Intrinsic Evidence for Set-Theoretic Truth

The incompleteness of ZFC

Sadly, the "standard" axioms ZFC for set theory are incomplete

They fail to answer many interesting questions in set theory:

Continuum Hypothesis?
Singular cardinal hypothesis?
Is there a definable, non-measurable set of reals?
Suslin's hypothesis?
Whitehead's conjecture?
Borel conjecture?
Are non-Borel analytic sets Borel isomorphic?
Does Square hold at singular cardinals?
Do large cardinals exist?

Intrinsic Evidence for Set-Theoretic Truth

To solve this problem we should add *new axioms* to ZFC

Example. Consider the following new axiom:

Min. Every set is constructible in the sense of Gödel and there are no transitive models of ZFC

ZFC + *Min* is consistent (assuming ZFC is).
ZFC + *Min* is a very powerful theory; it solves all of the above problems.

But we want our new axioms to be *true*!

Question. Is the *Min* axiom true?

When can we say that an axiom is true?

Intrinsic Evidence for Set-Theoretic Truth

Let's return to ZFC for a moment:

Claim. The axioms of ZFC are true.

1. The iterative concept of set (Zermelo - von Neumann)
The universe of sets contains the empty set and is closed under iteration of the powerset operation through the ordinals.

Thus V (the universe of set) is the union of the V_α's, α an ordinal

2. Gödel reflection: Any definable property of V with set parameters also holds for one of its initial segments V_α

Conclusions: The axioms of ZFC are true.
The *Min* axiom is false. $\square$

Intrinsic Evidence for Set-Theoretic Truth

[*Remark.* Set-theorists accept the above proof of the truth of the ZFC axioms, but not all philosophers of set theory do.
My opinion is that all such objections belong to the 20th century, not the 21st.]

The above argument is based on *intrinsic evidence* about the nature of sets and the universe of all sets.

Key question. Is there *intrinsic* evidence for the truth of new axioms of set theory which resolve interesting problems like CH or the existence of large cardinals?

Until now there has been very little progress on this question, for a number of reasons:

Intrinsic Evidence for Set-Theoretic Truth

Feferman's vagueness argument (pessimism):
"I have been led to the view that the statement CH is inherently vague and that it is meaningless to speak of its truth value; the fact that no remotely plausible axioms of higher set theory serve to settle CH only bolsters my conviction."

Shelah's pluralist view (love-affair with ZFC):
"My feeling is that ZFC exhausts our intuition except for things like consistency statements, so a proof means a proof in ZFC."
"I do not feel a universe of ZFC is like the sun, it is rather like a human being of some fixed nationality."

However:

"Some believe that compelling, additional axioms for set theory which settle problems of real interest will be found or even have been found. It is hard to argue with hope ..."

Intrinsic Evidence for Set-Theoretic Truth

Balaguer's full-blooded Platonism (FBP) (dismissing the problem):
"According to FBP, both ZFC and ZF+ not-C [negation of AC] truly describe parts of the mathematical realm; but there is nothing wrong with this, because they describe different parts of that realm. This might be expressed by saying that ZFC describes the universe of sets1, while ZF+not-C describes sets2, where sets1 and sets2 are different kinds of things."

"What FBP says is that there are so many different kinds of sets that every consistent theory is true of an actual universe of sets."

Hamkins goes even further:
"... the continuum hypothesis is a settled question; it is incorrect to describe the CH as an open problem ... the most important and essential facts about CH are deeply understood, and these facts constitute the answer to the CH question."

Intrinsic Evidence for Set-Theoretic Truth

Maddy's naturalism (deferral to set-theoretic practice):
"What, then, does naturalism suggest for the case of the CH? First, that we needn't concern ourselves with whether or not the CH has a determinate truth value ... Instead, we need to assess the prospects of finding a new axiom that is well-suited to the goals of set theory and also settles CH."

But there are at least some set-theorists other than myself who advocate the search for intrinsic justifications for new axioms:

Intrinsic Evidence for Set-Theoretic Truth

Absoluteness principles

For any infinite cardinal number κ, $H(\kappa)$ denotes the union of all transitive sets of size less than κ.
V (the universe of all sets) is the union of the $H(\kappa)$'s

The *theory of* $H(\kappa)$ is the set of all sentences true in $H(\kappa)$

The sentences of set theory are classified by the *Lévy hierarchy*:

$\Sigma_1 \subseteq \Sigma_2 \subseteq \cdots$
$\bigcup_n \Sigma_n =$ all sentences

The Σ_n *theory of* $H(\kappa)$ is the set of Σ_n sentences true in $H(\kappa)$

Intrinsic Evidence for Set-Theoretic Truth

We write $M \sqsubseteq N$ if $M \subseteq N$ are transitive models of ZFC with the same ordinals.

Trivial absoluteness: If $M \sqsubseteq N$ are models of ZFC then the theory of $H(\omega)$ is the same in M and N.

Lévy absoluteness: If $M \sqsubseteq N$ are models of ZFC then the Σ_1 theory of $H(\omega_1)$ is the same in M and N.

Non-absoluteness I: There are models $M \sqsubseteq N$ of ZFC such that the Σ_2 theory of $H(\omega_1)$ is not the same in M and N.

Can we fix this problem by strengthening ZFC?

Trivially yes: If $M \sqsubseteq N$ are models of ZFC $+ V = L$ then $M = N$.

But can we do this compatibly with large cardinals?

Intrinsic Evidence for Set-Theoretic Truth

Non-absoluteness II: For any recursive first-order theory T containing ZFC which is compatible with large cardinals there are models $M \sqsubseteq N$ of T such that the Σ_2 theory of $H(\omega_1)$ is not the same in M and N.

Woodin has tried to circumvent this problem by severely restricting the relation $M \sqsubseteq N$:

$M \sqsubseteq^{set-generic} N$ iff $M \subseteq N$ are transitive models of ZFC with the same ordinals *and N is a SET-GENERIC extension of M*

By a result of Bukovsky:
$M \sqsubseteq^{set-generic} N$ iff $M \sqsubseteq N$ and for some cardinal κ of M, every function in N on a set in M into M is contained in a multi-valued function in M with fewer than κ values for each argument.

Intrinsic Evidence for Set-Theoretic Truth

Woodin set-generic absoluteness I: If $M \sqsubseteq^{set-generic} N$ are models of ZFC + large cardinals then the theory of $H(\omega_1)$ is the same in M and N.

Woodin set-generic absoluteness II: If $M \sqsubseteq^{set-generic} N$ are models of ZFC + large cardinals + CH then the Σ_1 theory of $H(\omega_2)$ (with parameter ω_1) is the same in M and N.

To go further one strengthens the relation $\sqsubseteq$ even more:

Intrinsic Evidence for Set-Theoretic Truth

$M \sqsubseteq^{stationary-preserving-set-generic} N$ iff $M \subseteq N$ are transitive models of ZFC with the same ordinals *and N is a STATIONARY-PRESERVING set-generic extension of M*

Viale stationary-preserving set-generic absoluteness: If $M \sqsubseteq^{stationary-preserving-set-generic} N$ are models of ZFC + large cardinals + MM^{+++} then the theory of $H(\omega_2)$ is the same in M and N.

Set-generic absoluteness results are impressive mathematically but they don't solve the philosophical problem; consider the following report of a panel discussion in which Paul Cohen took part:

Intrinsic Evidence for Set-Theoretic Truth

"Cohen said that he was surprised to see that generic extensions, or forcing extensions, were being used as fundamental notions in their own right, rather than just technical artifacts of his (Cohen's) method of proof."

I share Cohen's view: Set-genericity is a technical notion which is *not intrinsic* to set theory

There is however a more promising approach to absoluteness via *the Hyperuniverse Programme*. Recall:

Non-absoluteness II: For any recursive first-order theory T containing ZFC which is compatible with large cardinals there are models $M \sqsubseteq N$ of T such that the Σ_2 theory of $H(\omega_1)$ is not the same in M and N.

But there is some good news:

Intrinsic Evidence for Set-Theoretic Truth

Consistent Σ_2 absoluteness for $H(\omega_1)$: Assuming large cardinals there is a countable transitive model M of ZFC such that whenever $M \sqsubseteq N$ and N is a model of ZFC then the Σ_2 theory of $H(\omega_1)$ is the same in M and N.

So there do exist models with the desired degree of absoluteness; the problem is simply that this cannot be guaranteed by a reasonable *first-order* theory.

Thus absoluteness *cannot* be used to discover new axioms based on intrinsic *first-order* properties of the universe of sets, as were the axioms of ZFC

The *Hyperuniverse Programme* instead proposes that such axioms be discovered as *first-order consequences* of principles based on intrinsic *higher-order* properties of the universe of sets

Intrinsic Evidence for Set-Theoretic Truth

Rather than present the *Hyperuniverse Programme* in full, I'll illustrate it with some relevant examples:

Let $\mathbb{H}$ denote the set of countable transitive models of ZFC, the *Hyperuniverse*. The elements of $\mathbb{H}$ are called *universes*.

A universe V is *powerset maximal* if whenever $V \sqsubseteq W$ and φ is a first-order sentence true in some $W_0 \sqsubseteq W$ then φ is true in some $V_0 \sqsubseteq V$

Powerset maximality is an intrinsic property of a universe, expressible in a first-order way within the Hyperuniverse; it is however a *higher-order* property of the given universe

Theorem. Assuming large cardinals, powerset maximal universes exist. They all share the following first-order properties: projective determinacy is false and there are no large cardinals.

Intrinsic Evidence for Set-Theoretic Truth

Thus we see that *maximality* can unexpectedly conflict with the existence of large cardinals.

However we have not taken into account *ordinal maximality* (which I won't define here precisely). We write:

IMH = powerset maximality
$\text{IMH}^{\#}$ = powerset maximality for ordinal maximal universes

Theorem. Assuming large cardinals there are ordinal maximal universes which satisfy $\text{IMH}^{\#}$ together with the existence of large cardinals.

What about the Continuum Problem? Can we resolve it using higher-order intrinsic properties of set-theoretic universes?

I conjecture that we can:

Intrinsic Evidence for Set-Theoretic Truth

Strong powerset maximality: whenever $V \sqsubseteq W$ and φ is a first-order sentence with "absolute parameters" true in some $W_0 \sqsubseteq W$ then φ is true in some $V_0 \sqsubseteq V$

SIMH $=$ Strong powerset maximality
SIMH$^{\#}$ $=$ Strong powerset maximality for ordinal maximal universes

Theorem. CH is false in all universes which satisfy the SIMH$^{\#}$.

Conjecture. Assuming large cardinals, there are universes with large cardinals that satisfy SIMH$^{\#}$.

Verifying this Conjecture will provide a refutation of CH based on intrinsic higher-order principles for universes of set theory.

Intrinsic Evidence for Set-Theoretic Truth

The *Hyperuniverse Programme* is not only concerned with maximality principles.

It is also concerned with other philosophical principles which serve as motivation for the selection of *preferred universes*.

Two other such principles are *omniscience* (definability of the class of properties that can hold in larger universes) and *typicality* (motivated by Shelah's concept of typical universe)

But the programme is new, so the mathematical work remains to be done to formulate these principles as precise mathematical criteria for the selection of preferred universes and to synthesise these criteria with maximality criteria and with each other.

It will be interesting to see how the programme develops.

Happy Birthday, Gabriele!