Annals of Pure and Applied Logic 26 (1984) 103–122
North-Holland

MODEL THEORY FOR $L_{\infty\omega_1}$*

Sy D. FRIEDMAN

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Communicated by A. Nerode
Received 29 November 1981

Introduction

The development of compactness and completeness theorems for countable fragments of $L_{\infty\omega}$, initiated in [1], has had numerous applications throughout logic. (See for example, [2], [10], [5].) In contrast, the model theory of fragments of $L_{\infty\omega_1}$ is entirely undeveloped. The essential reason for this is that well-foundedness is expressible in $L_{\infty\omega_1}$, providing a serious barrier to any kind of general compactness result.

It is our work in [7] that led us to believe that compactness and completeness results do exist for (uncountable) fragments of $L_{\infty\omega_1}$ and that indeed these results have useful applications. In that paper we extended a result of Sacks [14] on countable admissible ordinals to the uncountable case. Sacks' result can be proved by either forcing or by compactness methods. We have discovered the uncountable analogue of the simplest of the forcing proofs, that based on Steel forcing [15]. Thus it was natural to search for an uncountable analogue of the compactness methods as well. It is by analyzing the essential nature of uncountable Steel forcing that we were led to our compactness theorem.

We begin our study in Section 1 with a discussion of elementary equivalence for $L_{\infty\omega_1}$. The Scott analysis for $L_{\infty\omega}$ has a natural analogue for $L_{\infty\omega_1}$ and thus leads to the notion of ω-Scott rank of a structure. Determining a canonical bound on this notion of rank leads to the definition of ω-HYP($\mathfrak{M}$), the smallest ω-admissible set above $\mathfrak{M}$, in analogy to Barwise's HYP($\mathfrak{M}$). Nadel's theorem [13] easily goes through in this context: ω-Scott rank$(\mathfrak{M}) \leq \omega$-HYP$(\mathfrak{M}) \cap \mathrm{ORD}$. We show that this bound is best possible in that for each ω-admissible α there is a structure $\mathfrak{M}$ such that ω-Scott rank$(\mathfrak{M}) = \omega$-HYP$(\mathfrak{M}) \cap \mathrm{ORD} = \alpha$. The proof is based on our generalization of Steel's forcing, though we present here a model-theoretic argument. This also gives a new and simpler proof of a result of Friedman [7]: If α is ω-admissible, $\omega_1 < \alpha < \omega_2$, then $\alpha =$ least X-admissible for some $X \subseteq \omega_1$.

* This research was supported by National Science Foundation Grant # MCS 7906084.
Work done during the Logic Year 1980–81 at the Institute for Advanced Studies, The Hebrew University of Jerusalem.

Other basic facts concerning HYP($\mathfrak{M}$) carry over naturally to ω-HYP($\mathfrak{M}$): The relations on $|\mathfrak{M}|$ which are ω-inductive over HC($\mathfrak{M}$) coincide with those $\Sigma_1(\omega$-HYP($\mathfrak{M}$)). And, ω-HYP($\mathfrak{M}$) $\cap$ ORD = least α such that $\mathfrak{M}$ is α-recursively saturated for $L_{\infty\omega_1} \cap L_\alpha$.

Our compactness result is presented in Section 2. It is obtained by abstracting some of the properties held by the theory of the structure used in Section 1 to show the optimality of the canonical bound on ω-Scott rank. The compactness theorem applies to certain $\Sigma_1(s)$-theories over ω-admissible L_α, where $s : L_\alpha \to L_\alpha$ is the function $s(x) = x^\omega$. We also restrict ourselves to *ω-closed* structures; structures $\mathfrak{M}$ where the countable limit of types realized in $\mathfrak{M}$ is realized in $\mathfrak{M}$. A sort of 'ordinal preservation' is obtained in the sense that the model $\mathfrak{M}$ produced obeys ω-HYP($\mathfrak{M}$) $\cap$ ORD $\leqslant \alpha$. We also provide in Section 2 generalized versions of the Barwise hard core theorem and the Scott isomorphism theorem.

Section 3 is devoted to applications of the results of Section 2. As an illustration we first reprove the main result of Section 1. In a second application we characterize those admissible sets of the form pure part (ω-HYP($\mathfrak{M}$)) as the ω-admissible, resolvable ones. This is in analogy with our earlier result in [6] characterizing pure part (HYP($\mathfrak{M}$)). Our final applications deal with uncountable versions of definability-theoretic results first obtained in [15] in the countable case. For example, we show that for some subset $T \subseteq \omega_1$ there are $\pi_1(\langle L_{\omega_1}, T\rangle)$-singletons $A, B \subseteq \omega_1$ such that A is not definable over $\langle L_{\omega_1}, T, B\rangle$ and B is not definable over $\langle L_{\omega_1}, T, A\rangle$.

Our paper ends with a discussion of some further results and open questions.

1. ω-HYP($\mathfrak{M}$)

We fix a structure $\mathfrak{M}$ for a language $\mathscr{L}$ of finite similarity type. The $\mathscr{L}_{\infty\omega_1}$-theory of $\mathfrak{M}$ can be expressed by a single sentence $\phi \in \mathscr{L}_{\infty\omega_1}$. We describe how this is done using an analogue of Scott's analysis of $\mathscr{L}_{\infty\omega}$-equivalence.

Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}', \ldots$ range over $|\mathfrak{M}|^{<\omega_1}$. We define a sequence of relations $\sim_\beta$ on members of $|\mathfrak{M}|^{<\omega_1}$ of the same length by induction:

$$\begin{aligned}
\boldsymbol{x} \sim_0 \boldsymbol{y} \quad &\text{iff} \quad \boldsymbol{x}, \boldsymbol{y} \text{ realize the same atomic type in } \mathfrak{M},\\
\boldsymbol{x} \sim_{\beta+1} \boldsymbol{y} \quad &\text{iff} \quad \forall \boldsymbol{x}'\, \exists \boldsymbol{y}'\, (\boldsymbol{x} * \boldsymbol{x}' \sim_\beta \boldsymbol{y} * \boldsymbol{y}')\\
&\qquad \text{and } \forall \boldsymbol{y}'\, \exists \boldsymbol{x}'\, (\boldsymbol{x} * \boldsymbol{x}' \sim_\beta \boldsymbol{y} * \boldsymbol{y}'),\\
\boldsymbol{x} \sim_\lambda \boldsymbol{y} \quad &\text{iff} \quad \boldsymbol{x} \sim_\beta \boldsymbol{y} \text{ for all } \beta < \lambda,\ \lambda \text{ limit.}
\end{aligned}$$

In the above $*$ denotes concatenation of sequences. Let

$$\omega\text{-rk}(\mathfrak{M}) = \text{least } \alpha \text{ s.t. } \forall \boldsymbol{x}\, \forall \boldsymbol{y}\, (\boldsymbol{x} \sim_\alpha \boldsymbol{y} \to \boldsymbol{x} \sim_{\alpha+1} \boldsymbol{y}).$$

As the $\sim_\beta$ define successively finer equivalence relations on $|\mathfrak{M}|^{<\omega_1}$ it is clear that ω-rk($\mathfrak{M}$) is a well-defined ordinal less than card($|\mathfrak{M}|^{<\omega_1}$)$^+$.

The $\mathscr{L}_{\infty\omega_1}$-theory of $\mathfrak{M}$ is captured by a sentence $\phi \in \mathscr{L}_{\kappa\omega_1}$ ($\kappa = \mathrm{card}(|\mathfrak{M}|^{<\omega_1})^+$) which describes the preceding induction. Thus we define formulas $\phi_{\boldsymbol{x}}^{\beta}(\boldsymbol{v})$ for each β, each $\boldsymbol{x} \in |\mathfrak{M}|^{<\omega_1}$ by

$$\phi_{\boldsymbol{x}}^{0}(\boldsymbol{v}) = \bigwedge \{\psi(\boldsymbol{v}) \mid \psi \text{ atomic or negatomic, } \mathfrak{M} \vDash \psi(\boldsymbol{x})\},$$

$$\phi_{\boldsymbol{x}}^{\beta+1}(\boldsymbol{v}) = \Big(\bigwedge_{\boldsymbol{y}} \exists \boldsymbol{w}\, \phi_{\boldsymbol{x}*\boldsymbol{y}}^{\beta}(\boldsymbol{v}, \boldsymbol{w})\Big) \wedge \Big(\forall \boldsymbol{w} \bigvee_{\boldsymbol{y}} \Phi_{\boldsymbol{x}*\boldsymbol{y}}^{\beta}(\boldsymbol{v}, \boldsymbol{w})\Big),$$

$$\phi_{\boldsymbol{x}}^{\lambda}(\boldsymbol{v}) = \bigwedge_{\beta<\lambda} \phi_{\boldsymbol{x}}^{\beta}(\boldsymbol{v}).$$

Then the *canonical ω-Scott sentence* of $\mathfrak{M}$, $\phi(\mathfrak{M})$, is defined to be:

$$\phi(\mathfrak{M}) = \phi_{\emptyset}^{\alpha} \wedge \bigwedge_{\boldsymbol{x}} \forall \boldsymbol{v}\, (\phi_{\boldsymbol{x}}^{\alpha}(\boldsymbol{v}) \to \phi_{\boldsymbol{x}}^{\alpha+1}(\boldsymbol{v})) \quad \text{where } \alpha = \omega - \mathrm{rk}(\mathfrak{M}).$$

Proposition 1. *Suppose $\mathfrak{N} \vDash \phi(\mathfrak{M})$. Then $\mathfrak{M}, \mathfrak{N}$ are $\mathscr{L}_{\infty\omega_1}$-equivalent.*

Proof. Note that $\mathfrak{M} \vDash \phi_{\boldsymbol{x}}^{\beta}(\boldsymbol{y})$ iff $\boldsymbol{x} \sim_{\beta} \boldsymbol{y}$ so certainly $\mathfrak{M} \vDash \phi(\mathfrak{M})$. We show that $\mathfrak{M} \vDash \phi(\boldsymbol{x})$ iff $\mathfrak{N} \vDash \phi(\boldsymbol{x}')$ whenever $\mathfrak{N} \vDash \phi_{\boldsymbol{x}}^{\alpha}(\boldsymbol{x}')$, by induction on ϕ. This is clear for atomic ϕ by definition of $\phi_{\boldsymbol{x}}^{0}$. The only interesting case of the induction is $\phi = \exists \boldsymbol{v}\, \psi(\boldsymbol{v})$. Suppose $\mathfrak{M} \vDash \exists \boldsymbol{v}\, \psi(\boldsymbol{v}, \boldsymbol{x})$ and $\mathfrak{N} \vDash \phi_{\boldsymbol{x}}^{\alpha}(\boldsymbol{x}')$. Then choose $\boldsymbol{y}$ so that $\mathfrak{M} \vDash \psi(\boldsymbol{y}, \boldsymbol{x})$. Now since $\mathfrak{N} \vDash \forall \boldsymbol{v}\, (\phi_{\boldsymbol{x}}^{\alpha}(\boldsymbol{v}) \to \phi_{\boldsymbol{x}}^{\alpha+1}(\boldsymbol{v}))$ and $\mathfrak{N} \vDash \phi_{\boldsymbol{x}}^{\alpha}(\boldsymbol{x}')$, there must be $\boldsymbol{y}'$ s.t. $\mathfrak{N} \vDash \phi_{\boldsymbol{x}*\boldsymbol{y}}^{\alpha}(\boldsymbol{x}' * \boldsymbol{y}')$. Then by induction $\mathfrak{N} \vDash \psi(\boldsymbol{y}', \boldsymbol{x}')$ and so $\mathfrak{N} \vDash \phi(\boldsymbol{x}')$. The converse if similar. Finally, since $\mathfrak{N} \vDash \phi_{\emptyset}^{\alpha}$, we have that $\mathfrak{M} \vDash \phi$ iff $\mathfrak{N} \vDash \phi$ for all $\mathscr{L}_{\infty\omega_1}$ sentences ϕ. $\square$

Note. Proposition 1 and related results can be found in [4].

Nadel [13] provides a natural bound on the Scott rank of $\mathfrak{M}$, $\mathrm{rk}(\mathfrak{M})$, in terms of admissible set theory: If $\mathrm{HYP}(\mathfrak{M})$ is the least admissible set above $\mathfrak{M}$, then $\mathrm{rk}(\mathfrak{M}) \leqslant \mathrm{O}(\mathfrak{M}) = \mathrm{HYP}(\mathfrak{M}) \cap \mathrm{ORD}$. We provide here the appropriate version of Nadel's result for ω-$\mathrm{rk}(\mathfrak{M})$.

We begin by examining the connection between ω-rank and saturation. Let us assume $V = L$ and that $\alpha \geqslant \omega_1$ is admissible. We say that $\mathfrak{M}$ is *α-recursively saturated for $\mathscr{L}_{\alpha\omega_1}$* ($= L_{\infty\omega_1} \cap L_{\alpha}$) if for any $\Delta_1(L_{\alpha})$-set Φ of formulas of $\mathscr{L}_{\alpha\omega_1}$, $\mathfrak{M}$ obeys:

$$\exists \boldsymbol{v} \bigwedge \Phi_0(\boldsymbol{v}, \boldsymbol{p}) \quad \text{for all } \Phi_0 \subseteq \Phi,\ \Phi_0 \in L_{\alpha} \to \exists \boldsymbol{v} \bigwedge \Phi(\boldsymbol{v}, \boldsymbol{p})$$

whenever $\boldsymbol{p} \in |\mathfrak{M}|^{<\omega_1}$. The next result is analogous to results of Nadel, Schlipf and Ressayre in the case of $\mathscr{L}_{\infty\omega}$ (see Barwise [2, p. 143]):

Proposition 2. *If $\mathfrak{M}$ is α-recursively saturated for $\mathscr{L}_{\alpha\omega_1}$, then ω-$\mathrm{rk}(\mathfrak{M}) \leqslant \alpha$.*

Proof. Suppose $\boldsymbol{x}_1 \sim_{\alpha} \boldsymbol{y}_1$ and we are given $\boldsymbol{x}_2$. We show that there exists $\boldsymbol{y}_2$ such that $\boldsymbol{x}_1 * \boldsymbol{x}_2 \sim_{\alpha} \boldsymbol{y}_1 * \boldsymbol{y}_2$.

Notice that in defining $z \sim_\beta y$ it would have sufficed to deal with sequences from $|\mathfrak{M}|$ of length ω (as any $<\omega_1$-sequence from $|\mathfrak{M}|$ can be rearranged in order-type ω). Thus the relations $x \sim_\beta y$ for $\beta < \alpha$ are uniformly expressible by sentences in $\mathscr{L}_{\alpha\omega_1}$.

Let $\Phi(u_1, u_2, v_1, v) = \{u_1 * u_2 \sim_\beta v_1 * v \mid \beta < \alpha\}$ and notice that for any $\Phi_0 \subseteq \Phi$, $\Phi_0 \in L_\alpha$, $\mathfrak{M} \models \exists v \bigwedge \Phi_0(x_1, x_2, y_1, v)$. Thus by the hypothesis on $\mathfrak{M}$, there is $y_2 \in |\mathfrak{M}|^{<\omega_1}$ such that $x_1 * x_2 \sim_\beta y_1 * y_2$ for each $\beta < \alpha$. So $x_1 * x_2 \sim_\alpha y_1 * y_2$. $\square$

Thus a bound on ω-rk$(\mathfrak{M})$ is obtained by considering the least α such that $\mathfrak{M}$ is α-recursively saturated for $\mathscr{L}_{\alpha\omega_1}$. In case this ordinal is greater than ω_1, we can generate it in a natural way using the theory of admissible sets. The next lemma assures us that this is indeed always the case.

Lemma 3. *No infinite structure is ω_1-recursively saturated for $\mathscr{L}_{\omega_1\omega_1}$.*

Proof. There is a formula $\phi(v_1, v_2, \ldots)$ of $\mathscr{L}_{\omega_1\omega_1}$ which asserts that $W(v_1, v_2, \ldots) = \{(n, m) \mid V_{2^n3^m+1} = v_{2^n3^m}\}$ is a well-ordering of ω. There are also formulas $\phi_\alpha(v_1, v_2, \ldots)$, $\alpha < \omega_1$, which assert that $W(v_1, v_2, \ldots)$ does not have ordertype α. Then $\Phi = \{\phi(v_1, v_2, \ldots)\} \cup \{\sim\phi_\alpha(v_1, v_2, \ldots) \mid \alpha < \omega_1\}$ is not satisfiable but every countable $\Phi_0 \subseteq \Phi$ is satisfiable. $\square$

Now for any structure $\mathfrak{M}$ (of finite similarity type) define a new structure $\mathfrak{M}^{<\omega_1}$ as follows: The universe of $\mathfrak{M}^{<\omega_1}$ is $M^{<\omega_1}$ where $M = \text{universe}(\mathfrak{M})$. If P is an n-ary relation of $\mathfrak{M}$, then P' is a relation of $\mathfrak{M}^{<\omega_1}$ where $P'(f_1, \ldots, f_n)$ iff $f_i \neq \emptyset$ for all i and $p(f_1(0), \ldots, f_n(0))$. If F is an n-ary function of $\mathfrak{M}$, then F' is a function of $\mathfrak{M}^{<\omega_1}$ where:

$$F'(f_1, \ldots, f_n) = \begin{cases} \langle F(f_1(0), \ldots, f_n(0))\rangle, & f_i \neq 0 \text{ for all } i, \\ \emptyset, & \text{otherwise.} \end{cases}$$

If c is a constant of $\mathfrak{M}$, then $\langle c\rangle$ is a constant of $\mathfrak{M}^{<\omega_1}$. Finally, we add the relation $R(f, g) \Leftrightarrow f \subseteq g$ to $\mathfrak{M}^{<\omega_1}$.

Definition. ω-HYP$(\mathfrak{M})$ = HYP$(\mathfrak{M}^{<\omega_1})$. $0_\omega(\mathfrak{M}) = 0(\mathfrak{M}^{<\omega_1}) = \text{HYP}(\mathfrak{M}^{<\omega_1}) \cap \text{ORD}$.

Proposition 4. $0_\omega(\mathfrak{M})$ = *least α s.t. $\mathfrak{M}$ is α-recursively saturated for $\mathscr{L}_{\alpha\omega_1}$.*

Proof. By a result of Schlipf (see [2, p. 143]), $0(\mathfrak{M}^{<\omega_1})$ = least α s.t. $\mathfrak{M}^{<\omega_1}$ is α-recursively saturated (for $\mathscr{L}_{\alpha\omega}$). Thus it suffices to show that $\mathfrak{M}$ is α-recursively saturated for $\mathscr{L}_{\alpha\omega_1}$ iff $\mathfrak{M}^{<\omega_1}$ is α-recursively saturated (for $\mathscr{L}_{\alpha\omega}$). An α-recursive set of $\mathscr{L}_{\alpha\omega_1}$ formulas Φ is easily converted into an α-recursive set of $\mathscr{L}_{\alpha\omega}$ formulas Φ' s.t. $\mathfrak{M} \models \Phi$ iff $\mathfrak{M}^{<\omega_1} \models \Phi'$. To do this, one need only notice that the relation $P(f, g) \Leftrightarrow f(0) = g(\beta)$ is definable over $\mathfrak{M}^{<\omega_1}$ by an $\mathscr{L}_{\alpha\omega}$-formula for each $\beta < \omega_1$. Thus $\mathfrak{M}^{<\omega_1}$ α-recursively saturated implies $\mathfrak{M}$ α-recursively saturated for $\mathscr{L}_{\alpha\omega_1}$.

The converse uses the fact that $\alpha > \omega_1$ to convert $\mathfrak{M}^{<\omega_1} \models \exists v \phi$ to $\mathfrak{M} \models \bigvee_{\alpha<\omega_1} \exists \langle v_\beta \mid \beta < \alpha \rangle \phi^*$ (where ϕ^* is the conversion of ϕ). This translates the truth of an α-recursive set of $\mathscr{L}_{\alpha\omega}$ formulas over $\mathfrak{M}^{<\omega_1}$ into that of an α-recursive set of $\mathscr{L}_{\alpha\omega_1}$ formulas over $\mathfrak{M}$. □

In a similar way, we can establish a characterizing property of ω-HYP($\mathfrak{M}$) by appealing to a known result about HYP($\mathfrak{M}$).

Proposition 5. *The relations on $|\mathfrak{M}|$ which are $\Sigma_1(\omega\text{-HYP}(\mathfrak{M}))$ are those which are defined by an $\mathscr{L}_{\omega_1\omega_1}$-positive inductive definition over* $\text{HC}(\mathfrak{M}) = \{x \in V_{\mathfrak{M}} \mid x$ *has countable transitive closure*$\}$.

Proof. By [3] the relations in question are those which are inductive over $\text{HF}(\mathfrak{M}^{<\omega_1})$. But it is routine to convert $\mathscr{L}_{\omega_1\omega_1}$-positive induction on HC($\mathfrak{M}$) into $\mathscr{L}_{\omega\omega}$-positive inductions on $\text{HF}(\mathfrak{M}^{<\omega_1})$ and conversely. □

Our main goal in this section is to show that our bound $0_\omega(\mathfrak{M})$ on ω-rk($\mathfrak{M}$) is optimal. The first step in accomplishing this is to first determine an important necessary condition on α so that $\alpha = 0_\omega(\mathfrak{M})$ for some $\mathfrak{M}$.

Definition. An admissible set $A_{\mathfrak{M}}$ is *ω-admissible* if $A_{\mathfrak{M}}$ is countably closed and is admissible with respect to the function $s(x) = x^\omega$. The ordinal α is *ω-admissible* if L_α is ω-admissible.

Remarks. (a) To say that $A_{\mathfrak{M}}$ is countably closed is to say that $(A_{\mathfrak{M}})^\omega \subseteq A_{\mathfrak{M}}$.

(b) The condition that $A_{\mathfrak{M}}$ is admissible with respect to s is taken to imply that $x \in A_{\mathfrak{M}} \to s(x) \in A_{\mathfrak{M}}$.

(c) In case $A_{\mathfrak{M}} = L(\alpha)_{\mathfrak{M}}$ it is enough that cofinality $(\alpha) > \omega$, $L(\alpha)_{\mathfrak{M}}$ admissible and closed under s in order that $L(\alpha)_{\mathfrak{M}}$ be ω-admissible. This is essentially shown in Lemma 4 of [7].

Proposition 6. *For any $\mathfrak{M}$, ω-HYP($\mathfrak{M}$) is ω-admissible.*

Proof. For any $X \subseteq \omega$-HYP($\mathfrak{M}$), let $H(X) = \Sigma_1$-Skolem hull of $X = \{y \in \omega$-HYP($\mathfrak{M}$) $\mid$ y is the unique solution to a Σ_1-formula over ω-HYP($\mathfrak{M}$) with parameters from $X\}$. Then for any $X \supseteq |\mathfrak{M}|^{<\omega_1}$ closed under pairing, $H(X) <_{\Sigma_1} \omega$-HYP($\mathfrak{M}$): Suppose $\exists w\, \phi(w, p_1, \ldots, p_n)$ is true in ω-HYP($\mathfrak{M}$) where $p_1, \ldots, p_n \in H(X)$ and ϕ is Δ_0. We may also choose Δ_0 formulas $\phi_1, \ldots, \phi_n$ and sequences $\boldsymbol{q}, \ldots, \boldsymbol{q}_n$ from $X^{<\omega}$ such that each p_i is the unique solution x_i to $\exists w_i\, \phi_i(w_i, \boldsymbol{q}_i, x_i)$. Any element of ω-HYP($\mathfrak{M}$) is definable over $L(|\mathfrak{M}|^{<\omega_1}, \beta)$ from $|\mathfrak{M}|^{<\omega_1}$ together with some element of $|\mathfrak{M}|^{<\omega_1}$, for some $\beta < 0_\omega(\mathfrak{M})$. So, choose $m_0 \in |\mathfrak{M}|^{<\omega_1}$ such that for some $\beta < 0_\omega(\mathfrak{M})$ $\exists w, w_1, \ldots, w_n, x_1, \ldots, x_n$ definable over $L(|\mathfrak{M}|^{<\omega_1}, \beta)$ from $\{|\mathfrak{M}|^{<\omega_1}, m_0\}$ such that $\phi(w, x_1, \ldots, x_n)$, $\phi_1(w_1, \boldsymbol{q}_1, x_1), \ldots, \phi_n(w_n, \boldsymbol{q}_n, x_n)$. Then $H(X)$ must contain the least such sequence $(\beta, w, w_1, \ldots, w_n, x_1, \ldots, x_n)$ in the canonical well-ordering of those elements of ω-HYP($\mathfrak{M}$) definable over $L(|\mathfrak{M}|^{<\omega_1}, \beta)$ from $\{|\mathfrak{M}|^{<\omega_1}, m_0\}$ for some β. Thus $H(X) \models \exists w\, \phi(w, p_1, \ldots, p_n)$.

Now we take $X = L(|\mathfrak{M}|^{<\omega_1}, \omega_1)$. Then $H(X)$ is transitive: Let $\pi : H(X) \xrightarrow{\sim} L(|\mathfrak{M}|^{<\omega_1}, \beta)$ be the transitive collapse of $H(X)$. π is the identity on X since X is transitive. But if $y \in H(X)$ is the unique solution in ω-HYP($\mathfrak{M}$) (and hence in $H(X)$) to some Σ_1-formula $\phi(y, p_1, \ldots, p_n)$ where $p_1, \ldots, p_n \in X$, then $\pi(y) = y$ since $\pi(y)$ must also be a solution to $\phi(y, p_1, \ldots, p_n)$. Now $H(X)$ is also admissible above $\mathfrak{M}^{<\omega_1}$ and hence $H(X) = \omega$-HYP($\mathfrak{M}$).

We can now prove the proposition. The fact $H(X) = \omega$-HYP($\mathfrak{M}$) gives a $\Sigma_1(\omega$-HYP($\mathfrak{M}$))-surjection f of X onto ω-HYP($\mathfrak{M}$). Moreover for any $y \in \omega$-HYP($\mathfrak{M}$) there is some $\beta_y < 0_\omega(\mathfrak{M})$ such that $y \subseteq H^{\beta_y}(X)$ where $H^{\beta}(X)$ is the Σ_1-Skolem hull of X in $L(|\mathfrak{M}|^{<\omega_1}, \beta)$. This gives a surjection f_y of X onto y, $f_y \in \omega$-HYP($\mathfrak{M}$). Now X is countably closed. Thus $g : \omega \to \omega$-HYP($\mathfrak{M}$) implies $g = f \circ \bar{g}$ where $\bar{g} : \omega \to X$ belongs to X. Since ω-HYP($\mathfrak{M}$) is admissible, $g \in \omega$-HYP($\mathfrak{M}$) and we have shown that ω-HYP($\mathfrak{M}$) is countably closed. Also for $y \in \omega$-HYP($\mathfrak{M}$), $y^\omega = \{f_y \circ g \mid g \in X^\omega\}$ showing $y^\omega \in \omega$-HYP($\mathfrak{M}$). Moreover the function $y \mapsto y^\omega$ is $\Delta_1(\omega$-HYP($\mathfrak{M}$)) since f_y can be found effectively from y. □

Proposition 7. *If $\mathfrak{M}$ has cardinality ω_1, then for some $X \subseteq \omega_1$, $0_\omega(\mathfrak{M})$ = least α s.t. $L_\alpha(X)$ is admissible.*

Proof. Let $\mathcal{P} \in \omega$-HYP($\mathfrak{M}$) be the partial-ordering for collapsing $|\mathfrak{M}|$ to ω_1. Thus a condition is a function $f : \gamma \xrightarrow{1\text{-}1} |\mathfrak{M}|$, $\gamma < \omega_1$ and conditions are ordered by extension. If G is $\mathcal{P}$-generic over ω-HYP($\mathfrak{M}$) then there is a unique structure $\mathfrak{N}$ on ω_1 s.t. $G : \mathfrak{N} \to \mathfrak{M}$ is an isomorphism. Let X code the relation and functions of $\mathfrak{N}$ as a subset of ω_1. The admissibility of $L_{0_\omega(\mathfrak{M})}(X)$ follows from that of $L_{0_\omega(\mathfrak{M})}(G)$. And, since $\mathfrak{M} \simeq \mathfrak{N} \in L_{0_\omega(\mathfrak{M})}(X)$ we see that $0_\omega(\mathfrak{M})$ must be the least α such that $L_\alpha(X)$ is admissible. Finally, the existence of G is guaranteed by the countable closure of ω-HYP($\mathfrak{M}$). □

Corollary 8. *For any $\mathfrak{M}$, $0_\omega(\mathfrak{M})$ is ω-admissible.*

Proof. It suffices, by absoluteness, to verify this when $\mathfrak{M}$ has cardinality ω_1 (as the statement is unchanged by passing to a countably closed elementary submodel of cardinality ω_1). By the main result of [7] the result follows in that case from the fact that $0_\omega(\mathfrak{M})$ = least α s.t. $L_\alpha(X)$ is admissible, some $X \subseteq \omega_1$. □

Thus not every admissible $\alpha > \omega_1$ can be of the form $0_\omega(\mathfrak{M})$ for some $\mathfrak{M}$. The remainder of this section is devoted to the proof of the following result.

Theorem 9. *If α is ω-admissible, then $\alpha = 0_\omega(\mathfrak{M})$ for some $\mathfrak{M}$, ω-rk($\mathfrak{M}$) $= \alpha$.*

Proof. Our argument here is modelled after the construction in Theorem 11 of

[6]. The desired structure $\mathfrak{M}$ is a certain tree easily defined from α. A *tag* is an object of the form ∞_γ or (β, γ) where $\beta < \alpha$ and $\gamma < \omega_1$. The tree $\mathcal{T}$ is defined by

$$\begin{aligned}\mathcal{T} = \{(t_0, \ldots, t_n) \mid & \text{ Each } t_i \text{ is a tag and for all } 0 \le i < j \le n \\ & \text{Either (a) } t_i = \infty_\gamma \text{ some } \gamma, \\ & \text{(b) } t_i = (\beta, \gamma), t_j = (\beta', \gamma'), \beta' < \beta, \\ & \text{or (c) } t_i = (\beta, \gamma), t_j = (\beta, \gamma'), \gamma' < \omega\}.\end{aligned}$$

Definition. Suppose $\mathcal{T}_1$, $\mathcal{T}_2$ are countable substructures of $\mathcal{T}$ and $\beta < \alpha$. Then we write $\mathcal{T}_1 \equiv_\beta \mathcal{T}_2$ if there is an isomorphism $j : \mathcal{T}_1 \xrightarrow{\sim} \mathcal{T}_2$ s.t. for all $(t_0, \ldots, t_n) \in |\mathcal{T}_1|$:

(i) $j((t_0, \ldots, t_n))$ has length $n+1$.

(ii) If $j((t_0, \ldots, t_n)) = (s_0, \ldots, s_n)$, then for all $i \le n$

$$(t_i)_0 < \beta \Leftrightarrow (s_i)_0 < \beta, \qquad (t_i)_0 < \beta \to (t_i)_0 = (s_i)_0.$$

Lemma 10. *If $\beta_1 < \beta_2 < \alpha$, $\mathcal{T}_1$, $\mathcal{T}_2$, $\bar{\mathcal{T}}_1$ are countable substructures of $\mathcal{T}$, then*

$$\mathcal{T}_1 \subseteq \bar{\mathcal{T}}_1, \mathcal{T}_1 \equiv_{\beta_2} \mathcal{T}_2 \to \exists \bar{\mathcal{T}}_2 \supseteq \mathcal{T}_2, \bar{\mathcal{T}}_1 \equiv_{\beta_1} \bar{\mathcal{T}}_2.$$

Proof. We define an isomorphism $k : \bar{\mathcal{T}}_1 \xrightarrow{\sim} \mathcal{T}$ such that $k \supseteq j : \mathcal{T}_1 \xrightarrow{\sim} \mathcal{T}_2$, where j witnesses $\mathcal{T}_1 \equiv_\beta \mathcal{T}_2$. Then $\bar{\mathcal{T}}_2$ will be defined as the substructure of $\mathcal{T}$ with universe Range(k).

First, if $\boldsymbol{t} \subseteq t \in |\mathcal{T}_1|$, then let $k(\boldsymbol{t}) = j(t) \restriction \text{length}(\boldsymbol{t})$. Otherwise there is a longest initial segment $\boldsymbol{t}_1$ of $\boldsymbol{t}$ such that $\boldsymbol{t} \subseteq t \in |\mathcal{T}_1|$ for some t, and we can write $\boldsymbol{t} = \boldsymbol{t}_1 * \boldsymbol{t}_2$ (where $*$ denotes concatenation). If $\boldsymbol{t}_2 = (s_0, \ldots, s_n)$ and $s_0 = (\delta, \gamma)$ where $\delta < \beta_2$, then we let $k(\boldsymbol{t}) = k(\boldsymbol{t}_1) * \boldsymbol{t}_2$. Otherwise we can choose a unique $\bar{\gamma} < \omega_1$ for each such $\boldsymbol{t}$ such that $\boldsymbol{t}_1 * \langle(\beta_1, \bar{\gamma})\rangle$ belongs to $\mathcal{T}$ and is not an initial segment of any $t \in |\mathcal{T}_1|$ and define $k(\boldsymbol{t}) = k(\boldsymbol{t}_1) * t_2$ where t_2 is obtained from $\boldsymbol{t}_2$ by replacing all components (δ, γ), $\delta \ge \beta_2$ by $(\beta_1, \bar{\gamma})$ and leaving other components unchanged. The existence of $\bar{\gamma}$ follows from the fact that each tag of the form (β_2, γ) can be followed on $\mathcal{T}$ by uncountably many different tags of the form $(\beta_1, \bar{\gamma})$.

Claim 1. $\omega\text{-rk}(\mathcal{T}) \ge \alpha$.

Proof. Lemma 10 implies that if $f, g : \omega \to \mathcal{T}$ are such that $j(f(n)) = g(n)$ defines a witness to Range$(f) \equiv_\beta$ Range(g) (when viewed as substructures of $\mathcal{T}$), then $f \sim_\beta g$. But for any $\beta < \alpha$ we can choose f, g such that Range$(f) \equiv_\beta$ Range(g), Range$(f) \not\equiv_{\beta+1}$ Range(g). Then $f \sim_\beta g$ but $(\mathcal{T}, f) \not\equiv_{\infty\omega_1} (\mathcal{T}, g)$, proving $\omega\text{-rk}(\mathcal{T}) > \beta$.

Claim 2. $0_\omega(\mathcal{T}) \le \alpha$.

Proof. We use Proposition 4. Thus it sufficies to show that $\mathcal{T}$ is α-recursively saturated for $\mathcal{L}_{\alpha\omega_1}$. Suppose $\Phi(v, \boldsymbol{p})$ is an α-recursive set of formulas in $\mathcal{L}_{\alpha\omega}$ and $\mathcal{T}^{<\omega_1} \models \exists v\, \Phi_0(v, \boldsymbol{p})$ whenever $\Phi_0 \subseteq \Phi$, $\Phi_0 \in L_\alpha$. We show that $\mathcal{T}^{<\omega_1} \models \exists v \bigwedge \phi(v, \boldsymbol{p})$, thus establishing the α-recursive saturation of $\mathcal{T}^{<\omega_1}$ (and hence the α-recursive saturation for $\mathcal{L}_{\alpha\omega_1}$ of $\mathcal{T}$). We assume that Φ proves the basic defining properties of $\mathcal{T}^{<\omega_1}$. Note that Φ is consistent in the usual logic for $\mathcal{L}_{\alpha\omega}$. Thus if $\Psi_0(v)$ is any set of formulas forming an element of L_α, Φ can be consistently extended to include either ψ or $\sim\psi$ for each $\psi\in\Psi_0$. Thus we can assume that Φ decides the length of the element of $|\mathcal{T}|^{<\omega_1}$ denoted by v as well as all sentences:

$$(*)\qquad v(\gamma_1)\subseteq v(\gamma_2),\qquad \text{length}(v(\gamma))=n.$$

Now for each $\gamma<\text{length}(v)$ and $i<\text{length}(v(\gamma))$ choose $\beta_{\gamma,i}<\alpha$ such that: if $\Phi\cup\{``(v(\gamma)(i))_0\neq\beta" \mid \beta<\alpha\}$ is inconsistent, then $\Phi\vdash \bigvee\{``(v(\gamma)(i))_0=\beta" \mid \beta<\beta_{\gamma,i}\}$. By admissibility of L_α (and since $\text{length}(v)<\omega_1$) we can obtain a bound $\beta<\alpha$ such that for any $\gamma<\text{length}(v)$, $i<\omega$:

$$\Phi\cup\{v(\gamma)(i) \text{ has tag } \infty_\delta \text{ some } \delta\} \text{ is consistent}$$
$$\textit{or } \Phi\vdash v(\gamma)(i) \text{ has tag } (\beta',\delta) \text{ for some } \beta'<\beta,\ \delta<\omega_1.$$

The above properties are expressible as to say that $v(\gamma)(i)$ has tag (β',δ) some δ is to say that $v(\gamma)\restriction(i+1)$ has ω-rank β' in $\mathcal{T}$(ω-rank$(\sigma)=|\sigma|$ is defined by: $|\sigma|=0$ if σ has only countably many immediate extensions; $|\sigma|=\sup\{\gamma+1 \mid \sigma$ has uncountably many immediate extensions of ω-rank $\gamma\}$, otherwise).

But now the possible tags of the nodes $v(\gamma)$, $\gamma<\text{length}(v)$ form a set in L_α. Hence we can assume that Φ actually specifies the value of $v(\gamma)(i)$ for each γ, i as either $(f(\gamma,i),\delta)$ or ∞_δ (for some $\delta<\omega_i$). Now it is clear that these conditions on $v(\gamma)$ can be satisfied by some f in $\mathcal{T}^{<\omega_1}$. But together with the sentences in $(*)$ provable from Φ this completely characterized the type of f in $\mathcal{T}^{<\omega_1}$. Thus $\mathcal{T}^{<\omega_1}\models\Phi(f,\boldsymbol{p})$ and we are done. $\square$

Remarks. (1) An alternate proof of Claim 2 is provided by a generalization of Steel forcing [15].

In this generalization one forces over L_α with countable subtrees of $\omega_1^{<\omega}$ tagged with tags of the form $\boldsymbol{\infty}$ or $\boldsymbol{\beta}$, $\beta<\alpha$. The further restriction is that a node tagged with a given tag must have infinitely many immediate extensions with the same tag. However, when a condition is extended, extensions of old nodes cannot be added unless they have smaller tag ($\boldsymbol{\infty}<\infty$, $\boldsymbol{\beta}<\infty$, $\boldsymbol{\beta_1}<\boldsymbol{\beta_2}$ iff $\beta_1<\beta_2$).

The resulting generic tree is isomorphic to $\mathcal{T}_\alpha$. A version of Steel's retagging lemma is needed to argue that a generic tree will preserve the admissibility of α. The argument is similar to the construction used in proving Claim 1.

(2) There is a version of Theorem 9 when ω_1 is replaced by κ, κ regular. The notion ω-admissible becomes $<\kappa$-admissible (defined in [7], page 2). The ordinal $0_\omega(\mathfrak{M})$ is replaced by $0_{<\kappa}(\lambda)=$ the least ordinal not in $<\kappa$-HYP$(\mathfrak{M})$, the least $<\kappa$-admissible set above $\mathfrak{M}$. The ordinal ω-rk$(\mathfrak{M})$ becomes the ordinal of the Scott analysis of $\mathfrak{M}$ in $\mathcal{L}_{\infty\kappa}$. Finally the tree used in Theorem 9 is replaced by the

following tree:

$$\mathcal{T} = \{(t_0, \ldots, t_n) \mid \text{Each } t_i = \infty_\gamma \text{ or } (\beta, \gamma, \delta) \text{ for some } \delta < \gamma < \kappa \text{ and}$$
$$0 \leqslant i < j \leqslant n \to \text{(a) } t_i = \infty_\gamma,$$
$$\text{or (b) } t_i = (\beta, \gamma, \delta) t_j = (\beta', \gamma', \delta'), \beta' < \beta,$$
$$\text{or (c) } t_i = (\beta, \gamma, \delta) t_j = (\beta, \gamma, \delta')\}.$$

One argument of Theorem 9 will carry over to the present context. Alternatively a proof can be given using an appropriate generalization of Steel forcing.

(3) In case α is a *successor* ω-admissible, $\alpha = \beta^+$, then $\mathcal{T}$ as defined in Theorem 9 is isomorphic to a β-recursive structure. See [9].

2. Compactness

Theorem 9 is suggestive that a reasonable model theory does exist for ω-admissible fragments of $\mathcal{L}_{\infty\omega_1}$. The difficulty of course when dealing with uncountable languages is that special care is needed to preserve consistency at a limit stage is a Henkin construction. The fact that the tree $\mathcal{T}$ of Theorem 9 can be built by an ω-closed forcing construction (see Remark 1 of the previous section) is suggestive that this difficulty can be overcome in special cases.

The close connection between Henkin constructions and forcing constructions was noted in [11]. The idea is that a Henkin argument can be viewed as the selection of a generic object over the canonical consistency property for the theory, viewed as a partial ordering. Our compactness theorem is proved by showing that for appropriate theories an ω-closed consistency property can be constructed, thus allowing for the existence of a 'generic' model (when considering fragments of size ω_1). The key to this ω-closure is the assumption that we are dealing with an 'effectively scattered' theory (defined below).

We now proceed to details. Fix an ω-admissible ordinal α of cardinality ω_1. Our compactness theorem deals with certain $\Sigma_1(s)$-theories over the fragment $\mathcal{L}_{\alpha\omega_1} = \mathcal{L}_{\infty\omega_1} \cap L_\alpha$, where s is the function with domain L_α given by $s(x) = x^\omega$. It will be important to consider structures which are closed under the formation of 'limit types', ω-*closed* structures. For the purpose of the next definition recall the equivalence relation from Section 1. For $\beta < \alpha$ it is definable by an $\mathcal{L}_{\alpha\omega_1}$-formula.

Definition. $\mathfrak{M}$ is ω-*closed* (for $\mathcal{L}_{\alpha\omega_1}$) if the following holds: Fix $\beta < \alpha$. Suppose we are given for each n a sequence $\boldsymbol{m}_n$ from $|\mathfrak{M}|$ of length $\gamma_n < \omega_1$. Suppose that for $i < j$, $\gamma_i < \gamma_j$ and $\boldsymbol{m}_i \sim_\beta \boldsymbol{m}_j \restriction \gamma_i$. Then there exists $\boldsymbol{m}$ of length $\gamma = \bigcup_i \gamma_i$ such that $\boldsymbol{m}_i \sim_\beta \boldsymbol{m} \restriction \gamma_i$ for all i.

Our model theory for $\mathcal{L}_{\alpha\omega_1}$ is based on ω-closed structures. Fortunately, in many cases ω-closed models can be found (see Section 3).

Definition. Suppose T is a theory in $\mathscr{L}_{\alpha\omega_1}$ and $\beta<\alpha$. A *T-consistent β-type* is a set of formulas $p(\boldsymbol{x})$ in $\mathscr{L}_{\beta\omega_1}$ such that: for any α-finite $T_0\subseteq T$ there is an ω-closed model $\mathfrak{M}$ of T_0, $\boldsymbol{m}$ from $|\mathfrak{M}|$ with $p(\boldsymbol{x})=\{\phi(\boldsymbol{x})\in\mathscr{L}_{\beta\omega_1}\mid\mathfrak{M}\models\phi(\boldsymbol{m})\}$. We let $S_\beta(T)=\{T\text{-consistent }\beta\text{-types}\}$.

Notice that our notion of type is defined semantically in terms of ω-closed structures. For appropriate T a syntactic definition (in terms of proof-theoretic consistency) is possible.

The main hypothesis we impose on T is that $S_\beta(T)$ be small, in an effective sense. This hypothesis comes in two parts. First we insist that $S_\beta(T)\in L_\alpha$ for each $\beta<\alpha$ and we can effectively determine an α-finite $T_0\subseteq T$ such that $S_\beta(T_0)=S_\beta(T)$. In the second part we say that the T-consistent β-types are all eventually isolated (in the sense of Cantor derivative) and the way in which this comes about coheres over a closed unbounded set of β's. The precise definitions follow.

Definition. Let $\langle T_\beta\mid\beta<\alpha\rangle$ be a $\Sigma_1(s)$-increasing sequence of α-finite subtheories of T such that $T=\bigcup_\beta T_\beta$. Then T is *thin* if:

(a) $S_\beta(T)\in L_\alpha$ for all $\beta<\alpha$ and the function $\beta\mapsto S_\beta(T)$ is $\Sigma_1(s)$.

(b) For some $\Sigma_1(s)$-function $g:\alpha\to\alpha$ we have $S_\beta(T)=S_\beta(T_{g(\beta)})$ for all $\beta<\alpha$.

Definition. $p\in S_\beta(T)$ is *isolated* if for some $\phi(\boldsymbol{x})\in p$, p is the only member of $S_\beta(T)$ such that $\phi\in p$. Then inductively define:

$$S^0_\beta(T)=\{p\in S_\beta(T)\mid p\text{ is isolated}\},$$
$$S^\gamma_\beta(T)=\{p\in S_\beta(T)\mid\text{for some }\phi(\boldsymbol{x})\in p(\boldsymbol{x})\text{ if }\phi(\boldsymbol{x})\in q(\boldsymbol{x})\in S_\beta(T)$$
$$\text{then either }q=p\text{ or }q\in S^{\gamma'}_\beta(T)\text{ some }\gamma'<\gamma\},\quad\gamma>0.$$

Then T is *β-scattered* if $S^\gamma_\beta(T)=S_\beta(T)$ for some γ. In this case, if $p\in S_\beta(T)$, then the *rank* p is $|p|=\text{least }\gamma$ s.t. $p\in S^\gamma_\beta(T)$ and the *β-rank* of T is $\sup\{|p|\mid p\in S_\beta(T)\}$.

The notion of scattered first appears in [12]. We shall need to assume that T is β-scattered in a very uniform way, captured by the next definition. If F is a finite set of variables and $p\in S_\beta(T)$ then $p\restriction F$ denotes that element of $S_\beta(T)$ consisting of those formulas in p all of whose free variables belong to F.

Definition. T is *effectively scattered* if T is thin and for some $\Delta_1(s)$-closed unbounded $C\subseteq\alpha$:

(a) $\beta\in C\to T$ is β-scattered with β-rank$<\omega_1$.

(b) $\beta_1<\beta_2$, both in $C\to$ each $p\in S_{\beta_1}(T)$ has a unique extension $q\in S_{\beta_2}(T)$ with the same free variables as p s.t. for all finite F, $|p\restriction F|=|q\restriction F|$.

Theorem 11. *Suppose α is ω-admissible and has cardinality ω_1, $s:L_\alpha\to L_\alpha$ is defined by $s(x)=x^\omega$. Suppose $T\subseteq\mathscr{L}_{\alpha\omega_1}$ is an effectively scattered $\Sigma_1\langle L_\alpha,s\rangle$-theory. If* *every α-finite $T_0 \subseteq T$ has an ω-closed model, then T has an ω-closed model $\mathfrak{M}$ such that $0_\omega(\mathfrak{M}) \leqslant \alpha$.*

[The second clause in the conclusion ($0_\omega(\mathfrak{M}) \leqslant \alpha$) is important in applications. It is our version of 'ordinal pinning' in this context (see [2, p. 105].]

Proof. As we remarked earlier our proof is a forcing argument. First note that there is a $\Delta_1(L_\alpha)$-collection of $\mathscr{L}_{\alpha\omega_1}$-sentences asserting that "the universe is an ω-closed structure". That is for each $\beta < \alpha$ we can form a sentence ϕ_β expressing the property displayed in the definition given above of an ω-closed structure and the operation $\beta \mapsto \phi_\beta$ is $\Delta_1(L_\alpha)$. By our hypothesis on T we can assume that T contains all of these sentences ϕ_β. Indeed, adding them to T does not disturb the existence of an ω-closed model for each α-finite $T_0 \subseteq T$ nor does it change the sets $S_\beta(T)$.

Fix an ω_1-sequence $d_0, d_1, \ldots, d_\gamma, \ldots$ of new constants and let $\bar{\mathscr{L}}$ be the language obtained from $\mathscr{L}$ by adjoining them.

Let C be as in the definition of effectively scattered for T. We assume that $\beta_1 < \beta_2$ in $C \to S_{\beta_1}(T) \in L_{\beta_2}$ (see definition of thin, part (a)). For any $\beta < \alpha$, $p \in S_\beta(T)$ we let $p(\boldsymbol{d})$ denote the collection of $\bar{\mathscr{L}}_{\beta\omega_1}$ formulas obtained by replacing each variable x_γ by the constant d_γ in some formula $\phi(\boldsymbol{x}) \in p$. Then $\mathscr{P} = \{p(\boldsymbol{d}) \mid p \in S_\beta(T)$ for some $\beta \in C\}$. Given $p \in S_\beta(T)$ let $\boldsymbol{x}(p)$ denote exactly those free variables x_γ appearing in $p(\boldsymbol{x})$. Then we write $q(\boldsymbol{d}) \leqslant p(\boldsymbol{d})$ if:

(a) $p(\boldsymbol{d}), q(\boldsymbol{d}) \in \mathscr{P}$, $p \in S_{\beta_1}(T)$, $q \in S_{\beta_2}(T)$, $\beta_1 \leqslant \beta_2$.

(b) $q \restriction \boldsymbol{x}(p)$ is the unique extension of p in $S_{\beta_2}(T)$ with the same free variables as p s.t. for all finite F, $|q \restriction (\boldsymbol{x}(p) \cap F)| = |p \restriction F|$.

Thus when a condition is extended more information is given about a possibly larger collection of constants, subject to 'preserving rank' on any finite set of constants. We use $\mathscr{P}(\beta)$ to denote $\{p(\boldsymbol{d}) \in \mathscr{P} \mid p \in S_\beta(T)\}$ and $\mathscr{P}_\beta = \bigcup_{\beta' < \beta} \mathscr{P}_{\beta'}$.

Lemma 12. *If $p(\boldsymbol{d}) \in \mathscr{P}(\beta)$, $\beta < \bar{\beta} \in C$ and d_γ is a constant, then there is $q(\boldsymbol{d}) \leqslant p(\boldsymbol{d})$, $q(\boldsymbol{d}) \in \mathscr{P}(\bar{\beta})$ such that d_γ is mentioned in $q(\boldsymbol{d})$.*

Proof. By (b) in the definition of effectively scattered for T, we can choose $\hat{q}(\boldsymbol{d}) \leqslant p(\boldsymbol{d})$ in $\mathscr{P}(\bar{\beta})$. Recall in the definition of thin for T, part (b), that there is an ordinal $g(\boldsymbol{\beta})$ s.t. $S_{\bar{\beta}}(T) = S_{\bar{\beta}}(T_{g(\bar{\beta})})$. Since $\hat{q} \in S_{\bar{\beta}}(T)$ we can pick a model $\mathfrak{M}$ of $T_{g(\bar{\beta})}$ and $\boldsymbol{m}$ from $|\mathfrak{M}|$ s.t. $\mathfrak{M} \models \phi(\boldsymbol{m})$ for each $\phi(\boldsymbol{x}) \in \hat{q}$. If x_γ is not included in $\hat{q}$ also choose an arbitrary $m_\gamma \in |\mathfrak{M}|$. Then let $q(\boldsymbol{x}) = \{\psi(\boldsymbol{x}) \in \mathscr{L}_{\bar{\beta}\omega_1} \mid \mathfrak{M} \models \psi(\boldsymbol{m}, m_\gamma)\}$. Then $q(\boldsymbol{d}) \leqslant p(\boldsymbol{d})$, $q(\boldsymbol{d}) \in \mathscr{P}(\bar{\beta})$ and d_γ is mentioned in $q(\boldsymbol{d})$. $\square$

Lemma 13. *$\mathscr{P}$ is countably closed. (Hence, there exist $\mathscr{P}$-generic sets over L_α.)*

Proof. This is where part (b) in the definition of extension is crucial. Suppose $p_1(\boldsymbol{d}) \geqslant p_2(\boldsymbol{d}) \geqslant \cdots$ and let $\boldsymbol{d}_n$ = those constants in $\boldsymbol{d}$ appearing in $p_n(\boldsymbol{d})$. Let $p_n(\boldsymbol{d}) \in S_{\beta_n}(T)$ and $\beta = \bigcup_n \beta_n$. then $\beta \in C$. We show that $\bigcup_n p_n(\boldsymbol{d}) \in S_\beta(T)$.

Consider $p_1(\boldsymbol{d}_1)$. By (b) in the definition of effectively scattered for T, $p_1(\boldsymbol{d}_1)$ has a unique extension $\hat{p}_1(\boldsymbol{d}_1) \in S_\beta(T)$ s.t. $|p_1 \upharpoonright F| = |\hat{p}_1 \upharpoonright F|$ for all finite F. Consider $p_n \upharpoonright \boldsymbol{d}_1$. Again, $p_n \upharpoonright \boldsymbol{d}_1$ has a unique extension $\hat{p}_n(\boldsymbol{d}_1) \in S_\beta(T)$ s.t. $|p_n \upharpoonright (\boldsymbol{x}(p_1) \cap F)| = |\hat{p}_n \upharpoonright F|$ for all finite F. But since $p_n \leqslant p_1$ we have $|p_n \upharpoonright (\boldsymbol{x}(p_1) \cap F| = |p_1 \upharpoonright F|$. thus $\hat{p}_n(\boldsymbol{d}_1) = \hat{p}_1(\boldsymbol{d}_1)$. Thus $p(\boldsymbol{d}_1) = \bigcup_n p_n \upharpoonright \boldsymbol{d}_1$ is an element of $S_\beta(T)$, $p \leqslant p_n \upharpoonright \boldsymbol{d}_1$ for all n.

For each $n > 1$ let $\boldsymbol{x}_n = \boldsymbol{x}(p_n) - \boldsymbol{x}(p_{n-1})$. Then for each n the sentence $\exists \boldsymbol{x}_2 \cdots \exists \boldsymbol{x}_n \bigwedge p_n(d_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)$ must belong to p (as it belongs to $p_n \upharpoonright \boldsymbol{d}_1$). But T contains axioms which assert that the universe is ω-closed. Thus since any α-finite $T_0 \subseteq T$ has a model $\mathfrak{M}$ with $\boldsymbol{m}_1$ from $|\mathfrak{M}|$ s.t. $\mathfrak{M} \models \bigwedge p(\boldsymbol{m}_1)$, there is such a model also satisfying the βth instance of ω-closure, producing $\boldsymbol{m}_2, \boldsymbol{m}_3, \ldots$, from $|\mathfrak{M}|$ s.t. $\mathfrak{M} \models \bigwedge_n p_n(\boldsymbol{m}_1, \ldots, \boldsymbol{m}_n)$. (It is easily checked that $\boldsymbol{m} \sim_\beta \boldsymbol{n} \to (\mathfrak{M}, \boldsymbol{m}) \equiv_{\beta\omega_1} (\mathfrak{M}, \boldsymbol{n})$.) Thus $\bigcup_n p_n$ is an element of $S_\beta(T)$. But as before it follows that for any n, $|p_n \upharpoonright F| = |p \upharpoonright \boldsymbol{x}(p_n) \cap F|$ for any finite F. Thus $p \leqslant p_n$ for all n. $\square$

It is now clear that if G is $\mathcal{P}$-generic over L_α, then $\bigcup G$ defines a complete, Henkinized theory in $\bar{\mathcal{L}}_{\alpha\omega_1}$ containing T. Thus $\bigcup G$ gives rise in the usual way (by taking equivalence classes of constants) to a model $\mathfrak{M}_G$ of T.

Let G be $\mathcal{P}$-generic over L_α. We conclude the proof of Theorem 11 by showing that $L_\alpha[G]$ is admissible. (This suffices as $\mathfrak{M}_G^{\leqslant\omega_1} \in L_\alpha[G]$.) As is typical of class forcing constructions, it is important to bound the amount of the partial ordering needed to decide sentences of bounded rank. Terms, ranked formulas, formulas, forcing are defined in the usual way (an elegant treatment is to be found in [15]).

Lemma 14. *Suppose $p, q \in \mathcal{P}$ and $p \cap \bar{\mathcal{T}}_{\beta_\gamma\omega_1} = q \cap \bar{\mathcal{T}}_{\beta_\gamma\omega_1}$ where $\beta_\gamma = \gamma$th element of C. If $\phi(\mathbf{G})$ has rank $<\gamma$, then $p \Vdash \phi(\mathbf{G}) \Leftrightarrow q \Vdash \Phi(\mathbf{G})$.*

Proof. If $\gamma = 1$, then $\phi(\mathbf{G})$ has rank 0. A typical such ϕ asserts that $\psi(\boldsymbol{d}) \in \mathbf{G}$ for some $\psi \in \bar{\mathcal{L}}_{\omega\omega}$. But $p \cap \bar{\mathcal{L}}_{\omega\omega} = q \cap \bar{\mathcal{L}}_{\omega\omega}$ so we are done.

The proof is by induction on $\operatorname{rank}(\phi(\mathbf{G}))$. The only interesting case is $\phi(\mathbf{G}) = \sim\psi(\mathbf{G})$. Suppose $p \Vdash \sim\psi(\mathbf{G})$ but $q \nVdash \sim\psi(\mathbf{G})$. Pick $\gamma' < \gamma$ such that $\operatorname{rank}(\psi(\mathbf{G})) < \gamma'$. As $q \nVdash \sim\psi(\mathbf{G})$ we can choose $r \leqslant q$, $r \Vdash \psi(\mathbf{G})$. Let $r = r(\boldsymbol{d}_1, \boldsymbol{d}_2)$ where $\boldsymbol{d}_1$ are the constants mentioned in q. Also let $r_0 = r \cap \mathcal{L}_{\beta_{\gamma'}\omega_1}$. Now $q(\boldsymbol{d}_1)$ contains the sentence $\exists \boldsymbol{x}_2 \bigwedge r_0(\boldsymbol{d}_1, \boldsymbol{x}_2)$. This sentence belongs to $\bar{\mathcal{L}}_{\beta_\gamma\omega_1}$ since $S_{\beta_{\gamma'}}(T) \in L_{\beta_\gamma}$. By hypothesis p contains this sentence. Then we can choose $t \leqslant p$ such that $t \cap \mathcal{L}_{\beta_{\gamma'}\omega_1} = r_0$. But by induction we must have $t \Vdash \psi(\mathbf{G})$, contradicting $p \Vdash \sim\psi(\mathbf{G})$. $\square$

For simplicity we write $p \equiv_\gamma q$ if $p \cap \bar{\mathcal{L}}_{\beta_\gamma\omega_1} = q \cap \bar{\mathcal{L}}_{\beta_\gamma\omega_1}$ ($\beta_\gamma = \gamma$th element of C). An immediate consequence of the preceding lemma is that the relation $p \Vdash \phi(\mathbf{G})$ where $\phi(\mathbf{G})$ is ranked is $\Sigma_1\langle L_\alpha, s\rangle$. For, this relation is given by an induction which is Σ_1 with the possible exception of the negation case:

$$p \Vdash \sim\phi \Leftrightarrow \forall q \leqslant p \sim(q \Vdash \phi).$$

But by Lemma 14, if rank$(\phi)<\gamma$, then we can write:

$$p \Vdash \sim\phi \Leftrightarrow \forall q \in \mathscr{P}_{\beta_\gamma}(q \leq p \rightarrow \sim q \Vdash \phi)$$

and the function $\gamma \mapsto \mathscr{P}_\beta$ is $\Sigma_1\langle L_\alpha, s\rangle$.

Finally we are prepared to show that $L_\alpha[G]$ is admissible. It suffices to show that if ϕ is Δ_0 and $p \Vdash \forall x\, \exists y\, \phi(x, y)$, then $p \Vdash \forall x_\beta\, \exists y_\beta\, \phi(x_\beta, y_\beta)$ for some $\beta<\alpha$. Let $\mathscr{T}_\beta$ consist of all terms for naming elements of $L_\beta[G]$. Choose γ_0 so that $\gamma_0 = \beta_{\gamma_0}$, p together with the parameter in ϕ belong to L_{γ_0} and rank$(\phi)<\gamma_0$. We may define a continuous $\Sigma_1\langle L_\alpha, s\rangle$ sequence $\gamma_0<\gamma_1<\cdots<\gamma_\delta<\cdots$ of length ω_1 so that $\gamma_\delta = \beta_{\gamma_\delta}$ for $\delta \geq 0$ and:

(*) For all $\sigma \in \mathscr{T}_{\gamma_\delta}$, $q<p$, $q \in \mathscr{P}_{\gamma_\delta}$ there are $\tau \in \mathscr{T}_{\gamma_{\delta+1}}$, $r \leq q$, $r \in \mathscr{P}_{\gamma_{\delta+1}}$ s.t. $r \vdash \phi(\sigma, \tau)$.

The reason for this is that $p \Vdash \forall x\, \exists y\, \phi(x, y)$ and the relation $p \Vdash \phi$ for ranked ϕ is $\Sigma_1(s)$. Let $\beta = \bigcup_\delta \gamma_\delta < \alpha$.

Claim. $p \Vdash \forall x_\beta\, \exists y_\beta\, \phi(x_\beta, y_\beta)$.

Proof. Choose $\sigma \in \mathscr{T}_\beta$, $q \leq p$. Let $\bar{q} = q \cap \bar{\mathscr{L}}_{\beta\omega_1}$. It is enough to prove:

Subclaim. For a closed unbounded set of $\delta<\omega_1$, $\bar{q} \leq \bar{q} \cap \bar{\mathscr{L}}_{\gamma_\delta\omega_1} \leq p$.

For, given the subclaim choose $\delta<\omega_1$ so that $\bar{q} \leq q_0 = \bar{q} \cap \bar{\mathscr{L}}_{\gamma_\delta\omega_1} \leq p$ and $\sigma \in \mathscr{T}_{\gamma_\delta}$. Then by (*) there is $r_0 \leq q_0$, $r_0 \in \mathscr{P}_{\gamma_{\delta+1}}$ such that $r_0 \Vdash \phi(\sigma, \tau)$. Choose $\bar{r} \leq r_0$, $r \in S_\beta(T)$. Then $\bar{r} \restriction \boldsymbol{x}(q) = \bar{q}$, as both $\bar{q}$ and $\bar{r} \restriction \boldsymbol{x}(q)$ are extensions of q_0 in $S_\beta(T)$ with the same free variables as q_0 and preserving rank on finite subsequences. But then if we write $r_0 = r_0(\boldsymbol{x}_1, \boldsymbol{x}_2)$, $\boldsymbol{x}_1 = \boldsymbol{x}(q)$, then $\exists \boldsymbol{x}_2 \bigwedge r_1(\boldsymbol{x}_1, \boldsymbol{x}_2) \in \bar{q} \subseteq q$, where $r_1 = (\bar{r} \cap \mathscr{L}_{\gamma_{\delta+1}}) \restriction (\boldsymbol{x}_1, \boldsymbol{x}_2)$. So $q \cup r_1$ is contained in some T-consistent type $r \leq q$. Now $r_1(\boldsymbol{d}) \leq r_0(\boldsymbol{d})$ and $\phi(\sigma, \tau)$ has rank $<\gamma_{\delta+1}$. As $r \equiv_{\gamma_{\delta+1}} r_1$ and $r_1 \Vdash \phi(\sigma, \tau)$ we have $r \Vdash \phi(\sigma, \tau)$. We have shown $p \Vdash \forall x_\beta\, \exists y_\beta\, \phi(x_\beta, y_\beta)$.

Proof of Subclaim. For any $\delta<\omega_1$, $\bar{q} \cap \bar{\mathscr{L}}_{\gamma_\delta\omega_1} \leq p$. For, p has some extension $\bar{p} \leq p$, $\bar{p} \in S_{\gamma_\delta}(T)$ and $\bar{p}$ has some extension $\bar{\bar{p}} \in S_{\bar{\beta}}(T)$, (where $q \in S_{\bar{\beta}}(T)$). But by uniqueness (see part (b) of effectively scattered), $\bar{\bar{p}} \restriction \boldsymbol{x}(p) = q \restriction \boldsymbol{x}(p)$ and so $\bar{p} \restriction \boldsymbol{x}(p) = (\bar{q} \cap \mathscr{L}_{\gamma_\delta\omega_1}) \restriction \boldsymbol{x}(p)$. So $\bar{q} \cap \bar{\mathscr{L}}_{\gamma_\delta\omega_1} \leq p$.

To prove the first part of the subclaim it suffices to show that for any finite $F \subseteq \boldsymbol{x}(\bar{q})$ there are closed unboundedly many $\delta<\omega_1$ such that $|\bar{q}_\delta \restriction F| = |\bar{q} \restriction F|$, where $\bar{q}_\delta = \bar{q} \cap \mathscr{L}_{\gamma_\delta\omega_1}$. (For then if one lets $D \subseteq \omega_1$ be the intersection of countably many closed unbounded sets, one for each finite $F \subseteq \boldsymbol{x}(\bar{q})$, then $\delta \in D \rightarrow \bar{q} \leq \bar{q}_\delta$.) By the definition of effectively scattered, part (a), $|\bar{q} \restriction F| < \omega_1$.

We prove by induction on $\gamma<\omega_1$ that if r is a β-type in a finite set of free variables of rank γ, then r_δ is a γ_δ-type of rank γ for a closed unbounded set of δ's. First note that for δ large enough we must have that r_δ has γ_δ-rank $\leq\gamma$. Otherwise for unboundedly many δ we can choose $s_{\delta+1} \in S_{\gamma_{\delta+1}}(T)$ of γ_σ-rank γ such that $s_\delta = r_\delta$, $s_{\delta+1} \neq r_{\delta+1}$. Then $s_{\delta+1}$ has an extension to a β-type $\bar{s}_{\delta+1}$ of rank γ.

This shows that r is a limit of β-types of rank γ, contradicting the hypothesis that $\operatorname{rank}(r)=\gamma$.

So we may assume that $\gamma>0$. For each $\gamma'<\gamma$ and $\delta<\omega_1$ choose a β-type $s^{\gamma'}_{\delta+1}$ of rank γ' such that $r_\delta\subseteq s^{\gamma'}_{\delta+1}$. By induction we can choose closed unbounded sets $C^{\gamma'}_\delta$ such that $\mu\in C^{\gamma'}_\delta\to|s^{\gamma'}_{\delta+1}\cap\mathcal{L}_{\gamma_\mu\omega_1}|=\gamma'$. Let $C^{\gamma'}$ be the diagonal intersection of the sets $C^{\gamma'}_\delta$, and $\bar{C}=\bigcap_{\gamma'}C^{\gamma'}$. Then if μ is a limit point of $\bar{C}$ we have that $|r_{\gamma_\mu}|\geqslant\gamma$. By taking an appropriate final segment of $\bar{C}$ we obtain our desired closed unbounded set.

This completes the proof of Theorem 11. $\square$

Corollary 15 (to proof). *Suppose T is a $\Sigma_1(s)$ effectively scattered theory over $\mathcal{L}_{\alpha\omega_1}$ with an ω-closed model. Suppose $x\in\omega$-HYP($\mathfrak{M}$) whenever $\mathfrak{M}$ has cardinality ω_1, $\mathfrak{M}$ is an ω-closed model of T, $0_\omega(\mathfrak{M})\leqslant\alpha$. Then $x\in L_\alpha$.*

Proof. Build two mutually generic sets G_1, G_2 for the forcing described in the proof of Theorem 11. Then $L_\alpha=L_\alpha[G_1]\cap L_\alpha[G_2]$. But any pure set in ω-HYP($\mathfrak{M}_{G_1}$) must belong to $L_\alpha[G_1]$ (as $\mathfrak{M}^{<\omega_1}_{G_1}$ does); similarly for G_2. Since $\mathfrak{M}_{G_1}$, $\mathfrak{M}_{G_2}$ are ω-closed models of T and $0_\omega(\mathfrak{M}_{G_1}), 0_\omega(\mathfrak{M}_{G_2})\leqslant\alpha$ we are done. $\square$

This is our version for $\mathcal{L}_{\alpha\omega_1}$ of the Barwise hard core theorem (see [2, p. 113]). It will be of importance in our characterization theorem for pure part (ω-HYP($\mathfrak{M}$)) given in the next section.

The Scott isomorphism theorem says that the isomorphism type of a countable structure is determined by its $\mathcal{L}_{\infty\omega}$ theory. The analogous result fails for structures of size ω_1: There are $\mathcal{L}_{\infty\omega_1}$-equivalent, non-isomorphic structures of size ω_1. (The first example is due to Morley. A simple example is: take any two non-isomorphic normal ω_1-trees.)

However this cannot happen for ω-local structures.

Definition. A structure $\mathfrak{M}$ is *ω-local* if whenever $\boldsymbol{x}=(x_1,x_2,\ldots)$ and $\boldsymbol{y}=(y_1,y_2,\ldots)$ are ω-sequences from $|\mathfrak{M}|$:

$$\langle\mathfrak{M},\boldsymbol{x}\restriction F\rangle\equiv_{\infty\omega_1}\langle\mathfrak{M},\boldsymbol{y}\restriction F\rangle\quad\text{for all finite } F\to\langle\mathfrak{M},\boldsymbol{x}\rangle\equiv_{\infty\omega_1}\langle\mathfrak{M},\boldsymbol{y}\rangle$$

Proposition 16. *If $\mathfrak{M}\equiv_{\infty\omega_1}\mathfrak{N}$, $\mathfrak{M}$ ω-local, $\mathfrak{M}$ and $\mathfrak{N}$ have cardinality ω_1, then $\mathfrak{M}\simeq\mathfrak{N}$.*

Proof. Let $\alpha=\omega\text{-rk}(\mathfrak{M})$. For $\boldsymbol{m}\in|\mathfrak{M}|^{<\omega_1}$ and $\boldsymbol{n}\in|\mathfrak{N}|^{<\omega_1}$ define $\boldsymbol{m}\sim\boldsymbol{n}$ if $\mathfrak{N}\models\phi^\alpha_{\boldsymbol{m}}(\boldsymbol{n})$ (where $\phi^\alpha_{\boldsymbol{m}}$ is defined in Section 1). Then

$$\boldsymbol{m}\sim\boldsymbol{n}\to\forall\boldsymbol{m}'\,\exists\boldsymbol{n}'\,(\boldsymbol{m},\boldsymbol{m}')\sim(\boldsymbol{n},\boldsymbol{n}')\quad\text{and}\quad\forall\boldsymbol{n}'\,\exists\boldsymbol{m}'\,(\boldsymbol{m},\boldsymbol{m}')\sim(\boldsymbol{n},\boldsymbol{n}').$$

Note that $\mathfrak{N}$ is ω-local since $\mathfrak{N}\equiv_{\infty\omega_1}\mathfrak{M}$ (and the (∞,ω_1)-type of $\langle\mathfrak{M},\boldsymbol{x}\rangle$ is expressible by a single $\mathcal{L}_{\infty\omega_1}$ sentence). Now define sequences $\boldsymbol{m}_0\subseteq\boldsymbol{m}_1\subseteq\cdots\subseteq\boldsymbol{m}_\gamma\subseteq\cdots$ and $\boldsymbol{n}_0\subseteq\boldsymbol{n}_1\subseteq\cdots\subseteq\boldsymbol{n}_\gamma\subseteq\cdots$ of length ω_1 such that $\boldsymbol{m}_\gamma\sim\boldsymbol{n}_\gamma$ for all γ and $\bigcup_\gamma\boldsymbol{m}_\gamma=$

$|\mathfrak{M}|$, $\bigcup_\gamma \boldsymbol{n}_\gamma = |\mathfrak{N}|$. This is possible since by ω-locality, $\boldsymbol{m}_\gamma \sim \boldsymbol{n}_\gamma$ for all $\gamma < \lambda \to \bigcup_{\gamma<\lambda} \boldsymbol{m}_\gamma \sim \bigcup_{\gamma<\lambda} \boldsymbol{n}_\gamma$, for limit $\lambda < \omega_1$. Then we have defined an isomorphism between $\mathfrak{M}$ and $\mathfrak{N}$ by corresponding $\boldsymbol{m}_\gamma$ to $\boldsymbol{n}_\gamma$. □

The property ω-local blends nicely with Theorem 11. Define $\mathfrak{M}$ to be *ω-local for $\mathcal{L}_{\alpha\omega_1}$* if $\mathfrak{M}$ satisfies the definition of ω-local with $\equiv_{\infty\omega_1}$ replaced by $\equiv_{\alpha\omega_1}$. Then Theorem 11 goes through if one replaces 'ω-closed for $\mathcal{L}_{\alpha\omega_1}$' everywhere with '$\omega$-closed and ω-closed for $\mathcal{L}_{\alpha\omega_1}$'. Moreover the model produced in the conclusion is in fact ω-local as it has ω-rk$\leqslant\alpha$. In addition, condition (b) in the definition of effectively scattered can be replaced by the simpler:

(b′) $\beta_1 < \beta_2$, both in $C \to$ each $p \in S_{\beta_1}(T)$ in finitely many free variables has a unique extension $q \in S_{\beta_2}(T)$, with the same free variables as p, of the same rank.

The only reason for not presenting Theorem 11 in this way is to allow for the possibility of applications where ω-local models cannot be obtained.

3. Applications

There are a number of interesting examples of effectively scattered theories. We use Theorem 11 in this section to study them.

Example 1. Consider the structure $\mathcal{T}$ used in the proof of Theorem 9. We show how $\mathcal{T}$ can be described by an effectively scattered theory. For any tree $\mathcal{S}$ the ω-Cantor rank of $\sigma \in S$ is defined by

$|\sigma|_\omega = 0$ if σ has at most countably many immediate extensions in $\mathcal{S}$,

$|\sigma|_\omega = \sup\{\beta + 1 \mid \sigma$ has uncountably many immediate extensions of ω-Cantor rank $\beta\}$, otherwise.

If $|\sigma|_\omega$ is not defined by this definition for $\sigma \in \mathcal{S}$, then we write $|\sigma|_\omega = \infty$. For each $\beta < \alpha$ there is a formula $\phi_\beta(x)$ (in the language of $\leqslant$) such that if $\mathcal{S}$ is a tree, then $\mathcal{S} = (S, \leqslant) \models \phi_\beta(x)$ if and only if $|x|_\omega = \beta$. The axioms of the theory T_1 are:

(a) $\leqslant$ defines a tree of depth ω, with a unique top node $\emptyset$.

(b) $\phi_\beta(x) \to \exists$ exactly $\aleph_0$-many s.t. $\phi_\beta(y)$ and y is an immediate extension of x, for each $\beta < \alpha$.

(c) $\sim\phi_\beta(\emptyset)$, for each $\beta < \alpha$.

T_1 is effectively scattered: the β-type of a countable sequence $\langle x_\gamma \mid \gamma < \gamma_0\rangle$ is determined by the formulas "x_γ is on level n", $n \in \omega$; "$x_{\gamma_1} \leqslant x_{\gamma 2}$", $\gamma_1, \gamma_2 < \gamma_0$; $\phi_{\beta'}(x_\gamma)$, $\beta' < \beta$, $\gamma < \gamma_0$. From this, the thinness of T_1 follows easily. The β-type of a single element x has rank 0 if $\gamma_{\beta'}(x)$ holds for some $\beta' < \beta$; otherwise, the β-type of x has rank 1. Given the β_1-type p of $\langle x_\gamma \mid \gamma < \gamma_0\rangle$ and $\beta_2 > \beta_1$ we can define the unique β_2-type q of $\langle x_\gamma \mid \gamma < \gamma_0\rangle$ satisfying (b) in the definition of effectively scattered as follows: q contains p, q contains $\sim\phi_\beta(x_\gamma)$, $\beta < \beta_2$, whenever $\sim\Phi_\beta(x_\gamma)$ belongs to p for all $\beta<\beta_1$. (In this way if $p \upharpoonright x_\gamma$ had rank 1 then so does $q \upharpoonright x_\gamma$.) The closed unbounded set C consists of all limit ordinals.

Of course $\mathcal{T}$ as defined in Theorem 9 is a model of T_1. But any model $\mathcal{S}$ of T_1 of cardinality ω_1 such that $0\omega(\mathcal{S})\leqslant\alpha$ must be isomorphic to $\mathcal{T}$. Indeed, $\emptyset$ must have uncountably many immediate extensions σ such that $|\sigma|_\omega=\infty$, as otherwise ω-HYP($\mathcal{S}$) contains (as an element) the set of all nodes σ on level one s.t. $|\sigma|_\omega=\beta$ for some $\beta<\alpha$. The function $\sigma\mapsto|\sigma|_\omega$ on this set is $\Sigma_1(\omega\text{-HYP}(\mathcal{S}))$ and so is bounded. But this contradicts the assumption $0_\omega(\mathcal{S})\leqslant\alpha$. An identical argument shows that any node $\sigma\in\mathcal{S}$ s.t. $|\sigma|_\omega=\infty$ has uncountably many immediate extensions with this property. So $\mathcal{S}\simeq\mathcal{T}$.

Theorem 11 implies that T_1 has a model $\mathcal{S}$ such that $0_\omega(\mathcal{S})\leqslant\alpha$; so $0_\omega(\mathcal{T})\leqslant\alpha$. This gives another proof of the first part of Theorem 9.

Example 2. We define the *pure part* of ω-HYP($\mathfrak{M}$) to be ω-HYP($\mathfrak{M}$) $\cap V$ ($=\{x\in\omega\text{-HYP}(\mathfrak{M})\mid$ Transitive Closure of x contains no urelements$\}$). Suppose $A=\text{pp}(\omega\text{-HYP}(\mathfrak{M}))$. Then A is *resolvable*; i.e., there is a function $f:\text{ORD}(A)\to A$ s.t. $\langle A,\varepsilon,f\rangle$ is admissible and $A=\bigcup\text{Range}(f)$. For example, we can let $f(\beta)=L_\beta(\mathfrak{M})\cap V$. A is also closed under the function $s(x)=x^\omega$ and is in fact ω-admissible.

Theorem 17. $A=\text{pp}(\omega\text{-HYP}(\mathfrak{M}))$ *for some* $\mathfrak{M}$ *if and only if* A *is* ω*-admissible and resovable.*

Proof. We can assume that A has cardinality ω_1. Choose f so that $\langle A,\varepsilon,f\rangle$ is admissible, $f:\text{ORD}(A)\to A$, $A=\bigcup\text{Range}(f)$. We also assume that $f(0)\neq\emptyset$, the $f(\beta)$'s, $\beta<\alpha=\text{ORD}(A)$, are transitive, increasing, closed under $s(x)=x^\omega$ and β limit $\to f(\beta)=\bigcup\{f(\beta')\mid\beta'<\beta\}$. Now Theorem 11 and Corollary 15 apply equally well to $\langle A,\varepsilon,f\rangle$ as to $\langle L_\alpha,\varepsilon,s\rangle$. (The ω-admissibility of L_α together with its resolvability sufficed.) So by Corollary 15 (applied to $\langle A,\varepsilon,f\rangle$ it suffices to define a $\Sigma_1\langle A,\varepsilon,f\rangle$-theory T_2 (over $A\cap L_{\infty\omega_1}$) such that:

(a) T_2 is effectively scattered and has an ω-closed model.

(b) Whenever $\mathfrak{M}_1$, $\mathfrak{M}_2$ are models of T_2 of cardinality ω_1 $0_\omega(\mathfrak{M}_1)$, $0_\omega(\mathfrak{M}_2)$ both $\leqslant\alpha$, then $\mathfrak{M}_1\simeq\mathfrak{M}_2$.

(c) $A\subseteq\omega$-HYP($\mathfrak{M}$) whenever $\mathfrak{M}\models T_2$.

For then, let $\mathfrak{M}$ be a model of T_2, $0_\omega(\mathfrak{M})\leqslant\alpha$. By (c), $A\subseteq\text{pp}(\omega\text{-HYP}(\mathfrak{M}))$. By (a), (b) and Corollary 15, $\text{pp}(\omega\text{-HYP}(\mathfrak{M}))\subseteq A$.

We now define T_2. Let $\mathcal{T}$ be a tree. For $\beta<\alpha$ and $x\in f(\beta)$ we define $|\sigma|=(\beta,x)$ and $|\sigma|=\beta$ by induction on $\beta+\text{rk}(x)$:

$|\sigma|=0$ iff $|\sigma|=(0,\emptyset)$ iff σ has only countably many immediate extensions on $\mathcal{T}$,

$|\sigma|=\beta$ iff every immediate extension τ of σ has $|\tau|=\gamma$ for some $\gamma\leqslant\beta$, and σ has uncountably many immediate extensions τ s.t. $|\tau|=\gamma$ iff $\gamma<\beta$,

$|\sigma| = (\beta, x)$ iff every immediate extension τ of σ has $|\tau| = \gamma$ (some $\gamma < \beta$) or $|\tau| = (\beta, y)$ (some $y \in x$), and σ has uncountably many immediate extensions τ s.t. $|\tau| = \gamma$ iff $\gamma < \beta$ and s.t. $|\tau| = (\beta, y)$ iff $y \in x$.

Thus the relations $|\sigma| = \beta$, $|\sigma| = (\beta, x)$ can be described by formulas $\phi_\beta(\sigma)$, $\phi_{(\beta,x)}(\sigma)$ in $\mathscr{L}_{\infty\omega_1} \cap A$. We write $|\sigma| = \infty$ if σ is not defined by the above definition. the axioms of T_2 are:

(1) $\leqslant$ defines a tree of depth ω, with a unique top node $\emptyset$.

(2) $\phi_\beta(\sigma) \to \exists$ exactly $\aleph_0$-many immediate extensions τ of σ s.t. $\phi_\beta(\tau)$, for each $\beta < \alpha$.

(3) $\phi_{(\beta,x)}(\sigma) \to$ (every immediate extension τ of σ satisfies $\phi_\gamma(\tau)$ or $\phi_{(\beta,y)}(\tau)$ for some $\gamma < \beta$, $y \in x$), each $\beta < \alpha$, $x \in f(\beta)$.

(4) $\emptyset$ has uncountably many immediate extensions τ s.t. $\phi_\beta(\tau)$ and uncountably many immediate extensions τ s.t. $\phi_{(\beta,x)}(\tau)$, for each $\beta < \alpha$, $x \in f(\beta)$.

(5) If $|\sigma|_\omega = \beta$, then $\phi_{\beta'}(\sigma)$ or $\phi_{(\beta',x)}(\sigma)$ holds for some $\beta' \leqslant \beta$, $x \in f(\beta')$, each $\beta < \alpha$.

A model of T_2 is described as follows: Define a *tag* to be something of the form (β, δ), ∞_δ, (β, x, δ) for some $\delta < \omega_1$, $\beta < \alpha$, $x \subset f(\beta)$. Then

$\mathscr{T}_2 = \{(t_0, \ldots, t_n) \mid$ For all i, t_i is a tag and for $0 \leqslant i < j \leqslant n$:

$$t_i = \infty_\delta \text{ for some } \delta \to t_j = (\beta, \delta) \text{ for some } \delta,$$

$$t_i = (\beta, \delta) \to t_j = (\beta', \delta') \text{ for some } \beta' < \beta \text{ or } (\beta, \delta') \text{ for some } \delta' < \omega,$$

$$t_i = (\beta, x, \delta) \to t_j = (\beta', \dot{\delta}) \text{ for some } \beta' < \beta \text{ or } (\beta, y, \delta') \text{ for some } y \in x\}.$$

β-types are defined over $\mathscr{L}_{\infty\omega_1} \cap f(\beta)$. Assume $\mathrm{ORD}(f(\beta)) = \beta$. The β-type of a countable sequence $\langle x_\gamma \mid \gamma < \gamma_0 \rangle$ is determined by the formulas "x_γ is on level n", $n \in \omega$; "$x_{\gamma_1} \leqslant x_{\gamma_2}$", $\gamma_1, \gamma_2 < \gamma_0$; $\phi_{\beta'}(x_\gamma)$, $\beta' < \beta$, $\gamma < \gamma_0$; $\phi_{(\beta',x)}(x_\gamma)$, $\beta' < \beta$, $x \in f(\beta')$, $\gamma < \gamma_0$. From this follows the thinness of T_2. The β-type of σ has rank 0 if $\phi_{\beta'}(\sigma)$ or $\phi_{(\beta',x)}(\sigma)$ holds for some $\beta' < \beta$; otherwise the β-type of σ has rank 1. Assume $\beta_1 < \beta_2$ and $\mathrm{ORD}(f(\beta_1)) = \beta_1$, $\mathrm{ORD}(f(\beta_2)) = \beta_2$. Let p be the β_1-type of $\langle x_\gamma \mid \gamma < \gamma_0 \rangle$. To satisfy (b) in the definition of effectively scattered define a β_2-type $q \supseteq p$ as follows: q contains $\sim\phi_\beta(x_\gamma)$, $\sim\phi_{(\beta,x)}(x_\gamma)$ for all $\beta < \beta_2$, $x \in f(\beta)$ whenever $\sim\phi_\beta(x_\gamma)$, $\sim\phi_{(\beta,x)}(x_\gamma)$ belong to p for all $\beta < \beta_1$, $x \in f(\beta)$. The closed unbounded set C consists of all limit ordinals β such that $\mathrm{ORD}(f(\beta)) = \beta$.

Thus property (a) is satisfied. Property (c) follows as the ε-structure of x is coded into $\mathscr{T}$ below any node σ satisfying $\phi_{(\beta,x)}(\sigma)$, for any $\mathscr{T} \models T_2$. The decoding can be done inside ω-HYP($\mathscr{T}$).

Lastly we must show that $\mathfrak{M}_1 \simeq \mathfrak{M}_2$ whenever $\mathfrak{M}_1$, $\mathfrak{M}_2$ are models of T_2 of cardinality ω_1 satisfying $0_\omega(\mathfrak{M}_1)$, $0_\omega(\mathfrak{M}_2) \leqslant \alpha$. We in fact show that $\mathfrak{M}_1 \simeq \mathscr{T}_2$. Axiom (5) of T_2 guarantees that $\mathfrak{M}_1$, $\mathscr{T}_2$ have the same structure on nodes σ s.t. $|\sigma| \neq \infty$. Note that in $\mathfrak{M}_1$, $\emptyset$ must have uncountably many immediate extensions σ s.t. $|\sigma| = \infty$, as otherwise ω-HYP($\mathfrak{M}_1$) contains (as an element) the set of all nodes σ on level 1 such that $|\sigma| = \beta$ or (β, x) for some $\beta < \alpha$, $x \in f(\beta)$. The function

$\sigma \mapsto |\sigma|$ on this set is $\Sigma_1(\omega\text{-HYP}(\mathfrak{M}_1))$ and hence bounded. This contradicts the assumption $0_\omega(\mathfrak{M}_1) \leq \alpha$. Similarly, any node $\sigma \in \mathfrak{M}_1$ s.t. $|\sigma| = \infty$ has uncountably many immediate extensions with this property. So $\mathfrak{M}_1 \simeq \mathcal{T}_2$. This completes the proof of Theorem 17.

Example 3. In [15] it is shown: For each $\lambda < \omega_1$ there is $T \subseteq \omega$ and $\Pi_1^0(T)$-singletons f, g such that $f \notin L_\lambda(T, g)$ and $g \notin L_\lambda(T, f)$. We show now how to obtain an analogous result one cardinal higher. Given $\lambda < \omega_2$ we wish to obtain a tree $T \subseteq \omega_1^{<\omega}$ and subtrees T_0, T_1 of T such that T_0, T_1 are each unique solutions to $\Pi_1\langle L_{\omega_1}, T\rangle$-formulas but $T_0 \notin L_\lambda(T, T_1)$ and $T_1 \notin L_\lambda(T, T_0)$. It seems to be necessary in this context to work with subtrees T_0, T_1 rather than single paths f_0, f_1 (as in Steel's result).

We pick λ to be ω-admissible, $\lambda < \omega_2$. Our tree T in this example will be isomorphic to the $\mathcal{T}$ of Theorem 9. A tree $\mathcal{S}$ is *full* if each node of $\mathcal{S}$ has exactly ω_1 immediate extensions. Our desired subtrees T_0, T_1 will be full subtrees of $\mathcal{T}$. However to guarantee that T_0, T_1 are $\Pi_1\langle L_{\omega_1}, T\rangle$-singletons we need to introduce more structure into our tree.

Definition. Let $\sim$ be an equivalence relation on a set S with ω_1 equivalence classes each of size ω_1: One equivalence class S_β for each $\beta < \lambda$ and one equivalence class S_∞. We define a tree $\mathcal{T}_3$ as follows: A tag is something of the form (s, n), $s \in S$, $n \in \omega$.

$$\mathcal{T}_3 = \Big\{(t_0, \ldots, t_n) \mid \text{each } t_i \text{ is a tag and if } i < j, \text{ then:}$$
$$\text{(a) } t_i = (s, n),\ s \in S_\infty,$$
$$\text{or (b) } t_i = (s, n),\ s \in S_\beta \text{ and } t_j = (s, m) \text{ or } (s', n') \text{ where } s' \in \bigcup_{\beta' < \beta} S_{\beta'} \Big\}.$$

Then $\mathcal{T}_3^* = (\mathcal{T}_3, \leq, S, f_i)_{i<\omega}$ where $f_i((s_0, n_0), \ldots, (s_m, n_m)) = s_i$.

Note that we do not put the equivalence relation $\sim$ into $\mathcal{T}_3^*$. Now let $T_0 \subseteq |\mathcal{T}_3|$ be defined by $T_0 = \{(t_0, \ldots, t_n) \in |\mathcal{T}_3| \mid t_i = (s, n) \text{ for some } s \in S_{\infty'} \text{ for all } i\}$.

Claim. *T_0 is the unique subset of $|\mathcal{T}_3|$ such that:*

(a) *T_0 defines a full subtree of $\mathcal{T}_3$*

(b) $\sigma = (t_0, \ldots, t_n)$, $f_n(\sigma) \in \text{Range}(f_0 \restriction T_0) \rightarrow \sigma \in T_0$.

Proof. (a) implies that we at least are talking about a subset of T_0. But (b) implies that all $\sigma = (t_0, \ldots, t_n)$ with $f_i(\sigma) \in S_\infty$ for all i must be present (as any such σ can be extended to one obeying the hypothesis of (b).). □

We can also define $\mathcal{T}_3^{**}$ just like $\mathcal{T}_3^*$ except this time we have ∞_0, ∞_1 each with different equivalence classes. Moreover we insist that $f_i(\sigma) \in S_\infty \rightarrow f_{i+1}(\sigma) \in S_{\infty_1} \cup \bigcup_{\beta<\lambda} S_\beta$ and $f_i(\sigma) \in S_{\infty_0} \rightarrow f_{i+1}(\sigma) \in S_{\infty_0} \cup \bigcup_{\beta<\lambda} S_\beta$. Then we get T_0, T_1, each uniquely characterized by properties as in the claim.

Now the structure $(\mathscr{T}_3^*, T_0)$ can be described by a $\Sigma_1(s)$-effectively scattered theory over $L_{\lambda\omega_1}$. This is done in a way similar to our earler examples. Moreover any two models $\mathfrak{M}_1$, $\mathfrak{M}_2$ of this theory of size ω_1 such that $0_\omega(\mathfrak{M}_1)$, $0_\omega(\mathfrak{M}_2) \leqslant \lambda$ are isomorphic. An example of such a model is $(\mathscr{T}_3^{**}, T_0)$.

However notice that for any term $\tau(T)$ in $\mathscr{L}_{\lambda\omega_1}$, $(\mathscr{T}_3^*, T_0) \models$ "If $\tau(T_0)$ is a full subtree of $\mathscr{T}_3$, then $\tau(T_0) \subseteq T_0$". So we may as well have added these sentences to our effectively scattered theory; thus these sentences are true in $(\mathscr{T}_3^{**}, T_0)$ (since this new theory has a model $\mathfrak{M}$ of cardinality ω_1 s.t. $0_\omega(\mathfrak{M}) \leqslant \lambda$ by Theorem 11, and since $(\mathscr{T}_3^{**}, T_0)$ is isomorphic to this model). Thus $T_1 \notin L_\lambda((\mathscr{T}_3^{**}, T_0))$. By symmetry $T_0 \notin L_\lambda(\mathscr{T}_3^{**}, T_0))$.

Finally collapse $\mathscr{T}_3^{**}$ to a structure on ω_1, generically over $L_\lambda((\mathscr{T}_3^{**}, T_0, T_1))$, as in Proposition 7. This produces a tree T_G on $\omega_1^{<\omega}$, functions $f_i^G : T_G \to \omega_1$ and full subtree T_G^0, T_G^1. Pick $\sigma_0 \in T_G^0$ and $\sigma_1 \in T_G^1$, σ_0 and σ_1 of length >0. Then T_G^0 is the *unique* full subtree T of T_G s.t. $\sigma_0 \in T$ and $\sigma = (t_0, \dots, t_n) \in T_g$, $f_n^G(\sigma) \in \text{Range}(f_0 \restriction T) \to \sigma \in T$. (Similarly for T_G^1.) This is easily seen to be a $\Pi_2\langle L_{\omega_1}, T_G\rangle$-condition on T. Thus T_G^0, T_G^1 are $\Pi_2\langle L_{\omega_1}, T_G\rangle$-singletons. The generic collapse of $\mathscr{T}_3^{**}$ to ω_1 did not damage the properties $T_1 \notin L_\lambda((\mathscr{T}_3^{**}, T_0))$, $T_0 \notin L_\lambda((\mathscr{T}_3^{**}, T_1))$ (see the proof of Theorem 12 in [6]).

To get $\Pi_1\langle L_{\omega_1}, T_G\rangle$-singletons use the fact that any $\Pi_2\langle L_{\omega_1}, T_G\rangle$-singleton X can be made into a $\Pi_1\langle L_{\omega_1}, T_G\rangle$-singleton by considering $\langle X, f\rangle$ where f is the natural Skolem function for the Π_2-property characterizing X. Thus we have proved:

Theorem 18. *For any $\lambda < \omega_2$ there are $T \subseteq \omega_1$ and $\Pi_1\langle L_{\omega_1}, T\rangle$-singletons $T_0, T_1 \subseteq \omega_1$ such that $T_0 \notin L_\lambda(T, T_1)$ and $T_1 \notin L_\lambda(T, T_0)$.*

Example 4. We provide only a sketch of how the method of Example 3 can be amplified to obtain an analogue for ω_1 of Steel's result that $\boldsymbol{\Delta}_1^1$-CA $\not\to \boldsymbol{\Sigma}_1^1$-AC. $\boldsymbol{\Delta}_1^1$-CA for subsets of ω_1, and $\boldsymbol{\Sigma}_1^1$-AC$_{\omega_1}$ are the natrual generalizations to ω_1 of these principles.

Example 3 constructed two independent full subtrees. One could equally well construct ω_1-many full subtrees, one below each node on a full subtree of the entire tree T. These subtrees are very independent in that given countably many of them $T_0, T_1, \dots$, the only full subtrees of T in ω-HYP$(T, T_0, T_1, \dots)$ are subtrees of $\bigcup_i T_i$. Moreover the only full subtrees of T in ω-HYP$(T, T_0, T_1, \dots)$ satisfying a condition (b^*) like (b) in Example 3 must contain some T_i.

Once again we generically collapse T together with its 'full subtree' of full subtrees to ω_1, to get $T_G, T_0^G, T_1^G, \dots$ as subtrees of $\omega_1^{<\omega}$. (Everything is done so as to preserve the ω-admissibility of $\alpha = \omega_1^+ =$ least admissible $>\omega_1$.) If we let $M = (\bigcup \{L_\alpha(T_G, \mathscr{S}) \mid \mathscr{S}$ is a countable collection of trees $T_{\gamma_0}^G, T_{\gamma_1}^G, \dots\}) \cap \mathscr{P}(\omega_1)$, then $M \models \forall\gamma < \omega_1$ there exists a function $f : \gamma \xrightarrow{1-1}$ (Full Subtrees of T_G obeying (b^*)), but $M \models \sim\exists f : \omega_1 \xrightarrow{1-1}$ (Full Subtrees of T_G obeying (b*)). So $M \models \sim\boldsymbol{\Sigma}_1^1$-AC$_{\omega_1}$.

But as in [6] we can use Theorem 11 to show that any $X \subseteq \omega_1$ which is $\boldsymbol{\Sigma}_1^1(M)$ in parameters $m_0, m_1, \ldots$ is $\boldsymbol{\Sigma}_1^1(L_\alpha(T_G, m_0, m_1, \ldots) \cap \mathscr{P}(\omega_1))$ and hence $M \models \boldsymbol{\Delta}_1^1$-CA. Thus our result is:

Theorem 19. *There is $M \subseteq \mathscr{P}(\omega_1)$ which satisfies $\boldsymbol{\Delta}_1^1$-comprehension for subsets of ω_1 and full countable choice (M is closed under countable joins) but $\boldsymbol{\Sigma}_1^1$-AC_{ω_1} fails in M.*

4. Further results and open questions

(1) As is shown in [9], the $\pi_1(L_{\omega_1})$-singletons (in L) are constructed cofinally in the least *stable* ordinal $\alpha > \omega_1$. This suggests that the optimal form for Theorem 18 is: If $T \subseteq \omega_1$ and $\lambda <$ first T-stable greater than ω_1, then there exist $\Pi_1(L_{\omega_1}, T\rangle$-singletons T_0, T_1 such that $T_0 \notin L_\lambda(T, T_1)$, $T_1 \notin L_\lambda(T, T_0)$. Is this true?

(2) [9] provides, for each cardinal κ, a κ-recursive structure $\mathfrak{M}$ of $\mathscr{L}_{\infty\kappa}$-rank $\kappa^+ =$ next admissible after κ.

(3) What is the proper definition of $<\kappa$-HYP($\mathfrak{M}$) when κ is singular? What is the model-theoretic proof based on it of the result of [8] giving a sufficient (and necessary) condition for an admissible α of cardinality $\aleph_\omega$ to be the first admissible relative to some $X \subseteq \aleph_\omega$?

References

[1] J. Barwise, Infinitary logic and admissible sets, J. Symbolic Logic 34 (1969) 226–252.
[2] J. Barwise, Admissible Sets and Structures (Springer-Verlag, New York, 1975).
[3] J. Barwise, R.O. Gandy and Y.N. Moschovakis, The next admissible set, J. Symbolic Logic 36, 108–120.
[4] M.A. Dickman, Large Infinitary Languages (North-Holland, Amsterdam, 1975).
[5] H. Friedman, Countable models of set theories, in: Proceedings of the Cambridge 71 Logic Meeting, Lecture Notes in Math. 337 (Springer-Verlag, Berlin, 1973).
[6] S.D. Friedman, Steel Forcing and Barwise compactness, Annals Math. Logic 22 (1982) 31–46.
[7] S.D. Friedman, Uncountable admissibles I: Forcing, Trans. AMS 270 (1) (1982).
[8] S.D. Friedman, Uncountable admissibles II: Compactness, Israel J. Math. 40 (2) (1981).
[9] S.D. Friedman and S. Shelah, Tall α-recursive structures, Proc. AMS 88 (4) (1983).
[10] J. Keisler, Model Theory for Infinitary Logic (North-Holland, Amsterdam, 1971).
[11] J. Keisler, Forcing and the Omitting Types Theorem, Studies in Model Theory (MAA, New York, 1973).
[12] M. Morley, The number of countable models, J. Symbolic Logic 35 (1970) 14–18.
[13] M. Nadel, Scott sentences and admissible sets, Annals Math. Logic 7 (1974) 267–294.
[14] G. Sacks, Countable admissible ordinals and hyperdegrees, Advances in Math. 20 (1976).
[15] J. Steel, Forcing with tagged trees, Annals Math. Logic 15 (1978) 55–74.