ESI Workshop

ESI = Erwin Schrödinger Institut, Vienna

ESI WORKSHOP ON LARGE CARDINALS AND DESCRIPTIVE SET THEORY
June 14–25, 2009

1st week: June 14–18 Emphasis on Large Cardinals
2nd week: June 21–25 Emphasis on Descriptive Set Theory

All are welcome; no registration fee

For further information:

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Internal Consistency

φ is *internally consistent* iff φ is true in some inner model

Assumption: There are inner models of V with large cardinals

A new type of (relative) consistency result.

$\mathrm{Con}(\mathrm{ZFC} + \varphi) = \mathrm{ZFC} + \varphi$ is consistent

$\mathrm{ICon}(\mathrm{ZFC} + \varphi) = \mathrm{ZFC} + \varphi$ holds in some inner model

Internal Consistency

Consistency result:
$\mathrm{Con}(\mathrm{ZFC} + \mathrm{LC}) \rightarrow \mathrm{Con}(\mathrm{ZFC} + \varphi)$,
where LC is a large cardinal axiom

Internal consistency result:
$\mathrm{ICon}(\mathrm{ZFC} + \mathrm{LC}) \rightarrow \mathrm{ICon}(\mathrm{ZFC} + \varphi)$

Internal Consistency

Internal consistency is stronger than consistency

Example

$\mathrm{Con(ZFC)} \rightarrow \mathrm{Con(ZFC + GCH\ fails\ at\ all\ regular\ cardinals)}$
(Force with an Easton product over L)

$\mathrm{ICon(ZFC} + 0^{\#}\ \mathrm{exists)} \rightarrow$
$\mathrm{ICon(ZFC + GCH\ fails\ at\ all\ regular\ cardinals)}$
(Force with a reverse Easton iteration over L, build a generic using the Silver indiscernibles)

Proving Internal Consistency *demands new techniques*

2 Types of Internal Consistency Results

Two types of internal consistency results:

Type 1. $\text{ICon}(\text{ZFC} + 0^\# \text{ exists}) \rightarrow \text{ICon}(\text{ZFC} + \varphi)$

Build generics by cohering partial generics along Silver indiscernibles.
Two techniques:
Easier cases: *Generic modification* (F-Ondrejović)
Harder cases: *Partial master conditions* (F-Thompson)

Key to Type 2 results: Show that the relevant forcings preserve measurability

2 Types of Internal Consistency Results

Type 2. First show:

($*$) ZFC+ LC$\rightarrow$ In some set-forcing extension, φ holds in V_κ for some measurable κ

Then we have:

$\text{ICon}(\text{ZFC}+\text{LC}) \rightarrow \text{ICon}(\text{ZFC}+\varphi$ holds in V_κ, κ measurable)
(Force with a countable p.o. over an inner model)

and also:

$\text{ICon}(\text{ZFC}+\varphi$ holds in V_κ, κ measurable) $\rightarrow \text{ICon}(\text{ZFC}+\varphi)$
(Iterate the measure to ∞)

So we conclude:

$\text{ICon}(\text{ZFC}+\text{LC}) \rightarrow \text{ICon}(\text{ZFC}+\varphi)$

2 Types of Internal Consistency Results

How do we show $(*)$?

$(*)$ ZFC+ LC$\longrightarrow$ In some set-forcing extension, φ holds in V_κ for some measurable κ

In easier cases: *Master conditions* (Silver), *Partial master conditions* (Magidor, F-Honzík) or *Generic modification* (Woodin)
In harder cases: κ-*Tree forcings* (F-Thompson for κ-Sacks products, Dobrinen-F for κ-Sacks iterations, F-Zdomskyy for κ-Miller iterations)

Examples of Internal Consistency

Some Internal Consistency Results

Cardinal Exponentiation: F-Ondrejović, F-Honzík

Costationarity of the Ground Model: Dobrinen-F

Global Domination: F-Thompson

Tree Property: Dobrinen-F

Embedding Complexity: F-Thompson

Cofinality of the Symmetric Group: F-Zdomskyy

Internal Consistency: Cardinal Exponentiation

Cardinal Exponentiation

Easton function:
$F : \text{Reg} \to \text{Card}$, F nondecreasing, $\text{cof}(F(\kappa)) > \kappa$ for all $\kappa \in \text{Reg}$

Easton: F a provably definable Easton function. Then
$\text{Con(ZFC)} \to \text{Con}(\text{ZFC} + 2^\kappa = F(\kappa)$ for all regular $\kappa)$

Easton used an Easton product

This gives *no* internal consistency result

Internal Consistency: Cardinal Exponentiation

F-Ondrejović: Instead use Easton iteration of Easton products and *generic modification*

Theorem

F a provably definable Easton function. Then
$ICon(ZFC + 0^{\#}$ exists$) \rightarrow ICon(ZFC + 2^{\kappa} = F(\kappa)$ for all regular $\kappa)$

Type 2 result (F-Honzík): F a provably definable Easton function, κ is $H(F(\kappa))$-hypermeasurable witnessed by j with $j(F)(\kappa) \geq F(\kappa)$. Then in a set-generic extension, κ is measurable and F is realised below κ (in fact, everywhere). Sample corollary:

Theorem

$ICon(ZFC+$ There is a $P_2\kappa$ hypermeasurable$) \rightarrow$
$ICon(ZFC + 2^{\kappa} = \kappa^{++}$ for all regular $\kappa +$ There is a proper class of Ramsey cardinals$)$

Internal Consistency: Global Domination

Global Domination

κ an infinite regular cardinal
Suppose $f, g : \kappa \to \kappa$
f *dominates* g iff $f(\alpha) > g(\alpha)$ for sufficiently large $\alpha < \kappa$
$\mathcal{F}$ is a *dominating family* iff
every $g : \kappa \to \kappa$ is dominated by some f in $\mathcal{F}$
$d(\kappa)$ = the smallest cardinality of a dominating family

Fact: $\kappa < d(\kappa) \leq 2^\kappa$ for all infinite regular κ

Global Domination: $d(\kappa) < 2^\kappa$ for all infinite regular κ

Internal Consistency: Global Domination

Cummings-Shelah: Global Domination is consistent
Proof uses κ-Cohen and κ-Hechler forcings
Corollary to their proof:
ICon(ZFC+ a supercompact cardinal) $\rightarrow$
ICon(ZFC+ Global Domination)

F-Thompson: Instead use κ-Sacks product (and *tuning forks*)

Theorem

ICon(ZFC $+ 0^{\#}$ *exists)* $\rightarrow$ *ICon(ZFC+ Global Domination)*

Theorem

If κ *is* $P_2\kappa$ *hypermeasurable then in a set-generic extension,* κ *is measurable and global domination holds below* κ *(in fact, everywhere).*

Internal Consistency: Global Domination

In the previous two theorems, $(d(\kappa), 2^\kappa) = (\kappa^+, \kappa^{++})$

What about other possibilities for $(d(\kappa), 2^\kappa)$?

Global Domination Pair (d, F): For regular κ, $\kappa < d(\kappa) \leq F(\kappa)$, $d(\kappa)$ regular, F an Easton function

Realising arbitrary global domination pairs seem to require very large cardinals:

Definition. ∞ is *super-Woodin* iff for any class $A \subseteq \mathrm{Ord}$ there is κ such that for any λ, some j witnessing that κ is λ-supercompact satisfies $j(A) \cap \lambda = A \cap \lambda$
(Follows from a stationary class of almost-huge cardinals)

Internal Consistency

Theorem

Assume GCH and ∞ super-Woodin. Then if φ defines a global domination pair there are inner models $W_0 \subseteq W_1$ such that φ defines in W_0 the global domination pair realised in W_1.

Uses κ-Cohen and κ-Hechler forcing, as in Cummings-Shelah, but preserving the measurability of κ

Internal Consistency: The Tree Property

The Tree Property

κ regular
A κ-*Aronszajn tree* is a tree of height κ with no κ-branch
κ has the *tree property* iff there is no κ-Aronszajn tree

Mitchell: Con(ZFC+ Proper class of weakly compact cardinals) $\rightarrow$ Con(ZFC $+ \alpha^{++}$ has the tree property for all inaccessible α)

Proof uses "Mitchell forcing"

Corollary to proof:
ICon(ZFC+ a supercompact cardinal) $\rightarrow$
ICon(ZFC $+ \alpha^{++}$ has the tree property for all inaccessible α)

Internal Consistency The Tree Property

Dobrinen-F: Instead use iterated κ-Sacks forcing

Theorem

$ICon(ZFC + 0^{\#}$ exists$) \to$
$ICon(ZFC + \alpha^{++}$ has the tree property for all inaccessible $\alpha)$

Theorem

If κ is weakly compact hypermeasurable then in a set-generic extension, κ remains measurable and the tree property holds at κ^{++}.

Theorem

$ICon(ZFC+$ There is a weakly compact hypermeasurable$) \to$
$ICon(ZFC+$ The tree property holds at α^{++} for inaccessible α and there is a proper class of Ramsey cardinals$)$

Internal Consistency. The Tree Property

A related consistency result:

Foreman:
Con(ZFC+ supercompact + a larger weak compact) $\rightarrow$
Con(ZFC+ Tree Property at λ^{++} for a singular λ)

Theorem

(F-Halilović-Magidor) Con(ZFC + κ weakly compact hypermeasurable) $\rightarrow$ Con(Tree property at $\aleph_{\omega+2}$)

Internal Consistency: Embedding Complexity

Embedding Complexity

$\alpha \leq \kappa$ infinite and regular

$G(\alpha, \kappa) =$ Set of graphs of size κ which omit α-cliques

Embedding complexity of $G(\alpha, \kappa) = \mathrm{ECG}(\alpha, \kappa)$:
Smallest size of a $U \subseteq G(\alpha, \kappa)$ such that every graph in $G(\alpha, \kappa)$ embeds into some element of U (as a subgraph)

What are the possibilities for $\mathrm{ECG}(\alpha, \kappa)$ as a function of α and κ?

Internal Consistency: Embedding Complexity

Complexity triple (a, c, F):
a, c, F : Reg $\rightarrow$ Card
F is an Easton function
$a(\kappa) \leq \kappa < c(\kappa) \leq F(\kappa)$ for all κ

Theorem

(Džamonja-F-Thompson) Suppose that (a, c, F) is a provably definable complexity triple. Then $Con(ZFC) \rightarrow Con(ZFC + ECG(a(\kappa), \kappa) = c(\kappa)$ and $2^\kappa = F(\kappa)$ for all $\kappa \in Reg)$

Internal Consistency: Embedding Complexity

Theorem

(F-Thompson) Suppose that (a, c, F) is a provably definable complexity triple. Then $ICon(ZFC + 0^{\#}$ exists$) \rightarrow$ $ICon(ZFC + ECG(a(\kappa), \kappa) = c(\kappa)$ and $2^{\kappa} = F(\kappa)$ for all $\kappa \in Reg)$

The generic is built using *partial master conditions*

Consistency with measurability: Looks difficult. Need a "tree-like" forcing to control embedding complexity

Internal Consistency: Cofinality of the Symmetric Group

Cofinality of the Symmetric Group

κ regular.
$\mathrm{Sym}(\kappa)$ = the symmetric group on κ

$\mathrm{cof}(\mathrm{Sym}(\kappa))$ = the length of the shortest chain of proper subgroups of $\mathrm{Sym}(\kappa)$ whose union is all of $\mathrm{Sym}(\kappa)$

Internal Consistency: Cofinality of the Symmetric Group

Theorem

(F-Zdomskyy) Con(ZFC $+ \kappa$ *is* $P_2\kappa$ *hypermeasurable)* $\rightarrow$
Con(ZFC $+$ *cof(Sym(*κ*))* $= \kappa^{++}$ *for a measurable* κ*)*

Uses an iteration of a special version of κ-Miller forcing

Theorem

(F-Zdomskyy) ICon(ZFC $+ 0^{\#}$ *exists)* $\rightarrow$
ICon(ZFC $+$ *cof(Sym(*α*))* $= \alpha^{++}$ *for all inaccessible* α*).*

Uses the *partial master conditions* of F-Thompson.

Internal Consistency: Open Problems

Type 1 results

What global patterns can be realised in inner models of $L[0^\#]$ for the following characteristics?
Easton functions with parameters
Dominating pairs (d, F)
Sym(κ)
Tree Property (κ)
Stationary reflection at κ
$\Box_\kappa$

Type 2 results

General open problem: How can one preserve measurability with *iteration* of "κ-Cohen like" forcings? Is there a general method for converting these into "κ-Tree like" forcings?