta$ and $\bar{A} = \mathrm{even}(\bar{B}^R)$. We assume that α is projectible in $J_{\eta+1}[\bar{B}^R - (\omega_\omega)^{J_\eta[R]}]$ and therefore let k^R denote the least k so that α is projectible in

$$\Sigma_k(\langle J_{\bar{\beta}}[\bar{B}^R - (\omega_\omega)^{J_\eta[R]}], C^{\bar{B}^R - (\omega_\omega)^{J_\eta[R]}} \rangle).$$

We require that $J_{\eta+1}[R] \models$ "$\bar{B}^R - (\omega_\omega)^{J_\eta[R]}$, $\langle \eta, k^R \rangle$ gives rise to a canonical ω-sequence of quasiconditions $p_0 \geqslant p_1 \geqslant \cdots$ in $\mathcal{P}^{\bar{A}}$ and R satisfies $\bar{p}_0$." The preceding will be defined later when we specify which sequences $p = \langle p(0), p(\omega), p(\omega_1), \ldots \rangle$ of generalized trees are to be put into $\mathcal{P}^A$.

For R to code a type A fusion we require that η, $\bar{B}^R$, k^R as above are defined and that $0 \in R^*$ (as defined in Case 4). In this case set $t^R = \bigcap \{\bar{p}_n(0) \mid n < \omega\}$.

We also consider *type* B *fusions*. For R to code a type B fusion the following conditions must be met. Let η be least so that α is not regular in $J_{\eta+1}[R]$. Now decode from R the canonical sequence of ω-trees $\langle t^R_\beta \mid \beta < \alpha \rangle = x^R$. We require that $J_\eta[R] \models$ for some $\beta_0 < \omega_2^{J_\eta[R]}$, x^R is a path through the generalized tree $T^R = T^{x^R}_{\beta_0}$ and R is R^{T^R}-generic over $L[A \cap \alpha, T^R]$. (β_0 is uniquely determined.) We assume that α is singular in $J_{\eta+1}[A \cap \alpha, T^R]$ and that $k \geq 2$ where k is least so that α is $\Sigma_k(J_\eta[A \cap \alpha, T^R])$-projectible. Let $\mathcal{A} = \Sigma_{k-2}$-Master Code structure for $J_\eta[A \cap \alpha, T^R]$ and we suppose that C has ordertype ω, where C consists of all $\alpha' < \alpha$ such that $\alpha' \notin H_{\alpha'} = \Sigma_1$-Skolem hull of $\alpha' \cup \{p\}$ in $\mathcal{A}$, $p =$ least p such that $\mathcal{A}$ is Σ_1-projectible to $\rho_1^{\mathcal{A}} = \alpha$ with parameter p. Let $\mathcal{A}' = \Sigma_{k-1}$-Master Code structure for $J_\eta[A \cap \alpha, T^R]$ and $h : \alpha \to \omega$ a canonical $\Sigma_1(\mathcal{A}')$-injection.

For R to code a type B fusion we consider the sequence i_k defined by $i_k =$ least $i < \omega$ such that $\alpha_{2^k 3^i} + 1 \in B^R$ where $C = \{\alpha_0 < \alpha_1 < \alpha_2 < \cdots\}$. We insist that i_k is defined and equal to a fixed $\hat{i}$ for k sufficiently large. The key requirement then is that the $\Sigma_1(\mathcal{A}')$-set defined by the $\mathbf{\Sigma}_1$-formula with index $h^{-1}(\hat{i})$ is a fusion sequence $t_0 \geq_1 t_1 \geq_2 t_2 \geq_3 t_3 \cdots$ of conditions in $R^0_{<\alpha}$. If in addition $0 \in R^*$ (as defined in Case 4), then we set $t^R = \bigcap \{t_i \mid i < \omega\}$.

Add t to $\bar{R}^0_\alpha$ if $t = t^R$ for every $R \in [t]$ where t^R is defined as above. Obtain R^0_α by closing $R^0_{<\alpha} \cup \bar{R}^0_\alpha$ under $(*)$, $(**)$.

This completes the construction of $R^0 = \bigcup \{R^0_\alpha \mid \alpha < \omega_1\}$. R^0 has properties analogous to those held by R^* in Section 2. In particular define γ^R_α inductively by: $\gamma^R_0 =$ least p.r. closed ordinal greater than ω, $\gamma^R_\alpha =$ least $\gamma > \sup\{\gamma^R_\beta \mid \beta < \alpha\}$ such that γ is p.r. closed and $L_\gamma[R] \models \text{card}(\alpha) \leq \omega$ (for $\alpha > 0$). Then t^R_α is uniformly definable as an element of $L_{\gamma^R_\alpha}[R]$ (whenever t^R_α is defined).

Type 1 extension is defined just as in Section 2; these are the extensions which arise from any of the cases in the construction of R^0 with the exception of the latter parts of Case 5 (the fusion cases). We then have the following.

Lemma 3.1. *Suppose $t \in \bar{R}^0$, $k \in \omega$, σ is acceptable and $\hat{\alpha} \leq \alpha(t)$ is 0 or a limit ordinal. Also suppose that $R \in [t] \to B^R(\hat{\alpha} + 4\gamma + 1) = \sigma(R)(\gamma)$ for $4\gamma + 1 < \min(|\sigma|, \alpha(t) - \hat{\sigma})$. Then if $\alpha > \alpha(t)$, there exists $t^* \in \bar{R}^0_\alpha$ such that $t^* \leq_k t$, $t^* \leq t$ is a type* 1 *extension and*

$$R \in [t^*] \to B^R(\hat{\alpha} + 4\gamma + 1) = \sigma(R)(\gamma) \quad \textit{for} \quad 4\gamma + 1 < \min(|\sigma|, \alpha - \hat{\alpha}).$$

Proof. Much like the proof of Lemma 2.1. As some of the details are different we give a complete proof. By induction on α we define the notion "$b \subseteq [\alpha(t), \alpha)$

is t-special at α" and prove not only the existence of such b but also that for such b there exists t^* as desired so that $R \in [t^*] \to B^R$, b agree on $[\alpha(t), \alpha) - \{4\gamma + 1 \mid \gamma \in \mathrm{ORD}\}$, $B^R(\hat{\alpha} + 4\gamma + 1) = \sigma(R)(\gamma)$ for $4\gamma + 1 < \min(|\sigma|, \alpha - \hat{\alpha})$.

Suppose $\alpha = \beta + 1$. Then b is t-special at α iff $b \cap \beta$ is t-special at β and $\beta = 2\gamma \to (\beta \in B$ iff $\gamma \in A)$. The existence of t^* is clear by induction unless $\beta = 4\gamma + 1$. In that case the Sublemma to Lemma 2.1 shows that the desired t^* exists, using Case 2 of the construction of R^0.

If $\alpha = \beta + \omega$, β limit or 0, then $b \subseteq [\alpha(t), \alpha)$ is t-special at α if $\alpha(t) < \beta \to b \cap \beta$ is t-special at β. Now it is clear that the desired $t^* \leq_k t$ exists, using Case 2 of the construction of R^0.

Finally suppose α is a limit of limit ordinals. Given $b \subseteq [\alpha(t), \alpha)$ define R^* as in Case 4 with B^R replaced by b; for b to be t-special at α we first require that $0 \notin R^*$ and R^* codes an ordinal $\geq \alpha$. Set $\alpha_0' < \alpha_1 < \cdots$ equal to the $L[R^*]$-least ω-sequence of limit ordinals cofinal in α. Let $\alpha_0 < \alpha_1 < \cdots$ be the final segment of $\alpha_0' < \alpha_1' < \cdots$ defined by $\alpha_0 =$ least α_i' greater than $\alpha(t)$. We also require that $b \cap \alpha_0$ is t-special at α_0. Let $t_0 = L[A \cap \alpha, R^*]$-least $t' \leq_k t$ in $\bar{R}^0_{\alpha_0}$ such that $S \in [t'] \to B^S$, b agree on $[\alpha(t), \alpha_0) - \{4\gamma + 1 \mid \gamma \in \mathrm{ORD}\}$ and $B^S(\hat{\alpha} + 4\gamma + 1) = \sigma(S)(\gamma)$ for $4\gamma + 1 < \min(|\sigma|, \alpha_0 - \hat{\alpha})$. We require that $b \cap [\alpha_0, \alpha_1)$ is t_0-special at α_1. Then define $t_1 \leq_{k+1} t_0$ to be least so that $\alpha(t_1) = \alpha_1$ and $S \in [t_1] \to B^S$, b agree on $[\alpha_0, \alpha_1) - \{4\gamma + 1 \mid \gamma \in \mathrm{ORD}\}$, $B^S(\hat{\alpha} + 4\gamma + 1) = \sigma(S)(\gamma)$ for $4\gamma + 1 < \min(|\sigma|, \alpha_1 - \hat{\sigma})$. Continue in this way for ω steps. This completes the definition of 't-special at α' as well as the proof that the desired t^* exists, using Case 4 of the construction of R^0.

We must show that a t-special at α set exists when $t \in \bar{R}^0$, σ is acceptable, $\hat{\alpha}$ is limit or 0, $\hat{\alpha} \leq \alpha(t) < \alpha$. The argument is as in the proof of Lemma 1.2: The result is clear inductively except when α is a limit of limit ordinals. In that case first correctly define $b \cap \{2\gamma \mid \gamma \in \mathrm{ORD}\}$ and define $b \cap \{\lambda + 4n + 3 \mid \lambda$ limit, $n \in \omega\}$ so as to yield an R^* as in Case 4. Note that the latter positive commitments on b can be restricted to an ω-sequence. Thus it is easy to fill in $b \cap \alpha_n$ successively as in the preceding paragraph (where $\alpha_0 < \alpha_1 < \cdots$ arises from R^*, $\alpha(t)$ as in Case 4) so as to guarantee that $b \cap \alpha_{n+1}$ is t_n-special at α_{n+1}, while honoring earlier commitments to $b \cap \{\lambda + 4n + 3 \mid \lambda$ limit, $n \in \omega\}$. As in the proof of Lemma 1.2 it is important to not include any new ordinals of the form $4(\lambda + \bar{m}) + 3$ in $b \cap \alpha_{m+1}$ for $\bar{m} \leq m$, so as to not alter the definition of R^* from b. □

And as in Section 2 we also have:

Lemma 3.2. *Suppose $t \geq t^*$ is type 1 and $\alpha \in [\alpha(t), \alpha(t^*)]$. Then there is a unique $t' \in \bar{R}^0_\alpha$ such that $t \geq t' \geq t^*$.*

This enables us to define an equivalence relation $\sim$ just as in Section 2.

Generalized ω_n-trees

Our goal now is to extend the notions ω-tree, R^0, type 1 extension in $\tilde{R}^0$, $\sim$ on $\tilde{R}^0$ to generalized ω_n-tree, R^n, type 1 extension in $\tilde{R}^n$, $\sim$ on $\tilde{R}^n$ for $0<n<\omega$. The case $n=1$ will be treated very similarly to the way we handled the definition of R^A in Section 2. We begin with that case.

A *generalized ω_1-tree* is a sequence $T=\langle (f_i, g_i) \mid i<\omega_1\rangle$ obeying the definition of generalized tree in Section 2. And as in Section 2 we have:

Lemma 3.3 (Extendibility for R^T). *Suppose $t \in R_i^T$, $k \in \omega$ and $i \leqslant j < \omega_1$. Then there exists $t' \leqslant_k t$, $t' \in R_j^T$.*

Define $T_0 \leqslant T_1$ iff $R^{T_0} \subseteq R^{T_1}$, for generalized ω_1-trees T_0, T_1. As in Section 2 we have:

Lemma 3.4. *Any ω-sequence $T_0 \geqslant T_1 \geqslant \cdots$ of generalized ω_1-trees has a greatest lower bound.*

And paths through generalized ω_1-trees are defined as in Section 2. What is different now is the definition of R^1, which is obtained from that of R^A in Section 2 by introducing some new fusion sequences. We shall again need the $\square$-sequences $\langle C_\alpha^y \mid \alpha$ limit, $\omega_1 \leqslant \alpha < (\omega_2)^{L[y]}$, $y \subseteq \alpha\rangle$ as we did there and also the operation $(***)$ which introduces $T(a, t, T_0, T_1)$ when $T_0, T_1 \leqslant T$ are generalized ω_1-trees, $t \in \tilde{R}^{T_0} \cap \tilde{R}^{T_1}$ and $a \in 2^{<\omega}$.

In addition we must generalize the notion of acceptable term to *acceptable ω_1-term.* An *ω_1-path* is a function $x:\omega_1 \rightarrow \tilde{R}^0$ such that for some real R, $x(\alpha)=t_\alpha^R$ for all $\alpha<\omega_1$. Acceptable ω_1-terms are certain functions $\sigma:\omega_1$-paths$\rightarrow 2^{<\omega_2}$ such that for some fixed limit $|\sigma|<\omega_2$, length$(\sigma(x))=|\sigma|$ for all ω_1-paths x. They are defined inductively by:

(a) Any constant ω_1-term $\sigma(x)=s_0$, s_0 a fixed element of $2^{<\omega_2}$ of limit length, is acceptable.

(b) If σ_1, σ_2 are acceptable, $|\sigma_1|=|\sigma_2|$ and τ is an acceptable *ω-term,* then σ is acceptable where $\sigma(x)=\sigma_1(x)$ if $t \in \text{Range}(x)$, $R \in [t] \rightarrow B^R(4\gamma+1)=\tau(R)(\gamma)$ for $4\gamma+1<\min(|\tau|, \alpha(t))$; $\sigma(x)=\sigma_2(x)$ otherwise.

(c) If σ_1 is acceptable and σ_2 is acceptable, then $\sigma_1 * \sigma_2$ is acceptable where $\sigma_1 * \sigma_2(x)=\sigma_1(x) * \sigma_2(x)$ and $*$ denotes concatenation.

(d) If $\sigma_1, \sigma_2, \ldots$ are acceptable and $\sigma_i \subseteq \sigma_j$ for all $i<j<\omega_1$ (i.e., $\sigma_i(x) \subseteq \sigma_j(x)$ for all x), then σ is acceptable where $\sigma(x)=\bigcup\{\sigma_i(x) \mid i<\omega_1\}$.

As in the proof of Lemma 2.1 we have the following.

Lemma 3.5. *Let $\mathcal{P}_{\omega_1}$ denote the collection of ω_1-paths and for any acceptable ω-term τ let $\mathcal{P}_{\omega_1}(\tau)$ denote the collection of all ω_1-paths x such that $t \in \mathrm{Range}(x)$, $R \in [t] \to B^R(4\gamma+1) = \tau(R)(\gamma)$ for $4\gamma+1 < \min(|\tau|, \alpha(t))$. If σ is an acceptable ω_1-term, $\alpha < |\sigma|$, then $\{x \in \mathcal{P}_{\omega_1} \mid \sigma(x)(\alpha) = 0\}$ can be written as an finite Boolean combination of sets of the form $\mathcal{P}_{\omega_1}(\tau)$, τ an acceptable ω-term.*

Proof. Clear, by induction on σ. $\square$

Now we give the construction of R^1.

Case 1: $\alpha \leq \omega_1$. R^1_α contains only the generalized ω_1-tree $T_\emptyset$ defined by $T_\emptyset = \langle (f_i, g_i) \mid i < \omega_1 \rangle$ where $f_0(2^{<\omega}) = g_0(2^{<\omega}) = \emptyset$ and $f_i(t)(R) = \langle 0, 0, \ldots \rangle$, $g_1(t)(R) = \langle 1, 0, 0, \ldots \rangle$ for $i > 0$, $t \in \tilde{R}_i^{T_\emptyset}$.

Case 2: $\alpha = \beta + 1 > \omega_1$. Let $T = \tilde{R}^1_\beta$. First define generalized ω_1-trees T_0, $T_1 \leq T$ exactly as in Case 2 of the construction of R^A in Section 2. Now we define T_i^k for $k < \omega_1$, $i < k$ by induction on k. To define $\langle T_i^k \mid i < \omega_1 \rangle$ first fix a list of all finite or cofinite subsets F of the set of acceptable ω-terms in the sequence $\langle F_i \mid i < \omega_1 \rangle$. For each $i < k$ choose $T(i) \leq_k T$ so that no $T(i)$ shares a path with any $T(j)$, $j \neq i$ nor with any $T_j^{k'}$, $k' < k$ and so that (a) $\sigma \in F_i$, $t \in \tilde{R}_k^T$, $R \in [t] \to B^R(4\gamma+1) = \sigma(R)(\gamma)$ for $\gamma < \alpha(t)$, $x \in [T(i)]$, $t \in \mathrm{Range}(x) \to x \in [T_0]$, (b) $\sigma \in F_i$, $t \in \tilde{R}_k^T$, $R \in [t] \to B^R(4\gamma+1) \neq \sigma(R)(\gamma)$ for some $\gamma < \alpha(t)$, $x \in [T(i)]$, $t \in \mathrm{Range}(x) \to x \in [T_1]$. If β is not even, then add all the $T(i)$, $i < k$, to $\tilde{R}^1_\alpha$ and if $\beta = 2\gamma$, then add $T(i)$ to $\tilde{R}^1_\alpha$ iff ($F_i = \emptyset$, $\gamma \in A$ or $F_i =$ all acceptable ω-terms, $\gamma \notin A$). Set $T_i^k = T(i)$.

Then R^1_α is obtained from $R^1_{<\alpha}$ by including all T_i^k as above for all trees $T \in \tilde{R}^1_\beta$ and closing under $(***)$.

Case 3: α limit, $\alpha > \omega_1$ but not of the form $\omega_i \cdot \lambda$, λ limit. Write $\alpha = \beta + \delta$ where $0 < \delta \leq \omega_1$ and ω_1 divides β. Given an ω_1-path choose countable ordinals $\hat{\hat{\alpha}} \leq \hat{\hat{\beta}} < \delta$ ($\hat{\hat{\alpha}}$ limit or 0), the acceptable ω_1-term $\hat{\sigma}$ of *countable* length $|\hat{\sigma}|$ and the countable ordinal $\hat{k}$. In that case we set $\hat{\alpha} = \beta + \hat{\hat{\alpha}}$, $\hat{\beta} = \beta + \hat{\hat{\beta}}$. Let $T_0 = T^x_{\hat{\beta}}$. If T_γ is defined, then let $T_{\gamma+1} \leq_{\hat{k}+\gamma} T_\gamma$ be canonical and least in $L[A \cap \alpha, T_0]$ such that $y \in [T_{\gamma+1}] \to B^y$, B^x agree at $\hat{\beta} + \gamma$ (if $\hat{\beta} + \gamma$ is not of the form $\hat{\alpha} + 4\gamma' + 1$) or $B^y(\hat{\alpha} + 4\gamma' + 1) = \hat{\sigma}(y)(\gamma')$ (if $\hat{\beta} + \gamma = \hat{\alpha} + 4\gamma' + 1$, $\gamma' < |\hat{\sigma}|$), and such that $\alpha(T_{\gamma+1}) = \hat{\beta} + \gamma + 1$. If T_γ is defined for all $\gamma < \lambda$ where $\lambda \leq \alpha - \hat{\beta}$ is a limit ordinal, then let $T_\lambda =$ greatest lower bound to $\langle T_\gamma \mid \gamma < \lambda \rangle$. Define $T^x(\hat{\alpha}, \hat{\beta}, \hat{\sigma}, \hat{k}) = \bigcap \{T_\gamma \mid \gamma < \alpha - \hat{\beta}\}$ if all the T_γ, $\gamma < \alpha - \hat{\beta}$, are defined.

Include in $\tilde{R}^1_\alpha$ all generalized ω_1-trees T such that for some $(\hat{\alpha}, \hat{\beta}, \hat{\sigma}, \hat{k})$, $T^x(\hat{\alpha}, \hat{\beta}, \hat{\sigma}, \hat{k}) = T$ for all $x \in [T]$. Then $R^1_\alpha = (***)$-closure of $R^1_{<\alpha} \cup \tilde{R}^1_\alpha$.

Case 4: $\alpha = \omega_1 \cdot \lambda$, λ limit and $L_\alpha[A] \nvDash \omega_1$ is the largest cardinal. In this case we guarantee extendibility using $\square$. Given an ω_1-path x define $x^* = \{\delta <$

$\alpha \mid 4 \cdot \langle \delta, \delta' \rangle + 3 \in B^x$ for unboundedly many such ordinals $<\alpha\}$. Then $\alpha'_0 < \alpha'_1 < \cdots$ is defined if $0 \notin x^*$ and $\alpha < (\omega_2)^{L[x^*]}$. Thus $C^{x^*}_\alpha$ is defined as a closed subset of α; if it is unbounded in α, then let $\alpha'_0 < \alpha'_1 < \cdots$ enumerate $C^{x^*}_\alpha$. If $C^{x^*}_\alpha$ is bounded in α, then let $\alpha'_0 < \alpha'_1 < \cdots$ be the increasing enumeration of $C^{x^*}_\alpha$ followed by the $L[x^*]$-least ω-sequence $\beta_0 < \beta_1 < \cdots$ cofinal in α such that $\bigcup C^{x^*}_\alpha < \beta_0$ and each β_i is divisible by ω_1.

Choose $\hat{\alpha}$, $\hat{\beta}$ so that $\hat{\alpha} \leqslant \hat{\beta} < \alpha$ ($\hat{\alpha}$ limit, $\hat{\alpha} \geqslant \omega_1$), an acceptable ω_1-term $\hat{\sigma}$ and a countable ordinal $\hat{k}$. In this case set $\hat{T} = T^x_{\hat{\beta}}$. Also let $\alpha_0 < \alpha_1 < \cdots$ be the final segment of $\alpha'_0 < \alpha'_1 < \cdots$ defined by $\alpha_0 =$ least α'_i greater than $\hat{\beta}$. Let $\gamma_0 = \text{ordertype}\{\alpha_0 < \alpha_1 < \cdots\}$.

Now define $\langle T_\gamma \mid 0 \leqslant \gamma < \gamma_0 \rangle$ as follows. $T_0 = L[A \cap \alpha, x^*]$-least $T \leqslant_{\hat{k}} \hat{T}$ in $\tilde{R}^1_{\alpha_0}$ so that $y \in [T] \rightarrow B^y$, B^x agree on $[\hat{\beta}, \alpha_0) - \{4\gamma + 1 \mid \gamma \in \text{ORD}\}$ and $B^y(\hat{\alpha} + 4\gamma + 1) = \hat{\sigma}(y)(\gamma)$ for $4\gamma + 1 < \min(\alpha_0 - \hat{\alpha}, |\hat{\sigma}|)$. If T_γ is defined as an element of $\tilde{R}^1_{\alpha_\gamma}$, then let $T_{\gamma+1}$ be the $L[A \cap \alpha, x^*]$-least $T \leqslant_{\hat{k}+\gamma} T_\gamma$ in $\tilde{R}^1_{\alpha_{\gamma+1}}$ such that $y \in [T] \rightarrow B^y$, B^x agree on $[\alpha_\gamma, \alpha_{\gamma+1}) - \{4\gamma' + 1 \mid \gamma' \in \text{ORD}\}$ and $B^y(\hat{\alpha} + 4\gamma' + 1) = \hat{\sigma}(y)(\gamma')$ for $\gamma' < \min(\alpha_{\gamma+1} - \hat{\alpha}, |\hat{\sigma}|)$. For limit $\gamma \leqslant \gamma_0$ let $T_\gamma =$ greatest lower bound to $\langle T_{\gamma'} \mid \gamma' < \gamma \rangle$. If all the T_γ, $\gamma \leqslant \gamma_0$, are defined then let $T^x(\hat{\alpha}, \hat{\beta}, \hat{\sigma}, \hat{k}) = T_{\gamma_0}$.

Include in $\tilde{R}^1_\alpha$ all trees T such that for some $(\hat{\alpha}, \hat{\beta}, \hat{\sigma}, \hat{k})$, $T^x(\hat{\alpha}, \hat{\beta}, \hat{\sigma}, \hat{k}) = T$ for all $x \in [T]$. Obtain R^1_α by closing $R^1_{<\alpha} \cup \tilde{R}^1_\alpha$ under $(***)$.

Case 5: $\alpha = \omega_1 \cdot \lambda$, λ limit and $L_\alpha[A] \vDash \omega_1$ is the largest cardinal. First add conditions to $\tilde{R}^1_\alpha$ exactly as in Case 4. We now describe the other canonical conditions to be added, needed for fusion.

Given an ω_1-path x we define what it means for x to code a *type* A *fusion*. For this to be defined we require the following. Let $\eta \geqslant \alpha$ be least so that α is not regular in $J_{\eta+1}[x]$; we assume that η exists and that x codes a predicate $\bar{B}^x \subseteq [\alpha, (\omega_{\omega+1})^{J_\eta[x]})$ via the index i_0 for decoding B^R from $\mathscr{P}^A$-generic reals R, using the parameter $\langle \bar{\omega}_n \mid n \in \omega \rangle$, $\bar{\omega}_n = (\omega_n)^{J_\eta[x]}$.

Let $\bar{\alpha} = (\omega_\omega)^{J_\eta[A]}$, $\bar{\beta} = (\omega_{\omega+1})^{J_\eta[x]} \leqslant \eta$ and $\bar{A} = \text{even}(\bar{B}^x)$. We assume that α is not regular in $J_{\eta+1}[\bar{B}^x - \bar{\alpha}]$ and then let k^x denote the least k so that α is projectible in $\Sigma_k(\langle J_{\bar{\beta}}[\bar{B}^x - \bar{\alpha}], C^{\bar{B}^x - \bar{\alpha}}_{\bar{\beta}} \rangle)$. We require that $J_{\eta+1}[x] \vDash$ "$\bar{B}^x - \bar{\alpha}$, $\langle \eta, k^x \rangle$ gives rise to a canonical γ-sequence of quasiconditions $\langle \bar{p}_i \mid i < \gamma \rangle$ in $\mathscr{P}^{\bar{A}}$ where γ is a limit ordinal $\leqslant (\omega_1)^{J_\eta[x]}$, and x satisfies $\bar{p}_0$." This will be defined later when we complete the definition of $\mathscr{P}^A$.

For x to code a type A fusion we require that η, $\bar{B}^x$, k^x as above are defined and $0 \in x^*$ (as defined in Case 4). In this case set $T^x = \bigcap \{\bar{p}_i(\omega) \mid i < \gamma\}$.

We also consider *type* B *fusions*. For x to code a type B fusion we must have the following. Let η be least so that α is not regular in $J_{\eta+1}[x]$. Now decode from x the canonical α-sequence of generalized ω_1-trees $\langle T^x_\beta \mid \beta < \alpha \rangle = S^x$. We require that $J_\eta[x] \vDash$ for some β_0, S^x is a path through the generalized $(\omega_2)^{J_\eta[x]}$-tree $T^x = T^{S^x}_{\beta_0}$ and x is R^T-generic over $L[A \cap \alpha, T]$, $T = T^{S^x}_{\beta_0}$.

We assume that α is singular in $J_{\eta+1}[A \cap \alpha, T^X]$ and that $k \geqslant 2$ where k is least so that α is $\Sigma_k(J_\eta[A \cap \alpha, T^x])$-projectible. Let $\mathscr{A} = \Sigma_{k-2}$-Master Code structure for $J_\eta[A \cap \alpha, T^x]$ and we suppose that $\rho_1^{\mathscr{A}} = \alpha$ and C has ordertype $\leqslant \omega_1$ where C consists of all $\alpha' < \alpha$ such that $\alpha' \notin H_{\alpha'} = \Sigma_1$-Skolem hull of $\alpha' \cup \{p\}$ in $\mathscr{A}$, p = least p such that $\mathscr{A}$ is Σ_1-projectible to α with parameter p. Let $\mathscr{A}' = \Sigma_{k-1}$-Master Code structure for $J_\eta[A \cap \alpha, T^x]$ and $h: \alpha \to \omega_1$ a canonical $\Sigma_1(\mathscr{A}')$-injection.

For x to code a type B fusion we consider the sequence δ_i defined by δ_i = least $\delta < \omega_1$ such that $4(\alpha_i + \delta) + 3 \in B^x$ where $C = \{\alpha_0 < \alpha_1 < \cdots\}$. We insist that δ_i equals a fixed $\hat{\delta}$ for i sufficiently large. The key requirement then is that the $\Sigma_1(\mathscr{A}')$-set defined by the $\mathbf{\Sigma}_1$-formula with index $h^{-1}(\hat{\delta})$ is a fusion sequence $T_0 \geqslant_{\hat{i},1} T_1 \geqslant_{\hat{i},2} T_2 \geqslant_{\hat{i},3} \cdots$ of length γ_0 = ordertype(C), where $T_i \in R^1_{<\alpha}$. If in addition $0 \in x^*$ (as defined in Case 4), we set $T^x = \bigcap \{T_i \mid i < \gamma_0\}$.

Add T to $\tilde{R}^1_\alpha$ if $T = T^x$ for each $x \in [T]$ where T^x is defined as above. Obtain R^1_α by closing $R^1_{<\alpha} \cup \tilde{R}^1_\alpha$ under (***).

This completes the construction of $R^1 = \bigcup \{R^1_\alpha \mid \alpha < \omega_2\}$. For $T \in R^1$ let $\alpha(T)$ denote the unique α such that $T \in R^1_\alpha - R^1_{<\alpha}$. As in Section 2 if $x \in [T]$ for some $T \in \tilde{R}^1_\alpha$, then $T = T^x_\alpha$ is uniquely determined and can be recovered uniformly from x.

Type 1 extensions are defined for elements of $\tilde{R}^1$ just as they were for $\tilde{R}^0$; we give now the precise inductive definition. The trivial extension $T \leqslant T$ is type 1. If $\alpha(T) = \beta + 1$, then $T \leqslant T^*$ is type 1 if there is a sequence $T \leqslant T' \leqslant T^*$ where $\alpha(T') = \beta$ and $T' \leqslant T^*$ is type 1. Finally, if $\alpha(T)$ is a limit ordinal, then $T \leqslant T^*$ is type 1 if $T' \geqslant T^*$ where T arises from T' as in Cases 3, 4 or the first part of Case 5 (not involving fusion), but also where the extensions $T' \geqslant T_0 \geqslant T_1 \geqslant \cdots$ arising there are all type 1. We then have the following.

Lemma 3.6. *Suppose $T \in \tilde{R}^1$, $k \in \omega_1$, σ is an acceptable ω_1-term and $\hat{\alpha} \leqslant \alpha(T)$ is a limit ordinal $\geqslant \omega_1$. Also suppose that $x \in [T] \to B^x(\hat{\alpha} + 4\gamma + 1) = \sigma(x)(\gamma)$ for $4\gamma + 1 < \min(|\sigma|, \alpha(T) - \hat{\alpha})$. Then if $\alpha > \alpha(T)$ there exists $T^* \in \tilde{R}^1_\alpha$ such that $T^* \leqslant_k T$, $T^* \leqslant T$ is a type 1 extension and $x \in [T^*] \to B^x(\hat{\alpha} + 4\gamma + 1) = \sigma(x)(\gamma)$ for $4\gamma + 1 < \min(|\sigma|, \alpha - \hat{\alpha})$.*

Proof. Similar to the proof of Lemma 2.7, using ideas from the proof of Lemma 3.1. We define the notion "$b \subseteq [\alpha(T), \alpha)$ is T-special at α" exactly as in Lemma 2.7 and prove that for such b there exists T^* as desired so that $x \in [T^*] \to B^x$, b agree on $[\alpha(T), \alpha) - \{4\gamma + 1 \mid \gamma \in \mathrm{ORD}\}$.

If $\alpha \leqslant \omega_1$, then the result is trivial.

Suppose $\alpha = \beta + 1 > \omega_1$. Then $b \subseteq [\alpha(T), \alpha)$ is T-special at α iff $b \cap \beta$ is T-special at β and $\beta = 2\gamma \to (\beta \in b$ iff $\gamma \in A)$. The existence of T^* now follows from induction, Lemma 3.5 and the construction in Case 2 of the definition of R^1.

Suppose $\alpha = \omega_1 \cdot \gamma + \delta$ where $0 < \delta \leqslant \omega_1$ is a limit ordinal. Choose some $\hat{T} \leqslant_k T$ in $\tilde{R}^1_{\omega_1 \cdot \gamma}$ (if $\alpha(T) < \omega_1 \cdot \gamma$; $\hat{T} = T$ otherwise) such that $x \in [\hat{T}] \to B^x$, b agree on $[\alpha(T), \omega_1 \cdot \gamma) - \{4\gamma + 1 \mid \gamma \in \mathrm{ORD}\}$, $B^x(\hat{\alpha} + 4\gamma + 1) = \sigma(x)(\gamma)$ for $\gamma <$

$\min(|\sigma|, \omega_1 \cdot \gamma - \hat{\alpha})$. It is now clear that the desired $T^* \leq_k \hat{T}$ in $\tilde{R}^1_\alpha$ exists in this case, using Case 3 of the construction of R^1.

Finally suppose $\alpha = \omega_1 \cdot \lambda$, λ limit. Define x^* as in Case 4 with B^x replaced by b. Note that $0 \notin x^*$ and $\alpha < (\omega_2)^{L[x^*]}$. Define $\alpha'_0 < \alpha'_1 < \cdots$ as in Case 4. Let $\alpha_0 < \alpha_1 < \cdots$ be the final segment of $\alpha'_0 < \alpha'_1 < \cdots$ determined by $\sigma_0 =$ least α'_i greater than $\alpha(T)$. Let $\gamma_0 = \text{ordertype}\{\alpha_0 < \alpha_1 < \cdots\}$.

We note that for $i < \gamma_0$, x^* has the same value if in its definition B^x, α are replaced by $b \cap \alpha_i$, α_i. Now define $\langle T_\gamma \mid 0 \leq \gamma \leq \gamma_0 \rangle$ as in Case 4 with B^x, $\hat{\beta}$, $\hat{k}$, $\hat{T}$ replaced by b, $\alpha(T)$, k, T. We have that $b \cap [\alpha_i, \alpha_{i+1})$ is T_i-special at α_{i+1} for each $i < \gamma_0$. Note that for i limit, $T_i =$ greatest lower bound to $\langle T_j \mid j < i \rangle$ does belong to $\tilde{R}^1_{\alpha_i}$ as $C^{x^*}_\alpha \cap \alpha_i = C^{x^*}_{\alpha_i}$, and x^* has the same value 'at α_i' as it does 'at α'. Thus we have proved the existence of the desired $T^* \leq_k T$. $\square$

As in Section 2 we also have:

Lemma 3.7. *Suppose $T \geq T^*$ is a type 1 extension in $\tilde{R}^1$ and $\alpha \in [\alpha(T), \alpha(T^*)]$. Then there is a unique $T' \in \tilde{R}^1_\alpha$ such that $T \geq T' \geq T^*$.*

This enables us to define an analogue (for elements of $\tilde{R}^1$) of the equivalence relation $\sim$ of Section 2. If $T_1, T_2 \in \tilde{R}^1$ then $T_1 \sim_{\omega_1} T_2$ provided $\alpha(T_1) = \alpha(T_2)$ and either $T_1 = T_2$ or there are $\bar{T}_1, \bar{T}_2 \in \tilde{R}^1$ such that:

(a) $\bar{T}_1 \geq T_1$, $\bar{T}_2 \geq T_2$ are type 1 extensions,

(b) $\bar{T}_1 \sim_{\omega_1} \bar{T}_2$,

(c) $x, y \in [T_1] \cup [T_2] \to B^x, B^y$ agree on $[\alpha(\bar{T}_1), \alpha(T_1)) - \{4\gamma + 1 \mid \gamma \in \text{ORD}\}$.

Before going on to the case of generalized ω_n-trees for arbitrary finite n we discuss the notion of generalized ω_2-tree and establish extendibility for $\tilde{R}^T$ when T is a generalized ω_2-tree. Then the generalization of all of the above for arbitrary finite n will indeed be straightforward.

We now will use lower case letters $t_0, t_1, \ldots$ for generalized ω_1-trees and upper case letters $T_0, T_1, \ldots$ for generalized ω_2-trees. A *generalized ω_2-tree* is an ω_2-sequence $T = \langle (f_i, g_i) \mid i < \omega_2 \rangle$ where:

(a) $f_i, g_i : \tilde{R}^T_i \to$ Acceptable ω_1-Terms, where $\tilde{R}^T_i =$ the canonical elements of $R^T_i - R^T_{<i}$ (defined below).

(b) $|f_i(t)| = |g_i(t)|$ is divisible by ω_1 for all $t \in \tilde{R}^T_i$ and all i. In addition $f_0(t_\emptyset) = g_0(t_\emptyset)$ (where $t_\emptyset \geq t$ for all $t \in \tilde{R}^1$) and for $i > 0$, $t \in \tilde{R}^T_i : f_i(t)(x)(0) = 0$, $g_i(t)(x)(0) = 1$ for all ω_1-paths x.

(c) If $t_1 \sim_{\omega_1} t_2$ belong to $\tilde{R}^T_i$, then $f_i(t_1) = f_i(t_2)$, $g_i(t_1) = g_i(t_2)$.

We define R^T_i, $\tilde{R}^T_i$ by induction on $i < \omega_2$ for a given generalized ω_2-tree T. If x is an ω_1-path, define $S^x \cap \alpha = \{\gamma \mid 4\gamma + 1 \in B^x\} \cap \alpha$ provided t^x_β is defined for all $\beta <$ the limit ordinal α. Now $\tilde{R}^T_i = \{t_\emptyset\} = R^T_i$ for $i \leq \omega_1$. If R^T_i, $\tilde{R}^T_i$ are defined, then $t' \in \tilde{R}^T_{i+1}$ if for some $t \in \tilde{R}^T_i$, $t \geq t' \in \tilde{R}^1_{\alpha(t)+\eta}$ where $\eta = |f_i(t)|$ and either $x \in [t'] \to S^x(\alpha(t) + \gamma) = f_i(t)(x)(\gamma)$ for $\gamma < \eta$ or $x \in [t'] \to S^x(\alpha(t) + \gamma) = g_i(t)(x)(\gamma)$ for $\gamma < \eta$. To obtain R^T_{i+1} close $R^T_i \cup \tilde{R}^T_{i+1}$ under (***). Finally $\tilde{R}^T_\lambda$ for limit λ consists of all $t^* \in \tilde{R}^1$ which can be written $t^* = \text{glb}\{t_i \mid i < \lambda\}$ where $t_0 \geqslant t_1 \geqslant \cdots$, $\bigcup\{\alpha(t_i) \mid i < \lambda\} = \alpha$, $t_i \in R^T_{<\lambda}$ for all $i < \lambda$ and $\lambda = \bigcup\{j \mid t_i \in R^T_j - R^T_{<j}$ for some $i < \lambda\}$. $R^T_\lambda = (***)$-closure of $R^T_{<\lambda} \cup \tilde{R}^T_\lambda$. For $t \in R^T$ let $|t| =$ the unique i such that $t \in R^T_i - R^T_{<i}$.

Lemma 3.8 (Extendibility for R^T, T a generalized ω_2-tree). *Suppose $t \in R^T_i$, $m \in \omega_1$ and $i \leqslant j < \omega_2$. Then there exists $t' \leqslant_m t$ such that $|t'| = j$.*

Proof. It suffices to consider $t \in \tilde{R}^T_i$. We show that there is a type 1 extension $t' \leqslant_k t$ with $t' \in \tilde{R}^T_j$, by induction on j. If $j = i$, there is nothing to show. If $j > i$ is a successor ordinal, then the result is clear by induction and Lemma 3.6.

Suppose $j > i$ is a limit ordinal and choose $j_0 < j_1 < \cdots$ cofinal in j of length $\gamma_0 \leqslant \omega_1$. Now by induction we can choose $t_0, t_1, \ldots$ so that $t_{k+1} \leqslant t_k \leqslant_m t$ for each k and $t_k \in \tilde{R}^T_{j_k}$ (we do not insist that $t_l \leqslant t_k$ when $l - k$ is infinite). In fact we claim that we can choose the t_i's so that in addition $k < l \rightarrow B^{x_l}, B^{x_k}$ agree on $[\alpha(t), \alpha(t_k)) - \{4\gamma + 1 \mid \gamma \in \text{ORD}\}$ for $x_k \in [t_k]$, $x_l \in [t_l]$. To see this proceed as follows. Let $t_0 =$ any $t' \leqslant_m t$ in $\tilde{R}^T_{j_0}$ so that $\langle 1, \gamma\rangle \in \tilde{b}_0 = \{\delta \mid 4 \cdot \langle \delta, \delta'\rangle + 3 \in B^x$ for unboundedly many such ordinals $< \alpha(t_0)$, for all $x \in [t_0]\}$ iff $\gamma = \alpha(t)$. If t_k is defined, then choose $t_{k+1} \leqslant_m t_k$ in $\tilde{R}^T_{j_{k+1}}$ so that $\langle 1, \gamma\rangle \in \tilde{b}_{k+1}$ iff $\gamma \in \{\alpha(t), \alpha_0, \alpha_1, \ldots, \alpha_k\}$ (where $\tilde{b}_{k+1}$ is defined like $\tilde{b}_0$ with t_0 replaced by t_{k+1}). For limit $k < \gamma_0$ note that $\bigcup\{\tilde{b}_{k'} \mid k' < k\} = \tilde{b}_k$ has the property that $\bigcup\{\alpha(t_{k'}) \mid k' < k\} = \alpha_k$ has cardinaltiy ω_1 in $L[\tilde{b}_k]$. Thus we can choose a t-special at α_k set b_k so that $k' < k \rightarrow b_k, B^x$ agree on $[\alpha(t), \alpha(t_{k'})) - \{4\gamma + 1 \mid \gamma \in \text{ORD}\}$ for $x \in [t_{k'}]$. By Lemma 3.6 choose t_k so that $t_k \leqslant_m t$ and $x \in [t_k] \rightarrow B^x, b_k$ agree on $[\alpha(t), \alpha_k) - \{4\gamma + 1 \mid \gamma \in \text{ORD}\}$ and $B^x(\alpha(t) + 4\gamma + 1) = \sigma_k(x)(\gamma)$ for $\gamma < \alpha_k - \alpha(t)$ (where $\sigma_k = \bigcup\{\sigma_{k'} \mid k' < k\}$ and $x \in [t_{k'}] \rightarrow B^x(\alpha(t) + 4\gamma + 1) = \sigma_{k'}(x)(\gamma)$ for $\gamma < \alpha(t_{k'}) - \alpha(t)$). Thus $t_k \in \tilde{R}^T_{j_k}$ as desired.

Finally, note that the last part of the preceding paragraph applied to $k = \gamma_0$ as well and so we have proved that there exists $t' = t_{\gamma_0} \leqslant_m t$ as desired. □

We can now proceed to the general case of finite $n \geqslant 2$. Generalized ω_n-trees are defined just like generalized ω_2-trees with ω_2, Acceptable ω_1-Terms, divisible by ω_1, ω_1-paths, $\sim\omega_1$ replaced by ω_n, Acceptable ω_{n-1}-Terms, divisible by ω_{n-1}, ω_{n-1}-paths, $\sim\omega_{n-1}$ respectively. The collection of generalized ω_n-trees is $\leqslant\omega_n$-closed. An ω_n-path is a function $x : \omega_n \rightarrow \tilde{R}^{n-1}$ such that $x(\alpha) = t^y_\alpha$ for all $\alpha < \omega_n$, where y is an ω_{n-1}-path. An ω_n-path x is a path through a generalized ω_n-tree T is $x(\alpha) \in \tilde{R}^T$ for unboundedly many $\alpha < \omega_n$. $[T] =$ collection of all paths through T. And B^x for an ω_n-path x is defined as follows. Suppose T^x_α is defined (this makes sense after the construction of R^n is given). For $\beta < \omega_n$ choose $t^x_\beta \in \text{Range}(x) \cap \tilde{R}^{T^x_\alpha}_\beta$. Then x goes left at β on T^x_α if $y \in [x(\alpha(t^x_\beta + 2))] \rightarrow \alpha(t^x_\beta) + 1 \notin B^y$, x goes right at β on T^x_α otherwise. Then $\alpha \in B^x$ iff x goes right at β on T^x_α for sufficiently large $\beta < \omega_n$.

We define a version of the operation $(***)$ for generalized ω_n-trees. We use capital letters $T_0, T_1, \ldots$ for generalized ω_n-trees and small letters $t_0, t_1, \ldots$ for ω_{n-1}-trees. If $t \in \bar{R}^T$ define $T(t)$ by: $\alpha(t) \leqslant \alpha(t')$, $[t] \cap [t'] \neq \emptyset \rightarrow t' \in T(t)$; $t_0 \sim_{\omega_{n-1}} t_1$, $t_0 \in T(t) \rightarrow t_1 \in T(t)$. Now if $T_0, T_1 \leqslant T$, $t \in \bar{R}^{T_0} \cap \bar{R}^{T_1}$, τ an acceptable ω_{n-2}-term define $T^* = T(\tau, t, T_0, T_1) = \langle (f_i^*, g_i^*) \mid i < \omega_2 \rangle$ as follows. Suppose (f_i^*, g_i^*) is defined for $i < \gamma$. Pick $t^* \in T(t)$, $t^* \in \bar{R}_\gamma^{T^*} \cap \bar{R}^{T_0} \cap \bar{R}^{T_1}$ and canonically choose type 1 extensions t_0, t_1 of t^* in $\bar{R}^{T_0}$, $\bar{R}^{T_1}$ so that $\alpha(t_0) = \alpha(t_1)$, with corresponding acceptable ω_{n-1}-terms σ_0, σ_1 so that $\sigma_0(x)(0) = 0$, $\sigma_1(x')(0) = 0$ for $x \in [t_0]$, $x' \in [t_1]$. (That is, $x \in [t_0] \rightarrow B^x(\alpha(t^*) + 4\gamma + 1) = \sigma_0(x)(\gamma)$ for $\gamma < \alpha(t_0) - \alpha(t^*)$; similarly for t_1.) Then $f_\gamma^*(t^*)$ is defined to be σ where $\sigma(x) = \sigma_0(x)$ if $x \in \mathcal{P}_{\omega_{n-1}}(\tau)$, $= \sigma_1(x)$ otherwise. ($\mathcal{P}_{\omega_{n-1}}(\tau) =$ all ω_{n-1}-paths x such that $z \in \text{Range}(x)$, $q \in [z] \rightarrow B^q(4\gamma + 1) = \tau(q)(\gamma)$ for $4\gamma + 1 < \min(|\tau|, \alpha(x))$.) Define g_γ^* similarly. If $t^* \notin \bar{R}^{T_i}$ but $t^* \in T(t)$, define $(f_\gamma^*(t^*), g_\gamma^*(t^*))$ to agree with T_{1-i} and if $t^* \notin T(t)$, then define it so as to agree with T.

Acceptable ω_n-terms are defined inductively just as in the definition of acceptable ω_1-term, with $2^{<\omega_2}$, ω-term, $\bigcup \{\alpha_i(x) \mid i < \omega_1\}$ replaced $2^{<\omega_{n+1}}$, ω_{n-1}-term, $\bigcup \{\sigma_i(x) \mid i < \omega_n\}$.

The construction of R^n, $\bar{R}^n$ is perfectly analogous to that of R^1, $\bar{R}^1$ using $\square$-sequences $\langle C_\alpha^y \mid \alpha$ limit, $\omega_n \leqslant \alpha < (\omega_{n+1})^{L[y]}$, $y \subseteq \alpha \rangle$. The cases are: $\alpha \leqslant \omega_n$; $\alpha = \beta + 1 > \omega_n$; α limit, $\alpha > \omega_n$, α not of the form $\omega_n \cdot \lambda$ (λ limit); $\alpha = \omega_n \cdot \lambda$, λ limit and $L_\alpha[A] \nvDash \omega_n$ is the largest cardinal; otherwise. In the part of Case 5 discussing type A fusions we insist that the sequence $\langle \bar{p}_i \mid i < \gamma \rangle$ have limit length $\gamma \leqslant (\omega_n)^{J_\eta[x]}$. In the type B fusions the set C should have ordertype $\leqslant \omega_n$. As before we can define type 1 extension, prove extendibility and define an equivalence relation on elements of $\bar{R}^n$. Then extendibility for $\bar{R}^{\mathcal{J}}$, $\mathcal{J}$ a generalized ω_{n+1}-tree can be carried out just as in the case $n = 1$. This completes our definition of R^n, $\bar{R}^n$ for finite n.

The forcing $\mathcal{P}^A$

Now we build the desired forcing for minimally coding A by a real. Conditions are of the form $p = \langle p(\omega_n) \mid n < \omega \rangle$ where $p(\omega_n)$ is a generalized ω_n-tree and $p(\omega_n) \in R^{p(\omega_{n+1})}$ (these are called *quasiconditions*). Set $p \leqslant q$ if $p(\omega_n) \leqslant q(\omega_n)$ for all n. A *path through* p is a sequence $\langle x_n \mid n < \omega \rangle$ such that each x_n is a path through $p(\omega_n)$ and for all n, $x_{n+1}(\alpha) = t_\alpha^{x_n}$ for $\alpha < \omega_{n+1}$. An ω_ω-path is a path through $p_\emptyset$, the weakest quasicondition.

The construction $\mathcal{P}^A$ requires the use of both $\square$ and $\diamondsuit$. We assume for convenience that ω_ω divides $\alpha \rightarrow L_\alpha(A) \vDash \omega_\omega$ is the largest cardinal. As in Section 2 we have a natural system of $\square$-sequences $\langle C_\alpha^y \mid \alpha$ limit, $\omega_\omega \leqslant \alpha < (\omega_{\omega+1})^{L[y]}$, $y \subseteq \alpha \rangle$ and it will be useful below to define $E = \{\langle \alpha, y \rangle \mid C_\alpha^y$ is defined and bounded in α, α is p.r. closed$\}$. Also select a system of $\diamondsuit(E)$-sequences $\langle D_\alpha^y \mid \langle \alpha, y \rangle \in E \rangle$. This system has the following property: For α p.r. closed let $\nu(\alpha, y) =$ largest ordinal ν such that α is regular in $L_\nu[y]$. Then $X \subseteq \alpha$, $X \in L_{\nu(\alpha,y)}[y] \rightarrow X \cap \beta = D_\beta^{y \cap \beta}$ for some $\langle \beta, y \cap \beta \rangle \in E$, $\beta < \alpha$. This is possible to arrange because E is stationary in the sense that $\langle \alpha, y\rangle$ as above, $C \subseteq \alpha$ closed unbounded in α, $C \in L_{\nu(\alpha,y)}[y] \rightarrow \langle \beta, y \cap \beta\rangle \in E$ for some $\beta \in C$.

We are now almost ready to define $\mathscr{P}^A = \bigcup \{\mathscr{P}^A_\alpha \mid \alpha < \omega_{\omega+1}\}$. To each $p \in \mathscr{P}^A$ will be assigned a pair $\langle \lambda(p), B(p)\rangle$ such that $B(p) \subseteq \lambda(p)$, $L_{\lambda(p)}[B(p)] \models \omega_\omega$ is the largest cardinal. Also set $B^*(p) = \{\delta < \lambda(p) \mid 4 \cdot \langle \delta, \delta'\rangle + 3 \in B(p)$ for unboundedly many such ordinals $<\lambda(p)\}$. The condition p will then be $\Sigma_2(\langle L_{\lambda(p)}[B(p)], C^{B^*(p)}_{\lambda(p)}\rangle)$, when $C(p) = C^{B^*(p)}_{\lambda(p)}$ is unbounded in $\lambda(p)$. If $\lambda(p) = \lambda' + 1$, then $C(p) = \emptyset$. When $C^{B^*(p)}_{\lambda(p)}$ is bounded in $\lambda(p)$, $\lambda(p)$ limit, we let $C(p) = \{\omega_\omega \cdot \lambda' + \gamma' \mid \gamma' < \gamma\}$ if $\lambda(p)$ is of the form $\omega_\omega \cdot \lambda' + \gamma$, $\gamma \leqslant \omega_\omega$ and otherwise $C(p) =$ an ω-sequence cofinal in $\lambda(p)$ such that $\bigcup C^{B^*(p)}_{\lambda(p)} < \min C(p)$ and Lemma 6.41 of Beller–Jensen–Welsch [1] holds for $\langle \lambda(p), B^*(p)\rangle$. If x is an ω_ω-path, then x *satisfies* p, $x \in [p]$, if $x(n) \in [p(\omega_n)]$ for each n. The generic ω_ω-path x will canonically give rise to conditions p^x_α for $\alpha < \omega_{\omega+1}$. We define B^x as follows. For each $n \in \omega$, $\alpha < \omega_{\omega+1}$ let $X^\alpha_n = \Sigma_1$-Skolem hull of $\omega_n \cup \{\omega_\omega\}$ in $\langle L_{\lambda(p)}[B(p)], C(p)\rangle$ where $p = p^x_\alpha$, and let $\gamma^\alpha_n = X^\alpha_n \cap \omega_{n+1}$. Then $\alpha \in B^x$ iff $4\gamma^\alpha_n + 3 \in B^{x(n)}$ for sufficiently large $n < \omega$. Of course we will require that $2\alpha \in B^x$ iff $\alpha \in A$ for all $\alpha < \omega_{\omega+1}$.

If p, q are quasiconditions, then $p \leqslant_n q$ if $p \leqslant q$ and $p(\omega_m) = q(\omega_m)$ for $m \leqslant n$.

Case 1: $\alpha \leqslant \omega_\omega$. $\mathscr{P}_\alpha$ contains only the weakest quasicondition $p_\emptyset$.

Case 2: $\alpha = \beta + 1 > \omega_\omega$. Let $p \in \mathscr{P}_\beta - \mathscr{P}_{<\beta}$. We assume inductively that for sufficiently large $n < \omega$, $|p(\omega_n)| = X_n \cap \omega_{n+1}$ where $X_n = \Sigma_1$-Skolem hull of $\omega_n \cup \{\omega_\omega\}$ in $\langle L_{\lambda(p)}[B(p)], C(p)\rangle$. We begin by defining $\hat{p}_i \leqslant p$ to be the least quasicondition so that $x \in [\hat{p}_i] \rightarrow B^{x(n)}(4\gamma^\beta_n + 3) = i$ for all n such that $|p(\omega_n)| = X_n \cap \omega_{n+1} = \gamma^\beta_n$ (for $i = 0, 1$). Clearly $\hat{p}_i$ is $\Sigma_2\langle L_{\lambda(p)}[B(p)], C(p)\rangle$.

Now we build a sequence of quasiconditions $\hat{p}_i = p_0 \geqslant p_1 \geqslant \cdots$ such that p_n is $\Sigma_{n+2}(\langle L_{\lambda(p)}[B(p), C(p)\rangle)$, uniformly in n. To define p_1 first let $A(p)$ be a Σ_1-Master Code for $\langle L_{\lambda(p)}[B(p)], C(p)\rangle$ and for each pair $m < n$ let $X^{\beta,2}_{m,n} = \Sigma_1$-Skolem hull of $\omega_m \cup \{\bar{p}\}$ in $\langle L_{\omega_n}[A], A(p) \cap \omega_n\rangle$, $\bar{p} =$ standard parameter for $\langle L_{\omega_n}[A], A(p) \cap \omega_n\rangle$. Now define $\hat{p}_i = p^0_1 \geqslant p^1_1 \geqslant \cdots$ successively as follows: $p^{k+1}_1 = L[A]$-least quasicondition $q \leqslant p^k_1$ such that $q(\omega_l) = p^k_1(\omega_l)$ for $l > k + 1$ and for $l \leqslant k$: $q(\omega_{l+1})(q(\omega_l)) = q'(q(\omega_l))$ where q' is canonical and $q(\omega_l)$ reduces all predense $\mathscr{D} \subseteq R^{q(\omega_{l+1})}$ which belong to $X^{\beta,2}_{l,k+1} \cap L_{\alpha(q(\omega_{l+1}))}[q(\omega_{l+1})(q(\omega_l))]$. (See Corollary 3.11 for a discussion of reduction of predense sets.) We also require that $p^{k+1}_1(\omega) \leqslant_k p^k_1(\omega)$. Then set $p_{1,0} = \text{glb}\{p^k_1 \mid k \in \omega\}$. The type A fusion part of Case 5 in the construction of the forcings R^n, $n \in \omega$, will guarantee that $p_{1,0}$ is a quasicondition, as we now specify that $B(p)$, $\langle \nu(\lambda(p), B(p)), 2\rangle$ gives rise to the canonical ω-sequence $p^0_1 \geqslant p^1_1 \geqslant p^2_1 \geqslant \cdots$ of quasiconditions.

Having defined $p_{1,0}$ we now describe how to obtain $p_{1,1} \leqslant p_{1,0}$. Choose a canonical listing $\langle D_i \mid i < \omega_\omega\rangle$ of all predense $D \subseteq \mathscr{P}(p_{1,0})$, $D \in \Sigma_1(\langle L_{\lambda(p)}[B(p)], C(p)\rangle)$ where $\mathscr{P}(p_{1,0}) = \{$quasiconditions $q \leqslant p_{1,0} \mid q(\omega_n) = p_{1,0}(\omega_n)$ for sufficiently large $n\}$. Also list all finite $q \restriction \omega_n$ for $q \in \mathscr{P}(p_{1,0})$ as

$\langle q_i \mid i<\omega_\omega\rangle$. Then $p_{1,1}$ is built from $p_{1,0}$ exactly as was $p_{1,0}$ built from p_0, with the exception that $A(p)$ is replaced by a Σ_1-Master code for $\langle L_{\lambda(p)}[B(p)], C(p), p_{1,0}\rangle$ and we require that $p_{1,1}\leqslant$ some element of D_0. More generally define $p_{1,i+1}$ from $p_{1,i}$ by using $\langle L_{\lambda(p)}[B(p)], C(p), p_{1,i}\rangle$ and requiring $p_{1,i+1}$ to obey the following: Suppose $i=\langle i_0, i_1\rangle$ where q_{i_0} has domain $\{\omega, \omega_1, \ldots, \omega_n\}$, $i<\omega_{n+1}$ and $q_{i_0}(\omega_n)\in R^{p_{1,i}(\omega_{n+1})}$. Also suppose that there exists $q\leqslant$ some element of D_{i_1} such that $q\leqslant p_{1,i}$ and $q(\omega_j)=q_{i_0}(\omega_j)$ for $j\leqslant n$. Then let $p_{1,i+1}\leqslant p_{1,i}$ be least so that $p_{1,i+1}$ agrees with such a q above ω_n and $p_{1,i+1}\leqslant_n p_{1,i}$. For limit λ, $p_{1,\lambda}=\text{glb}\{p_{1,i}\mid i<\lambda\}$ and we specify that $B(p)$, $\langle \nu(\lambda(p), B(p)), 2\rangle$ gives rise to the sequence $\langle p_{1,i}\mid i<\lambda\rangle$, for $\lambda<\omega_\omega$. For $\lambda=\omega_\omega$ we have the sequence $\langle p_{1,\omega_n}\mid n<\omega\rangle$. Finally define $p_2=\text{glb}\langle p_{1,i}\mid i<\omega_\omega\rangle$. Note that we have arranged that p_2 reduces all predense $D\in\Sigma_1(\langle L_{\lambda(p)}[B(p)], C(p)\rangle)$.

Having defined p_1 we now describe how to obtain $p_2\leqslant p_1$. The construction is perfectly analogous to that of p_1. Define $X^{\beta,3}_{m,n}=\Sigma_1$-Skolem hull of $\omega_m\cup\{\bar p\}$ in $\{L_{\omega_n}[A], A^2(p)\cap\omega_n\rangle$ where $a^2(p)$ is a Σ_2-Master Code for $\langle L_{\lambda(p)}[B(p)], C(p)\rangle$ and $\bar p$ is the standard parameter. Then $p_2^0=p_1$ and $p_2^{k+1}=L[A]$-least quasicondition $q\leqslant p_2^k$ such that $q(\omega_l)=p_2^k(\omega_l)$ for $l<k+1$ and for $l\leqslant k$: $q(\omega_{l+1})(q(\omega_l))=q'(q(\omega_l))$ where q' is canonical and $q(\omega_l)$ reduces all predense sets for $R^{q(\omega_{l+1})}$ in $X^{\beta,3}_{l,k+1}\cap L_{\alpha(q(\omega_{l+1}))}[q(\omega_{l+1})(q(\omega_l))]$. Also require that $p_2^{k+1}(\omega)\leqslant_k p_2^k(\omega)$ and set $p_{2,0}=\text{glb}\{p_2^k\mid k\in\omega\}$. We specify that $B(p)$, $\langle \nu(\lambda(p), B(p)), 3\rangle$ gives rise to the canonical ω-sequence $p_1=p_2^0\geqslant p_2^1\geqslant\cdots$ of quasiconditions. Then obtain $p_{2,0}\geqslant p_{2,1}\geqslant\cdots$ as before, but using $\Sigma_2(\langle L_{\lambda(p)}[B(p)], C(p)\rangle)$-predense sets.

Let $p_i^*=\text{glb}\{p_n\mid n\in\omega\}$. We specify that $B(p)*i$, $\langle \nu(\lambda(p), B(p))+1, 1\rangle$ gives rise to $\hat p_i\geqslant p_1\geqslant p_2\geqslant\cdots$; note that $|p_i^*(\omega_n)|=\gamma_n^\alpha$ for sufficiently large n where $\gamma_n^\alpha=X_n^\alpha\cap\omega_{n+1}$, $X_n^\alpha=\Sigma_1$-Skolem hull of $\omega_n\cup\{\omega_\omega\}$ in $\langle L_{\nu(\lambda(p),B(p))+1}[B(p)], C\rangle$, $C=\emptyset$. Now we put p_0^*, p_1^* into $\mathcal{P}^A_\alpha$ if β is not even and if $\beta=2\gamma$, then $p_0^*\in\mathcal{P}^A_\alpha$ iff $\gamma\notin A$, $p_1^*\in\mathcal{P}^A_\alpha$ iff $\gamma\in A$. Clearly p_i^* is $\Sigma_2\langle L_{\lambda(p_i^*)}[B(p_i^*)], C(p_i^*)\rangle$ where $\lambda(p_i^*)=\nu(\lambda(p), B(p))+1$, $B(p_i^*)=B(p)*i$ and $C(p_i^*)=C$.

To complete the description of $\mathcal{P}^A_\alpha$, repeat the above construction for each $n\in\omega$, where all the extensions $p_k\leqslant\hat p_i$ (and $\hat p_0$, $\hat p_1$ as well) are required to obey $p_k(\omega_m)=p(\omega_m)$ for $m\leqslant n$. Also take care to arrange that the resulting extensions have no paths in common, for the purpose of guaranteeing that each ω_ω-path goes through at most one of them. This completes the definition of $\mathcal{P}^A_\alpha$ in this case.

Case 3: $\alpha>\omega_\omega$ is a limit ordinal not divisible by $\omega_\omega\cdot\omega$. Write $\alpha=\beta+\delta$ where $0<\delta\leqslant\omega_\omega$ and ω_ω divides β. Given an ω_ω-path x and ordinal $\hat\beta\in[\beta,\alpha)$, $\hat k\in\omega$ proceed as follows. Let $p_0=p^x_{\hat\beta}$. If p_γ is defined, then let $p_{\gamma+1}\leqslant p_\gamma$ be $L[A\cap\alpha, p_0]$-least in $\mathcal{P}^A_{\hat\beta+\gamma+1}$ such that $p_{\gamma+1}\leqslant_{\hat k+n}p_\gamma$, $\omega_n=\text{card}(\gamma)$, and $y\in[p_{\gamma+1}]\to B^y$, B^x agree at $\hat\beta+\gamma$. If p_γ is defined for $\gamma<\lambda$ limit, then let $p_\gamma=$ greatest lower bound of $\langle p_\gamma\mid\gamma<\lambda\rangle$. We specify that $B(p_\lambda)$ ($=\bigcup\{B(p_\gamma)\mid\gamma<\lambda\}$), $\langle \nu(\lambda(p_0), B(p_0))+\lambda, 1\rangle$ gives rise to the λ-sequence $p_0\geqslant$

$p_1 \geq \cdots \geq p_\gamma \geq \cdots$, $\gamma < \lambda$. Define $p^x(\hat{\beta}, \hat{k}) = \text{g.l.b}\{p_\gamma \mid \gamma < \delta\}$. If $\delta < \omega_\omega$, then specify that $B(p^x(\hat{\beta}, \hat{k}))$, $\langle v(\lambda(p_0), B(p_0)) + \delta, 1\rangle$ gives rise to $\langle p_\gamma \mid \gamma < \delta\rangle$. If $\delta = \omega_\omega$, then the former gives rise to $\langle p_{\omega_n} \mid n < \omega\rangle$.

Now include in $\mathcal{P}^A_\alpha$ all $p^x(\hat{\beta}, \hat{k})$ as above. For such a condition p we have $B(p) = \bigcup \{B(p_\gamma) \mid \gamma < \delta\}$, $\lambda(p) = v(\lambda(p_0), B(p_0)) + \delta$.

Case 4: $\alpha = \omega_\omega \cdot \lambda$, λ limit and α is not p.r. closed. We are not concerned here with E or with building in any special fusions; only with guaranteeing extendibility. Given an ω_ω-path x define $x^* = \{\delta < \alpha \mid 4 \cdot \langle \delta, \delta'\rangle + 3 \in B^x$ for unboundedly many such ordinals $<\alpha\}$. then $\alpha'_0 < \alpha'_1 < \cdots$ is defined if $0 \notin x^*$ and $\alpha < (\omega_{\omega+1})^{L[x^*]}$. If $C^{x^*}_\alpha$ is unbounded in α, then let $\alpha'_0 < \alpha'_1 < \cdots$ enumerate $C^{x^*}_\alpha$. If not, then let $\alpha'_0 < \alpha'_1 < \cdots$ enumerate C, where C is defined from α, x^* as was $C(p)$ defined from $\lambda(p)$, $B^*(p)$.

Choose $\hat{\beta} < \alpha$ and $\hat{k} \in \omega$. Also define $\alpha_0 < \alpha_1 < \cdots$ to be the final segment of $\alpha'_0 < \alpha'_1 < \cdots$ determined by $\alpha_0 =$ least α'_i greater than $\hat{\beta}$. Now define $\langle p_\gamma \mid 0 \leq \gamma < \gamma_0\rangle$ as follows where $\gamma_0 =$ ordertype of $\{\alpha_0 < \alpha_1 < \cdots\}$. $p_0 = L[A \cap \alpha, x^*]$-least $p \leq_{\hat{k}} p^x_{\hat{\beta}}$ in $\mathcal{P}^A_{\alpha_0}$ so that $y \in [p] \to B^y, B^x$ agree on $[\hat{\beta}, \alpha_0)$. If p_γ is defined in $\mathcal{P}^A_{\alpha_\gamma}$, then let $p_{\gamma+1}$ be the $L[A \cap \alpha, x^*]$-least $p \leq_{\hat{k}+n} p_\gamma$, $\omega_n = \text{card}(\gamma)$, such that $p \in \mathcal{P}^A_{\alpha_{\gamma+1}}$ and $y \in [p] \to B^y, B^x$ agree on $[\alpha_\gamma, \alpha_{\gamma+1})$. If p_γ is defined for $\gamma < \lambda \leq \gamma_0$, λ limit, then let $p_\lambda =$ greatest lower bound of $\langle p_\gamma \mid \gamma < \lambda\rangle$. We specify that $B(p_\lambda)$, $\langle v(\alpha_\lambda, B(p_\lambda)), 1\rangle$ gives rise to the λ-sequence $\langle p_\gamma \mid \gamma < \lambda\rangle$. Define $p^x(\hat{\beta}, \hat{k}) = p_{\gamma_0}$.

Include in $\mathcal{P}^A_\alpha$ all $p^x(\hat{\beta}, \hat{k})$ as above. For such a condition p we have $B(p) = \bigcup \{B(p_\gamma) \mid \gamma < \gamma_0\}$, $\lambda(p) = \alpha$.

Case 5: α is p.r. closed. First add conditions as in Case 4 but with an important restriction: If p is added to $\mathcal{P}^A_\alpha$ and $\langle \alpha, B(p)\rangle \in E$, $D^B_\alpha(p)$ is a dense subset of $\mathcal{P}(\bar{p}, B(p)) = \{q \in \mathcal{P}^A_{<\alpha} \mid q \leq \bar{p}, B(q)$ and $B(p)$ agree on $[\lambda(\bar{p}), \lambda(q))\}$, for some $\bar{p} \geq p$, $\bar{p} \in \mathcal{P}^A_{<\alpha}$, then insist that $p \leq q \leq \bar{p}$ for some q such that $r \leq q \to \exists r' \leq r$ $(r' \in D^{B(p)}_\alpha$, $r'(\omega_l) = r(\omega_l)$ for $l \geq n)$, for some n. It will be easy to verify that this restriction will not injure extendibility.

Now we also add conditions for the sake of anticipating certain fusions. Given an ω_ω-path x define what it means for x to code a fusion sequence $p_0 \geq p_1 \geq \cdots$ of length $<\omega_\omega$. Let (η, k) be least so that α is $\Sigma_k(L_\eta[A \cap \alpha])$-projectible. We suppose that (η, k) exists and $k \geq 2$. Let $\mathcal{A} = \Sigma_{k-2}$-Master Code structure for $L_\eta[A \cap \alpha]$; we suppose that $\rho^{\mathcal{A}}_1 = \alpha$ and C has ordertype $<\omega_\omega$ where C consists of all $\alpha' < \alpha$ such that $\alpha' \notin H_{\alpha'} = \Sigma_1$-Skolem hull of $\alpha' \cup \{\omega_\omega, \bar{p}\}$ in $\mathcal{A}$, $\bar{p} =$ least parameter $\bar{p}$ such that $\mathcal{A}$ is Σ_1-projectible to α with parameter p. Also let $\mathcal{A}' = \Sigma_{k-1}$-Master Code structure for $L_\eta[A \cap \alpha]$ and choose a canonical $\Sigma_1(\mathcal{A}')$-injection $f: \mathcal{A}' \to \omega_\omega$.

Now enumerate C as $\alpha_0 < \alpha_1 < \cdots$ and consider the sequence $\delta_0 < \delta_1 < \cdots$ defined by $\delta_i =$ least $\delta < \omega_\omega$ such that $4(\alpha_1 + \delta) + 3 \in B^x$. We require that δ_i has a constant value $\langle 0, \hat{\delta}\rangle$ for i sufficiently large. (Note that this implies $0 \in x^*$). For x

to code a fusion sequence $p_0 \geqslant p_1 \geqslant \cdots$ we must have that the $\Sigma_1(\mathscr{A}')$-set with defining parameter $f^{-1}(\hat{\delta})$ is a sequence of conditions $p_0 \geqslant p_1 \geqslant \cdots$ where $p_\lambda = \text{glb}\{p_i \mid i < \lambda\}$ for limit λ, $\alpha(p_i)$ (= least α' such that $p_i \in \mathscr{P}^A_{\alpha'+1}$) is at least α_i and $i > \omega_n \rightarrow p_i \leqslant_n p_{\omega_n}$. Now add the greatest lower bound p for all such fusion sequences $p_0 \geqslant p_1 \geqslant \cdots$ to $\mathscr{P}^A_\alpha$ provided $x \in [p] \rightarrow x$ codes this fusion sequence as above and provided for all $i <$ ordertype(C), p_{i+1} reduces all $D \in H^{\mathscr{A}_i}_{\alpha_i} = \Sigma_1$-Skolem hull of $\alpha_i \cup \{\bar{p}\}$ in $\mathscr{A}_i$, where $\langle \mathscr{A}_i \mid i < \text{ordertype } C \rangle$ is a $\Sigma_2(\mathscr{A})$-approximation to $\mathscr{A}$ and D is predense on $\mathscr{P}^{A \cap \alpha} = \mathscr{P}^A_{<\alpha} \cap L_\alpha[A]$. We also specify for limit λ that $B(p_\lambda) = A \cap \alpha_\lambda$, $\langle \nu(\alpha_\lambda, B(p_\lambda)), 1\rangle$ gives rise to $\langle p_i \mid i < \lambda \rangle$ and that $B(p) = A \cap \alpha$, $\langle \nu(\alpha, B(p)), 1 \rangle$ gives rise to $\langle p_i \mid i < \text{ordertype } C \rangle$. This completes Case 5.

Finally close each $\mathscr{P}^A_\alpha$ under: $p \in \mathscr{P}^A_\alpha$, $q(\omega_n) = p(\omega_n)$ for all sufficiently large $n \rightarrow q \in \mathscr{P}^A_\alpha$. This completes the construction of $\mathscr{P}^A = \bigcup \{\mathscr{P}^A_\alpha \mid \alpha < \omega_{\omega+1}\}$. We now prove a series of lemmas which ultimately will show that a $\mathscr{P}^A$-generic real minimally codes A. First we establish fusion for the forcings R^T, $T \in \tilde{R}^{n+1}$. If t, t' are generalized n-trees for $n \geqslant 1$, then we write $t' \leqslant_{l,l'} t$ $(l, l' < \omega_m)$ provided $t' \leqslant_l t$ and $u \in \tilde{R}^t_{l'} \rightarrow u \in \tilde{R}^{t'}_{l'}$ unless $u \in t(u_i)$ for some i, $\omega_{n-1} \cdot i \geqslant l'$ (where $\langle u_j \mid j < \omega_n \rangle$ is a fixed canonical enumeration of $\tilde{R}^{n-1}$). And $\mathscr{D} \subseteq R^T$ is l, l'-dense below $t \in R^T$ if $t' \leqslant_l t \rightarrow \exists t'' \leqslant_{l,l'} t'$ such that $t'' \in \mathscr{D}$.

Lemma 3.9 (Fusion for R^T, $T \in \tilde{R}^{n+1}$). *Suppose $t \in R^T$, $T \in \tilde{R}^{n+1}$, $l < \omega_n$ and $D_{l,l'}$ is l, l'-dense below t for each $l' < \omega_n$. Also suppose that for some $\gamma < (\omega_{n+2})^{L[T]}$, $\mathscr{D}_{l,l'} \in L_\gamma[T]$ for each $l' < \omega_n$. Then there exists $t' \leqslant_l t$ such that $t' \in D_{l,l'}$ for each $l' < \omega_n$.*

Proof. Here we use the type B fusions from Case 5 of the construction of R^n. The hypothesis implies that $\langle D_{l,l'} \mid l' < \omega_n \rangle$ belongs to $L[T]$, as $A \cap \omega_{n+1} \in L[T]$. Suppose the lemma fails and choose $\hat{\eta} < (\omega_{n+2})^{L[T]}$ to be least so that there is a counterexample t, $\langle D_{l,l'} \mid l' < \omega_n \rangle$ definable over $L_{\hat{\eta}}[T]$ where $D_{l,l'} \in L_{\hat{\eta}}[T]$ for all $l' < \omega_n$. Choose $\hat{k} \geqslant 2$ so that this counterexample is $\Sigma_{\hat{k}-1}(L_{\hat{\eta}}[T])$ with parameter ω_n and let $\hat{\mathscr{A}} = \Sigma_{\hat{k}-2}$-Master Code structure for $L_{\hat{\eta}}[T]$.

Note that $\rho_1^{\hat{\mathscr{A}}} = \omega_{n+1}$ and let p be the standard parameter for $\hat{\mathscr{A}}$. Let C consist of the first ω_n ordinals $\alpha' > \omega_n$ such that $\alpha' \notin H_{\alpha'} = \Sigma_1$-Skolem hull of $\alpha' \cup \{\omega_{n+1}\} \cup \{p\}$ in $\hat{\mathscr{A}}$ and write $C = \{\alpha_0 < \alpha_1 < \cdots\}$. Now let $\alpha = \bigcup C$, $\mathscr{A} =$ transitive collapse of H_α and $\mathscr{A}' = \Sigma_1$-Master Code structure for $\mathscr{A} = \Sigma_{\hat{k}-1}$-Master Code structure for $L_\eta[T \upharpoonright \alpha]$ (for an appropriate η), $h: \alpha \rightarrow \omega_n$ a canonical $\Sigma_1(\mathscr{A}')$-injection.

We are precisely in the type B situation of Case 5 of the construction of R^n. Now pick any $\hat{\delta} < \omega_n$. Attempt to build the sequence $t_0 \geqslant_{l,1} t_1 \geqslant_{l,2} t_2 \geqslant_{l,3} \cdots$ as follows. Let $t_0 = t$. If t_i has been chosen, then let $t_{i+1} \leqslant_{l,i+1} t_i$ be least in $L[A]$ so that $t_{i+1} \in D_{l,i}$ and $\delta_i = \hat{\delta}$ where δ_i is least so that $4(\alpha_i + \delta_i) + 3 \in B^x$ for all $x \in [t_{i+1}]$. Also insist that $\alpha_i + 4(\langle 0, \delta_i \rangle) + 3 \in B^x$ for all $x \in [t_{i+1}]$ (to guarantee that $0 \in x^*$). As $\langle D_{l,i} \mid i < \omega_n \rangle$ is $\Sigma_{\hat{k}-1}(L_{\hat{\eta}}[T])$ it follows that $t_{i+1} \in R^T_{<\alpha_{i+1}}$. For limit λ let $t_\lambda = \text{glb}\{t_i \mid i < \lambda\}$.

We claim that $\hat{\delta}$ can be chosen so that $\langle t_i \mid i < \omega_n \rangle$ is well-defined and $\text{glb}\{t_i \mid i < \omega_n\} = t'$ is a condition in R^T. To see this let $h' : \alpha \to \alpha$ be $\Sigma_1(\mathscr{A}')$ and so that $h'(\delta)$ is an index for the above sequence $\langle t_i \mid i < \omega_n \rangle$ where $\hat{\delta} = h(\delta)$. By the recursion theorem we can choose $\hat{\delta} = h(\delta)$ so that $\delta, h'(\delta)$ define the same sequence $\langle t_i \mid i < \omega_n \rangle$, in which case by Case 5 of the definition of R^n, $\langle t_i \mid i < \omega_n \rangle$ is totally defined. For the latter claim we need to arrange that α is regular in $L_\eta[x]$ for $x \in [t']$.

Due to the leastness of $\hat{\eta}$ it suffices to arrange that $x \in [t'] \to x$ is $R^{T \restriction \alpha}$-generic over $L_\eta[T \restriction \alpha]$. To do so it would suffice to arrange that $x \in [t_{i+1}] \to x$ is $R^{T \restriction \alpha_{i+1}}$-generic over $L_{\eta_{i+1}}[T \restriction \alpha_{i+1}]$ (for the appropriate η_{i+1}). For simplicity suppose $\hat{k} = 2$ and let p_{i+1} be the standard parameter for $\hat{\mathscr{A}}_{i+1} = L_{\eta_{i+1}}[T \restriction \alpha_{i+1}]$. Approximate η_{i+1} by a $\Sigma_1(\hat{\mathscr{A}}_{i+1})$-sequence $\langle \eta_j^* \mid j < j_0 \rangle$ and let $\bar{\mathscr{A}}_j = L_{\bar{\eta}_j}[T \restriction \bar{\alpha}_j]$ be the transitive collapse of $H_j = \Sigma_1$-Skolem hull of $\alpha_i \cup \{\alpha_i\} \cup \{p_{i+1}\}$ in $L_{\eta_j^*}[T \restriction \alpha_{i+1}]$. Thus the $\bar{\mathscr{A}}_j$'s approximate $\hat{\mathscr{A}}_{i+1}$. If we can successively extend t_i to $\bar{t}_j$ such that $x \in [\bar{t}_j] \to x$ is $R^{T \restriction \alpha_j}$-generic over $\bar{\mathscr{A}}_j$, then after j_0 steps we have the desired t_{i+1}. Thus assuming t_{i+1} as desired does not exist, some $\bar{t}_j$ as desired does not exist. We can repeat this argument now for $\bar{t}_j$, leading ultimately to an infinite descending sequence of ordinals. Thus t_{i+1} can be found as desired and thus t' can be constructed as desired, contradicting our choice of counterexample. □

As in Section 2 we can now infer the following.

Corollary 3.10 ($<\omega_n$-Distributivity for R^T, $T \in \tilde{R}^{n+1}$). *Suppose $t \in R^T$, $T \in \tilde{R}^{n+1}$ and D_i is open, dense below t for each $i < \omega_{n-1}$. Also suppose $\langle D_i \mid i < \omega_{n-1} \rangle \in L[T]$. Then there exists $t' \leq t$ such that $t' \in D_i$ for each $i < \omega_{n-1}$.*

Corollary 3.11 (Density Reduction for R^T, $T \in \tilde{R}^{n+1}$). *Suppose $t \in R^T$, $T \in \tilde{R}^{n+1}$ and D_i is open, dense below t for $i < \omega_n$. Also suppose that $\langle D_i \mid i < \omega_n \rangle \in L[T]$. Then there exists $t' \leq t$ such that t' **reduces** each D_i, $i < \omega_n$ (i.e., for each $i < \omega_n$, $\omega_{n-1} \cdot i < j \to t'(u_j) = t''(u_j)$ for some $t'' \in D_i$, where $\langle u_j \mid j < \omega_n \rangle$ is a fixed canonical enumeration of $\tilde{R}^{n-1}$).*

We can now attack the proof of extendibility for $\mathscr{P}^A$.

Lemma 3.12 (Extendibility for $\mathscr{P}^A$). *Suppose $p \in \mathscr{P}^A$ and $\alpha(p) < \alpha < \omega_{\omega+1}$. Then there exists $q \leq p$, $\alpha(q) \geq \alpha$.*

Proof. By induction on α we show that there exists a type 1 extension (no fusion involved) $q \leq_n p$ such that $\alpha(q) = \alpha$ and $x \in [q] \to B^x$, b agree on $[\alpha(p), \alpha)$, for a given $b \subseteq [\alpha(p), \alpha)$ which is 'p-special at α'. We can assume $\alpha > \omega_\omega$.

Suppose $\alpha = \beta + 1$. We can assume that $\alpha(p) = \beta$. For simplicity assume that $n = 0$. Now consider the quasiconditions $p_0 \geq p_1 \geq \cdots$ built in Case 2 of the construction of $\mathscr{P}^A$. We must show the p_i's as defined are in fact quasiconditions;

the main thing to check is that $p_i(\omega_n) \in R^n$. (Given the existence of p_{i-1} as a quasicondition, each p_i^k can be constructed using Corollary 3.11 finitely many times.) Now the verification that $p_i(\omega_n) \in R^n$ follows from the fact that type A fusions were added in Case 5 of the construction of R^n together with the fact we specified that $B(p)$, $\langle \nu(\lambda(p), B(p)), i+2 \rangle$ givees rise to the sequence $p_{i+1}^0 \geq p_{i+1}^1 \geq \cdots, p_{i+1}^0 = p_i$. (Similarly for $\langle p_{i+1,j} \mid j < \omega_\omega \rangle$.) The only thing to check is that $x \in [p_{i+1,0}(\omega_n)] \rightarrow \bar{\alpha} = \alpha(p_{i+1,0}(\omega_n))$ is regular in $L_\eta[x]$ where the transitive collapse of $H_n^{i+2} = \Sigma_{i+2}$-Skolem hull of $\bar{\alpha} \cup \{\langle \omega_m \mid m < \omega \rangle\}$ in $\langle L_{\lambda(p)}[B(p)], C(p) \rangle$ is equal to $\langle L_{\bar{\beta}}[\bar{B}], \bar{C} \rangle$ and η is least so that α is singular in $L_{\eta+1}[\bar{B}]$. We assume that p reduces all predense $D \subseteq \mathcal{P}(\bar{p}, B(p))$, $D \in L_{\nu(\lambda(p), B(p))}[B(p)]$ (if p arises as in the first part of Case 5) or all predense $D \subseteq \mathcal{P}^{A \cap \lambda(p)}$, $D \in L_{\nu(\lambda(p), A \cap \lambda(p))}[A \cap \lambda(p)]$ (if p arises as in the second part of Case 5). (These assumptions are justified by Lemmas 3.13, 3.14.) But now we see that $x \in [p_{i+1,0}(\omega_n)] \rightarrow x$ is generic over $L_\eta[\bar{B}]$ using the above density reductions and the density reduction built into the definition of $\langle p_{i+1}^k \mid k \in \omega \rangle$. Moreover, the forcings $\mathcal{P}(\bar{p}, B(p))$, $\mathcal{P}^{A \cap \lambda(p)}$ are cardinal preserving and hence so are their transitive collapses. It follows that x preserves cardinals over $L_\eta[\bar{B}]$ and hence $\bar{\alpha}$ is regular in $L_\eta[x]$, as desired. A similar argument applies to $\langle p_{i+1,j} \mid, j < \omega_\omega \rangle$.

Suppose $\alpha < \omega_\omega$ is a limit ordinal not divisible by $\omega_\omega \cdot \omega$. We can assume that $\alpha(p) = \hat{\beta} \geq \beta$ where $\alpha = \beta + \delta$, $0 < \delta \leq \omega_\omega$, ω_ω divides β. Now define the sequence $\langle p_\gamma \mid \gamma < \delta \rangle$ as in Case 3 where $p_0 = p$, $\hat{k} = n$. By induction we need only show that p_λ is well-defined at limit stages $\lambda \leq \delta$. As in the previous case the fact that we specified that $B(p_\lambda)$, $\langle \nu(\lambda(p), B(p)) + \lambda, 1 \rangle$ gives rise to $\langle p_\gamma \mid \gamma < \lambda \rangle$ guarantees this provided we check that $x \in [p_\lambda(\omega_n)] \rightarrow \bar{\alpha} = \alpha(p_\lambda(\omega_n))$ is regular in $L_\eta[x]$ where in the present case $\langle L_\eta[\bar{B}], \bar{C} \rangle$ = transitive collapse of the Σ_1-Skolem hull of $\bar{\alpha} \cup \{\bar{p}\}$ in $\langle L_{\lambda(p_\lambda)}[B(p_\lambda)], C(p_\lambda) \rangle$, $\bar{p}$ = standard parameter for $\langle L_{\lambda(p_\lambda)}[B(p_\lambda)], C(p_\lambda) \rangle$. But by induction it is clear that x is generic for the collapse of $\mathcal{P}(p, B(p_\lambda))$. Thus the desired cardinal preservation follows from fusion for $\mathcal{P}(p, B(p_\lambda))$, which is established in Lemma 3.13.

Next suppose $\alpha = \omega_\omega \cdot \lambda$, λ limit, but α is not p.r. closed. Then the argument is identical to the one used in the proceding paragraph. Note that we can assume that $\alpha(p) \geq \beta$ = greatest p.r. closed ordinal less than α, and hence $\nu(\lambda(p_\gamma), B(p_\gamma)) = \lambda(p_\gamma)$ for all p_γ considered. Also one needs the fact that $C(p_\gamma) = C(p_{\gamma_0}) \cap \lambda(p_\gamma)$ for limit $\gamma < \gamma_0$ to verify that p_γ is a condition.

Finally, suppose α is p.r. closed. Then build $\langle p_\gamma \mid \gamma \leq \gamma_0 \rangle$ as in Case 4, obeying the proviso set forth at the start of Case 5. The only difference between this case and the previous is that for limit $\lambda \leq \gamma_0$ we need to know that $x \in [p_\lambda(\omega_n)] \rightarrow \bar{\alpha} = \alpha(p_\lambda(\omega_n))$ is regular in $L_\eta[x]$ but now possibly $\eta > \bar{\beta}$ where $\langle L_{\bar{\beta}}[\bar{B}], \bar{C} \rangle$ is the transitive collapse of the Σ_1-Skolem hull of $\bar{\alpha} \cup \{\bar{p}\}$ in $\langle L_{\lambda(p_\lambda)}[B(p_\lambda)], C(p_\lambda) \rangle$, $\bar{p}$ = standard parameter for $\langle L_{\lambda(p_\lambda)}[B(p_\lambda)], C(p_\lambda) \rangle$. In this case we need (as in the successor case $\alpha = \beta + 1$) the reduction by p_λ of all predense $D \subseteq \mathcal{P}(p, B(p_\lambda))$, $D \in L_{\nu(\lambda(p_\lambda), B(p_\lambda))}[B(p_\lambda)]$. This follows from Lemma 3.13. □

In the preceding proof we have made extensive use of the next two lemmas, which in fact are established by a simultaneous induction with Lemma 3.12.

Lemma 3.13 (Chain Condition and Density Reduction for $\mathcal{P}(p, B)$). *Suppose $p \in \mathcal{P}^A$, q is a type 1 extension of p and $B = B(q)$. Then $\mathcal{P}(p, B) = \{p' \leq p \mid B(p'), B$ agree on $[\alpha(p), \alpha(p'))\}$ obeys the $\omega_{\omega+1}$-cc in $L_{\nu(\lambda(q),B(q))}[B(q)]$ and q **reduces** all predense $D \subseteq \mathcal{P}(p, B)$, $D \in L_{\nu(\lambda(q),B(q))}[B(q)]$. (I.e., for some n, $r \leq q \to \exists r' \leq r$ ($r' \leq$ some element of D, $r'(\omega_m) = r(\omega_m)$ for all $m > n$).)*

Proof. The fact that $\mathcal{P}(p, B)$ obeys the chain condition follows from the use of $\diamondsuit(E)$ in the first part of Case 5 of the construction of $\mathcal{P}^A$. Indeed, if $D \in L_{\nu(\lambda(q),B(q))}[B(q)]$ is predense, then $\{\alpha < \lambda(q) \mid D \cap L_\alpha[B(q)]$ is predense on $\mathcal{P}(p, B) \cap L_\alpha[B(q)]\}$, is CUB in $\lambda(q)$, belongs to $L_{\nu(\lambda(q),B(q))}[B]$ and hence contains an α such that $\langle \alpha, B \cap \alpha\rangle \in E$. Then Case 5 reveals that $D \subseteq L_\alpha[B]$.

To show that q reduces all predense $D \in L_{\nu(\lambda(q),B)}[B]$ it therefore suffices to consider $D \in L_{\lambda(q)}[B(q)]$. But then we can assume that $\alpha(q)$ is a successor ordinal in which case predensity reduction follows from Case 2 of the construction of $\mathcal{P}^A$. □

Lemma 3.14 (Density Reduction for $\mathcal{P}^{A\cap\alpha}$). *Suppose $p \in \mathcal{P}^A_\alpha$ is not of type 1 (i.e., p arises from the second part of Case 5). Then p reduces all predense $D \subseteq \mathcal{P}^{A\cap\alpha} = \mathcal{P}^A \cap L_\alpha[A]$, $D \in L_{\nu(\alpha,A\cap\alpha)}[A \cap \alpha]$.*

Proof. First note that we can assume $\nu(\alpha, A \cap \alpha) > \alpha$ as otherwise we can choose $\beta < \alpha$ so that $D \in L_{\beta+1}[A]$ and choose $q \geq p$, $\lambda(q) = \beta + 1$; then the result follows from Lemma 3.13.

Now we can apply induction, using the second part of Case 5 of the construction of $\mathcal{P}^A$. Clearly p was constructed there so as to reduce all predense $D \in L_{\nu(\alpha,A\cap\alpha)}[A \cap \alpha]$. The only question is whether or not p_λ is well-defined at limit stages λ. But this is clear using the distributivity of $\mathcal{P}^{A\cap\alpha}$ (see the next lemma) and the fact that we specified that $B(p_\lambda)$, $\langle \nu(\alpha_\lambda, B(p_\lambda)), 1\rangle$ gives rise to $\langle p_i \mid i < \lambda\rangle$. □

Lemma 3.15 (Distributivity and Cardinal Preservation). (a) *Suppose $\langle D_i \mid i < \omega_n\rangle$ are n-predense on $\mathcal{P}(p, B)$ (i.e., $\forall p' \exists q' \leq_n p'$ ($q' \leq$ some element of D_i)) and $\langle D_i \mid i < \omega_n\rangle \in L_{\nu(\lambda(q),B(q))}[B(q)]$ (using the notation of Lemma 3.13). Then $p' \in \mathcal{P}(p, B) \to \exists q' \leq_n p'$ ($q' \leq$ some element of D_i for all $i < \omega_n$).*

(b) *Suppose $\langle D_i \mid i < \omega_n\rangle$ are n-predense on $\mathcal{P}^{A\cap\alpha}$, $\langle D_i \mid i < \omega_n\rangle \in L_{\nu(\alpha,A\cap\alpha)}[A \cap \alpha]$. Then $\forall p \exists q \leq_n p$ ($q \leq$ some element of D_i for all $i < \omega_n$).*

(c) *The forcings $\mathcal{P}(p, B)$, $\mathcal{P}^{A\cap\alpha}$ are cardinal-preserving over $L_{\nu(\lambda(q),B(q))}[B(q)]$, $L_{\nu(\alpha,A\cap\alpha)}[A \cap \alpha]$, respectively.*

Proof. (a) By the Chain Condition we are reduced to the case where $\alpha(q)$ is a successor ordinal. But then the result is clear using the construction of Case 2.

(b) By induction on α. We use Case 5 of the construction o $\mathcal{P}^A$. Suppose the property fails and choose η to be least so that there is a counterexample $\langle D_i \mid i<\omega_n\rangle, p$ definable over $L_\eta[A\cap\alpha]$. Choose $k\geq 2$ so that this counterexample is $\Sigma_{k-1}(L_\lambda[A\cap\alpha])$ with parameter α and let $\mathcal{A}=\Sigma_{k-2}$-Master Code structure for $L_\lambda[A\cap\alpha]$. Then $\rho_1^{\mathcal{A}}=\alpha$ and we let $\alpha_0<\alpha_i<\cdots$ be the first ω_n ordinals α' such that $\alpha'\notin H_{\alpha'}=\Sigma_1$-Skolem hull of $\alpha'\cup\{\bar{p}\}$ in $\mathcal{A}, \bar{p}=$ standard parameter for $\mathcal{A}$. Let $\hat{\alpha}=\bigcup\{\alpha_i \mid i<\omega_n\}$ and choose a canonical $\Sigma_1(\mathcal{A}')$-injection $f:\alpha\to\omega_\omega$, $\mathcal{A}'=\Sigma_{k-1}$-Master Code structure for $L_\eta[A\cap\alpha]$.

Now build the sequence $p_0\geq p_1\geq\cdots$ as follows, for any given $\hat{\delta}<\omega_\omega$. Let $p_0=p$. If p_i is defined, let $p_{i+1}\leq_{\omega_n}p_i$ be least so that $p_{i+1}\leq$ some element of D_i, p_{i+1} reduces all $D\in H_{\bar{\alpha}_i}^{\mathcal{A}_i}=\Sigma_1$-Skolem hull of $\alpha_i\cup\{\bar{p}\}$ in $\mathcal{A}_i$, where $\langle\mathcal{A}_i \mid i\in\omega_n\rangle$ is a $\Sigma_2(\mathcal{A})$-approximation to $\bar{\mathcal{A}}=$ transitive collapse $(H_{\hat{\alpha}})$ and D is predense on $\mathcal{P}^{A\cap\hat{\alpha}}$, and so that $x\in[p_{i+1}]\to\hat{\delta}=$ least δ such that $4(\alpha_i+\langle 0,\delta\rangle)+3\in B^x$. For limit λ let $p_\lambda=\text{g.l.b}\langle p_i \mid i<\lambda\rangle$. By the recursion theorem we can choose $\hat{\delta}$ so that $\langle p_i \mid i<\omega_n\rangle$ has index $f^{-1}(\hat{\delta})$ as a $\Sigma_1(\mathcal{A}')$-sequence, contradicting the choice of counterexample.

(c) This is an immediate consequence of (a), (b) and Corollary 3.11. □

Corollary 3.16. *Suppose R is $\mathcal{P}^A$-generic. Then R preserves cardinals and $A\in L[R]$.*

Proof. Clear from Lemma 3.15(c) and Lemma 3.14. □

Finally we must establish:

Lemma 3.17. *If R is $\mathcal{P}^A$-generic, the R is V-minimal.*

Proof. Suppose $p\Vdash \boldsymbol{x}\subseteq\text{ORD}$, $\boldsymbol{x}\notin V$. Suppose $q\leq p$. It suffices to show the following.

Claim. *These exists α, q_0, q_1 such that $q_0(\omega_n)=q_1(\omega_n)$ for all $n>0$, $q_0, q_1\leq q$ and $q_0\Vdash\alpha\notin\boldsymbol{x}$, $q_1\Vdash\alpha\in\boldsymbol{x}$.*

Given the Claim we can build a fusion sequence $q_0\geq q_1\geq q_2\geq\cdots$ so that $\langle q_i \mid i<\omega\rangle$ has a greatest lower bound q^* and s, t incompatible elements of $q^*(\omega)\to(q^*(\omega)_s, q^*(\omega_1),\ldots)$ and $(q^*(\omega)_t, q^*(\omega_1),\ldots)$ force different facts about $\boldsymbol{x}$. So $q^*\Vdash R\in V[\boldsymbol{x}]$.

Proof of Claim. Choose $q_0, q_1\leq q$ and α such that $q_0\Vdash\alpha\notin\boldsymbol{x}$, $q_1\Vdash\alpha\in\boldsymbol{x}$. Replace $q_0(\omega_1)$, $q_1(\omega_1)$ by $q(\omega_1)$ $(a, 2^{<\omega}, q_0(\omega_1), q_1(\omega_1))$, where we assume $a\in q_0(\omega)-q_1(\omega)$. Then replace $q_0(\omega_2)$, $q_1(\omega_2)$ by $q(\omega_2)(\tau, t_\emptyset, q_0(\omega_1), q_1(\omega_2))$ where $t_\emptyset=$

weakest element of R^1, τ is an acceptable ω-term such that $[q_0(\omega_1)] \subseteq \mathcal{P}_{\omega_1}(\tau)$, $[q_1(\omega_1)] \cap \mathcal{P}_{\omega_1}(\tau) = \emptyset$. If we continue for finitely many steps, we still have a pair q_0, $q_1 \leq q$ such that $q_0 \Vdash \alpha \notin \boldsymbol{x}$, $q_1 \Vdash \alpha \in \boldsymbol{x}$. By ω-distributivity there in fact exists a pair of conditions q_0, $q_1 \leq q$ such that $q_0 \Vdash \alpha \notin x$, $q_1 \Vdash \alpha \in \boldsymbol{x}$ but $q_0(\omega_n) = q_1(\omega_n)$ for all $n > 0$. □

This completes the proof of Minimal Coding when $A \subseteq \omega_{\omega+1}$.

4. The general case

In this section we extend the ideas of Section 3 to establish the full result. There are some new ideas here involving coding at inaccessible cardinals but most of the ideas required for the proof are implicit in Section 3.

We assume that $V = L[A]$ where $A \subseteq \mathrm{ORD}$, $2^\kappa \subseteq L_{\kappa^+}[A]$ for every infinite cardinal κ and in addition for convenience that $\kappa \cdot \omega$ divides $\lambda \in (\kappa, \kappa^+] \rightarrow L_\lambda[A] \vDash \kappa$ is the largest cardinal. Conditions in the desired forcing $\mathcal{P}^A$ for minimizing coding A are certain functions $p : \mathrm{Dom}(p) \rightarrow V$ of the form $p(\gamma) = (p_\gamma, \bar{p}_\gamma)$ where $\mathrm{Dom}(p)$ is an initial segment of CARD $= \{0\} \cup$ Infinite Cardinals. Each p_γ is a (generalized) γ^+-tree ($0^+ = \omega$) and $\bar{p}_\gamma$ effects a restraint on $B^x \cap$ Even Ordinals, for $x \in [\langle p_{\gamma'} \mid \gamma' \in \gamma \rangle]$. We shall define $R^\gamma =$ the appropriate candidates for p_γ as well as the notion of γ^+-tree by induction on γ. In addition, we define $[p_\gamma] =$ the γ^+-paths through p_γ as well as B^x for $x \in [p_\gamma]$. The need for $\bar{p}_\gamma$ is to deal with coding A at inaccessibles, as in Beller–Jensen–Welch [1]. A forcing $R^{p_{\gamma^+}}$ will be defined as well, for coding γ^{++}-paths through p_{γ^+} by a subset of γ^+ (using γ^+-trees p_γ). For limit cardinals $\kappa > \omega$ a forcing $\mathcal{P}^\kappa$ will also be considered; κ^+-trees are built from elements of $\mathcal{P}^\kappa$. To each $x \in [p_\gamma]$ will be associated a canonical sequence of γ^+-trees $\langle p^x_\alpha \mid \gamma^+ \leq \alpha \leq \alpha(p_\gamma) \rangle$ where $x \in [p^x_\alpha]$ and $\alpha(p_\gamma) < \gamma^{++}$. Similarly elements of $\mathcal{P}^\kappa$ are certain functions $p^\kappa : \mathrm{CARD} \cap \kappa \rightarrow V$ and a path through p^κ is a function $x : \mathrm{CARD} \cap \kappa \rightarrow V$ such that (among other things) $x(\gamma) \in [p^\kappa(\gamma)]$ and $x(\gamma^+) = \langle p^{x(\gamma)}_\alpha \mid \gamma^+ \leq \alpha < \gamma^{++} \rangle$ for $\gamma < \kappa$. To each path x through p^κ ($x \in [p^\kappa]$) will also be associated a canonical sequence $\langle p^x_\alpha \mid \kappa \leq \alpha < \alpha(p^\kappa) \rangle$ of conditions of $\mathcal{P}^\kappa$ such that $x \in [p^x_\alpha]$. The final generic will yield a sequence $\langle G(\gamma) \mid \gamma \in \mathrm{CARD} \rangle$ of γ^+-paths so that $G(\gamma^+) = \langle p^{G(\gamma)}_\alpha \mid \gamma^+ \leq \alpha < \gamma^{++} \rangle$ and for limit cardinals $\gamma < \omega$, $\langle G(\gamma') \mid \gamma' < \gamma \rangle$ codes $G(\gamma)$. Moreover we have $B^{G(\gamma)} \cap [\gamma^+, \gamma^{++})$ codes $A \cap [\gamma^+, \gamma^{++})$ by: $\alpha \in A$ iff $2\langle \alpha, \beta \rangle \in B^{G(\gamma)}$ for unboundedly many $\beta < \gamma^{++}$.

We now begin the inductive definition of R^γ, $\gamma \in \mathrm{CARD}$, and the related notions. We also define $\mathcal{P}^\gamma$ when $\gamma > \omega$ is a limit cardinal. We have $R^\gamma = \bigcup \{R^\gamma_\alpha \mid \alpha < \gamma^{++}\}$, $\mathcal{P}^\gamma = \bigcup \{\mathcal{P}^\gamma_\alpha \mid \alpha < \gamma^+\}$. In all cases we make use of an index i_0 describing how $A \cap \gamma^+$ is decoded (uniformly in γ) from the general real R using the parameter $\mathrm{CARD} \cap \gamma^+$, as a $\Sigma_1(L_{\gamma^+}[R])$-procedure. Also we define $\bar{R}^\gamma_\alpha$, $\bar{\mathcal{P}}^\gamma_\alpha =$ the 'canonical' elements of $R^\gamma_\alpha - R^\gamma_{<\alpha}$, $\mathcal{P}^\gamma_\alpha - \mathcal{P}^\gamma_{<\alpha}$ respectively. Then R^γ_α, $\mathcal{P}^\gamma_\alpha$ is obtained from $R^{\gamma}_{<\alpha} \cup \tilde{R}^{\gamma}_{\alpha}$, $\mathcal{P}^{\gamma}_{<\alpha} \cup \tilde{\mathcal{P}}^{\gamma}_{\alpha}$ using the operation (+) to be described below.

Definition of R^0

First define (+) to be the operation that takes ω-trees $t_0, t_1 \leqslant t$, $a \in 2^{<\omega}$ and produces $t(a, t_0, t_1) = (t_1 - (t_1)_a) \cup (t_0)_a$, where it is understood that if $a \notin t_i$, then $(t_i)_a = 0$. We will only define $\tilde{R}^0_\alpha$ for $\alpha < \omega_1$ as then $R^0_\alpha = [(+)\text{-closure of } R^0_{<\alpha} \cup \tilde{R}^0_\alpha]$.

Case 1: $\alpha = 0$. $\tilde{R}^0_0 = \{2^{<\omega}\}$.

Case 2: $\alpha = \beta + 1$. Let $t \in \tilde{R}^0_\beta$. We define the canonical extensions of t in $\tilde{R}^0_\alpha$. First fix a listing $\langle (k_i, a_i) \mid i < \omega \rangle$ of all pairs (k, a) such that $k \in \omega$, $a \in 2^{<\omega}$ and $k \geqslant \text{length(a)}$. Also define $t^k_j = \{a \mid a \subseteq f(b)$ for some $b \in 2^{<\omega}$, $b(2i) = j$ for $k < 2i < \text{length}(b)\}$ where $f: 2^{<\omega} \to \text{Split}(t)$ is bijective and $f(b * 0) \supseteq f(b) * 0$ for all $b \in 2^{<\omega}$. Now inductively define $t^0, t^1, \ldots$ as follows: t^{2i} is chosen so that $t^{2i} \leqslant_{k_i} t(a_i, t^{k_i}_0, t^{k_i}_1)$ and t^{2i} shares no path with any $t^{i'}$, $i' < 2i$; t^{2i+1} is chosen so that $t^{2i+1} \leqslant_{k_i} t(a_i, t^{k_i}_1, t^{k_i}_0)$ and t^{2i+1} shares no path with any $t^{i'}$, $i' \leqslant 2i$. Then we add all resulting t^i to $\tilde{R}^0_\alpha$, for each $t \in \tilde{R}^0_\beta$.

Case 3: α limit, α not divisible by $\omega \cdot \omega$. Write $\alpha = \beta + \omega$ where β is 0 or a limit ordinal. Choose $t \in \tilde{R}^0_{\hat{\beta}}$ for some $\hat{\beta} \in [\beta, \alpha)$ as well as an acceptable term $\hat{\sigma}$, an ordinal $\hat{\alpha} \leqslant \hat{\beta}$ ($\hat{\alpha}$ limit or 0) and an integer $\hat{k}$. We now describe a canonical extension $\hat{t} \leqslant_{\hat{k}} t$ in $\tilde{R}^0_\alpha$ which 'follows the term $\hat{\sigma}$, starting at $\hat{\alpha}$'.

Let $t_0 = t$ and if t_n is defined as an element of $\tilde{R}^0_{\hat{\beta}+n}$, then define $t_{n+1} \in \tilde{R}^0_{\hat{\beta}+n+1}$ to be $L[t]$-least so that $t_{n+1} \leqslant_{\hat{k}+n} t_n$ and $R, S \in [t_{n+1}] \to B^R, B^S$ agree on $[\hat{\beta}, \hat{\beta}+n] - \{4\gamma + 1 \mid \gamma \in \text{ORD}\}$ and $B^R(\hat{\alpha} + 4\gamma + 1) = \hat{\sigma}(R)(\gamma)$ for $4\gamma + 1 < \min(|\hat{\sigma}|, \hat{\beta} + n + 1 - \hat{\alpha})$. Define $\hat{t} = \text{glb}\{t_n \mid n \in \omega\}$.

Include all $\hat{t}$ as above in $\tilde{R}^0_\alpha$.

Case 4: α divisible by $\omega \cdot \omega$. In this case we add conditions both for extendibility and for (two types of) fusion.

We consider extendibility first. To do so we define what it means for $b \subseteq \alpha$ to be *special* at α. For any ordinal $\beta \leqslant \alpha$ divisible by $\omega \cdot \omega$ define $b^*_\beta = \{\delta < \beta \mid 4 \cdot \langle \delta, \delta' \rangle + 3 \in b$ for unboundedly many such ordinals $< \beta\}$, where $\langle \cdot, \cdot \rangle$ is a fixed pairing function on the ordinals such that $n < \omega \to \langle n, \delta \rangle < \delta + \omega$. For $b \subseteq \alpha$ to be special at α we insist that $0 \notin b^*_\beta$, β is countable in $L[b^*_\beta]$ and $b^*_\beta = b^*_\alpha \cap \beta$ for all ordinals $\beta \leqslant \alpha$ divisible by $\omega \cdot \omega$. Clearly, if $b \subseteq \alpha$ is special at α and $\beta \leqslant \alpha$ is divisible by $\omega \cdot \omega$, then $b \cap \beta$ is special at β. Also there exists $b \subseteq \beta$ which is special at α.

Now pick $t \in \tilde{R}^0_{\hat{\beta}}$, $\hat{\beta} < \alpha$ as well as an acceptable term $\hat{\sigma}$, an ordinal $\hat{\alpha} \leqslant \hat{\beta}$ ($\hat{\alpha}$ limit or 0), an integer $\hat{k}$ and $\hat{b} \subseteq \alpha$ which is special at α. We now define a canonical extension $\hat{t} \leqslant_{\hat{k}} t$ in $\bar{R}^0_\alpha$ which 'follows $\hat{\sigma}$, $\hat{\alpha}$ and $\hat{b}$'. We let $\alpha'_0 < \alpha'_1 < \cdots$ be the $L[\hat{b}^*_\alpha]$-least ω-sequence cofinal in α and let $\alpha_0 < \alpha_1 < \cdots$ be the final segment of $\alpha'_0 < \alpha'_1 < \cdots$ determined by $\alpha_0 =$ least α'_i greater than $\hat{\beta}$.

Now define $\langle t_n \mid 0 \leqslant n < \omega \rangle$ as follows: $t_0 =$ the $L[\hat{b}^*_\alpha]$-least $t_0 \leqslant_{\hat{k}} t$ in $\bar{R}^0_{\alpha_0}$ such that $R \in [t_0] \to B^R$, $\hat{b}$ agree on $[\hat{\beta}, \alpha_0) - \{4\gamma + 1 \mid \gamma \in \mathrm{ORD}\}$ and $B^R(\hat{\alpha} + 4\gamma + 1) = \hat{\sigma}(R)(\gamma)$ for $4\gamma + 1 < \min(|\hat{\sigma}|, \alpha_0 - \hat{\alpha})$. If t_n is defined, then t_{n+1} is $L[\hat{b}^*_\alpha]$-least in $\bar{R}^0_{\alpha_{n+1}}$ such that $t_{n+1} \leqslant_{\hat{k}+n} t_n$ and $R \in [t_{n+1}] \to B^R$, $\hat{b}$ agree on $[\alpha_n, \alpha_{n+1}] - \{4\gamma + 1 \mid \gamma \in \mathrm{ORD}\}$, $B^R(\hat{\alpha} + 4\gamma + 1) = \hat{\sigma}(R)(\gamma)$ for $4\gamma + 1 < \min(|\hat{\sigma}|, \alpha_{n+1} - \hat{\alpha}_0)$. Set $\hat{t} = \mathrm{glb}\{t_n \mid n \in \omega\}$.

Include in $\bar{R}^0_\alpha$ all $\hat{t}$ as above, for some choice of t, $\hat{\sigma}$, $\hat{\alpha}$, $\hat{k}$, $\hat{b}$.

Next we turn to type A fusions. We put those t into $\bar{R}^0_\alpha$ which can be written $t = \mathrm{glb}\{t_n \mid n \in \omega\}$ where $t_0 \geqslant_1 t_1 \geqslant_2 t_2 \geqslant_3 t_3 \geqslant_4 \cdots$ has the property that each $R \in [t]$ *codes* $\langle t_n \mid n \in \omega \rangle$. The latter is defined as follows. Let $\eta \geqslant \alpha$ be least so that α is not regular in $L_{\eta+1}[R]$. We require that η exists and that R codes a predicate $G^R \subseteq L_{\kappa^+}[R]$, $\kappa^+ = (\kappa^+)^{L_\eta[R]}$, via the index i_0 for decoding $\mathcal{P}^A$-generics from reals R, using the parameter $(\mathrm{CARD} \cap \kappa^+)^{L_\eta[R]}$, where $L_\eta[R] \models \kappa$ is the largest limit cardinal. Let $\bar{A}$ be decoded in $L_\eta[R]$ from R as A is decoded from $\mathcal{P}^A$-generic reals and let $\mathcal{P}^{\bar{A}}$ denote the $L_\eta[R]$ version of $\mathcal{P}^A$, with $\bar{A}$ playing the role of A. We require that $L_{\eta+1}[R] \models G^r \restriction \kappa$, $\langle \eta, k, \gamma \rangle$ give rise to the canonical ω-sequence of $\mathcal{P}^{\bar{A}}$ quasiconditions $\bar{p}_0 \geqslant \bar{p}_1 \geqslant \cdots$ for some k, γ and $\alpha = \bigcup \{|\bar{p}_n(0)| \mid n \in \omega\}$. If in addition $0 \in (B^R \cap \alpha)^*_\alpha$, then we say that R *codes* the type A fusion $\langle \bar{p}_n(0) \mid n \in \omega \rangle$.

Finally we consider the type B fusions. As in the type A case we put those t into $\bar{R}^0_\alpha$ which can be written $t = \mathrm{glb}\{t_n \mid n \in \omega\}$ where $t_0 \geqslant_1 t_1 \geqslant_2 t_2 \geqslant_3 \cdots$ has the property that each $R \in [t]$ *codes* $\langle t_n \mid n \in \omega \rangle$. The latter is defined as follows. Let η be least so that α is not regular in $L_{\eta+1}[R]$. We require that η is defined and $L_\eta[R] \models \omega_1$ is the largest cardinal. Now let $x^R = \langle t^R_\beta \mid \beta < \alpha \rangle$ and we then require that $L_\eta[R] \models$ for some β_0, x^R is a path through the ω_1-tree $T = T^{x^R}_{\beta_0}$ and R is R^T-generic over $L[A \cap \alpha, T]$.

We also require that α is $\Sigma_k(L_\eta[A \cap \alpha, T])$-projectible for some least k s.t. $k \geqslant 2$. Now let $\mathcal{A} = \Sigma_{k-2}$-Master Code structure for $L_\eta[A \cap \alpha, T]$ and we require that C has ordertype ω where C consists of all $\alpha' < \alpha$ such that $\alpha' \notin H_{\alpha'} = \Sigma_1$-Skolem hull of $\alpha' \cup \{p\}$ in $\mathcal{A}$, $p =$ standard parameter for $\mathcal{A}$. Let $\mathcal{A}' = \Sigma_1$-Master Code structure for $\mathcal{A}$ and $h: \alpha \to \omega$ a canonical $\Sigma_1(\mathcal{A}')$-injection.

For R to code the type B fusion sequence $\langle t_n \mid n \in \omega \rangle$ we require that $0 \in (B^R \cap \alpha)^*_\alpha$ and $h^{-1}(\hat{\imath})$ is a $\Sigma_1(\mathcal{A}')$-index for $\langle t_n \mid n \in \omega \rangle$, $t_{n+1} \leqslant_{n+1} t_n$ for all n where $\hat{\imath}$ is defined as follows: List $C = \{\alpha_0 < \alpha_1 < \cdots\}$ and let $i_k =$ least $i < \omega$ such that $4(\alpha_k + i) + 3 \in B^R$. We require that i_k is defined and equal to $\hat{\imath}$ for k large enough.

This completes Case 4 and also the construction of $R^0 = \bigcup \{R^0_\alpha \mid \alpha < \omega_1\}$. For $t \in R^0$ we set $\alpha(t) =$ least α such that $t \in R^0_\alpha - R^0_{<\alpha}$, where $R^0_{<\alpha} = \bigcup \{R^0_\beta \mid \beta < \alpha\}$.

For any real R, t^R_α denotes the unique $t \in \bar{R}^0_\alpha$ such that $R \in [t]$; t^R_α is defined provided $R \in [t]$ for some $t \in \bar{R}^0_\alpha$. Also B^R is defined as follows. For any ω-tree t,

$\mathrm{Split}(t) = \{s \in t \mid s*0, s*1 \in t\}$. For $s \in t$, $\|s\|$ denotes the cardinality of $\{s' \subseteq s \mid s' \in \mathrm{Split}(t)\}$. If $R \in [t]$, then R goes right at the nth level of t if $s*1 \subseteq R$ where $s \in \mathrm{split}(t)$, $\|s\| = n$. Finally, $\alpha \in B^R$ iff R goes right at the sufficiently large even levels of t^R_α.

Terms are $L[\boldsymbol{R}, R_0]$-nemes for functions from 2^ω into $2^{<\omega_1}$, for some real R_0. The class of acceptable terms is defined inductively by:

(a) Any constant term $\sigma(R) = s_0$, so a fixed element of $2^{<\omega_1}$ of limit ordinal length, is acceptable. We set $|\sigma| = |s_0|$.

(b) If σ_1, σ_2 are acceptable, $|\sigma_1| = |\sigma_2|$ and $a \in 2^{<\omega}$, then σ is acceptable where $\sigma(R) = \sigma_1(R)$ if $a \subseteq R$, $=\sigma_2(R)$ if $a \not\subseteq R$. We set $|\sigma| = |\sigma_1| = |\sigma_2|$.

(c) If σ_1, σ_2 are acceptable, then so is $\sigma_1 * \sigma_2$ where $\sigma_1 * \sigma_2(R) = \sigma_1(R) * \sigma_2(R)$ and $*$ denotes concatenation. Set $|\sigma_1 * \sigma_2| = |\sigma_1| + |\sigma_2|$.

(d) If $\sigma_1 \subseteq \sigma_2 \subseteq \cdots$ are acceptable ($\sigma \subseteq \tau$ if $\sigma(R) \subseteq \tau(R)$ for all R), then $\sigma = \bigcup \{\sigma_n \mid n \in \omega\}$ is acceptable, where $\sigma(R) = \bigcup \{\sigma_n(R) \mid n \in \omega\}$.

If $t' \leq t$ belong to $\bar{R}^0 = \bigcup \{\bar{R}^0_\alpha \mid \alpha < \omega_1\}$, then $t' \leq t$ is a type 1 extension if for some $\alpha \geq \alpha(t')$, $b \subseteq \alpha$ which is special at α, b^*_α does not contain 0 as an element and $R \in [t'] \to B^R$, b agree on $[\alpha(t), \alpha(t')) - \{4\gamma + 1 \mid \gamma \in \mathrm{ORD}\}$. This means that t' is an extension of t which arises as in the construction of R^0 but without the use of the fusions of Case 4. An equivalence relation $\sim$ on elements of $\bar{R}^0$ is defined inductively by: $t_1 \sim t_2$ if $\alpha(t_1) = \alpha(t_2)$ and either $t_1 = t_2$, or there are type 1 extensions $t_1 \leq \bar{t}_1$, $t_2 \leq \bar{t}_2$ where $\bar{t}_1 \sim \bar{t}_2$ and $R \in [t_1]$, $S \in [t_2] \to B^R, B^S$ agree on $[\alpha(\bar{t}_1), \alpha(t_1)) - \{4\gamma + 1 \mid \gamma \in \mathrm{ORD}\}$, $B^R(4\gamma + 1) = \sigma(R)(\gamma)$ and $B^S(4\gamma + 1) = \sigma(S)(\gamma)$ for some fixed term σ and all $4\gamma + 1 \in [\alpha(\bar{t}_1), \alpha(t_1))$.

This completes our present discussion of R^0. The remaining notions from the construction of R^0 which are yet to be defined will be clarified by our upcoming definitions of R^γ, $\mathcal{P}^\gamma$ for $\gamma > \omega$. We also note here that in future reference to ω-tree, acceptable ω-term, $\sim_\omega$ we are referring to the notions tree, acceptable term, $\sim$ discussed above.

Definition of R^γ, $\gamma \geq \omega$

If γ is a limit cardinal $>\omega$, then $\mathcal{P}^\gamma$ has been defined by induction. If $\gamma = \kappa^+$, $\kappa \in \mathrm{CARD}$, then define $\mathcal{P}^\gamma = R^\kappa$. In any event we have inductively defined B^x for $x \in [p]$, $p \in \mathcal{P}^\gamma$ as well as the equivalence relation $\sim_\gamma$ on elements of $\bar{\mathcal{P}}^\gamma$.

T is a γ^+-tree if $T = \langle (f_i, g_i) \mid i < \gamma^+ \rangle$ where:

(a) f_i, $g_i : \hat{\bar{R}}^T_i \to$ Acceptable γ-Terms, where $\hat{\bar{R}}^T_i$ is defined below.

(b) $|f_i(t)| = |g_i(t)|$ for all $t \in \hat{\bar{R}}^T_i$ and all $i < \gamma^+$. In addition $f_0(t_\emptyset) = g_0(t_\emptyset)$ where $t_\emptyset =$ weakest element of $\bar{\mathcal{P}}^\gamma$ and for $i > 0$: $f_i(t)(x)(0) = 0$, $g_i(t)(x)(0) = 1$ for all $x \in [t]$, $t \in \hat{\bar{R}}^T_i$.

(c) If $t_1 \sim_\gamma t_2$ belong to $\hat{\bar{R}}^T_i$, then $f_i(t_1) = f_i(t_2)$, $g_i(t_1) = g_i(t_2)$.

We define $\hat{\bar{R}}^T_i$, $\hat{R}^T_i$ by induction on i. First we have $\hat{R}^T_0 = \hat{\bar{R}}^T_0 = \{t_\emptyset\}$ where $t_\emptyset =$ weakest element of $\mathcal{P}^\gamma$. If $\hat{\bar{R}}^T_i$, $\hat{R}^T_i$ have been defined, then $t' \in \hat{\bar{R}}^T_{i+1}$ if for some $t \in \hat{R}^T_i$, $t' \leq t$ in $\bar{\mathcal{P}}^\gamma$, $\alpha(t') = \alpha(t) + \eta$ where $\eta = |f_i(t)|$ and either for all

$x \in [t']$, $B^x(\alpha(t) + 4\eta' + 1) = f_i(t)(x)(\eta')$ for $\eta' < \eta$ or for all $x \in [t']$, $B^x(\alpha(t) + 4\eta' + 1) = g_i(t)(x)(\eta')$ for $\eta' < \eta$. Then $\hat{R}^T_{i+1}$ is the $(+_\gamma)$-closure of $\hat{\hat{R}}^T_{i+1} \cup \hat{R}^T_i$. To define $\hat{\hat{R}}^T_\lambda$ for limit λ we take all $t \in \bar{\mathcal{P}}^\gamma$ which can be written $t = \mathrm{glb}\langle t_i \mid i < \delta \rangle$, δ limit where $t_0 \geq t_1 \geq \cdots$ belong to $\hat{R}^T_{<\lambda}$ and $\alpha(t) = \bigcup \{\alpha(t_i) \mid i < \sigma\}$, $\lambda = \bigcup \{\lambda' \mid t_i \in \hat{R}^T_{\lambda'} - \hat{R}^T_{<\lambda'}$ for some $i < \delta\}$. And $\hat{R}^T_\lambda = (+_\gamma)$-closure of $\hat{R}^T_{<\lambda} \cup \hat{\hat{R}}^T_\lambda$. For $t \in \hat{R}^T$, $|t|$ denotes the unique i such that $t \in \hat{R}^T_i - \hat{R}^T_{<i}$. Also set $\hat{R}^T = \bigcup \{\hat{R}^T_i \mid i < \gamma^+\}$.

The reason we use the symbol $\hat{}$ above is that we want to define R^T slightly differently. For any $\delta < \gamma^+$ let $u(\delta) = \{2\langle \delta, \delta' \rangle \mid \delta' < \gamma^+\}$. A condition in R^T is a pair $(p, \bar{p})$ where $p \in \hat{R}^T$ and $\bar{p}$ is a subset of $\gamma^+ - A$ of cardinality $\leq \gamma$. Then $(p_0, \bar{p}_0) \leq (p_1, \bar{p}_1)$ iff $p_0 \leq p_1$ in $\mathcal{P}^\gamma$, $\bar{p}_0 \subseteq \bar{p}_1$ and $x \in [p_0] \to B^x(\eta) = 0$ for all $\eta \in [\alpha(p_1), \alpha(p_0))$ which belong to $u(\delta)$ for some $\delta \in \bar{p}_1$. We also write $x \in [(p, \bar{p})]$ if $x \in [p]$ and $B^x(\eta) = 0$ for all $\eta \in [\alpha(p), \gamma^+)$ which belong to $u(\delta)$ for some $\delta \in \bar{p}$. Of course the idea is that generically $\delta \in A$ iff $u(\delta) \cap B^{G \restriction \gamma}$ is unbounded in γ^+.

If t_0, t_1 are γ^+-trees (we now use lower case letters), then we write $t_0 \leq t_1$ if $R^{t_0} \subseteq R^{t_1}$. A useful fact is the $(\leq\gamma)$-closure of the collection of γ^+-trees.

Lemma 3.1. *Suppose $\langle t_i \mid i < \lambda \rangle$ are γ^+-trees, $i < j < \lambda \to t_j \leq t_i$ and $\lambda \leq \gamma$. Then $\langle t_i \mid i < \lambda \rangle$ has a greatest lower bound.*

Proof. This is clear, given extendibility for R^t, t a γ^+-tree. The latter is established later. □

We define the operation $(+_{\gamma^+})$ on γ^+-trees. First inductively define, for $t \in \hat{R}^T$ and T a γ^+-tree, the set $T(t)$: $\alpha(t) \leq \alpha(t')$, $[t] \cap [t'] \neq 0$, $t' \in \hat{\hat{R}}^T \to t' \in T(t)$; $t_0 \in T(t)$, $t_1 \sim_\gamma t_0 \to t_1 \in T(t)$. It is easily verified that $t' \in \hat{\hat{R}}^T \to t' \in T(t)$ for at most γ-many t. Now, if $T_0, T_1 \leq T$, $t \in \hat{\hat{R}}^{T_0} \cap \hat{\hat{R}}^{T_1}$ and τ is an acceptable κ-term for some $\kappa < \gamma$, then define $T^* = T(\tau, t, T_0, T_1) = \langle (f^*_i, g^*_i) \mid i < \gamma^+ \rangle$ as follows. Suppose (f^*_i, g^*_i) is defined for $i < \delta$. Pick $t^* \in T(t)$, $t^* \in \hat{\hat{R}}^{T^*}_\delta \cap \hat{\hat{R}}^{T_0} \cap \hat{\hat{R}}^{T_1}$ and canonically choose type 1 extensions t_0, t_1 of t^* in $\hat{\hat{R}}^{T_0}$, $\hat{\hat{R}}^{T_1}$ so that $\alpha(t_0) = \alpha(t_1)$, with corresponding acceptable γ-terms σ_0, σ_1 so that $\sigma_0(x)(0) = \sigma_1(x')(0) = 0$ for $x \in [t_0]$, $x' \in [t_1]$ (that is, $x \in [t_i] \to B^x(\alpha(t^*) + 4\gamma + 1) = \sigma_i(x)(\gamma)$ for $4\gamma + 1 < \alpha(t_i) - \alpha(t^*)$). Then define $f^*_\delta(t^*) = \sigma_0(x)$ if $x \in \mathcal{P}_\gamma(\tau)$, $= \sigma_1(x)$ otherwise. Here we are using $\mathcal{P}_\gamma(\tau) = \{\gamma\text{-paths } x \mid x$ is coded by some κ^+-path $\bar{x}$ such that $z \in \mathrm{Range}(\bar{x})$, $q \in [z] \to B^q(4\gamma + 1) = \tau(q)(\gamma)$ for $4\gamma + 1 < \min(|\tau|, \alpha(z))\}$. Define g^*_δ similarly. If $t^* \notin \hat{\hat{R}}^{T_i}$ but $t^* \in T(t)$, then define $(f^*_\delta(t^*), g^*_\delta(t^*))$ to agree with T_{1-i} and if $t^* \notin T(t)$, then define $(f^*_\delta(t^*), g^*_\delta(t^*))$ so as to agree with T.

Acceptable γ^+-terms are defined as follows. A γ^+-path is a function $x : \gamma^+ \to \bar{\mathcal{P}}^\gamma$ such that for some γ-path y, $x(\alpha) = t^y_\alpha$ for all $\alpha < \gamma^+$. Acceptable γ^+-terms are certain (names for) functions $\sigma : \gamma^+\text{-paths} \to 2^{<\gamma^{++}}$, defined inductively as follows:

(a) Any constant γ^+-term $\sigma(x) = s_0$, s_0 a fixed element of $2^{<\gamma^{++}}$ of limit length, is acceptable.

(b) If σ_1, σ_2 are acceptable, $|\sigma_1| = |\sigma_2|$ and τ is an acceptable κ-term for some $\kappa \leq \gamma$, then σ is acceptable where $\sigma(x) = \sigma_1(x)$ if x is coded by some κ^+-path $\bar{x}$ such that $t \in \text{Range}(x)$, $y \in [t] \to B^y(4\gamma + 1) = \tau(y)(\gamma)$ for $4\gamma + 1 < \min(|\tau|, \alpha(t))$; $\sigma(x) = \sigma_2(x)$ otherwise.

(c) σ_1, σ_2 acceptable $\to \sigma_1 * \sigma_2$ is acceptable.

(d) $\sigma_0 \subseteq \sigma_1 \subseteq \cdots \subseteq \sigma_i \subseteq \cdots$ acceptable for $i < \delta$ (where $\delta \leq \gamma^+$) $\to \bigcup_i \sigma_i$ is acceptable.

If x is a γ^+-path, then $x \in [t]$, t a γ^+-tree, if $x(\alpha) \in \hat{\bar{R}}^t$ for unboundedly many $\alpha < \gamma^+$. In this case we say that x goes right at β on t ($\beta < \gamma^+$) if $\gamma \in [x(\alpha(t^x_\beta + 2))] \to \alpha(t^x_\beta) + 1 \in B^y$, where $t^x_\beta \in \text{Range}(x) \cap \hat{\bar{R}}^t_\beta$. Below we shall define canonical γ^+-trees t^x_α for an initial segment of $\alpha < \gamma^{++}$, for any γ^+-path x. We then define B^x by; $\alpha \in B^x$ iff x goes right at $\beta + 1$ on t^x_α for sufficiently large $\beta < \gamma^{++}$.

And we shall need the tree form of $\square_{\gamma^+}$. We fix a canonical system $\langle C^y_\alpha \mid \alpha$ limit, $\gamma^+ \leq \alpha < (\gamma^{++})^{L[y]}$, $y \subseteq \alpha \rangle$ with the properties that C^y_α is closed and uniformly definable in $L_\mu[y]$ whenever $L_\mu[y] \models \text{card}(\alpha) \leq \gamma^+$ and $\beta \in C^y_\alpha \to C^{y\cap\beta}_\beta = C^y_\alpha \cap \beta$, ordertype $(C^y_\alpha) \leq \gamma^+$ and $\bigcup C^y_\alpha = \alpha$ unless $L[y] \models \text{cof}(\alpha) = \omega$.

We now give the construction of R^γ. We will only define $\bar{R}^\gamma_\alpha$ for $\alpha < \gamma^{++}$ as then R^γ_α is obtained by taking the $(+_{\gamma^+})$-closure of $\bar{R}^\gamma_\alpha \cup R^\gamma_{<\alpha}$.

Case 1: $\alpha = 0$. $\bar{R}^\gamma_0 = \{t_\emptyset\}$ where $t_\emptyset$ is the γ^+-tree defined by $t_\emptyset = \langle (f_i, g_i) \mid i < \gamma^+ \rangle$, $f_0(t^\gamma_\emptyset) = g_0(t^\gamma_\emptyset) = \emptyset$ ($t^\gamma_\emptyset$ = weakest element of $\bar{\mathscr{P}}^\gamma$), $f_i(t)(x) = \langle 0 \rangle$ and $g_i(t)(x) = \langle 1 \rangle$ for all $t \in \hat{\bar{R}}^{t_\emptyset}_i$ and γ-paths x.

Case 2: $\alpha = \beta + 1$. Let $t \in \bar{R}^\gamma_\beta$. We describe the extensions of t in $\bar{R}^\gamma_\alpha$. First we define two special extensions t^k_0, $t^k_1 \leq t$, for each $k < \gamma^+ : t^k_i$ is characterized by the properties that $t^k_i \leq_{\omega \cdot k} t$ and $x \in [t^k_i] \to x$ goes (right if $i = 1$, left if $i = 0$) on t at $\beta + 1$, for all $\beta \geq \omega \cdot k$. Now fix a listing $\langle (k_i, \tau_i) \mid i < \gamma^+ \rangle$ of all pairs (k, τ) such that $k < \gamma^+$ and τ is an acceptable γ-term of length $|\tau| < k$. Now inductively define $t^0, t^1, \ldots$ as follows: t^{2i} is chosen so that $t^{2i} \leq_{k_i} t(\tau_i, \emptyset, t^{k_i}_0, t^{k_i}_1)$ and t^{2i} shares no path with $t^{i'}$, $i' < 2i$; t^{2i+1} is chosen so that $t^{2i+1} \leq_{k_i} t(\tau_i, \emptyset, t^{k_i}_1, t^{k_i}_0)$ and t^{2i+1} shares no path with $t^{i'}$, $i' \leq 2i$. Then add all resulting t^i to $\bar{R}^\gamma_\alpha$, for each $t \in \bar{R}^\gamma_\beta$.

Case 3: α limit, α not divisible by $\gamma^+ \cdot \omega$. Write $\alpha = \beta + \delta$ where $0 < \delta \leq \gamma^+$ and γ^+ divides β. Choose $t \in \bar{R}^\gamma_{\hat{\beta}}$ for some $\hat{\beta} \in [\beta, \alpha)$ as well as an acceptable γ^+-term $\hat{\sigma}$, an ordinal $\hat{\alpha} \leq \hat{\beta}$ ($\hat{\alpha}$ limit or 0) and an ordinal $\hat{k} < \gamma^+$. We now describe a canonical extension $\hat{t} \leq_{\hat{k}} t$ in $\bar{R}^\gamma_\alpha$ which 'obeys $\hat{\sigma}$, $\hat{\alpha}$'.

Let $t_0 = t$ and if t_i is defined in $\bar{R}^\gamma_{\hat{\beta}+i}$, then define $t_{i+1} \in \bar{R}^\gamma_{\hat{\beta}+i+1}$ to be $L[t]$-least so that $t_{i+1} \leq_{\hat{k}+i} t_i$ and $x, y \in [t_{i+1}] \to B^x$, B^y agree on $[\hat{\beta}, \hat{\beta} + i] - \{4j + 1 \mid j \in \text{ORD}\}$ and $B^x(\hat{\alpha} + 4j + 1) = \hat{\sigma}(x)(j)$ for $4j + 1 < \min(|\hat{\sigma}|, \hat{\beta} + i + 1 - \hat{\alpha})$. Define $t_\lambda = \text{glb}\langle t_i \mid i < \lambda \rangle$ for limit $\lambda \leq \alpha - \hat{\beta}$. And $\hat{t} = t_{\alpha - \hat{\beta}}$.

Include all $\hat{t}$ as above in $\bar{R}^\gamma_\alpha$.

Case 4: α divisible by $\gamma^+ \cdot \omega$. In this case we add conditions both for extendibility and for two types of fusion.

First consider extendibility. We define what it means for $b \subseteq \alpha$ to be special at α. For any $\beta \leqslant \alpha$, β divisible by $\gamma^+ \cdot \omega$ define $b^*_\beta = \{\delta < \beta \mid 4 \cdot \langle \delta, \delta' \rangle + 3 \in b$ for unboundedly many such ordinals $<\beta\}$, where $\langle \cdot, \cdot \rangle$ is a fixed paring on ORD such that $\delta < \gamma^+ \rightarrow \langle \delta, \delta' \rangle < \delta' + \gamma^+$. For b to be special at α we insist that $0 \notin b^*_\beta$, β has cardinality γ^+ in $L[b^*_\beta]$ and $b^*_\beta = b^*_\alpha \cap \beta$, for all $\beta \leqslant \alpha$ which are divisible by $\gamma^+ \cdot \omega$.

Now pick $t \in \tilde{R}^\gamma_{\hat{\beta}}$, $\hat{\beta} < \alpha$ as well as an acceptable γ^+-term $\hat{\sigma}$, an ordinal $\hat{\alpha} \leqslant \hat{\beta}$ ($\hat{\alpha}$ limit or 0), an ordinal $\hat{k} < \gamma^+$ and $\hat{b} \subseteq \alpha$ which is special at α. We define a canonical extension $\hat{t} \leqslant_{\hat{k}} t$ in $\tilde{R}^\gamma_\alpha$ which 'obeys $\hat{\sigma}$, $\hat{\alpha}$ and $\hat{b}$'.

If $C^{\hat{b}^*}_\alpha$ is unbounded in α, then let $\alpha'_0 < \alpha'_1 < \cdots$ enumerate it (where $\hat{b}^* = \hat{b}^*_\alpha$). Otherwise $\alpha'_0 < \alpha'_1 < \cdots$ is the increasing enumeration of $C^{\hat{b}^*}_\alpha$ followed by the $L[\hat{b}^*]$-least ω-sequence cofinal in α such that $\bigcup C^{\hat{b}^*}_\alpha < \beta_0$. And $\alpha_0 < \alpha_1 < \cdots$ is the final segment of $\alpha'_0 < \alpha'_1 < \cdots$ defined by $\alpha_0 =$ least α'_i greater than $\hat{\beta}$.

Now define $t_0 = t$ and if t_i is defined, then let $t_{i+1} \leqslant_{\hat{k}+1} t_i$ be $L[\hat{b}^*]$-least in $\tilde{R}^\gamma_{\alpha_{i+1}}$ such that $x \in [t_{i+1}] \rightarrow B^x$, $\hat{b}$ agree on $[\hat{\beta}, \alpha_{i+1}) - \{4j + 1 \mid j \in \text{ORD}\}$ and $B^x(\hat{\alpha} + 4j + 1) = \hat{\sigma}(x)(j)$ for $4j + 1 < \min(|\hat{\sigma}|, \alpha_{i+1} - \hat{\alpha})$. Set $t_\lambda = \text{glb}\langle t_i \mid i < \lambda \rangle$ for limit λ and $\hat{t} = t_{\lambda_0}$ where $\lambda_0 = \text{ordertype}(\alpha_0 < \alpha_1 < \cdots)$.

Include all $\hat{t}$ as above in $\tilde{R}^\gamma_\alpha$.

We now turn to type A fusions. We put those t into $\tilde{R}^\gamma_\alpha$ which can be written $t = \text{glb}\langle t_i \mid i < \lambda \rangle$ where $t_0 \geqslant_1 t_1 \geqslant_2 t_2 \geqslant_3 t_3 \geqslant_4 \cdots$ has the property that each $x \in [t]$ ***codes*** $\langle t_i \mid i < \lambda \rangle$. The latter is defined as follows. Let η be least so that α is not regular in $L_{\eta+1}[x]$. We require that η exists and that x codes a predicate $G^x \subseteq L_{\kappa^+}[x]$, $\kappa^+ = (\kappa^+)^{L_\eta[R]}$, via the index i_0 for decoding G from $x^G = G(\gamma)$ for $\mathscr{P}^A$-generic G, using the parameter $(\text{CARD} \cap \kappa^+)^{L_\eta[R]}$ where $L_\eta[x] \vDash \alpha = \gamma^{++}$ and $\kappa =$ largest limit cardinal. Let $\bar{A}$ be decoded in $L_\eta[x]$ from x as A is decoded from $x^G = G(\gamma)$ for $\mathscr{P}^A$-generic G and let $\mathscr{P}^{\bar{A}}$ denote the $L_\eta[x]$ version of $\mathscr{P}^A$, with $\bar{A}$ playing the role of A. We require that $L_{\eta+1}[x] \vDash G^x \restriction \kappa$, $\langle \eta, k, \delta \rangle$ give rise to the canonical λ-sequence of $\mathscr{P}_{\bar{A}}$-quasiconditions $\bar{p}_0 \geqslant \bar{p}_1 \geqslant \cdots$ for some k, δ, limit λ and $\alpha = \bigcup \{|\bar{p}_i(\gamma)| \mid i < \lambda\}$. If in addition $0 \in (B^x \cap \alpha)^*_\alpha$, then we say that x codes the type A fusion $\langle \bar{p}_i(\gamma) \mid i < \lambda \rangle$.

Finally we consider type B fusions. As in the type A case we put those t into $\tilde{R}^\gamma_\alpha$ which can be written $t = \text{glb}\langle t_i \mid i < \lambda \rangle$ where $t_0 \geqslant_1 t_1 \geqslant_2 t_2 \geqslant_3 \cdots$ has the property that each $x \in [t]$ ***codes*** $\langle t_i \mid i < \lambda \rangle$. The latter is defined as follows. Let η be least so that α is not regular in $L_{\eta+1}[x]$. We require that η is defined and $L_\eta[x] \vDash \gamma^{++}$ is the largest cardinal. Now let $S^x = \langle t^x_\beta \mid \beta < \alpha \rangle$ and we then require that $L_\eta[x] \vDash$ for some β_0, S^x is a path through the γ^{++}-tree $T = T^{s^x}_{\beta_0}$ and x is R^T-generic over $L[A \cap \alpha, T]$.

We also require that α is $\Sigma_k(L_\eta[A \cap \alpha, T])$-projectible for some least k and $k \geqslant 2$. Now let $\mathscr{A} = \Sigma_{k-2}$-Master Code structure for $L_\eta[A \cap \alpha, T]$ and we require that C has ordertype $\leqslant \gamma^+$ where C consists of all $\alpha' < \alpha$ such that $a' \notin H_{\alpha'} = \Sigma_1$-Skolem hull of $\alpha' \cup \{p\}$ in $\mathscr{A}$, $p =$ standard parameter for $\mathscr{A}$. Let $\mathscr{A}' = \Sigma_1$-Master Code structure for $\mathscr{A}$ and $h : \alpha \rightarrow \gamma^+$ a canonical $\Sigma_1(\mathscr{A}')$-injection.

For x to code the type B fusion $t_0 \geqslant_{\hat{\imath},1} t_1 \geqslant_{\hat{\imath},2} t_2 \geqslant_{\hat{\imath},3} \cdots$ we require that $0 \in (B^x \cap \alpha)^*_\alpha$ and $h^{-1}(\hat{\imath})$ is a $\boldsymbol{\Sigma}_1(\mathscr{A}')$-index for $t_0, t_1, \ldots$, where $\hat{\imath}$ is defined as follows; List $C=\{\alpha_0<\alpha_1<\cdots\}$ and let $i_k=$ least $i<\gamma^+$ such that $4(\alpha_k+i)+3\in B^x$. We require that i_k is defined and equal to $\hat{\imath}$ for sufficiently large $k<\text{ordertype}(C)$.

This completes the construction of $R^\gamma=\bigcup\{R^\gamma_\alpha \mid \alpha<\gamma^{++}\}$. For $t\in R^\gamma$ we set $\alpha(t)=$ least α such that $t\in R^\gamma_\alpha-R^\gamma_{<\alpha}$, where $R^\gamma_{<\alpha}=\bigcup\{R^\gamma_\alpha \mid \beta<\alpha\}$.

If $t'\leqslant t$ belong to $\tilde{R}^\gamma$, then $t'\leqslant t$ is a type 1 extension if for some $\alpha\geqslant\alpha(t')$ there exists $b\subseteq\alpha$ which is special at α, $0\notin b^*_\alpha$ such that $x\in[t']\rightarrow B^x$, b agree on $[\alpha(t),\alpha(t'))-\{4j+1 \mid j\in\text{ORD}\}$. The equivalence relation $\sim_{\gamma^+}$ on elements of $\tilde{R}^\gamma$ is defined inductively by: $t_1\sim t_2$ if $\alpha(t_1)=\alpha(t_2)$ and either $t_1=t_2$ or there are type 1 extensions $t_1\leqslant\bar{t}_1$, $t_2\leqslant\bar{t}_2$ where $\bar{t}_1\sim\bar{t}_2$ and for some acceptable γ^+-term σ, $x\in[t_1]$, $y\in[t_2]\rightarrow B^x, B^y$ agree on $[\alpha(\bar{t}_1),\alpha(t_1))-\{4j+1 \mid j\in\text{ORD}\}$, $B^x(4j+1)=\sigma(x)(j)$ and $B^y(4j+1)=\sigma(y)(j)$ for all $4j+1\in[\alpha(\bar{t}_1),\alpha(t_1))$.

This completes our present discussion of R^γ, $\gamma\leqslant\omega$.

Definition of $\mathcal{P}^\gamma$, γ an uncountable limit cardinal

A quasicondition is a sequence $p=\langle p(\bar{\gamma}) \mid \bar{\gamma}\in\text{CARD}\cap\gamma\rangle$ where $p(\bar{\gamma})=(p_{\bar{\gamma}},\bar{p}_{\bar{\gamma}})\in R^{p_{\bar{\gamma}^+}}$ for $\bar{\gamma}$ not a limit cardinal and $p(\bar{\gamma})=(p_{\bar{\gamma}},\bar{p}_{\bar{\gamma}},\bar{\bar{p}}_{\bar{\gamma}})$ where $(p_{\bar{\gamma}},\bar{p}_{\bar{\gamma}})\in R^{p_{\bar{\gamma}^+}}$ and $(p\restriction\bar{\gamma},\bar{\bar{p}}_{\bar{\gamma}})\in R^{p_{\bar{\gamma}}}$ for $\bar{\gamma}$ a limit cardinal (we have $\omega=0^+$ is a successor cardinal for these purposes). In the latter case we write $(p_{\bar{\gamma}},\bar{p}_{\bar{\gamma}},\bar{\bar{p}}_{\bar{\gamma}})\leqslant(q_{\bar{\gamma}},\bar{q}_{\bar{\gamma}},\bar{\bar{q}}_{\bar{\gamma}})$ if $(p_{\bar{\gamma}},\bar{q}_{\bar{\gamma}})\leqslant(q_{\bar{\gamma}},\bar{q}_{\bar{\gamma}})$ and $\bar{\bar{p}}_{\bar{\gamma}}\supseteq\bar{\bar{q}}_{\bar{\gamma}}$. Then $p\leqslant q$ if $p(\bar{\gamma})\leqslant q(\bar{\gamma})$ for all $\bar{\gamma}$. And x is a path through p, $x\in[p]$, if $x=\langle x(\bar{\gamma}) \mid \bar{\gamma}\in\text{CARD}\cap\gamma\rangle$ where $x(\bar{\gamma})\in[(p_{\bar{\gamma}},\bar{p}_{\bar{\gamma}})]$ for all $\bar{\gamma}\in\text{CARD}\cap\gamma$, $x\restriction\bar{\gamma}\in[(p\restriction\bar{\gamma},\bar{\bar{p}}_{\bar{\gamma}})]$ for limit $\bar{\gamma}\in\text{CARD}\cap\gamma$ and in addition $x(\bar{\gamma}^+)(\alpha)=t^{x(\bar{\gamma})}_\alpha$ for $\alpha<\bar{\gamma}^{++}$ and all $\bar{\gamma}\in\text{CARD}\cap\gamma$, $x(\bar{\gamma})(\alpha)=t^{x\restriction\bar{\gamma}}_\alpha$ for $\alpha<\bar{\gamma}^+$ and all limit $\bar{\gamma}\in\text{CARD}\cap\gamma$. A γ-path is a path through $p_\emptyset$, the weakest quasicondition.

Acceptable γ-terms are certain (names for) functions $\sigma:\gamma\text{-paths}\rightarrow 2^{<\gamma^+}$ and are defined inductively as follows:

(a) Any constant term $\sigma(x)=s_0$, s_0 a fixed element of $2^{<\gamma^+}$, is acceptable.

(b) If σ_1, σ_2 are acceptable, $|\sigma_1|=|\sigma_2|$ and τ is an acceptable $\bar{\gamma}$-term for some $\bar{\gamma}\in\text{CARD}\cap\gamma$, then σ is acceptable where σ is defined by: $\sigma(x)=\sigma_1(x)$ if $t\in\text{Range}(x(\bar{\gamma}))$, $y\in[t]\rightarrow B^y(4j+1)=\tau(y)(j)$ for $4j+1<\min(|\tau|,\alpha(t))$; $\sigma(x)=\sigma_2(x)$ otherwise.

(c) σ_1, σ_2 acceptable $\rightarrow\sigma_1*\sigma_2$ is acceptable.

(d) $\sigma_0\subseteq\sigma_1\subseteq\sigma\cdots\subseteq\sigma_i\subseteq\cdots$ acceptable for $i<\gamma'$ (where $\gamma'\leqslant\gamma)\rightarrow$ $\bigcup\{\sigma_i \mid i<\gamma\}$ is acceptable.

To each condition $p\in\mathcal{P}^\gamma$ will be assigned a pair $(\lambda(p),D(p))$ such that $D(p)\subseteq L_{\lambda(p)}$ and $L_{\lambda(p)}[D(p)]\models\gamma$ is the largest cardinal. The condition p will then be $\Sigma_2(\mathcal{A}(p))$ where $\mathcal{A}(p)=\langle L_{\gamma(p)}[D(p)],D(p)\rangle$. Moreover Σ_1-projectum$(\mathcal{A}(p))=\gamma$ and Σ_1-cofinality$(\mathcal{A}(p))\leqslant\gamma$. Each $x\in[p]$ will canonically give rise to a sequence of conditions $\langle p^x_\alpha \mid \alpha\leqslant\alpha(p)\rangle$ where $p^x_{\alpha(p)}=p$. We define B^x as follows: For each condition q, $n\leqslant\omega$, and $\bar{\gamma}\in\text{CARD}\cap\gamma$ let $X^q_n(\bar{\gamma})=\Sigma_n$-Skolem hull of $\bar{\gamma}\cup\{\gamma\}$ in $\mathcal{A}(q)$ and let $\delta^q_n(\bar{\gamma})=X^q_n(\bar{\gamma})\cap\bar{\gamma}^+$. Then $\alpha\in B^x$ iff

$4\delta_\omega^{p_\alpha^x}(\bar\gamma^+)+3\in B^{x(\bar\gamma)}$ for sufficiently large $\bar\gamma<\gamma$ (if γ is $\Sigma_n(\mathscr{A}(p_\alpha^x))$-singular for some n) and $\alpha\in B^x$ iff $4\delta_n^{p_\alpha^x}(\bar\gamma^+)+3\in B^{x(\bar\gamma)}$ for sufficiently large $\bar\gamma<\gamma$, for sufficiently large n (if γ is $\Sigma_n(\mathscr{A}(p_\alpha^x))$-regular for all n).

Write $p\leqslant_{\bar\gamma}q$ for $\bar\gamma\in\mathrm{CARD}\cap\gamma$ if $p\leqslant q$ and $p(\bar{\bar\gamma})=q(\bar{\bar\gamma})$ for $\bar{\bar\gamma}\leqslant\bar\gamma$. We shall define $\mathscr{P}^\gamma=\bigcup\{\mathscr{P}_\alpha^\gamma\mid\alpha<\gamma^+\}$ in γ^+ stages where $\mathscr{P}_\alpha^\gamma=(+)_\gamma$-closure$(\tilde{\mathscr{P}}_\alpha^\gamma\cup\mathscr{P}_{<\alpha}^\gamma)$ and where $(+)_\gamma$ is the closure operation: Given p, q and $\bar\gamma\in\mathrm{CARD}\cap\gamma$ form r by $r(\bar{\bar\gamma})=p(\bar{\bar\gamma})$ for $\bar{\bar\gamma}\geqslant\bar\gamma$ and $r(\bar{\bar\gamma})=q(\bar{\bar\gamma})$ for $\bar{\bar\gamma}<\bar\gamma$ (provided r is a quasicondition). Thus it will only be necessary to define $\tilde{\mathscr{P}}_\alpha^\gamma$ for $\alpha<\gamma^+$.

As before conditions will be added both for extendibility and fusion. In the former case we will make use of the tree forms of $\Box$ and $\Diamond$. Thus as in Chapter 6 of Beller–Jensen–Welch [1] we have a system $\langle\hat C_\alpha^y\mid\gamma\leqslant\alpha<(\gamma^+)^{L[y]},\ \gamma\cdot\omega$ divides α, $y\subseteq\alpha$ and $L_\alpha[y]\vDash\gamma$ is the largest cardinal$\rangle$ where $\hat C_\alpha^y$ is a closed subset of α of ordertype $\leqslant\gamma$, $\beta\in\hat C_\alpha^y\to\hat C_\beta^{y\cap\beta}$ is defined and equal to $\hat C_\alpha^y\cap\beta$, $\hat C_\alpha^y$ is uniformly $\Sigma_{n+1}(L_\nu[y])$ where (ν,n) is least so that α is $\Sigma_n(L_\nu[y])$-projectible, $\hat C_\alpha^y$ bounded in $\alpha\to\Sigma_{n+1}(L_\nu[y])$-cofinality$(\alpha)=\omega$ and if

$$f:\langle L_{\bar\alpha}[\bar y],\bar C\rangle\xrightarrow{\Sigma_1}\langle L_\alpha[y],\hat C_\alpha^y\rangle,$$

then $\hat C_{\bar\alpha}^{\bar y}$ is defined (where γ above is replaced by $\bar\gamma=$ largest $L_{\bar\alpha}[\bar y]$-cardinal) and equal to $\bar C$. We can also define C_α^y (for the same pairs (α,y)) to have the same properties with the exception that $\beta\in C_\alpha^y\to C_\beta^{y\cap\beta}=C_\alpha^y\cap\beta$ only for β a limit of elements of C_α^y, but now C_α^y is also unbounded in α (and $C_\alpha^y=\hat C_\alpha^y$ when the latter is unbounded in α). It will be convenient to also define $C_\alpha^y=$ the interval $(\gamma\cdot\beta,\alpha)$ when $\alpha=\gamma\cdot\beta+\delta$, $0<\delta\leqslant\gamma$ for all $y\subseteq\alpha$, $L_\alpha[y]\vDash\gamma$ is the largest cardinal.

Let $E=\{(\alpha,y)\mid\hat C_\alpha^y$ is defined and bounded in $\alpha\}$. Then E is stationary in the sense that if C_α^y is defined, $\nu(\alpha,y)=$ least ν such that α is $\Sigma_{n(\alpha,y)}(L_\nu[y])$-projectible for some least $n(\alpha,y)<\omega$ and $\nu(\alpha,y)>\alpha$, then whenever $C\in L_{\nu(\alpha,y)}[y]$ is closed unbounded in α there exists $\beta\in C$ such that $(\beta,y\cap\beta)$ belongs to E. Using this we can define a $\Diamond(E)$ system $\langle D_\alpha^y\mid(\alpha,y)\in E\rangle$ with the properties that $\nu(\alpha,y)>\alpha$, $X\in L_{\nu(\alpha,y)}[y]$, $X\subseteq\alpha\to X\cap\beta=D_\beta^y$ for some $(\beta,y\cap\beta)\in E$ and that D_α^y is uniformly definable as an element of $L_{\nu(\alpha,y)}[y]$.

We shall also make an implicit use of 'singularizing' sequences, like the C_s^* of Theorem 6.46 of Beller–Jensen–Welch [1]. However we will define these explicitly during the construction, rather than specify them in advance.

Now we turn to the construction of $\tilde{\mathscr{P}}_\alpha^\gamma$, $\alpha<\gamma^+$.

Case 1: $\alpha=0$. $\tilde{\mathscr{P}}_0^\gamma=\{p_\emptyset\}$ where $p_\emptyset$ is the weakest quasicondition. $\lambda(p_\emptyset)=\gamma$ and $D(p_\emptyset)=\emptyset$.

Case 2: $\alpha=\beta+1$. Let $p\in\tilde{\mathscr{P}}_\beta^\gamma$. We describe the extensions of p in $\tilde{\mathscr{P}}_\alpha^\gamma$. We assume inductively that for sufficiently large $\bar\gamma<\gamma$, $\alpha(p_{\bar\gamma})=\delta_1^p(\bar\gamma^+)$ and for limit $\bar\gamma$, $\alpha(\langle p_{\bar{\bar\gamma}}\mid\bar{\bar\gamma}<\bar\gamma\rangle)=\delta_1^p(\bar\gamma)$.

Case 2A: γ is $\Sigma_2(\mathscr{A}(p))$-singular. Choose canonically a continuous cofinal sequence $\langle\gamma_i \mid i<\delta\rangle$ of cardinals less than γ such that $\delta<\gamma_0$, $x\in L_{\gamma_0}[A]$ where x is the least parameter defining p and $\langle\gamma_i \mid i<\delta\rangle$ as $\Sigma_1(\mathscr{A}(p)^*)$-sequences, $\mathscr{A}(p)^* = \Sigma_1$-Master Code structure for $\mathscr{A}(p)$. We assume as well that $\langle\gamma_i \mid i<\lambda\rangle$ is $\Sigma_1(\mathscr{A}^*(p)\restriction\lambda_\lambda)$ for limit $\lambda\leqslant\delta$. Also let $\mathscr{A}(p,q)^{n+1}$ denote the Σ_n-Master Code structure for $\langle\mathscr{A}(p)^*, q\rangle$ where $q\subseteq L_\gamma[A]$.

We will now build an ω-sequence of quasiconditions $p=p_0\geqslant p_1\geqslant p_2\geqslant\cdots$ such that p_n is $\Sigma_{n+2}(\mathscr{A}(p))$. Assuming that p_n has been defined we turn to the definition of $p_{n+1}=\text{glb}\ \langle p^j_{n+1}\mid j<\gamma\rangle$. Define $p^0_{n+1}=p_n$. To define p^{j+1}_{n+1} from p^j_{n+1}, proceed as follows. Set $\hat{p}^{j+1,0}_{n+1}=p^j_{n+1}$. For $\mu<\gamma_k$ let $X^{n+2}_{\mu,k}$ equal the Σ_1-Skolem hull of $\mu+1$ in $\mathscr{A}(p,p^j_{n+1})^{n+1}\restriction\gamma_k$. Then $\hat{p}^{j+1,i+1}_{n+1}$ is the least quasicondition $q\leqslant\hat{p}^{j+1,i}_{n+1}$ such that $(q)_{\gamma_i^+}=(\hat{p}^{j+1,i}_{n+1})_{\gamma_i^+}$ (where $(r)_\mu=r-r\restriction\mu$) and for $\lambda<\mu\in\text{CARD}$, $\mu\leqslant\gamma_i: q_{\mu^+}(q_\mu)=q'(q_\mu)$ where q' is canonical and $q(\mu)$ reduces all predense $D\subseteq R^{q_{\mu^+}}$ which belong to $X^{n+2}_{\mu,i+1}\cap L_{\alpha(q_{\mu^+})}[q_{\mu^+}(q_\mu)]$. Also require that for limit cardinals μ as above, $q_\mu(q\restriction\mu)=q'(q\restriction\mu)$ where q' is canonical and $q\restriction\mu$ reduces all predense $D\subseteq\mathscr{P}^{q_\mu}$ which belong to $X^{n+2}_{\mu,i+1}\cap L_{\alpha(q_\mu)}[q_\mu(q\restriction\mu)]$. For limit $i\leqslant\lambda$ we let $\hat{p}^{j+1,i}_{n+1}=\text{glb}\langle\hat{p}^{j+1,i'}_{n+1}\mid i'<i\rangle$. Also specify that y, $\langle\nu(\lambda(p),D(p)),n+1,\lambda\cdot j+i\rangle$ gives rise to $\langle\hat{p}^{j+1,i'}_{n+1}\mid i'<i\rangle$ for each γ-path $y\in[\hat{p}^{j+1,i}_{n+1}]$. Set $\hat{p}^{j+1}_{n+1}=\hat{p}^{j+1,\lambda}_{n+1}$.

We define p^{j+1}_{n+1} from $\hat{p}^{j+1}_{n+1}$ as follows. For any quasicondition r, $\mathscr{P}(r)=\{$quasiconditions $q\leqslant r\mid(q)_{\bar\gamma}=(r)_{\bar\gamma}$ for some $\bar\gamma<\gamma\}$. Choose a canonical listing $\langle(D_j,\bar q_j)\mid j<\gamma\rangle$ of all pairs $(D,\bar q)$ where $D\in\Sigma_1(\mathscr{A}(p)^n)$ is predense on $\mathscr{P}(p_n)$, $\bar q=q\restriction\bar\gamma$ for some $q\in\mathscr{P}(p_n)$, $\bar\gamma\in\text{CARD}\cap\gamma$. Now suppose $\bar q_j$ has domain $[0,\bar\gamma]\cap\text{CARD}$ and $\bar q_j(\bar\gamma)\in R^{r_{\bar\gamma^+}}$ where $r=\hat{p}^{j+1}_{n+1}$. Also suppose that there exists $q\leqslant$ some element of D_j such that $q\leqslant\hat{p}^{j+1}_{n+1}$ and $q\restriction\bar\gamma^+=\bar q_j$. Then let $p^{j+1}_{n+1}\leqslant\hat{p}^{j+1}_{n+1}$ be least so that p^{j+1}_{n+1} agrees with such a q above $\bar\gamma$ and with $\hat{p}^{j+1}_{n+1}$ below γ^+. Otherwise $p^{j+1}_{n+1}=\hat{p}^{j+1}_{n+1}$. For limit $j\leqslant\gamma$ we set $p^j_{n+1}=\text{glb}\langle p^{j'}_{n+1}\mid j'<j\rangle$ and specify that y, $\langle\nu(\lambda(p),D(p)),n+1,\lambda\cdot j\rangle$ gives rise to $\langle p^{j'}_{n+1}\mid j'<j\rangle$ for each γ-path $y\in[p^j_{n+1}]$. Then $p_{n+1}=p^\gamma_{n+1}$.

This completes the definition of the ω-sequence of quasiconditions $p=p_0\geqslant p_1\geqslant\cdots$ and we set $\hat p=\text{glb}\langle p_n\mid n\in\omega\rangle$. Also let $\hat p(i)$, $i=0$ or 1, denote the least quasicondition $\leqslant\hat p$ such that $4\delta^p_\omega(\bar\gamma^+)+3\in_i B^{\bar y}$ when $\bar y\in[p(i)_{\bar\gamma}]$, for sufficiently large $\bar\gamma\in\text{CARD}\cap\gamma$, where $\in_0=\notin$, $\in_1=\in$. Then we specify that y, $\langle\nu(\lambda(p),D(p)),\omega,1\rangle$ gives rise to the $(\omega+\omega)$-sequence consisting of $p_0\geqslant p_1\geqslant\cdots$ followed by the constant ω-sequence $\hat p(i)\geqslant\hat p(i)\geqslant\cdots$ for each $y\in[\hat p(i)]$, $i=0$ or 1.

To complete the description of $\bar{\mathscr{P}}^\gamma_\alpha$ in this subcase we repeat the above construction (given $p\in\bar{\mathscr{P}}^\gamma_\beta$) for each $\bar\gamma\in\text{CARD}\cap\gamma$ to obtain $\hat p(i,\bar\gamma)$, where we require that all extensions $\hat p^{j,i}_n$, p^j_n are $\leqslant_{\bar\gamma}p$; thus for example in defining $\hat p^{j+1,i+1}_{n+1}$ from $\hat p^{j+1,i}_{n+1}$ we only consider $\mu>\bar\gamma$, in defining p^{j+1}_{n+1} from $\hat p^{j+1}_{n+1}$ we only consider $\bar q_j$ with domain $[0,\bar{\bar\gamma}]$ for some $\bar{\bar\gamma}\geqslant\bar\gamma$ and in defining $\hat p(i)$ from $\hat p$ we only consider $4\delta^p_\omega(\mu^+)+3\in_i B^{\hat p(i)_\mu}$ for $\mu>\bar\gamma$. Also we choose p_0 not to be p but some $p'\leqslant_{\bar\gamma}p$ such that $(p')_\mu=(p)_\mu$ form some $\mu<\gamma$ and p' is incompatible with $\hat p(i,\bar{\bar\gamma})$ for all $\bar{\bar\gamma}<\bar\gamma$. This is easily arranged through a judicious choice of $p'_{\bar\gamma^+}$.

To define the extensions of p in $\bar{\mathscr{P}}^{\gamma}_{\alpha}$ we fix a listing $\langle(\bar{\gamma}_i, \tau_i) \mid i < \gamma\rangle$ of all pairs $(\bar{\gamma}, \tau)$ where $\bar{\gamma} \in \mathrm{CARD} \cap \gamma$ and τ is an acceptable $\bar{\bar{\gamma}}$-term for some $\bar{\bar{\gamma}} < \bar{\gamma}$. Now inductively define $q_0, q_1, \ldots$ as follows: q^{2i} is chosen so that $q^{2i} \leq_{\bar{\gamma}} p(\tau, \emptyset, \hat{p}(0, \bar{\gamma}), \hat{p}(1, \bar{\gamma}))$ and q^{2i} shares no path with $q^{i'}$, $i' < 2i$; q^{2i+1} is chosen so that $q^{2i+1} \leq_{\bar{\gamma}} p(\tau, \emptyset, \hat{p}(1, \bar{\gamma}), \hat{p}(0, \bar{\gamma}))$ and q^{2i+1} shares no path with $q^{i'}$, $i' \leq 2i$. To complete this definition we must define $p^* = p(\tau, \emptyset, \hat{q}^0, \hat{q}^1)$ when $p \geq \hat{q}^0, \hat{q}^1$ are quasiconditions and τ is an acceptable $\bar{\gamma}$-term, $\bar{\gamma} < \gamma$: we have $p^* \restriction \bar{\gamma} = p \restriction \bar{\gamma}$ and $p^*_{\bar{\bar{\gamma}}} = p_{\bar{\bar{\gamma}}}(\tau, \emptyset, \hat{q}^0_{\bar{\bar{\gamma}}}, \hat{q}^1_{\bar{\bar{\gamma}}})$, $\bar{p}^*_{\bar{\bar{\gamma}}} = \bar{q}^0_{\bar{\bar{\gamma}}} \cup \bar{q}^1_{\bar{\bar{\gamma}}}$, $\bar{\bar{p}}^*_{\bar{\bar{\gamma}}} = \bar{\bar{q}}^0_{\bar{\bar{\gamma}}} \cup \bar{\bar{q}}_{\bar{\bar{\gamma}}}$ (for $\bar{\bar{\gamma}}$ limit), for $\bar{\bar{\gamma}} \in \mathrm{CARD} \cap [\bar{\gamma}, \gamma)$.

Finally add all the q^i to $\bar{\mathscr{P}}^{\gamma}_{\alpha}$, for each choice of $p \in \bar{\mathscr{P}}^{\gamma}_{\beta}$ and set $\lambda(q^i) = \lambda(p) + 1$, $D(q^i) = D(p)$.

Case 2B: γ is $\Sigma_n(\mathscr{A}(p))$-regular for each $n \in \omega$. For any $n \in \omega$ let $C_n = \{\bar{\gamma} < \gamma \mid \bar{\gamma} = \gamma \cap H(\bar{\gamma})$ where $H(\bar{\gamma}) = \Sigma_1$-Skolem hull of $\bar{\gamma} \cup \{\gamma\}$ in $\mathscr{A}(p)^{n+1}\}$ where $\mathscr{A}(p)^{n+1} = \Sigma_{n+1}$-Master Code stucture for $\mathscr{A}(p)$. Then C_n is closed, unbounded in γ and we assume inductively that for sufficiently large $\bar{\gamma} \in C_0$, $\bar{\bar{p}}_{\bar{\gamma}} = \emptyset$.

Pick $i = 0$ or 1. We build an ω-sequence of quasiconditions $p = p_0 \geq p_1 \geq \cdots$ such that p_n is $\Sigma_{n+2}(\mathscr{A}(p))$, with the property that $\hat{p}(i) = \mathrm{glb}\langle p_n \mid n \in \omega\rangle$, $x \in [\hat{p}(i)] \to B^x(\beta) = i$. As before let $\mathscr{A}(p, q)^{n+1}$ denote the Σ_n-Master Code structure for $\langle\mathscr{A}(p)^*, q\rangle$ where $q \subseteq L_\gamma[A]$ and $\mathscr{A}(p)^* = \Sigma_1$-Master Code structure for $\mathscr{A}(p)$. Let $C_n(q)$ be defined like C_n, but using $\mathscr{A}(p, q)^{n+1}$ instead of $\mathscr{A}(p)^{n+1}$.

Assuming that p_n has been defined we now define p_{n+1}. First we define a sequence $\langle p^j_{n+1} \mid j < \gamma\rangle$. Set $p^0_{n+1} = p_n$ and to define p^{j+1}_{n+1} from p^j_{n+1} proceed as follows. Set $\hat{p}^{j+1,0}_{n+1} = p^j_{n+1}$. For $j < \mu < \bar{\gamma}$ let $X^{n+2}_{\mu,\bar{\gamma}}$ equal the Σ_1-Skolem hull of $\mu + 1$ in $\mathscr{A}(p, p^j_{n+1})^{n+1} \restriction \bar{\gamma}$. Then $\hat{p}^{j+1,\gamma^+}_{n+1}$ is the least quasicondition $q \leq \hat{p}^{j+1,\bar{\gamma}}_{n+1}$ such that $(q)_{\bar{\gamma}^+} = (\hat{p}^{j+1,\bar{\gamma}}_{n+1})_{\bar{\gamma}^+}$ and for $\mu \leq \bar{\gamma}$, $q_{\mu^+}(q_\mu) = q'(q_\mu)$ where q' is canonical and $q(\mu)$ reduces all predense $D \subseteq R^{q_{\mu^+}}$ which belong to $X^{n+2}_{\mu,\bar{\gamma}^+} \cap L_{\alpha(q_{\mu^+})}[q_{\mu^+}(q_\mu)]$. Also require that for limit cardinals μ, $\mu \leq \bar{\gamma}$, $\mu \notin C^{\bar{\gamma}^+}_n(p^j_{n+1})$ (= the set defined like $C_n(p^j_{n+1})$ but using $\mathscr{A}(p, p^j_{n+1})^{n+1} \restriction \bar{\gamma}^+$): $q_\mu(q \restriction \mu) = q'(q \restriction \mu)$ where q' is canonical and $q \restriction \mu$ reduces all predense $D \subseteq \mathscr{P}^{q_\mu}$ which belong to $X^{n+2}_{\mu,\bar{\gamma}^+} \cap L_{\alpha(q_\mu)}[q_\mu(q \restriction \mu)]$. For limit $\bar{\gamma} \leq \gamma$ we let $\hat{p}^{j+1,\bar{\gamma}}_{n+1} = \mathrm{glb}\langle\hat{p}^{j+1,\bar{\bar{\gamma}}}_{n+1} \mid \bar{\bar{\gamma}} \in \mathrm{CARD} \cap \bar{\gamma}\rangle$ and specify that $y, \langle v(\lambda(p), D(p)), n+1, \gamma \cdot j + \bar{\gamma}\rangle$ gives rise to $\langle\hat{p}^{j+1,\bar{\bar{\gamma}}}_{n+1} \mid \bar{\bar{\gamma}} \in \mathrm{CARD} \cap \bar{\gamma}\rangle$ for each γ-path $y \in [\hat{p}^{j+1,\bar{\gamma}}_{n+1}]$. Set $\hat{p}^{j+1}_{n+1} = \hat{p}^{j+1,\gamma}_{n+1}$.

Define p^{j+1}_{n+1} from $\hat{p}^{j+1}_{n+1}$ as follows. For any quasicondition r, $\mathscr{P}(r) = \{$quasiconditions $q \leq r \mid (q)_{\bar{\gamma}} = (r)_{\bar{\gamma}}$ for some $\bar{\gamma} < \gamma\}$. Choose a canonical listing $\langle(D_j, q_j) \mid j < \gamma\rangle$ of all pairs $(D, \bar{q})$ where $D \in \Sigma_1(\mathscr{A}(p)^n)$ is predense on $\mathscr{P}(p_n)$, $\bar{q} = q \restriction \bar{\gamma}$ for some $q \in \mathscr{P}(p_n)$, $\bar{\gamma} \in \mathrm{CARD} \cap \gamma$. Suppose $\bar{q}_j$ has domain $[0, \bar{\gamma}] \cap \mathrm{CARD}$ and $\bar{q}_j(\bar{\gamma}) \in R^{r_{\bar{\gamma}^+}}$ where $r = \hat{p}^{j+1}_{n+1}$. Also suppose that there exists $q \in$ some element of D_j such that $q \leq \hat{p}^{j+1}_{n+1}$ and $q \restriction \gamma^+ = \bar{q}_j$. Then let $(\hat{q}, C)$ be least so that $\hat{q} \leq \hat{p}^{j+1}_{n+1}$, $\hat{q}$ agrees with such a q above $\bar{\gamma}$, $\hat{q}$ agrees with $\hat{p}^{j+1}_{n+1}$ below $\bar{\gamma}^+$ and C is closed unbounded in γ, $\alpha \in C \to \bar{\bar{q}}_\alpha = \emptyset$. Otherwise let $\hat{q} = \hat{p}^{j+1}_{n+1}$. We define $p^{j+1}_{n+1} = \hat{q}$.

For limit $j \leq \gamma$ we set $p^j_{n+1} = \mathrm{glb}\langle p^{j'}_{n+1} \mid j' < j\rangle$ and specify that

y, $\langle \nu(\lambda(p), D(p)), n+1, \gamma \cdot j\rangle$ gives rise to $\langle p_{n+1}^{j'} \mid j' < j\rangle$ for each γ-path $y \in [p_{n+1}^j]$. Set $\bar{p}_{n+1} = p_{n+1}^{\gamma}$.

Now we describe how to obtain p_{n+1}. First define a sequence $\langle \bar{p}_{n+1}^{\bar{\gamma}} \mid \bar{\gamma} \in \gamma \cap \text{CARD}\rangle$ starting with $\bar{p}_{n+1}^0 = \bar{p}_{n+1}$ just like we defined $\langle \hat{p}_{n+1}^{j+1,\bar{\gamma}} \mid \bar{\gamma} \in \gamma \cap \text{CARD}\rangle$ starting with $\hat{p}_{n+1}^{j+1,0} = p_{n+1}^j$ but using $\mathscr{A}(p)^{n+2}$ in place of $\mathscr{A}(p, p_{n+1}^j)^{n+1}$ and specifying that y, $\langle (\lambda(p), D(p)), n+1, \gamma^2 + \bar{\gamma}\rangle$ gives rise to $\langle \bar{p}_{n+1}^{\bar{\bar{\gamma}}} \mid \bar{\bar{\gamma}} < \bar{\gamma}\rangle$ for each γ-path $y \in [\bar{p}_{n+1}^{\bar{\gamma}}]$ when $\bar{\gamma} < \gamma$ is an uncountable limit cardinal. Put $p_{n+1}^* = \text{glb}\langle \bar{p}_{n+1}^{\bar{\gamma}} \mid \bar{\gamma} \in \text{CARD} \cap \gamma\rangle$ and finally let $p_{n+1} \leq p_{n+1}^*$ be least so that $\alpha(p_{n+1} \restriction \bar{\gamma}) = \alpha(p_{n+1}^* \restriction \bar{\gamma})$, $\bar{p}_{n+1_{\bar{\gamma}}} = \emptyset$ for sufficiently large $\bar{\gamma} \in C_{n+1}$ and for sufficiently large $\bar{\gamma} \in \text{CARD} \cap \gamma$: $4\delta_{n+3}^p(\bar{\gamma}^+) + 3 \in_i B^{\bar{y}}$ when $\bar{y} \in [p_{n+1_{\bar{\gamma}}}]$ (where $\in_0 = \notin$, $\in_1 = \in$). We specify that y, $\langle \nu(\lambda(p), D(p)), n+2, 0\rangle$ gives rise to the $(\gamma + \omega)$-sequence consisting of $\langle \bar{p}_{n+1}^{\bar{\gamma}} \mid \bar{\gamma} \in \gamma \cap \text{CARD}\rangle$ followed by the constant ω-sequence $p_{n+1} \geq p_{n+1} \geq \cdots$ for each $y \in [p_{n+1}]$.

This completes the definition of the sequence $\langle p_n \mid n \in \omega\rangle$. Now let $p(i) = \text{glb}\langle p_n \mid n \in \omega\rangle$ and specify that y, $\langle \nu(\lambda(p), D(p), \omega, 0\rangle$ gives rise to $\langle p_n \mid n \in \omega\rangle$ for each $y \in [p(i)]$. Repeat this construction for each $\bar{\gamma} \in \text{CARD} \cap \gamma$ to obtain $p(i, \bar{\gamma}) \leq_{\bar{\gamma}} p$ and arrange that $\bar{\gamma} \neq \bar{\gamma}' \rightarrow p(i, \bar{\gamma})$ and $p(i, \bar{\gamma}')$ have no common path. Finally proceed as in Case 2A to define $\langle q^i \mid i < \gamma\rangle$ using $p(0, \bar{\gamma})$, $p(1, \bar{\gamma})$ in place of $\hat{p}(0, \bar{\gamma})$, $\hat{p}(1, \bar{\gamma})$ and add all q^i to $\bar{\mathscr{P}}_\alpha^\gamma$, for each choice of $p \in \bar{\mathscr{P}}_\beta^\gamma$, and define $\lambda(q^i) = \lambda(p) + 1$, $D(q^i) = D(p)$.

Case 2C: γ is $\Sigma_2(\mathscr{A}(p))$-regular but $\Sigma_n(\mathscr{A}(p))$-singular for some n. Let m be largest so that γ is $\Sigma_{m+1}(\mathscr{A}(p))$-regular. Then given $i \in \{0, 1\}$ and $p \in \bar{\mathscr{P}}_\beta^\gamma$ build $p_0 \geq p_1 \geq \cdots \geq p_m$ as in Case 2B. The only difference may be that when building p_m the closed unbounded set C_{m-1} may have ordertype $<\gamma$, which however causes no difficulty in the construction. Now build $p_m \geq p_{m+1} \geq \cdots \geq \hat{p}(i)$ as in Case 2A, observing that γ is $\Sigma_{m+2}(\mathscr{A}(p))$-singular. If we repeat this for each $\bar{\gamma} \in \gamma \cap \text{CARD}$ to obtain $\hat{p}(i, \bar{\gamma}) \leq_{\bar{\gamma}} p$, we can then define as before all the extensions of p in $\bar{\mathscr{P}}_\alpha^\gamma$ in this case.

Case 3: α a limit ordinal not divisible by $\gamma \cdot \omega$. Write $\alpha = \beta + \delta$ where $0 < \delta \leq \gamma$ and γ divides β. Choose $p \in \bar{\mathscr{P}}_{\hat{\beta}}^\gamma$ for some $\hat{\beta} \in [\beta, \alpha)$ as well as an acceptable γ-term $\hat{\sigma}$, a set $\hat{b} \subseteq \alpha$, an ordinal $\hat{\alpha} \leq \hat{\beta}$ ($\hat{\alpha}$ limit or 0) and $\hat{\gamma} \in \text{CARD} \cap \gamma$. We describe a canonical extension $\hat{p} \leq_{\hat{\gamma}} p$ in $\bar{\mathscr{P}}_\alpha^\gamma$.

Let $p_0 = p$ and if p_i has been defined, then let $p_{i+1} \leq_{\hat{\gamma}+\delta} p_i$ ($\delta = \text{card}(i)$) be $L[A \cap \gamma, p_0]$-least in $\bar{\mathscr{P}}_{\hat{\beta}+i+1}^\gamma$ such that $x \in [p_{i+1}] \rightarrow B^x$, $\hat{b}$ agree at $\hat{\beta} + i$ (if $\hat{\beta} + i$ is not of the form $\hat{\alpha} + 4j + 1$) and $B^x(\hat{\alpha} + 4j + 1) = \hat{\sigma}(x)(j)$ (if $\hat{\beta} + i = \hat{\alpha} + 4j + 1$). Define $p_\lambda = \text{glb}\langle p_i \mid i < \lambda\rangle$ if λ is a limit ordinal in which case we specify that y, $\langle \nu(\lambda(p), B(p)) + \lambda, 0, 0\rangle$ gives rise to $\langle p_i \mid i < \lambda\rangle$ for $y \in [p_\lambda]$. Define $\hat{p} = p_{\alpha - \hat{\beta}}$ and let $\lambda(\hat{p}) = \lambda(p) + (\alpha - \hat{\beta})$, $B(\hat{p}) = B(p) * D$ and $D(\hat{p}) = D(p) * D$ where $D \subseteq [\lambda(p), \lambda(\hat{p}))$ codes $\hat{b}$, $\hat{\sigma}$. Include all resulting $\hat{p}$ in $\bar{\mathscr{P}}_\alpha^\gamma$.

Case 4: α divisible by $\gamma \cdot \omega$. We add conditions both for extendibility and for fusion.

First consider extendibility. We define what it means for $b \subseteq \alpha$ to be special at α. For $\beta \leq \alpha$, β divisible by $\gamma \cdot \omega$ define $b^*_\beta = \{\delta < \beta \mid 4 \cdot \langle \delta, \delta' \rangle + 3 \in b$ for unboundedly many such ordinals $<\beta\}$. For b to be special at α we require that $0 \notin b^*_\beta$, β has cardinality γ in $L[b^*_\beta]$, $(b^*_\beta)_0$ codes an acceptable γ-term $\hat{\sigma}(b^*_\beta)$ and $b^*_\beta = b^*_\alpha \cap \beta$, for all $\beta \leq \alpha$ which are divisible by $\gamma \cdot \omega$.

Now pick $p \in \mathcal{P}^\gamma_{\hat{\beta}}$, $\hat{\beta} < \alpha$ as well as an acceptable γ-term $\hat{\sigma}$, an ordinal $\hat{\alpha} \leq \hat{\beta}$ ($\hat{\alpha}$ limit or 0), $\hat{\gamma} \in \mathrm{CARD} \cap \gamma$ and $\hat{b} \subseteq \alpha$ which is special at α where $\hat{\sigma} = \hat{\sigma}(b^*_\alpha)$. We define an extension $\hat{p} \leq_{\hat{\gamma}} p$ in $\tilde{\mathcal{P}}^\gamma_\alpha$.

Let $\alpha'_0 < \alpha'_1 < \cdots$ be the increasing enumeration of $C^{\hat{b}^*}_\alpha$ (where $\hat{b}^* = \hat{b}^*_\alpha$) and define $\alpha_0 < \alpha_1 < \cdots$ to be its final segment determined by $\alpha_0 =$ least α'_i greater than $\hat{\beta}$.

Now define $p_0 = p$ and if p_i is defined, then let $p_{i+1} \leq_{\hat{\gamma}+\delta} p_i$ ($\delta = \mathrm{card}(i)$) be $L[\hat{b}]$-least in $\tilde{\mathcal{P}}^\gamma_{\alpha_{i+1}}$ such that $x \in [p_{i+1}] \to B^x, \hat{b}$ agree on $[\hat{\beta}, \alpha_{i+1}) - \{4j+1 \mid j \in \mathrm{ORD}\}$ and $B^x(\hat{\alpha} + 4j + 1) = \hat{\sigma}(x)(j)$ for $4j+1 < \min(|\hat{\sigma}|, \alpha_{i+1} - \hat{\alpha})$. Set $p_\lambda = \mathrm{glb}\langle p_i \mid i < \lambda \rangle$ for limit λ and specify that $y, \langle \nu(\lambda(p_\lambda), B(p_\lambda)), 0, 0 \rangle$ gives rise to $\langle p_i \mid i < \lambda \rangle$ for each $y \in [p_\lambda]$. Define $\hat{p} = p_{\lambda_0}$ where $\lambda_0 = \mathrm{ordertype}(C^{\hat{b}^*}_\alpha)$ and define $\lambda(\hat{p}) = \lambda(p) + (\alpha - \hat{\beta})$, $B(\hat{p}) = \bigcup \{B(p_i) \mid i < \lambda_0\}$ and $D(\hat{p}) = D(p) * D$ where $D \subseteq [\lambda(p), \lambda(\hat{p}))$ codes $\hat{b}$, $C^{\hat{b}^*}_\alpha$.

If $(\alpha, B(\hat{p})) \notin E$, then add $\hat{p}$ to $\tilde{\mathcal{P}}^\gamma_\alpha$. Otherwise see if $D^{B(\hat{p})}_\alpha$ is a dense subset of $\mathcal{P}(\bar{p}, B(\hat{p})) = \{q \leq \bar{p} \mid B(q), B(\hat{p})$ agree on $[\lambda(\bar{p}), \lambda(q)), \alpha(q) < \alpha\}$ for some $\bar{p} \geq \hat{p}$. If not, then add $\hat{p}$ to $\tilde{\mathcal{P}}^\gamma_\alpha$. If so, then add $\hat{p}$ to $\tilde{\mathcal{P}}^\gamma_\alpha$ provided $\bar{p} \geq q \geq \hat{p}$ for some q which reduces $D^{B(\hat{p})}_\alpha$; i.e., $q \geq r \in \mathcal{P}(\bar{p}, B(\hat{p})) \to \exists r' \leq r$ ($r' \in D^{B(\hat{p})}_\alpha$ and $(r)_{\bar{\gamma}} = (r')_{\bar{\gamma}}$ for some $\bar{\gamma} \in \gamma \cap \mathrm{CARD}$).

Now we turn to type A fusions. We put those p into $\tilde{\mathcal{P}}^\gamma_\alpha$ which can be written $p = \mathrm{glb}\langle p_i \mid i < \lambda \rangle$ where $p_0 \geq p_1 \geq \cdots$ has the property that each $x \in [p]$ ***codes*** $\langle p_i \mid i < \lambda \rangle$. The latter is defined as follows. Let η be least so that α is not regular in $L_{\eta+1}[x]$. We require that η exists and that x codes a predicate $G^x \subseteq L_{\kappa^+}[x]$, $\kappa^+ = (\kappa^+)^{L_\eta[x]}$ via the index i_0 for decoding G from $x^G = G \upharpoonright \gamma$ for $\mathcal{P}^A$-generic G, using the parameter $(\mathrm{CARD} \cap \kappa^+)^{L_\eta[x]}$ where $L_\eta[x] \models \alpha = \gamma^+$ and $\kappa =$ the largest limit cardinal. Let $\bar{A}$ be decoded in $L_\eta[x]$ from x as A is decoded from $x^G = G \upharpoonright \gamma$ for $\mathcal{P}^A$-generic G and let $\mathcal{P}^{\bar{A}}$ denote the $L_\eta[x]$-version of $\mathcal{P}^A$, with $\bar{A}$ playing the role of A. We require that $L_{\eta+1}[x] \models G^x \upharpoonright \kappa, \langle \eta, \bar{\gamma}, \delta \rangle$ give rise to the canonical λ-sequence of $\mathcal{P}^{\bar{A}}$-quasiconditions $\bar{p}_0 \geq \bar{p}_1 \geq \cdots$ for some $\bar{\gamma}, \delta$, limit λ and $\alpha = \bigcup \{\alpha(\bar{p}_i \upharpoonright \gamma) \mid i < \lambda\}$. If in addition $0 \in (B^x \cap \alpha)^*_\alpha$, then we say that x codes the type A fusion $\langle \bar{p}_i \upharpoonright \gamma \mid i < \lambda \rangle$. Also let $\bar{\mathcal{A}} = \mathcal{A}(\bar{p})$ be defined for $\bar{p} = \mathrm{glb}\langle \bar{p}_i \mid i < \lambda \rangle$ (as in Cases 2, 3 or Extendibility, Type B Fusion parts of Case 4) and for $p = \mathrm{glb}\langle p_i \upharpoonright \gamma \mid i < \lambda \rangle$ we define $D(p)$ to be a subset of $\alpha = \lambda(p)$ that codes $\langle \bar{\mathcal{A}}_{\bar{\alpha}} \mid \bar{\alpha} < \alpha \rangle$ where $\bar{\mathcal{A}}_{\bar{\alpha}} =$ transitive collapse (Σ_1-Skolem hull of $\bar{\alpha} \cup \{\kappa\}$ in $\bar{\mathcal{A}}$). (We require that Σ_1-projectum$(\bar{\mathcal{A}}) = \kappa$.) We also let $B(p)$ equal $D(p)$.

Finally we consider type B fusions. We put those p into $\tilde{\mathcal{P}}^\gamma_\alpha$ which can be written $p = \mathrm{glb}\langle p_i \mid i < \lambda \rangle$ where $p_0 \geq p_1 \geq \cdots$ is ***coded*** by each $x \in [p]$. To define this notion of coding let η be least so that α is not regular in $L_{\eta+1}[x]$. We require that η is defined and $L_\eta[x] \models \alpha = \gamma^+$ is the largest cardinal. Now let $S^x = \langle p^x_\beta \mid \beta <$

$\alpha\rangle$ and then we require that $L_\eta[x] \models S^x$ is a path through the γ^+-tree $t = t^{S^x}_{\beta_0}$ where x is R^t-generic over $L[A \cap \alpha, t]$.

We also require that α is $\Sigma_l(L_\eta[A \cap \alpha, t])$-projectible for some least k and $k \geq 2$. Now let $\mathcal{A} = \Sigma_{k-2}$-Master Code structure for $L_\eta[A \cap \alpha, t]$ and we require that C has ordertype $\leq \gamma$ where C consists of all $\alpha' < \alpha$ such that $\alpha' \notin H_{\alpha'} = \Sigma_1$-Skolem hull of $\alpha' \cup \{q\}$ in $\mathcal{A}$, q = standard parameter for $\mathcal{A}$. Let $\mathcal{A}' = \Sigma_1$-Master Code structure for $\mathcal{A}$ and $h : \alpha \to \gamma$ a canonical $\Sigma_1(\mathcal{A}')$-injection.

For x to code the type B fusion $p_0 \geq p_1 \geq \cdots$ we require that $0 \in (B^x \cap \alpha)^*_\alpha$, $p = \mathrm{glb}\langle p_i \mid i < \mathrm{ordertype}(C)\rangle$ reduces all predense $D \subseteq R^t_{<\alpha}$, $D \in \mathcal{A}$ and $h^{-1}(\hat{i})$ is a $\boldsymbol{\Sigma}_1(\mathcal{A}')$ index for $p_0 \geq p_1 \geq \cdots$ where $\hat{i}$ is defined as follows: List $C = \{\alpha_0 < \alpha_1 < \cdots\}$ and let i_k = least $i < \gamma$ such that $4(\alpha_k + i) + 3 \in B^x$. We require that i_k is defined and equal to $\hat{i}$ for sufficiently large $k <$ ordertype(C).

Finally define $\mathcal{A}(p) = \mathcal{A}'$ for p as above (thus $D(p)$ is a Σ_1-Master Code for $\mathcal{A}$) and set $B(p) = D(p)$.

This completes the construction of $\mathcal{P}^\gamma = \bigcup\{\mathcal{P}^\gamma_\alpha \mid \alpha < \gamma^+\}$. For $p \in \mathcal{P}^\gamma$ we set $\alpha(p)$ = least α such that $p \in \mathcal{P}^\gamma_\alpha - \mathcal{P}^\gamma_{<\alpha}$, where $\mathcal{P}^\gamma_{<\alpha} = \bigcup\{\mathcal{P}^\gamma_\beta \mid \beta < \alpha\}$.

If $p' \leq p$ belong to $\bar{\mathcal{P}}^\gamma$, then $p' \leq p$ is a type 1 extension if for some $\alpha \geq \alpha(p')$ there exists $b \subseteq \alpha$ which is special at α, $0 \notin b^*_\alpha$ such that $x \in [p'] \to B^x$, b agree on $[\alpha(p), \alpha(p')) - \{4j+1 \mid j \in \mathrm{ORD}\}$. The equivalence relation $\sim_\gamma$ on elements of $\bar{\mathcal{P}}^\gamma$ is defined inductively by: $p_1 \sim_\gamma p_2$ if $\alpha(p_1) = \alpha(p_2)$ and either $p_1 = p_2$ or there are type 1 extensions $p_1 \leq \bar{p}_1$, $p_2 \leq \bar{p}_2$ where $\bar{p}_1 \sim_\gamma \bar{p}_2$ and for some γ-term σ, $(x \in [p_1], y \in [p_2]) \to B^x, B^y$ agree on $[\alpha(\bar{p}_1), \alpha(p_1)) - \{4j+1 \mid j \in \mathrm{ORD}\}$, $B^x(4j+1) = \sigma(x)(j)$ and $B^y(4j+1) = \sigma(y)(j)$ for all $4j+1 \in [\alpha(\bar{p}_1), \alpha(p_1))$.

This completes our definition of $\mathcal{P}^A = \bigcup\{\mathcal{P}^\gamma \mid \gamma$ an uncountable limit cardinal$\}$ (where $p \leq q$ if $p \upharpoonright \mathrm{Dom}(q) \leq q$). We now prove a series of lemmas which show that a $\mathcal{P}^A$-generic real minimally codes A. These lemmas concern fusion, chain conditions, extendibility and are established by a simultaneous induction.

We first consider distributivity and fusion for the forcings R^T, $T \in \bar{R}^{\gamma^+}$. If t, t' are γ^+-trees for $\gamma > 0$, then we write $t \leq_{l,l'} t'$ $(l, l' < \gamma^+)$ provided $t \leq_l t'$ (i.e., $\bar{R}^t_l = \bar{R}^{t'}_l$) and $u \in \bar{R}^{t'}_{l'} \to u \in \bar{R}^t_{l'}$ unless $u \in t'(u_i)$ for some i, $\gamma \cdot i \geq l'$. Here we are using a fixed canonical enumeration of $\mathcal{P}^\gamma$ of length γ^+. If $t = (t_0, \bar{t}_1)$, $t' = (t'_0, \bar{t}'_1) \in R^T$ $(T \in R^{\gamma^+})$, then $t \leq_{l,l'} t'$, $t \leq_l t'$ iff $t_0 \leq_{l,l'} t'_0$, $t_0 \leq_l t'_0$. And $D \subseteq R^T$ (for $T \in \bar{R}^{\gamma^+}$) is l, l'-dense below $t \in R^T$ if $t' \leq_l t \to \exists t'' \leq_{l,l'} t'$ $(t'' \in D)$.

Lemma 4.1 (Fusion for R^T, $T \in \bar{R}^{\gamma^+}$). *Suppose $t \in R^T$, $T \in \bar{R}^{\gamma^+}$, $l < \gamma^+$ and $D_{l,l'}$ is open and l, l'-dense below t for all $l' < \gamma^+$. Also suppose that $\langle D_{l,l'} \mid l' < \gamma^+\rangle \in L[T, A \cap \gamma^{++}]$. Then there exists $t' \leq_l t$ such that $t' \in D_{l,l'}$ for all $l' < \gamma^+$.*

Proof. We use the type B fusions of Case 4 of the construction of R^γ. Suppose the lemma fails and choose $\hat{\eta} < \gamma^{+++}$ (in the sense of $L[T, A \cap \gamma^{++}]$) to be least so that there is a counterexample t, $\langle D_{l,l'} \mid l' < \gamma^+\rangle$ definable over $L_{\hat{\eta}}[T, A \cap$

$\gamma^{++}]$. Choose $\hat{k} \geqslant 2$ so that this counterexample is $\Sigma_{\hat{k}-1}(L_{\hat{\eta}}[T, A \cap \gamma^{++}])$ with parameter γ^{++} and let $\hat{\mathscr{A}} = \Sigma_{\hat{k}-2}$-Master Code structure for $L_{\hat{\eta}}[T, A \cap \gamma^{++}]$. We also assume that $\alpha(T)$ is minimized among $T \in \tilde{R}^{\gamma^+}$ for which the lemma fails.

Note that $\rho_1^{\hat{\mathscr{A}}} = \gamma^{++}$ and let p be the standard parameter for $\hat{\mathscr{A}}$. Let $C = \{\alpha_0, \alpha_1, \ldots\}$ consist of the first γ^+ ordinals α' such that $\gamma^+ < \alpha' \notin H_{\alpha'} = \Sigma_1$-Skolem hull of $\alpha' \cup \{\gamma^{++}\} \cup \{p\}$ in $\hat{\mathscr{A}}$. Now let $\alpha = \cup C$, $\mathscr{A}$ = transitive collapse of H_α and $\mathscr{A}' = \Sigma_1$-Master Code structure for $\mathscr{A}$, $h : \alpha \to \gamma^+$ a canonical $\Sigma_1(\mathscr{A}')$-injection.

Now we are precisely in the type B situation of Case 4 of the construction of $\tilde{R}^{\gamma^+}$. Pick any $\hat{j} < \gamma^+$. We attempt to build the sequence $t_0 \geqslant_{l,0} t_1 \geqslant_{l,1} t_2 \geqslant_{l,2} \cdots$ as follows. Let $t_0 = t$. If t_i has been chosen, then let $t_{i+1} \leqslant_{l,i} t_i$ be least so that $t_{i+1} \in D_{l,i}$ and $\hat{j}_i = \hat{j}$ where $\hat{j}_i$ is least so that $4(\alpha_i + \hat{j}_i) + 3 \in B^x$ for all $x \in [t_{i+1}]$. Also insist that $\alpha_i + 4(\langle 0, \hat{j}_i \rangle) + 3 \in B^x$ for all $x \in [t_{i+1}]$ (to guarantee that $0 \in (B^x \cap \alpha)^*_\alpha$). As $\langle D_{l,i} \mid i < \gamma^+ \rangle$ is $\Sigma_{\hat{k}-1}(L_{\hat{\eta}}[T, A \cap \gamma^{++}])$ it follows that $t_{i+1} \in R^T_{<\alpha_{i+1}}$. For limit λ let $t_\lambda = \mathrm{glb}\langle t_i \mid i < \lambda \rangle$.

We claim that $\hat{j} < \gamma^+$ can be chosen so that $\langle t_i \mid i < \gamma^+ \rangle$ is well-defined and $\mathrm{glb}\langle t_i \mid i < \gamma^+ \rangle = t'$ is a well-defined condition in R^T. To see this let $h' : \alpha \to \alpha$ be $\Sigma_1(\mathscr{A}')$ and so that $h'(j)$ is a $\Sigma_1(\mathscr{A}')$-index for the above sequence $\langle t_i \mid i < \gamma^+ \rangle$ where we have chosen $\hat{j} = h(j)$. By the Recursion Theorem we can choose $\hat{j} = h(j)$ so that j and $h'(j)$ define the same sequence $\langle t_i \mid i < \gamma^+ \rangle$. But then Case 4 of the construction of R^{γ^+} shows that $\langle t_i \mid i < \gamma^+ \rangle$, t' are well-defined, provided we can verify: $x \in [t_\lambda]$, λ limit $\to \alpha_\lambda$ is regular in $L_{\eta_\lambda}[x]$ where $\eta_\lambda = \mathrm{ORD} \cap$ (transitive collapse of H_{α_λ}).

To arrange this last property, by the leastness of $\hat{\eta}$ it suffices to arrange that $x \in [t_\lambda] \to x$ is $R^{T \restriction \alpha_\lambda}$-generic over $L_{\eta_\lambda}[T \restriction \alpha_\lambda, A \cap \alpha_\lambda]$. But this can be arranged just as in the proof of Lemma 3.9.

Now the condition $t' \in \bigcap \{D_{l,i} \mid i < \gamma^+\}$ provides a counterexample to the choice of t, $\langle D_{l,i} \mid i < \gamma^+ \rangle$. □

Corollary 4.1A ($\leqslant\gamma$-Distributivity for R^T, $T \in \tilde{R}^{\gamma^+}$). *Suppose $t \in R^T$, $T \in \tilde{R}^{\gamma^+}$ and D_i is open, dense below t for each $i < \gamma$. Also suppose that $\langle D_i \mid i < \gamma \rangle \in L[T, A \cap \gamma^{++}]$. Then there exists $t' \leqslant t$ such that $t' \in D_i$ for all $i < \gamma$.*

Corollary 4.1B (Density Reduction for R^T, $T \in \tilde{R}^{\gamma^+}$). *Suppose $t \in R^T$, $T \in \tilde{R}^{\gamma^+}$ and D_i is open, dense below t for each $i < \gamma^+$, $\langle D_i \mid i < \gamma^+ \rangle \in L[T, A \cap \gamma^{++}]$. Then there exists $t' \leqslant t$ which* ***reduces*** *each D_i (i.e., for $\gamma \cdot i < j < \gamma^+$, $t'(u_j) = t''(u_j)$ for some $t'' \in D_i$, where $\langle u_j \mid j < \gamma^+ \rangle$ enumerates $\tilde{\mathscr{P}}^\gamma$). In particular, if D_i^* is open dense on $\{(t, u) \mid t \in R^T,\ u \in R^t\}$ and $\langle D_i^* \mid i < \gamma^+ \rangle \in L[T, A \cap \gamma^{++}]$, then there exists $t' \leqslant t$ such that $\{u \in R^{t'} \mid (t', u) \in D_i\}$ is dense on $R^{t'}$ for all $i < \gamma^+$.*

An entirely similar argument establishes the following.

Lemma 4.2 (Fusion for R^T, $T \in \tilde{R}^\gamma$, $\gamma > \omega$ a limit cardinal). *Suppose $p \in R^T$, $T \in \tilde{R}^\gamma$, γ an uncountable limit cardinal and for $\gamma_0 \leq i < \gamma$, $D_i \subseteq R^T$ is open and* card(i)*-dense below p (i.e., $q \leq_{\bar{\gamma}} p \to \exists r \leq_{\bar{\gamma}} q$ $(r \in D_i)$ where $\bar{\gamma} = \text{card}(i)$). Also suppose that $\langle D_i \mid \gamma_0 \leq i < \gamma\rangle \in L[T, A \cap \gamma^+]$. Then there exists $p' \leq_{\gamma_0} p$ such that $p' \in D_i$ for all $i \in [\gamma_0, \gamma)$.*

Proof. Use the type B fusion part of Case 4 of the construction of $\mathscr{P}^\gamma$, γ an uncountable limit cardinal. We need the form of density reduction stated in Corollary 4.2B below to guarantee that $x \in [t_\lambda] \to x$ is $R^{T \upharpoonright \alpha_\lambda}$-generic and hence α_λ is regular in $L_{\eta_\lambda}[x]$. □

Corollary 4.2A (Distributivity for R^T, $T \in \tilde{R}^\gamma$, γ an uncountable limit cardinal). *Suppose $p \in R^T$, $T \in \tilde{R}^\gamma$, γ an uncountable limit cardinal and $D_i \subseteq R^T$ is open and $\bar{\gamma}$-dense below p for $i < \bar{\gamma}$. Also suppose that $\langle D_i \mid i < \bar{\gamma}\rangle \in L[T, A \cap \gamma^+]$. Then there exists $q \leq p$ such that $q \in \bigcap \{D_i \mid i < \bar{\gamma}\}$.*

Corollary 4.2B (Density Reduction for R^T, $T \in \tilde{R}^\gamma$, γ an uncountable limit cardinal). *Suppose $p \in R^T$, $T \in \tilde{R}^\gamma$, γ an uncountable limit cardinal and $D_i \subseteq R^T$ is open and dense below p for each $i < \gamma$. Also suppose that $\langle D_i \mid i < \gamma\rangle \in L[T, A \cap \gamma^+]$. Then there exists $q \leq p$ which* **reduces** *each D_i (i.e., for each $i < \gamma$ there exists $\bar{\gamma} \in \text{CARD} \cap \gamma$ such that $r \leq q \to \exists r' \leq r$ $((r' \upharpoonright \bar{\gamma}) \cup (q)_{\bar{\gamma}} \in D_i)$.*

The proof of Lemma 4.1 implicity used the following.

Lemma 4.3 (Extendibility for R^T, $T \in \tilde{R}^{\gamma^+}$). *Suppose $t \in R^T_i$, $T \in \tilde{R}^{\gamma^+}$, $l < \gamma^+$ and $i < j < \gamma^{++}$. Then there exists $t' \leq_l t$, $t' \in R^T_j$.*

The proof of Lemma 4.3 depends on the next lemma.

Lemma 4.4 (Extendibility for $\tilde{R}^\gamma$). *Suppose $t \in \tilde{R}^\gamma$, $k < \gamma^+$, δ is an acceptable γ^+-term and $\hat{\alpha} \leq \alpha(t)$ is 0 or a limit ordinal. Also suppose that $x \in [t] \to B^x(\hat{\alpha} + 4j + 1) = \sigma(x)(j)$ for $4j + 1 < \min(|\sigma|, \alpha(t) - \hat{\alpha})$. Then if $\alpha > \alpha(t)$, $\alpha < \gamma^{++}$ there exists $t' \leq_k t$, $t' \in \tilde{R}^\gamma_\alpha$ such that $t' \leq t$ is a type 1 extension and $x \in [t'] \to B^x(\hat{\alpha} + 4j + 1) = \sigma(x)(j)$ for $4j + 1 < \min(|\sigma|, \alpha(t') - \hat{\alpha})$.*

Proof. First we note that if α is divisible by $\gamma^+ \cdot \omega$, then there exists a special $b \subseteq \alpha$ such that $(b^*_\alpha)_0$ codes the term σ. To see this we need only choose $X \subseteq \alpha$ such that $0 \notin X$, $L[(X)_1 \cap \gamma^+] \vDash \text{card}(\alpha) = \gamma^+$ and $(X)_0 \cap \beta$ codes $\sigma \upharpoonright \beta$ for $\beta < \alpha$ divisible by $\gamma^+ \cdot \omega$; then define b so that $4 \cdot \langle \delta, \delta' \rangle + 3 \in b$ iff $\delta \in X$. We see that $\beta < \alpha$, β divisible by $\gamma^+ \cdot \omega \to b^*_\beta = X \cap \beta$ and hence b is special at α.

The lemma is established by induction on α. We show that if α is divisible by $\gamma^+ \cdot \omega$, then for any $b \subseteq \alpha$ which is special at α such that $\sigma = \hat{\sigma}(b^*_\alpha)$ there exists t'

as in the lemma where in addition $x \in [t'] \to B^x$, b agree on $[\alpha(t), \alpha) - \{4j + 1 \mid j \in \text{ORD}\}$. If α is not divisible by $\gamma^+ \cdot \omega$, we show the same for any $b \subseteq \alpha$.

If $\alpha = 0$, there is nothing to show.

Suppose $\alpha = \beta + 1$. By induction first extend t to t'', $\alpha(t'') = \beta$ obeying the above for $b \cap \beta$. The extension of t'' to the desired $t' \in \tilde{R}^\gamma_\alpha$ is clear if β is not of the form $\hat{\alpha} + 4j + 1$ by Case 2 of the construction of R^γ. If $\beta = \hat{\alpha} + 4j + 1$, then we induct on the formation of the term σ, the important case being (b) in the definition of γ^+-term. But then the construction of the t^i in Case 2 shows that the desired extension exists.

Suppose α is a limit ordinal not divisible by $\gamma^+ \cdot \omega$ so we can write $\alpha = \beta + \delta$, $0 < \delta \leq \gamma^+$ and γ^+ divides β. By induction we can extend to $t'' \in \tilde{R}^\gamma_{\beta'}$ obeying the above, where $\beta' = \max(\beta, \alpha(t))$. But then make successive extensions $t'' \geqslant_k t''_1 \geqslant_k t''_2 \geqslant_k \cdots$ as in Case 3 of the construction of $\tilde{R}^\gamma$ where $\alpha(t''_i) = \beta' + i$. Finally $t' = t''_{\alpha - \beta'}$ is as desired, as the sequence $\langle t''_i \mid 0 \leq i \leq \alpha - \beta' \rangle$ is clearly well-defined at limit stages.

Finally suppose α is divisible by $\gamma^+ \cdot \omega$. Consider $C^{b^*_\alpha}_\alpha$. If it is bounded in α, then let C consist of $C^{b^*_\alpha}_\alpha \cup \{\beta_0, \beta_1, \ldots\}$ as in Case 4 of the construction of $\tilde{R}^\gamma$; if it is unbounded in α, then let $C = C^{b^*_\alpha}_\alpha$. We also let $\alpha_0 < \alpha_1 < \cdots$ enumerate the final segment of C determined by $\alpha_0 =$ least element of C greater than $\alpha(t)$. Now define $t = t_0 \geqslant_k t_1 \geqslant_k \cdots$ as in Case 4 where $\hat{k} = k$, $\hat{b} = b$, $\hat{\sigma} = \sigma$. The desired t' is $\hat{t}$ as defined there. The only thing to check is that $\langle t_i \mid i < \lambda_0 \rangle$ is well-defined at limit stages, $\lambda_0 = \text{ordertype}(\{\alpha_0, \alpha_1, \ldots\})$. But this is clear as β a limit point of $C \to \gamma^+ \cdot \omega$ divides β, $b^*_\beta = b^*_\alpha \cap \beta$ and hence $C^{b^*_\beta}_\beta = C \cap \beta$. □

Proof of Lemma 4.3. We first consider the case $t = (t_0, \bar{t}_1)$ where $\bar{t}_1 = \emptyset$. Thus we want to show that $t \in \hat{R}^T_i \to \exists t' \leqslant_l t \ (t' \in \hat{R}^T_j)$. It suffices to consider $t \in \hat{\hat{R}}^T_i$. By Lemma 4.4 we can assume that j is a limit ordinal. Choose $i = j_0 < j_i < \cdots$ cofinal in j of ordertype $\lambda \leq \gamma^+$. We inductively build a sequence $t = t_0, t_1, \ldots$ of length λ so that $t_k \leqslant_l t$ for each $k < \lambda$, $t_k \in \hat{\hat{R}}^T_{j_k}$ and $k_0 < k_1 \to B^{x_0}$, B^{x_1} agree on $[\alpha(t), \alpha(t_{k_0})) - \{4j + 1 \mid j \in \text{ORD}\}$ for $x_0 \in [t_{k_0}]$, $x_1 \in [t_{k_1}]$.

Define t_{k+1} from t_k so that $t_{k+1} \leqslant_l t_k$ and $t_{k+1} \in \hat{\hat{R}}^T_{j_{k+1}}$ by induction. Also we assume indictively that $x \in [t_k] \to B^x$, b_k agree on $[\alpha(t), \alpha(t_k)) - \{4j + 1 \mid j \in \text{ORD}\}$ where $b_k \subseteq \alpha(t_k)$ is special at $\alpha(t_k)$, $(b^*_k)_1 = \{\alpha(t_{k'})_1 \mid k' < k\}$ and in choosing t_{k+1} we can then also require the existence of a similar b_{k+1} such that $(b^*_{k+1})_1 = \{\alpha(t_{k'}) \mid k' \leq k\}$, $b^*_k = b^*_{k+1} \cap \alpha(t_k)$ and in addition $\delta < \alpha(t_k)$, $\delta \notin b^*_k \to 4 \cdot (\langle \delta, \delta' \rangle) + 3 \notin b^*_{k+1}$ for all $\delta' \in [\alpha(t_k), \alpha(t_{k+1}))$.

Now for limit $k \leq \lambda$ note that $b_k = \bigcup \{b_{k'} \mid k' < k\}$ is special at $\bigcup \{\alpha(t_{k'}) \mid k' < k\} = \alpha_k$. So by Lemma 4.4 we can choose $t_k \leqslant_l t$ such that $\alpha(t_k) = \alpha_k$ and $x \in [t_k] \to B^x$, b_k agree on $[\alpha(t), \alpha_k) - \{4j + 1 \mid j \in \text{ORD}\}$. We claim that $t_k \in \hat{\hat{R}}^T_{j_k}$. Indeed if for $k' < k$ we choose $t'_{k'} \geqslant t_{k'}$, $\alpha(t'_{k'}) = \alpha(t_{k'})$, then we see that $t'_{k'} \sim_{\gamma^+} t_{k'}$ and thus $f_{j_{k'}}(t'_{k'}) = f_{j_{k'}}(t_{k'})$, $g_{j_{k'}}(t'_{k'}) = g_{j_{k'}}(t_{k'})$ and so $t'_{k'} \in \hat{\hat{R}}^T_{j_{k'}}$ for $k' \leq k$. Thus t_λ is the desired extension of t in $\hat{\hat{R}}^T_j$.

Finally note that we can choose the above extensions so that $b_k \cap$ Even Ordinals $= \emptyset$ and therefore the case $\bar{t}_1 \neq \emptyset$ also follows. □

We now attack the proof of extendibility for $\mathscr{P}^A$.

Lemma 4.5 (Extendibility for $\bar{\mathscr{P}}^\gamma$, γ a limit cardinal). *Suppose γ is a limit cardinal, $p \in \bar{\mathscr{P}}^\gamma$ and $\alpha(p) < \alpha < \gamma^+$. Then there exists $q \leqslant p$, $\alpha(q) = \alpha$.*

Proof. For any p in $\bar{\mathscr{P}}^\gamma$ we show by induction on $\alpha \in (\alpha(p), \gamma^+)$ which are divisible by $\gamma \cdot \omega$ that for any $\bar{\gamma} \in \mathrm{CARD} \cap \gamma$, any $b \subseteq \alpha$ which is special at α and any $\hat{\alpha} \leqslant \alpha(p)$, $\hat{\alpha}$ limit or 0 there exists $q \leqslant_{\bar{\gamma}} p$, $\alpha(q) = \alpha$ such that $x \in [q] \to B^x$, b agree on $[\alpha(p), \alpha) - \{4j+1 \mid j \in \mathrm{ORD}\}$ and $B^x(\hat{\alpha} + 4j + 1) = \sigma(x)(j)$ for $4j + 1 < \min(|\sigma|, \alpha - \hat{\alpha})$ where $\sigma = \hat{\sigma}(b^*_\alpha)$. Also, if $\alpha \in (\alpha(p), \alpha(p) + \gamma \cdot \omega)$, we show the same without the restrictions that b is special or $\sigma = \hat{\sigma}(b^*)$. We also verify that each such $q \in \bar{\mathscr{P}}^\gamma$ reduces all predense $D \subseteq \mathscr{P}(p, b)$ which belong to $\hat{\mathscr{A}}_0(q) = \langle L_{\nu(\lambda(q), B(q))}[B(q)], B(q)\rangle$ when $\gamma \cdot \omega$ divides α, as well as other properties assumed inductively during the construction of $\mathscr{P}^\gamma$.

If $\alpha = 0$, there is nothing to show.

Suppose $\alpha = \beta + 1$. First we can extend p to $\bar{q} \in \bar{\mathscr{P}}^\gamma_\beta$ as above by induction. So we can assume that $\alpha(p) = \beta$. First suppose that γ is $\Sigma_2(\mathscr{A}(p))$-singular. We must verify that the sequences of quasiconditions built in Case 2A of the construction are well-defined. The existence of $\hat{p}^{j+1,i+1}_{n+1}$ follows by induction on γ: when $\gamma = \omega$th cardinal after 0 or the limit cardinal $\bar{\gamma}$ we use Density Reduction for R^T finitely many times together with the easily verified fact that $r \in R^\mu \to \exists r' \in \bar{R}^\mu$, r and r' are compatible. (The latter is needed to justify the condition $q_{\mu^+}(q_\mu) = q'(q_\mu)$, q' canonical.) For limit i we must also show that $\hat{p}^{j+1,i}_{n+1} = \mathrm{glb}\langle \hat{p}^{j+1,i'}_{n+1} \mid i' < i\rangle$ is a well-defined quasicondition. This follows from the fact that type A fusions were added (in Case 4 of the construction of R^0, R^γ, $\mathscr{P}^\gamma$) and the fact that we specify that y, $\langle \nu(\lambda(p), B(p)), n+1, \lambda \cdot j + 1\rangle$ gives rise to $\langle \hat{p}^{j+1,i'}_{n+1} \mid i' < i\rangle$ for $y \in [\hat{p}^{j+1,i}_{n+1}]$. The important thing to check however is that $\alpha(\hat{p}_\mu)$ is regular in $\bar{\mathscr{A}}[x]$ where $\bar{\mathscr{A}} =$ the transitive collapse of $X^{n+2}_{\mu^+,i}$, $\hat{p} = \hat{p}^{j+1,i}_{n+1}$ and $x \in [\hat{p}_\mu]$ (also we need that $\alpha(\hat{p} \restriction \mu)$ is regular in $\bar{\mathscr{A}}[x]$ where $\bar{\mathscr{A}} =$ transitive collapse of $X^{n+2}_{\mu,i}$, $\hat{p} = \hat{p}^{j+1,i}_{n+1}$ and $x \in [\hat{p} \restriction \mu]$, for limit cardinals $\mu \leqslant \gamma_i$). But we have (see Lemma 4.8 below) that p reduces all predense $D \subseteq \mathscr{P}(\bar{p}, B(p))$ (if p is of type 1) or all predense $D \subseteq \mathscr{P}^{\bar{A}}$ (if p arises from a type A fusion) or all predense $D \subseteq R^{t \restriction \alpha(p)} = \{q \in \mathscr{P}^\gamma_{<\alpha(p)} \mid q \in R^t\}$ (if p arises from a type B fusion), for any $D \in \hat{\mathscr{A}}_0(p)$, $\hat{\mathscr{A}}_1(p)$, $\hat{\mathscr{A}}_2(p)$ respectively. Thus we see that x as above is generic for (the image in the transitive collapse $(X^{n+2}_{\mu^+,i})$ of) one of the above forcings, using the above reductions together with those built into the definition of $\{p^{j+1,i'}_{n+1} \mid i' < i\rangle$. Moreover, these forcings are cardinal preserving by Corollary 4.10A. So we are done: $\hat{p}^{j+1}_{n+1}$ is well-defined. Exactly the same argument applies to verify that $p_{n+1} = \mathrm{glb}\langle p^j_{n+1} \mid j < \gamma\rangle$ is well-defined, using the built-in density reductions. We also get that $\langle p_n \mid n \in \omega\rangle$ is well-defined and the existence of $\hat{p}(i)$ follows by showing inductively that $\hat{p}(i) \upharpoonright \bar{\gamma}$ is a quasicondition for $\bar{\gamma} \in \mathrm{CARD} \cap \gamma$. The remaining part of Case 2A where q^i, $i < \gamma$, are defined, presents no problems and justifies the assertion that there exists $q \leq_{\bar{\gamma}} p$, $\alpha(q) = \alpha$, $x \in [q] \rightarrow B^x(\beta) = i$ (if $\beta \notin \{4j+1 \mid j \in \mathrm{ORD}\}$), $B^x(\hat{\alpha} + 4j + 1) = \sigma(x)(j)$ (if $\beta = \hat{\alpha} + 4j + 1$). Also note that q^i clearly reduces all predense $D \subseteq \mathcal{P}(p, B(q^i))$ belonging to $\hat{\mathcal{A}}_0(q^i)$ and that $\bar{\gamma} \in \mathrm{CARD} \cap \gamma \rightarrow \alpha(q^i_{\bar{\gamma}}) = \delta^{q^i}_1(\bar{\gamma}^+)$, $\alpha(q^i \upharpoonright \bar{\gamma}) = \delta^{q^i}_1(\bar{\gamma})$ for $\bar{\gamma}$ a sufficiently large limit cardinal $< \gamma$.

Suppose that γ is $\Sigma_n(\mathcal{A}(p))$-regular for all $n \in \omega$. We must verify that the quasiconditions defined in Case 2B of the construction of $\mathcal{P}^\gamma$ are well-defined. The existence of $\hat{p}^{j+1, \bar{\gamma}^+}_{n+1}$ follows by induction on γ as in the previous subcase. For limit $\bar{\gamma} \leq \gamma$ we must also verify that $\hat{p}^{j+1, \bar{\gamma}}_{n+1} = \mathrm{glb}\langle \hat{p}^{j+1, \bar{\bar{\gamma}}}_{n+1} \mid \bar{\bar{\gamma}} \in \mathrm{CARD} \cap \bar{\gamma} \rangle$ is well-defined. This follows as in the previus subcase from the fact that type A fusions were added (in Case 4 of the R^0, R^γ, $\mathcal{P}^\gamma$ constructions). The only new point to observe is that for $\mu \in C^{\bar{\gamma}}_n(p^j_{n+1}) = \bigcap \{C^{\bar{\bar{\gamma}}^+}_n(p^j_{n+1}) \mid \bar{\bar{\gamma}} \in \mathrm{CARD} \cap \bar{\gamma}\}$ we have that $\hat{p}^{j+1, \bar{\gamma}}_{n+1} \upharpoonright \mu$ has the same definition at μ as does $\hat{p}^{j+1, \bar{\gamma}}_{n+1} \upharpoonright \bar{\gamma}$ at $\bar{\gamma}$. Thus $\hat{p}^{j+1, \bar{\gamma}}_{n+1} \upharpoonright \mu$ does belong to R^q where $q = \hat{p}^{j+1, \bar{\gamma}}_{n+1} = p^j_{n+1_\mu}$. We must again check however that for $\mu \leq \bar{\gamma}$, $\alpha(\hat{p}_\mu)$ is regular in $\bar{\mathcal{A}}[x]$ where $\bar{\mathcal{A}}$ = transitive collapse of $X^{n+2}_{\mu^+, \bar{\gamma}^+}$, $\hat{p} = \hat{p}^{j+1, \bar{\gamma}}_{n+1}$ and $x \in [\hat{p}_\mu]$ (similarly for limit $\mu \leq \bar{\gamma}$, $\mu \notin C^{\bar{\gamma}}_n(p^j_{n+1})$, as in the previous subcase). Again by Lemma 4.8 we have that x is generic for the appropriate cardinal-preserving forcing (collapse of $\mathcal{P}(\bar{p}, B(p))$, $\mathcal{P}^{\bar{\mathcal{A}}}$ or $\mathcal{P}^t$) so we are done. The fact that p^{j+1}_{n+1} is well-defined follows, given the existence of the closed unbounded set C. But notice that we can choose $\hat{q}$ to belong to $\mathcal{P}(\hat{p}^{j+1}_{n+1})$ so we can let $C = a$ final segment of $C_n(p^j_{n+1})$.

The fact that $\bar{p}_{n+1}$ is well-defined follows as before using density reduction and closure under type A fusions. The construction of $p_{n+1} \leq p^*_{n+1} = \mathrm{glb}\langle \bar{p}^{\bar{\gamma}}_{n+1} \mid \bar{\gamma} \in \mathrm{CARD} \cap \gamma \rangle$ presents no new problems; again we must inductively verify that $p_{n+1} \upharpoonright \bar{\gamma}$ is a quasicondition for $\bar{\gamma} \in \mathrm{CARD} \cap \gamma$. Then we can define the $p(i, \gamma)$ and q^i. Notice that the verification that $q^i \leq p$ requires us to know that $\mu \in C_\omega = \bigcap \{C_n \mid n \in \omega\} \rightarrow \bar{\bar{p}}_\mu = \emptyset$, $p_\mu = t_\emptyset$ as we have specified $B^x(\alpha(q^i \upharpoonright \mu)) = B^x(\alpha(p \upharpoonright \mu))$ for $x \in [p \upharpoonright \mu]$. And for sufficiently large $\bar{\gamma} \in C_\omega$ we have $q^i_{\bar{\gamma}} = t_\emptyset$, $\bar{\bar{q}}^i_{\bar{\gamma}} = \emptyset$ and $\alpha(q^i \upharpoonright \bar{\gamma}) = \delta^{q^i}_1(\bar{\gamma})$ thereby preserving our induction hypotheses.

When γ is $\Sigma_2(\mathcal{A}(p))$-regular but $\Sigma_n(\bar{\mathcal{A}}(p))$-singular for some n, then we can combine the above two arguments to obtain the desired q.

Now suppose α is a limit ordinal not divisible by $\gamma \cdot \omega$. We can assume that $\hat{\beta} = \alpha(p) \geq \beta$ where $\alpha = \beta + \delta$, $0 < \delta \leq \gamma$ and γ divides β. The desired q arises from Case 3 of the construction of $\mathcal{P}^\gamma$. The condition p_λ in that construction is well-defined as before using type A fusions provided we check that p_λ reduces all predense $D \subseteq \mathcal{P}(p, B(p_\lambda))$ in $\mathcal{A}(p_\lambda)$, for $\lambda \leq \alpha - \hat{\beta}$. Induction proves this for $\lambda < \gamma$ and for $\lambda = \gamma$ note that even though $B(p_i) \neq B(p_{\alpha - \hat{\beta}}) \cap \lambda(p_i)$ for $i < \alpha - \hat{\beta}$ we can still verify the desired reduction by choosing i least so that $D \in \mathcal{A}(p_{i+1})$ and using Case 2 of the construction. Also note that if $\bar{\gamma} \in C_i \rightarrow p_{i_{\bar{\gamma}}} = t_\emptyset$, $\bar{\bar{p}}_{i_{\bar{\gamma}}} = \emptyset$ for $i < \alpha - \hat{\beta}$, then $\bar{\gamma} \in C \rightarrow \bar{\bar{p}}_{\bar{\gamma}} = \emptyset$, $\hat{p}_{\bar{\gamma}} = t_\emptyset$ where C = diagonal intersection of the C_i's (or just the ordinary intersection if $\alpha - \hat{\beta} < \gamma$).

Finally suppose that α is divisible by $\gamma \cdot \omega$. Then we must verify that the sequence $\langle p_i \mid i < \lambda_0 \rangle$ from Case 4 of the construction of $\mathcal{P}^\gamma$ is well-defined. But this follows from $C_{\alpha_\lambda}^{(\hat{b} \cap \alpha_\lambda)^*} = C_\alpha^{\hat{b}^*} \cap \alpha_\lambda$ provided we have that p_λ reduces all predense $D \subseteq \mathcal{P}(p, B(p_\lambda))$ in $L_{\nu(\lambda(p_\lambda), B(p_\lambda))}[B(p_\lambda)]$. The latter follows from Lemma 4.7 below. also note that the final restriction on when to add $\hat{p}$ to $\bar{\mathcal{P}}_\alpha^\gamma$ does not interfere with the desired extendibility. And in case γ is inaccessible in $\mathcal{A}(p_\lambda)$ we should note that $p_\lambda \restriction \bar{\gamma}$ is a quasicondition for $\bar{\gamma} \in \{\bar{\gamma} < \gamma \mid \bar{\gamma} = \gamma \cap (\Sigma_1$-Skolem hull of $\bar{\gamma} \cup \{\gamma\}$ in $\mathcal{A}(p_\lambda))\}$ thanks to the special collapsing properties of the $\square$-sequences $\langle C_\alpha^y \mid y \subseteq \alpha$, α divisible by $\gamma \cdot \omega$ and $L_\alpha[t] \models \gamma$ is the largest cardinal$\rangle$. $\square$

Lemma 4.6 (Extendibility for R^T, $T \in \tilde{R}^\gamma$, *γ an uncountable limit cardinal*). *Suppose $p \in R_i^T$, $T \in \tilde{R}^\gamma$, γ an uncountable limit cardinal, $\bar{\gamma} \in$ CARD $\cap$ γ and $i < j < \gamma^+$. Then there exists $q \leqslant_{\bar{\gamma}} p$, $q \in R_j^T$.*

Proof. Exactly like the proof of Lemma 4.3, using Lemma 4.5 now instead of Lemma 4.4. $\square$

We have made extensive use of the following lemmas, which in fact are established via *a* simultaneous induction with our earlier lemmas.

Lemma 4.7 (Chain Condition for $\mathcal{P}(p, B)$). *Suppose $p \in \bar{\mathcal{P}}^\gamma$, γ an uncountable limit cardinal and $q \leqslant p$ is a type* 1 *extension, $B = B(q)$. Then $\mathcal{P}(p, B) = \{p' \leqslant p \mid p' \in \bar{\mathcal{P}}_{\alpha(q)}^\gamma$, $B(p')$ and B agree on $[\alpha(p), \alpha(p'))\}$ obeys the γ^+-cc in $L_{\nu(\lambda(q), B)}[B]$.*

Proof. This follows from the use of the $\Diamond$-sequence in Case 4 of the construction of $\mathcal{P}^\gamma$. $\square$

Lemma 4.8 (Density Reduction for $\mathcal{P}(\bar{p}, B(p))$, $\mathcal{P}^{\bar{\mathcal{A}}}$, $R^{t \restriction \alpha(p)}$). *Suppose $p \in \bar{\mathcal{P}}^\gamma$ where γ is an uncountable limit cardinal.*

(a) *If $p \leqslant \bar{p}$ is a type* 1 *extension ($\bar{p} \neq p$), let $\hat{\mathcal{A}}_0(p) = L_{\nu(\lambda(p), B(p))}[B(p)]$. Then p reduces all predense $D \subseteq \mathcal{P}(\bar{p}, B(p))$, $D \in \hat{\mathcal{A}}_0(p)$.*

(b) *If p arises from a type* A *fusion, let $\hat{\mathcal{A}}_1(p) = \bar{\mathcal{A}}$ where $\bar{\mathcal{A}}$ and $\mathcal{P}^{\bar{\mathcal{A}}}$ arise from Case* 4 *of the construction of $\mathcal{P}^\gamma$. Then p reduces all predense $D \subseteq \mathcal{P}^{\bar{\mathcal{A}}}$, $D \in \hat{\mathcal{A}}_1(p)$.*

(c) *If p arises from a type* B *fusion, let $\hat{\mathcal{A}}_2(p) = \mathcal{A}$ where $\mathcal{A}$ and $R^{t \restriction \alpha(p)}$ arise from Case* 4 *of the construction of $\mathcal{P}^\gamma$. Then p reduces all predense $D \subseteq R^{t \restriction \alpha(p)}$, $D \in \hat{\mathcal{A}}_2(p)$.*

Proof. (a) is clear from the proof of Lemma 4.5. And (c) is clear from Case 4 of the construction of $\mathcal{P}^\gamma$. Lastly (b) follows from the definition of the canonical sequences in the construction of $\mathcal{P}^\gamma$. $\square$

Lemma 4.9 (Distributivity for $\mathcal{P}^A$). *Suppose $\langle D_i \mid i<\gamma\rangle$ is a definable sequence of open classes which are γ-dense below $p \in \mathcal{P}^A$. Then there exists $q \leqslant_\gamma p$, $q \in \bigcap \{D_i \mid i<\gamma\}$.*

Proof. Choose $\delta \in \mathrm{CARD}$ so that $p \in L_\delta[A] <_{\Sigma_n} L[A]$. Then each $D_i^* = D_i \cap L_\delta[A]$ is open dense below p^* on $\mathcal{P}(p^*)$ for some $p^* \in \mathcal{P}_0^\delta$ (i.e., $(p^*)_{\bar\delta}$ agrees with the weakest element of $\mathcal{P}^{\delta}$ for some $\bar\delta \in \mathrm{CARD} \cap \delta$). Then by definition, if $q \leqslant p^*$, $\alpha(q)=1$, then q reduces each D_i^*. Thus we see that D_i^* is predense on $\mathcal{P}^\delta$. So we can apply Corollary 4.2A to $T=$ the weakest element of $\bar R^\delta$ to obtain the desired q. □

Lemma 4.10 (Density Reduction for $\mathcal{P}^A$). *Suppose $p \in \mathcal{P}^A$ and $\langle D_i \mid i<\gamma\rangle$ is a definable sequence of open classes which are dense below p. Then there exists $q \leqslant p$ which reduces each D_i below γ.*

Proof. As in the proof of Lemma 4.9 it suffices to prove this with $\mathcal{P}^A$ replaced by R^T, $T \in \bar R^\delta$ for some limit cardinal $\delta > \gamma^+$ and $\langle D_i \mid i<\gamma\rangle \in L[T, A \cap \delta^+]$. Now apply Corollary 4.2A to reduce the D_i's below γ^+ and then Corollary 4.1B to reduce them below γ. □

Corollary 4.10A (Cardinal Preservation). *If R is a $\mathcal{P}^A$-generic, then R preserves cardinals.*

Proof. Immediate from Lemma 4.10, which also is needed to establish the definability of forcing. □

Lemma 4.11 (Minimal Coding). *If R is $\mathcal{P}^A$-generic, then A is $L[R]$-definable and R is V-Minimal.*

Proof. Extendibility for $\mathcal{P}^A$ follows from extendibility for $\mathcal{P}^\gamma$, Lemma 4.5. From this we can infer that A is $L[R]$-definable.

Now suppose $p \Vdash \boldsymbol{x} \subseteq \mathrm{ORD}, \boldsymbol{x} \notin L[A]$. It suffices to prove:

Claim. *There exist α, p_0, p_1 such that $p_0, p_1 \leqslant p$ and $p_0 \Vdash \alpha \notin \boldsymbol{x}$, $p_1 \Vdash \alpha \in \boldsymbol{x}$ and $(p_0)_\omega = (p_1)_\omega$.*

Given the Claim we can use Density Reduction for $\mathcal{P}^A$ to build a sequence $p_0 \geqslant p_1 \geqslant \cdots$ such that $p^* = \mathrm{glb}\langle p_i \mid i<\omega\rangle$ exists and s, t incompatible elements of $2^{<\omega} \rightarrow ((p_0^*(s), \bar p_0^*),\ p^*(1), \ldots)$ and $((p_0^*(t), \bar p_0^*),\ p^*(1), \ldots)$ force incompatible facts about $\boldsymbol{x}$. So $p^* \Vdash R \in V[\boldsymbol{x}]$.

Proof of Claim. Choose $p_0, p_1 \leqslant p$ and α as in the Claim but without the requirement $(p_0)_\omega = (p_1)_\omega$. Our operation $(+)$ allows us to modify p_0, p_1 so as to satisfy the Claim. By induction on $\gamma \in \mathrm{CARD}$ define p_0^γ, p_1^γ so as to satisfy the

Claim with $(p_0)_\omega = (p_1)_\omega$ replaced by $p_0^\gamma \restriction [\omega, \gamma) = p_1^\gamma \restriction [\omega, \gamma)$. For successor γ^+ we can use the operation $(+)_{\gamma^+}$ to define $\mathcal{P}^\gamma_{i_{\gamma^+}}$ to be $p^+_\gamma(\tau, t_\emptyset, p_{0_{\gamma^+}}, p_{1_{\gamma^+}})$ where τ is a $\bar{\gamma}$-term for some $\bar{\gamma} < \gamma$ such that $[p_{0_\gamma}] \subseteq \mathcal{P}_\gamma(\tau)$, $[p_{1_\gamma}] \cap \mathcal{P}_\gamma(\tau) = \emptyset$. Limit cardinals γ can be handled using γ-distributivity. □

This completes the proof of the Minimal Coding Theorem.

5. Further results

Theorem 5.1. *There exists a real $R \in L[0^\#]$ which is L-minimal but not set-generic over L.*

Proof. We need to produce a real R in $L[0^\#]$ which is weakly $\mathcal{P}^\emptyset$-generic over L, where by weakly generic we mean that G_R need only meet all predense $D \subseteq \mathcal{P}^\emptyset$, $D \in L$.

We proceed just as in Section 4.4 of Beller–Jensen–Welch [1]. Let I denote the Silver indiscernibles for L and for $i \in I$, $n \in \omega$ let $i(n)$ denote $i^+ \cap$ Skolem hull of $(i+1) \cup \{i_1, \ldots, i_n\}$ in L, where $i^+ = (i^+)^L$ and $i < i_1 < \cdots < i_n$ belong to I. Clearly this definition is independent of the choice of $i_1, \ldots, i_n$.

Now define, for each $n \in \omega$, sequences $\langle p^{ni} \mid i \in I\rangle$, $\langle t^{ni} \mid i \in I\rangle$ and $\langle u^{ni} \mid i \in I\rangle$ where $u^{ni} \in R^{i^+}$, $t^{ni} \in R^{u^{ni}}$, $p^{ni} \in R^{t^{ni}}$ and $u^{ni} = p^{nj}_i$ for $i < j$ in I. We define p^{0i}, t^{0i}, u^{0i} to be the weakest conditions obeying the preceding requirements. Then let $p^{(n+1)i} \leqslant p^{ni}$ be least in $R^{t^{(n+1)i}}$ such that $\alpha(p^{(n+1)i}) = i(n)$ where $t^{(n+1)i}$ is the least $t \in R^{u^{ni}}$, $t \leqslant t^{ni}$ which reduces all predense $D \in$ Skolem hull of $i^+ \cup \{i_1, \ldots, i_n\}$ in L, $i < i_1 < \cdots < i_n$ from I. We also insist that $p^{(n+1)i_0}$ meets the first n predense sets in Skolem hull of $\{i_0, i_1, \ldots, i_m\}$ in L, where $i_0 = \min(I)$ and $i_0 < i_1 < \cdots < i_n$ belong to I.

Clearly we have $n \leqslant m \to p^{mi} \leqslant p^{ni}$, $t^{mi} \leqslant t^{ni}$ and $u^{mi} \leqslant u^{ni}$ for $i \in I$ and $i < j \to p^{ni} = p^{nj} \restriction i$ for $n \in \omega$. Now let $\hat{G}$ consist of all conditions $p \in \mathcal{P}^\emptyset$ such that for some finite $F \subseteq I$ and $i \in I$, $n \in \omega$ we have that $F \subseteq i$ and $p \restriction (i - F) = p^{ni} \restriction (i - F)$, $p(j) = (p_j, \bar{p}_j, \bar{\bar{p}}_j)$ where $(p_j, \bar{p}_j) = t^{ni}$ for $j \in F$. Let $G = \{p \in \mathcal{P}^\emptyset \mid p \geqslant \hat{p}$ for some $\hat{p} \in \hat{G}\}$. Then G is a compatible class of conditions and by construction any predense $D \in L$ is reduced below i_0 by G. But then the requirement on $p^{(n+1)i_0}$ implies that in fact G meets D. So G is weakly $\mathcal{P}^\emptyset$-generic. □

Theorem 5.2. *There exists an L-definable forcing $\mathcal{P}$ for producing an L-minimal real which is not set-generic over L such that if $R \neq S$ are $\mathcal{P}$-generic over L, then (R, S) is $\mathcal{P} \times \mathcal{P}$-generic over L. In addition, $L[0^\#] \models \exists \mathcal{P}$-generic real.*

Proof sketch. We only deal with weak genericity; the modifications required for full genericity are as in Beller–Jensen–Welch [1, Lemma 5.3].

Modify the forcing $\mathcal{P}^\emptyset$ as follows: In the definitions of $\mathcal{P}^\gamma$ and R^γ restrict the type A, B fusions to a stationary set of $\alpha < \gamma^+$, avoiding a stationary set $E \subseteq \gamma^+$ on which lies a $\lozenge(E)$-sequence. Then when adding conditions to $\mathcal{P}^\gamma_\alpha$, R^γ_α for $\alpha \in E$ (for the sake of extendibility) make sure that any distinct pair $(p, q) \in \mathcal{P}^\gamma_\alpha \times \mathcal{P}^\gamma_\alpha$, $R^\gamma_\alpha \times R^\gamma_\alpha$ reduces a dense set on $\mathcal{P}^\gamma_{<\alpha} \times \mathcal{P}^\gamma_{<\alpha}$, $R^\gamma_{<\alpha} \times R^\gamma_{<\alpha}$ specified by the $\lozenge(E)$-sequence. This is possible provided $\mathcal{P}^\gamma_{<\alpha}$, $R^\gamma_{<\alpha}$ are subsets of L_α; we can arrange this by requiring (as in Section 1) that $\mathcal{P}^\gamma_\alpha$, $R^\gamma_\alpha \subseteq L_{\tilde{\alpha}}$ where $\tilde{\alpha} =$ least $\beta > \alpha$ such that β is admissible and $L_\beta \models \gamma$ is the largest cardinal. Then any two distinct $\mathcal{P}$-generics will reduce any given predense $D \subseteq \mathcal{P} \times \mathcal{P}$, $D \in L$. □

Open Questions. (1) Does there exist an L-minimal Π^1_2-singleton?

(2) Define $S \leqslant R$ if R is set-generic over $L[S]$ and $S \leqslant_L R$. Assume $0^\#$ exists. What are the finite initial segments of the resulting partial ordering of degrees ($R \sim S$ if $R \leqslant S$, $S \leqslant R$) below $0^\#$? Theorem 5.1 implies that there is a minimal such degree.

(3) Is there a K-minimal real which is not set-generic over $K =$ the core model?

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