Large cardinals and L-like universes

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There are many different ways to extend the axioms of ZFC. One way is to adjoin the axiom $V = L$, asserting that every set is constructible. This axiom has many attractive consequences, such as the generalised continuum hypothesis (GCH), the existence of a definable wellordering of the class of all sets, as well as strong combinatorial principles, such as $\diamondsuit$, $\square$ and the existence of morasses.

However $V = L$ adds no consistency strength to ZFC. As many interesting set-theoretic statements have consistency strength beyond ZFC, it is common in set theory to assume at least the existence of inner models of V which contain large cardinals.

Can we simultaneously have the advantages of both the axiom of constructibility and the existence of large cardinals? Unfortunately even rather modest large cardinal hypotheses, such as the existence of a measurable cardinal, refute $V = L$. We can however hope for the following compromise:

V is an "L-like" model containing large cardinals.

In this article we explore the possibilities for this assertion, for various notions of "L-like" and for various types of large cardinals.

There are two approaches to this problem. The first approach is via the

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Inner model program. Show that any universe with large cardinals has an L-like inner model with large cardinals.

The inner model program, through use of fine structure theory and the theory of iterated ultrapowers, has succeeded in producing very L-like inner models containing many Woodin cardinals.

An alternative approach is given by the

Outer model program. Show that any universe with large cardinals has an L-like outer model with large cardinals.

We will show that L-like outer models with extremely large cardinals can be obtained using the method of iterated forcing.

Large cardinals

A cardinal κ is *inaccessible* iff it is uncountable, regular and larger than the power set of any smaller cardinal. It is *measurable* iff there is a κ-complete, nonprincipal ultrafilter on κ.

Measurability is equivalent to a property expressed in terms of *embeddings*, and stronger large cardinal properties are also expressed in this way. As usual, V denotes the universe of all sets. Let M be an inner model, i.e., a transitive proper class that satisfies the axioms of ZFC. A class function $j : V \to M$ is an *embedding* iff it preserves the truth of formulas with parameters in the language of set theory and is not the identity. If j is an embedding then there is a least ordinal κ such that $j(\kappa) \neq \kappa$, called the *critical point* of j, which is a measurable cardinal.

For an ordinal α, $j : V \to M$ is *α-strong* iff V_α is contained in M. A cardinal κ is α-strong iff there is an α-strong embedding with critical point κ. *Strong* means α-strong for all α.

Kunen ([10]) showed that no embedding is strong. However a cardinal can be strong, as embeddings witnessing its α-strength can vary with α. Stronger properties are obtained by requiring $j : V \to M$ to have strength depending on the image under j of its critical point. For example, κ is *superstrong* iff there is a nontrivial elementary embedding $j : V \to M$ with critical point

κ which is $j(\kappa)$-strong. An important weakening of superstrength is the property that for each $f : \kappa \to \kappa$ there is a $\bar{\kappa} < \kappa$ closed under f and a nontrivial elementary embedding $j : V \to M$ with critical point $\bar{\kappa}$ which is $j(f)(\bar{\kappa})$-strong; such κ are known as *Woodin cardinals*. The consistency strength of the existence of a Woodin cardinal is strictly between that of a strong cardinal and a superstrong cardinal.

We can demand more than superstrength. A cardinal κ is *hyperstrong* iff it is the critical point of an embedding $j : V \to M$ which is $j(\kappa) + 1$-strong. For a finite $n > 0$, *n-superstrength* is obtained by requiring j to be $j^n(\kappa)$-strong, where $j^1 = j$, $j^{k+1} = j \circ j^k$. Finally, κ is *ω-superstrong* iff it is the critical point of an embedding $j : V \to M$ which is n-superstrong for all n. Kunen's result [10] shows that no embedding j with critical point κ is $j^\omega(\kappa) + 1$-strong, where $j^\omega(\kappa)$ is the supremum of the $j^n(\kappa)$ for finite n.

The inner model program

If κ is inaccessible, then κ is also inaccessible in L, the most L-like model of all. This is not the case for measurability, however if κ is measurable then κ is measurable in an inner model $L[U]$, where U is an ultrafilter on κ, which has a definable wellordering and in which GCH, $\diamondsuit$, $\square$ hold and gap 1 morasses exist. For a strong cardinal κ there is a similarly L-like inner model $L[E]$ in which κ is strong, where E now is not a single ultrafilter, but rather a sequence of generalised ultrafilters, called *extenders*. More recent work yields similar results for Woodin cardinals, and even for Woodin limits of Woodin cardinals (see [12]).

However, progress beyond that has been impeded by the so-called *iterability problem*.

The outer model program

How can we obtain L-like outer models with large cardinals? For inaccessibles one has the following result of Jensen (see [1]):

Theorem 1 *(L-coding) There is a generic extension $V[G]$ of V such that*

a. ZFC holds in $V[G]$.
b. $V[G] = L[R]$ for some real R.
c. Every inaccessible cardinal of V remains inaccessible in $V[G]$.

There are similar $L[U]$ and $L[E]$ coding theorems (see [6] for the former), providing outer models of the form $L[U][R]$ and $L[E][R]$, R a real, which are just as L-like as $L[U]$ and $L[E]$, preserving measurability and strength, respectively.

However the approach via coding is limited in its use. Obtaining L-like outer models via coding depends on the existence of L-like inner models, such as $L[U]$ or $L[E]$, which, as we have observed, are not known to exist beyond Woodin limits of Woodin cardinals. And there are problems with the coding method itself which arise already just past a strong cardinal.

A more promising approach is to use iterated forcing. To illustrate this, consider the problem of making the GCH true in an outer model. Begin with an arbitrary model V of ZFC. Using forcing, we can add a function from $\aleph_1$ onto $2^{\aleph_0}$ without adding reals, thereby making CH true. By forcing again, we add a function from (the possibly new) $\aleph_2$ onto (the possibly new) $2^{\aleph_1}$ without adding subsets of $\aleph_1$, thereby obtaining $2^{\aleph_1} = \aleph_2$. Continue this indefinitely (via a reverse Easton iteration) and the result is a model of the GCH.

Do we preserve large cardinal properties if we make GCH true in this way? The answer is Yes.

Theorem 2 *(Large cardinals and the GCH) If κ is superstrong then there is an outer model in which κ is still superstrong and the GCH holds. The same holds for hyperstrong, n-superstrong for finite n and ω-superstrong.*

Proof. First we describe in more detail the above iteration to make GCH true. By induction on α we define the iteration P_α of length α: P_0 is the trivial forcing. For limit λ, P_λ is the inverse limit of the P_α, $\alpha < \lambda$, if λ is singular and is the direct limit of the P_α, $\alpha < \lambda$, if λ is regular. For successor $\alpha + 1$, $P_{\alpha+1} = P_\alpha * Q_\alpha$, where Q_α is the forcing that collapses $2^{\aleph_\alpha}$ to $\aleph_{\alpha+1}$ using conditions of size at most $\aleph_\alpha$. The desired class forcing P is the direct limit of the P_α's. For any cardinal κ of the form $\beth_{\alpha+1}$, the entire iteration P can be factored as $P_\kappa * P^\kappa$, where P_κ has a dense subset of size κ and P^κ is κ^+-closed. In particular, strongly inaccessible cardinals remain strongly inaccessible after forcing with P.

Now suppose that κ is superstrong, witnessed by the embedding $j : V \to M$, and that G is P-generic. Let P^* denote M's version of P. To show that κ is superstrong in $V[G]$, it suffices to find a P^*-generic G^* which satisfies $G^*_{j(\kappa)} = G_{j(\kappa)} \cap V_{j(\kappa)}$ and which contains $j[G]$, the pointwise image of G under j, as a subclass. For then, we extend $j : V \to M$ to $j^* : V[G] \to M[G^*]$ by sending σ^G to $j(\sigma)^{G^*}$. The property $j[G] \subseteq G^*$ implies that j^* is well-defined and elementary. And $V_{j(\kappa)}$ of $V[G]$ is the same as $V_{j(\kappa)}[G_{j(\kappa)} \cap V_{j(\kappa)}] = V_{j(\kappa)}[G^*_{j(\kappa)}]$ and therefore belongs to $M[G^*]$, as $V_{j(\kappa)}$ belongs to M.

Now P^*_α is the same as P_α for $\alpha < j(\kappa)$, as j is a superstrong embedding. The first difference between P^* and P is at $j(\kappa)$: $P^*_{j(\kappa)}$ is the direct limit of the P_α, $\alpha < j(\kappa)$, as $j(\kappa)$ is inacessible in M; but $j(\kappa)$ is not necessarily regular in V and therefore it is possible that $P_{j(\kappa)}$ is the *inverse* limit of the P_α, $\alpha < j(\kappa)$. So we cannot simply choose $G^*_{j(\kappa)}$ to be $G_{j(\kappa)}$, as the latter is generic for the wrong forcing.

But this problem is easily fixed: As $j(\kappa)$ is in fact Mahlo in M, it follows that $P^*_{j(\kappa)}$ has the $j(\kappa)$-cc in M. So any $G^*_{j(\kappa)}$ contained in $P^*_{j(\kappa)}$ whose intersection with each P_α, $\alpha < j(\kappa)$, is P_α-generic must also be $P^*_{j(\kappa)}$-generic. It follows that we can take $G^*_{j(\kappa)}$ to simply be the intersection of $G_{j(\kappa)}$ with $P^*_{j(\kappa)}$. Notice that $G^*_{j(\kappa)}$ trivially contains the pointwise image of G_κ under j as j is the identity below κ.

Finally we must define a generic $G^{*,\, j(\kappa)}$ for the "upper part" $P^{*,\, j(\kappa)}$ of the P^* iteration, which starts at $j(\kappa)$ and is defined in the ground model $M[G^*_{j(\kappa)}]$. In addition, $G^{*,\, j(\kappa)}$ must contain the pointwise image of G^κ under j^*, where j^* is the lifting of j to $V[G_\kappa]$ and G^κ is generic for P^κ, the iteration starting at κ defined over the ground model $V[G_\kappa]$.

In fact, with a harmless additional assumption about j, this latter requirement completely determines $G^{*,\, j(\kappa)}$: We assume that $j : V \to M$ is given as an extender ultrapower embedding. This means that each element of M is of the form $j(f)(a)$, where a belongs to $V^M_{j(\kappa)} = V_{j(\kappa)}$ and f is a function (in V) with domain V_κ. This assumption is harmless, as if the initial $j : V \to M$ does not satisfy it, we can replace M by the transitive collapse $\bar{M}$ of H, the elementary submodel of M consisting of all $j(f)(a)$ of the above form, and replace j by $k \circ j$, where $k : H \simeq \bar{M}$.

Lemma 3 $j^*[G^\kappa]$ generates *a* $P^{*,\ j(\kappa)}$*-generic over* $M[G^*_{j(\kappa)}]$*, i.e., each predense subclass of* $P^{*,\ j(\kappa)}$ *which is definable over* $M[G^*_{j(\kappa)}]$ *has an element which is extended by a condition in* $j^*[G^\kappa]$*.*

Proof. We only consider predense *subsets* D of $P^{*,\ j(\kappa)}$ in $M[G^*_{j(\kappa)}]$; a similar argument works for predense subclasses.

As j is given as an extender ultrapower embedding, D is of the form $\sigma^{G^*_{j(\kappa)}}$ where the name σ can be written as $j(f)(a)$ with $f : V_\kappa \to V$ in V and $a \in V_{j(\kappa)}$. Now using the κ^+-closure of P^κ, choose a condition p in G^κ which extends an element of $f(\bar{a})$ whenever $\bar{a}$ belongs to V_κ and $f(\bar{a})^{G_\kappa}$ is predense on $P^{*,\ \kappa}$. Then $j^*(p)$ obviously belongs to $j^*[G^\kappa]$ and extends an element of $j(f)(a)^{G^*_{j(\kappa)}} = \sigma^{G^*_{j(\kappa)}} = D$, as desired. □ (Lemma 3)

This completes the construction of G^* and therefore the proof that P preserves superstrong cardinals.

Now suppose that κ is hyperstrong. Again we need to find a P^*-generic G^* containing $j[G]$ as a subclass. The forcings $P_{j(\kappa)+1} = P_{j(\kappa)} * Q_{j(\kappa)}$ and $P^*_{j(\kappa)+1}$ agree as $j(\kappa)$ is regular in V and M contains $V_{j(\kappa)+1}$. Also, $j^*[g_\kappa]$, where j^* is the lifting of j to $V[G_\kappa]$ and g_κ is the Q_κ-generic chosen by G at stage κ of the iteration, is a set of conditions in $Q_{j(\kappa)}$ which belongs to $M[G_{j(\kappa)}]$ and has size 2^κ there; therefore $j^*[g_\kappa]$ has a lower bound in $Q_{j(\kappa)}$. By choosing our generic G so that $g_{j(\kappa)}$ includes this lower bound, we can succeed in lifting j to $V[G_{\kappa+1}]$. We may assume that $j : V \to M$ is *given by a hyperextender*; this means that each element of M is of the form $j(f)(a)$ where f is a function in V with domain $V_{\kappa+1}$ and a is an element of $V_{j(\kappa)+1}$. Then we can use the argument of Lemma 3 to generate the entire generic G^* containing $j[G]$.

The case of n-superstrongs raises a new difficulty. We first treat the case $n = 2$. As in the superstrong case, P and P^* may take different limits at $j^2(\kappa)$, as the latter may be singular in V. As in that case, we can obtain a $P^*_{j^2(\kappa)}$-generic by intersecting $G_{j^2(\kappa)}$ with $P^*_{j^2(\kappa)}$. However we must also ensure that $G^*_{j^2(\kappa)}$ contain $j[G_{j(\kappa)}]$ as a subset. Write $P_{j(\kappa)}$ as $P_\kappa * P^\kappa_{j(\kappa)}$; it suffices to arrange that $G^{*,\ j(\kappa)}_{j^2(\kappa)}$ contain $j^*[G^\kappa_{j(\kappa)}]$ as a subset, where j^* is the lifting of j to $V[G_\kappa]$ and $G^\kappa_{j(\kappa)}$ is $P^\kappa_{j(\kappa)}$-generic over $V[G_\kappa]$.

We argue as follows. If $j[j(\kappa)]$ is bounded in $j^2(\kappa)$ then the set of conditions $j^*[G^\kappa_{j(\kappa)}]$ has a lower bound in $P^{*,\, j(\kappa)}_{j^2(\kappa)} \subseteq P^{j(\kappa)}_{j^2(\kappa)}$. (Actually, this first case can be ruled out by choosing $j^2(\kappa)$ minimally; see Lemma 10 below.) Otherwise $j^2(\kappa)$ is singular, so $P_{j^2(\kappa)}$ is an inverse limit and again the set of conditions $j^*[G^\kappa_{j(\kappa)}]$ has a lower bound in $P^{j(\kappa)}_{j^2(\kappa)}$. We therefore assume that our generic G has been chosen so that $G^{j(\kappa)}_{j^2(\kappa)}$ contains the greatest lower bound of $j^*[G^\kappa_{j(\kappa)}]$. Then we can take $G^*_{j^2(\kappa)}$ to be the intersection of $G_{j^2(\kappa)}$ with $P^*_{j^2(\kappa)}$ and thereby obtain $j[G_{j(\kappa)}] \subseteq G^*_{j^2(\kappa)}$. This allows us to lift j to $V[G_{j(\kappa)}]$. Then we can use the argument from Lemma 3 to generate the entire generic G^* containing $j[G]$.

For the case $n > 2$ the argument is similar; we must choose $G^{j^{n-1}(\kappa)}_{j^n(\kappa)}$ to contain the greatest lower bound of $j^*[G^{j^{n-2}(\kappa)}_{j^{n-1}(\kappa)}]$, where j^* is the lifting of j to the model $V[G_{j^{n-2}(\kappa)}]$.

Finally we consider ω-superstrength. Again we must choose G^* to be P^*-generic over M and to contain the pointwise image of G under j. Let $j^\omega(\kappa)$ denote the supremum of the $j^n(\kappa)$, $n \in \omega$. As before it suffices to find $G^*_{j^\omega(\kappa)}$ which is $P^*_{j^\omega(\kappa)}$-generic and contains $j[G_{j^\omega(\kappa)}]$ as a subset. Note that $j[G_\kappa] = G_\kappa$ is trivially contained in $G_{j^\omega(\kappa)}$ and $j^*[G^\kappa_{j^\omega(\kappa)}]$ has a lower bound in $P^{j(\kappa)}_{j^\omega(\kappa)}$ (as defined in $V[G_{j(\kappa)}]$); by choosing $G^{j(\kappa)}_{j^\omega(\kappa)}$ to contain this lower bound we can take $G^*_{j^\omega(\kappa)}$ to be $G_{j^\omega(\kappa)}$ and thereby obtain $j[G_{j^\omega(\kappa)}] \subseteq G^*_{j^\omega(\kappa)}$. And again we can use the argument of Lemma 3 to generate the entire generic G^* containing $j[G]$. So it only remains to show:

Lemma 4 *$G_{j^\omega(\kappa)} \cap P^*_{j^\omega(\kappa)}$ is $P^*_{j^\omega(\kappa)}$-generic over M.*

Proof. Suppose that $D \in M$ is dense on $P^*_{j^\omega(\kappa)}$ and write D as $j(f)(a)$ where f has domain $V_{j^\omega(\kappa)}$ and a belongs to $V_{j^{n+1}(\kappa)}$ for some n. (We may assume that every element of M is of this form.) Choose p in $G_{j^\omega(\kappa)}$ such that p *reduces* $f(\bar{a})$ below $j^n(\kappa)$ whenever $\bar{a}$ belongs to $V_{j^n(\kappa)}$ and $f(\bar{a})$ is open dense on $P_{j^\omega(\kappa)}$, in the sense that if q extends p then q can be further extended into $f(\bar{a})$ without changing q at or above $j^n(\kappa)$. Such a p exists using the $j^n(\kappa)^+$-closure of $P^{j^n(\kappa)}_{j^\omega(\kappa)}$ in $V[G_{j^n(\kappa)}]$. Then $j(p)$ belongs to $j[G_{j^\omega(\kappa)}]$ and reduces D below $j^{n+1}(\kappa)$. As $G_{j^{n+1}(\kappa)}$ is $P_{j^{n+1}(\kappa)}$-generic and P, P^* agree below $j^{n+1}(\kappa)$, it follows that $G_{j^\omega(\kappa)} \cap P^*_{j^\omega(\kappa)}$ intersects D, as desired. □ (Lemma 4)

This completes the proof of Theorem 2. $\Box$

Another important property of L is the existence of a definable wellordering of the universe.

Theorem 5 *(Large cardinals and definable wellorderings) If κ is superstrong then there is an outer model in which κ is still superstrong and there is a definable wellordering of the universe. The same holds for hyperstrong, n-superstrong for finite n and ω-superstrong.*

Proof. By Theorem 2 we may assume the GCH. Let κ have one of the large cardinal properties mentioned in the theorem, as witnessed by the embedding $j : V \to M$. Choose λ to be a cardinal greater than $j^\omega(\kappa)$. By the method of L-coding (see Theorem 1), we can enlarge V without adding subsets of λ to a universe of the form $L[A]$, A a subset of λ^+. By the argument of Lemma 3 the embedding j lifts to $L[A]$ and therefore κ retains its large cardinal properties.

Now we introduce a definable wellordering. Perform a reverse Easton iteration of length λ^+, indexed by successor cardinals greater than λ^+, where at the i-th successor cardinal, an i^+-Cohen set is added iff i belongs to A. The result is that i belongs to A iff not every subset of the successor of the i-th successor cardinal is constructible from a subset of the i-th successor cardinal. Now the result of this iteration is a model of the form $L[B]$ where B is a subset of $\lambda^{(\lambda^+)}$, the "λ^+-th cardinal greater than λ". Repeat this to code B using the next interval of successor cardinals. Continuing this indefinitely yields a model with a wellordering definable from the parameter λ.

To eliminate the parameter λ, use a pairing function $f : \mathrm{Ord} \times \mathrm{Ord} \to \mathrm{Ord}$ on the ordinals and arrange that the universe is of the form $L[C]$ where C is a class of ordinals and for any i, i is in C iff some subset of the successor to the $f(i,j)$-th successor cardinal is not constructible from a subset of the $f(i,j)$-th successor cardinal, for all sufficiently large j. $\Box$

Jensen's (global) $\Box$ principle asserts the existence of a sequence $\langle C_\alpha \mid \alpha$ singular$\rangle$ such that C_α has ordertype less than α for each α and $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$ whenever $\bar{\alpha} \in \mathrm{Lim}\ C_\alpha$. The following strengthens a result of Doug Burke [2]:

Theorem 6 *(Superstrong cardinals and $\Box$) If κ is superstrong then there is an outer model in which κ is still superstrong and $\Box$ holds.*

Proof. By Theorem 2, we may assume the GCH. Consider now the reverse Easton iteration P where at the regular stage α, Q_α is a P_α-name for the forcing which adds a $\Box$-sequence on the singular limit ordinals less than α. A condition in Q_α is a sequence $\langle C_\beta \mid \beta \leq \gamma,\ \beta \text{ singular}\rangle$, $\gamma < \alpha$, such that C_β has ordertype less than β for each β and $C_{\bar\beta} = C_\beta \cap \bar\beta$ whenever $\bar\beta$ belongs to Lim C_β.

Using the fact that P_α forces $\Box$-sequences of any regular length less than α, it is easy to verify by induction that any condition in Q_α can be extended to have arbitrarily large length less than α. Also Q_α, and indeed the entire iteration from stage α on, is α-distributive.

Let P^* denote M's version of P. We want to construct G^* to be P^*-generic over M, to agree with G strictly below $j(\kappa)$ and to contain $j[G]$ as a subclass. As in earlier arguments, P and P^* agree strictly below $j(\kappa)$ but not necessarily at $j(\kappa)$, which is regular in M but may be singular in V; as before we take $G^*_{j(\kappa)}$ to be $G_{j(\kappa)} \cap P^*_{j(\kappa)}$. Our new task is to define a $Q^*_{j(\kappa)}$-generic g over $M[G^*_{j(\kappa)}]$.

Lemma 7 *Assume GCH and let $j : V \to M$ witness the superstrength of κ with $j(\kappa)$ minimal. Then $j(\kappa)$ has cofinality κ^+.*

Proof. Let $\langle f_i \mid i < \kappa^+\rangle$ be a list of all functions from κ to κ. Then the sequence $\langle j(f_i) \mid i < \kappa\rangle$ belongs to M, as it equals $j(\langle f_i \mid i < \kappa^+\rangle) \restriction \kappa$. For any ordinal $\alpha < \kappa^+$ we can use a bijection between α and κ to similarly conclude that $\langle j(f_i) \mid i < \alpha\rangle$ belongs to M.

Now for each $\alpha < \kappa^+$ let κ_α be least so that κ_α is closed under each $j(f_i)$, $i < \alpha$. Then κ_α is less than $j(\kappa)$, as $j(\kappa)$ is regular in M. Let κ^* be the supremum of the κ_α's. It suffices to show that there is a superstrong embedding j^* with critical point κ such that $j^*(\kappa) = \kappa^*$; then by the minimality of $j(\kappa)$, we must have $j(\kappa) = \kappa^*$ and therefore $j(\kappa)$ is the supremum of a non-decreasing κ^+-sequence of ordinals strictly less than $j(\kappa)$. It follows that $j(\kappa)$ has cofinality κ^+.

To obtain j^* define $H = \{j(f)(a) \mid f : V_\kappa \to V,\ a \in V_{\kappa^*}\}$. Then H is an elementary submodel of M and $H \cap j(\kappa) = \kappa^*$. Let $\pi : H \simeq M^*$; then $j^* = \pi \circ j : V \to M^*$ satisfies $j^*(\kappa) = \pi(j(\kappa)) = \kappa^*$ and witnesses the superstrength of κ, as desired. $\square$

We can assume that j is given by an ultrapower, and therefore that j is continuous at κ^+. It follows that $(j(\kappa)^+)^M$ has cofinality κ^+ and $H(j(\kappa)^+)^M$ is closed under κ-sequences. Therefore we can write the collection of $(j(\kappa)^+)^M$-many open dense subsets of $Q^*_{j(\kappa)}$ as the union of κ^+ subcollections, each of which belongs to $M[G^*_{j(\kappa)}]$ and has size less than $j(\kappa)$ there. Now we can build g in κ^+ steps, using $j(\kappa)$-distributivity to meet fewer than $j(\kappa)$ open dense sets at each step (and defining the $\square$-sequence coherently at limit stages). We must also ensure that g extend g_κ; but this is easy to arrange as the latter is a condition in the forcing $Q^*_{j(\kappa)}$.

Finally the rest of G^* can be generated from $j[G]$ as before. $\square$

The proof of the previous theorem does not work for hyperstrong κ, and there is a good reason for this. κ is *subcompact* iff for any $B \subseteq H_{\kappa^+}$ there are $\mu < \kappa$, $A \subseteq H_{\mu^+}$ and an elementary embedding $j : (H_{\mu^+}, A) \to (H_{\kappa^+}, B)$ with critical point μ. (Note that by elementarity, j must send μ to κ.)

Proposition 8 *(a) If κ is hyperstrong then κ is subcompact. (b) (Jensen) If there is a subcompact cardinal then $\square$ (even when restricted to ordinals between κ and κ^+) fails.*

Proof. (a) Suppose that $j : V \to M$ witnesses hyperstrength. Then for all subsets B of $j(\kappa)^+$ in the range of j, j gives an elementary embedding of (H_{κ^+}, A) into $(H_{j(\kappa)^+}, B)$, where $j(A) = B$; moreover this embedding belongs to M as j is hyperstrong and $j \restriction H_{\kappa^+}$ belongs to $H_{j(\kappa)^+}$. As the range of j is an elementary submodel of M, it follows that there is an elementary embedding of some (H_{μ^+}, A) into $(H_{j(\kappa)^+}, B)$ (sending μ to $j(\kappa)$) which belongs to the range of j. So $j(\kappa)$ is subcompact in Range j and therefore by elementarity subcompact in M. As j is elementary, κ is subcompact in V.
(b) Suppose that κ is subcompact and $\vec{C} = \langle C_\alpha \mid \kappa < \alpha < \kappa^+,\ \alpha \text{ singular}\rangle$ has the properties of a $\square$-sequence. By thinning out the C_α's we can ensure that each has ordertype at most κ. Let j be an embedding from $(H_{\mu^+}, \vec{\vec{C}})$

to $(H_{\kappa^+}, \vec{C})$, sending μ to κ. Let α be the supremum of the ordinals in the range of j. Then α has cofinality μ^+. The ordinals in the range of j form a $< \mu$-closed and therefore ω-closed unbounded subset of α. And Lim C_α is a closed unbounded subset of α. Therefore the intersection D of these two sets is unbounded in α. By the coherence property of $\vec{C}$, the ordertype of C_β for sufficiently large β in D is at least μ. But as the ordertype of C_α is at most κ (in fact less than κ), the ordertype of C_β for all β in D is strictly less than κ. Thus there are β in $D \subseteq$ Range j with C_β of ordertype not in Range j, contradicting the elementarity of j. $\Box$

Remark. Cummings-Schimmerling [4] establish the optimality of Jensen's result above by proving the consistency of $\Box$ with 1-extendibility, a notion whose strength lies between that of superstrength and hyperstrength. (κ is 1-*extendible* iff there is an elementary embedding j with critical point κ from $V_{\kappa+1}$ into $V_{j(\kappa)+1}$). For their result, they perform the reverse Easton iteration of Theorem 6 and check that j lifts to $V_{\kappa+1}[G_\kappa]$, using the fact that j is the identity below κ.

For uncountable, regular κ, $\diamondsuit_\kappa$ says that there exists $\langle D_\alpha \mid \alpha < \kappa \rangle$ such that D_α is a subset of α for each α and for every subset D of κ, $\{\alpha < \kappa \mid D_\alpha = D \cap \alpha\}$ is stationary in κ. $\diamondsuit$ asserts that $\diamondsuit_\kappa$ holds for every uncountable, regular κ.

Theorem 9 *(Large cardinals and $\diamondsuit$) If κ is superstrong then there is an outer model in which κ is still superstrong and $\diamondsuit$ holds. The same holds for hyperstrong, n-superstrong for finite n and ω-superstrong.*

Proof. The proof combines the proofs of Theorems 2 and 6. As in the latter proof, we use a reverse Easton iteration P where at each regular stage α, Q_α is α-distributive (in fact, in the present context the entire iteration starting with α is α-closed). A condition in Q_α is a sequence $\langle D_\beta \mid \beta < \gamma \rangle$, $\gamma < \alpha$, such that D_β is a subset of β for each $\beta < \gamma$. It is easy to show that a Q_α-generic yields a $\diamondsuit_\alpha$-sequence, using the α-closure of Q_α.

The proof in the superstrong case is just as in Theorem 6, where we take $G^*_{j(\kappa)}$ to be the intersection of $G_{j(\kappa)}$ with $P^*_{j(\kappa)}$ and then build a $Q^*_{j(\kappa)}$-generic containing the condition g_κ. For hyperstrong κ (witnessed by $j : V \to M$), we need only observe that $j[g_{\kappa^+}]$ has a lower bound in the forcing $Q_{j(\kappa)^+}$

and choose $g^*_{j(\kappa)^+} = g_{j(\kappa)^+}$ to contain this lower bound. (This is where the argument with $\Box$ breaks down.)

For n-superstrongs, $1 < n$ finite, we can take $G^*_{j^n(\kappa)}$ to be the intersection of $G_{j^n(\kappa)}$ with $P^*_{j^n(\kappa)}$ (requiring the latter to contain the greatest lower bound of $j[G_{j^{n-1}(\kappa)}]$), but face the problem of defining a $Q^*_{j^n(\kappa)}$-generic containing the image of $g_{j^{n-1}(\kappa)}$ under (the lifting to $V[G_{j^{n-1}(\kappa)}]$ of) j. We use the following.

Lemma 10 *Suppose that n is greater than 1 and $j : V \to M$ witnesses the n-superstrength of κ, with $j^n(\kappa)$ chosen minimally. Then j is continuous at $j^{n-1}(\kappa)$ (i.e., the range of j is cofinal in $j^n(\kappa)$).*

Proof. Let κ^* be the supremum of the range of j intersect $j^n(\kappa)$. It suffices to show that there is an n-superstrong embedding j^* with critical point κ such that $j^* \restriction j^{n-1}(\kappa) = j \restriction j^{n-1}(\kappa)$ and $(j^*)^n(\kappa) = \kappa^*$.

Let H consist of all elements of M of the form $j(f)(a)$, where $f : V_\kappa \to V$ and a belongs to V_{κ^*}. Then H is an elementary submodel of M: If $M \vDash \varphi(y, j(f_1)(a_1), \ldots, j(f_n)(a_n))$ for some y in M, where $f_i : V_\kappa \to V$, $a_i \in V_{\kappa^*}$ for each i, then choose $g : V_\kappa \to V$ so that $g(\langle x_1, \ldots, x_n\rangle) = y$ is a solution to $\varphi(y, f_1(x_1), \ldots, f_n(a_n))$ in V (if there is such a solution y in V). Then by elementarity, $M \vDash \varphi(y, j(f_1)(a_1), \ldots, j(f_n)(a_n))$ where y equals $j(g)(\langle a_1, \ldots, a_n\rangle)$. The latter is an element of H.

Note that $H \cap j^n(\kappa) = \kappa^*$: If $j(f)(a)$ is less than $j^n(\kappa)$, where $f : V_\kappa \to V$ and $a \in V_{\kappa^*}$, then $j(f)(a)$ is less than the supremum of $j(f)[V_\alpha] \cap j^n(\kappa)$ where $\alpha \in \text{Range}\ (j) \cap \kappa^*$ is large enough so that V_α contains a; as the latter supremum belongs to Range (j), it follows that $j(f)(a)$ is less than κ^*.

Now let $\pi : H \simeq M^*$ be the transitive collapse of H and define $j^* = \pi j$. Then $j^* : V \to M^*$ is elementary and has critical point κ. As π sends $j^n(\kappa)$ to κ^* and is the identity on κ^*, it follows that $(j^*)^m(\kappa) = j^m(\kappa)$ for $m < n$ and $(j^*)^n(\kappa) = \kappa^*$. And j^* is n-superstrong as $V_{\kappa^*} = V^M_{\kappa^*} = V^{M^*}_{\kappa^*}$. $\Box$

We may assume that every element of M is of the form $j(f)(a)$ where $f : V_{j^{n-1}(\kappa)} \to V$ and a belongs to $V_{j^n(\kappa)}$. Now we claim that the image of $g_{j^{n-1}(\kappa)}$ under (the lifting to $V[G_{j^{n-1}(\kappa)}]$ of) j generates a generic for $Q^*_{j^n(\kappa)}$, in the sense that every dense subset of $Q^*_{j^n(\kappa)}$ which belongs to $M[G^*_{j^n(\kappa)}]$ is met by a condition in $j[g_{j^{n-1}(\kappa)}]$. For, if D is such a dense set, then D has a name of the form $j(f)(a)$ where a belongs to $V_{j(\alpha)}$ for some $\alpha < j^{n-1}(\kappa)$. By the $j^{n-1}(\kappa)$-distributivity of $Q_{j^{n-1}(\kappa)}$, there is a condition $\bar{p} \in g_{j^{n-1}(\kappa)}$ which meets all dense sets with names of the form $f(\bar{a})$, $\bar{a} \in V_\alpha$; then $j(\bar{p}) = p$ meets D.

Finally, ω-superstrength is handled just as in Theorem 2. □

Remark. The proof of Lemma 10 shows a bit more: Any n-superstrong embedding j with critical point κ has an "approximating" n-superstrong embedding j^* which agrees with j below $j^{n-1}(\kappa)$ and is continuous at $(j^*)^{n-1}(\kappa) = j^{n-1}(\kappa)$. A similar remark applies to the proof of Lemma 7.

The technique of the previous proof can also be used to force a weakened form of □, preserving very large cardinals. □ *holds at small cofinalities* iff the □ principle holds when restricted to singular ordinals of cofinality at most the least superstrong cardinal.

Theorem 11 *(Large cardinals and □ at small cofinalities) If κ is hyperstrong then there is an outer model in which κ is still hyperstrong and □ holds at small cofinalities. The same holds for n-superstrong for finite n and ω-superstrong.*

Proof. Perform a reverse Easton iteration where at each regular stage α, Q_α adds a □-sequence on the singular ordinals less than κ which have cofinality at most the least superstrong cardinal. If $j : V \to M$ witnesses that κ is hyperstrong, then we take $G^*_{j(\kappa)^+}$ to be $G_{j(\kappa^+)}$ and observe that $j[g_{\kappa^+}]$ does have a greatest lower bound in $Q_{j(\kappa)^+}$, because its supremum is an ordinal of cofinality κ^+, greater than the least superstrong cardinal of M. By choosing $g^*_{j(\kappa)^+} = g_{j(\kappa)^+}$ to contain this greatest lower bound, we can lift j to $V[G_{\kappa^++1}]$, and then to all of $V[G]$. If $j : V \to M$ witnesses the 2-superstrength of κ then similarly we get a greatest lower bound for $j[G_{j(\kappa)}]$ in $P_{j^2(\kappa)}$ as for each regular $\alpha \in (\kappa, j(\kappa))$, the supremum of $j[\alpha]$ is an ordinal of cofinality greater than κ, which is superstrong (and more) in M. Then we use the argument of the preceding proof to lift j to all of $V[G]$. A similar argument handles ω-superstrength. □

Remark. I mention some previous work on preserving weakened forms of $\square$ in the presence of very large cardinals. If κ is supercompact then Solovay proved that $\square_\lambda$ fails at all singular $\lambda > \kappa$. However Baumgartner showed that there is a forcing extension in which κ remains supercompact and $\square_\lambda$ does hold for all λ greater than κ, provided we restrict the square sequences to ordinals of cofinality less than some fixed $\mu < \kappa$. And Cummings-Foreman-Magidor [3] show that κ remains supercompact in some forcing extension in which for each singular cardinal λ of cofinality at least κ, the Weak Square principle $\square^*_\lambda$ holds (allowing λ-many clubs in each limit ordinal $\alpha < \lambda^+$ instead of just one).

Theorem 12 *(Large cardinals and Gap* 1 *morasses) If κ is superstrong then there is an outer model in which κ is still superstrong and gap* 1 *morasses exist at each regular cardinal. The same holds for hyperstrong, n-superstrong for finite n and ω-superstrong.*

Proof. For the definition of a gap 1 morass we refer the reader to [5]. Assume GCH and let κ be superstrong. We apply the reverse Easton iteration P where at each regular stage α, Q_α adds a gap 1 morass at α. A condition in Q_α is a size $< \alpha$ initial segment of a morass up to some top level, together with a map of an initial segment of this top level into α^+ which obeys the requirements of a morass map. To extend a condition, we end-extend the morass up to its top level and require that the map from the given initial segment of its top level into α^+ factor as the composition of a map into the top level of the stronger condition followed by the map given by the stronger condition into α^+. The forcing Q_α is α-closed and, using a Δ-system argument, is α^+-cc.

To obtain the desired G^*, we must build a $Q^*_{j(\kappa)}$-generic which extends the image under (the lifting to $V[G_\kappa]$ of) j of the Q_κ-generic g_κ. As in the case of $\square$ we use minimisation of $j(\kappa)$ to ensure that it has cofinality κ^+ and then build a $Q^*_{j(\kappa)}$-generic in κ^+ steps. Note that any condition in $j[g_\kappa]$ is extended by one which has top level κ and maps an initial segment of the top level into $j(\kappa)^+$ using j. Now given fewer than $j(\kappa)$ maximal antichains in $M[G^*_{j(\kappa)}]$, we can choose $\alpha < j(\kappa)^+$ of cofinality $j(\kappa)$ in M so that these maximal antichains are maximal when restricted to conditions which are "below α" in the sense that they map an initial segment of their top level into α. Moreover, there is a condition which serves as a lower bound to all conditions in $j[g_\kappa]$ which are below α in this sense. Therefore we can choose a condition below α meeting all of the given maximal antichains compatibly with the conditions in $j[g_\kappa]$ which are below α, and therefore compatibly with all conditions in $j[g_\kappa]$. Repeating this in κ^+ steps for increasingly large $\alpha < j(\kappa)^+$ of M-cofinality $j(\kappa)$ (taking unions at limit stages) yields the desired $Q^*_{j(\kappa)}$-generic. The remainder of the generic G^* can be generated using Lemma 3.

Now suppose that κ is hyperstrong. We must define a suitable $Q^*_{j(\kappa)^+}$-generic. We may assume that j is given by a hyperextender and therefore j is cofinal from κ^{++} into $j(\kappa)^{++}$ of M. Let S consist of those morass points at the top level (i.e., level κ^+) of g_{κ^+} which have cofinality κ^+. For each σ in S let $g_{\kappa^+} \restriction \sigma$ denote the set of conditions in g_{κ^+} which are below σ. Then $j[g_{\kappa^+} \restriction \sigma]$ has a greatest lower bound p_σ in $Q^*_{j(\kappa)^+}$.

The collection of maximal antichains of $Q^*_{j(\kappa)^+}$ which belong to $M[G^*_{j(\kappa)^+}]$ can be written as a union $\bigcup_{i<j(\kappa)^+} X_i$ where for each i and each σ in S, $X_i \restriction j(\sigma)$ (the subset of X_i consisting of those maximal antichains all of whose elements are below $j(\sigma)$) is a set of size at most $j(\kappa)$ in M. By induction on $\sigma \in S$ choose a condition q_σ extending p_σ and all q_τ, $\tau \in S \cap \sigma$, which meets all antichains in $X_0 \restriction j(\sigma)$. By hyperstrength, the sequence of $q_\tau \restriction j(\sigma)$, $\tau \in S$, has a greatest lower bound p^1_σ for each $\sigma \in S$. Now repeat this construction for $X_1, X_2, \ldots$ for $j(\kappa)^+$ steps, resulting in a set of conditions which generates a generic $g_{j(\kappa)^+}$ for $Q^*_{j(\kappa)^+}$. As before, the remainder of the generic G^* can be generated using Lemma 3.

The cases of n-superstrength, $2 \leq n$ finite, are handled as in the proof of Theorem 9. ω-superstrength is handled as in the proof of Theorem 2. □

Questions. 1. It is possible to force a definable wellordering of the universe over a model of GCH preserving the superstrength of all superstrong cardinals, at the cost of some cardinal collapsing. Is it possible to do this without cardinal collapsing? Is it possible to preserve the superstrength of all superstrong cardinals while forcing not only the universe but also each $H(\kappa)$, $\kappa > \omega_1$, to have a definable wellordering?
2. Is it consistent with a superstrong cardinal to have a gap 2 morass at

every regular cardinal?
3. To what extent are the condensation and hyperfine structural properties of L (see [9]) consistent with large cardinals? For the former, see the forthcoming [8].

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