Forcing, Combinatorics and Definability

Summary:

- Today: Definable Wellorders
  Large cardinals
  Forcing axioms
  Cardinal characteristics (new!)
  Other contexts
- Tuesday: Cardinal Characteristics on κ
  Rather new topic: Many open questions
  Continuum function 2^κ
  Dominating, bounding numbers
  Cofinality of the symmetric group
  Almost disjointness, splitting numbers
- Wednesday: Models of PFA, BPFA

Definable Wellorders

In ZF, AC is equivalent to:
$H(\kappa^+)$ can be wellordered for every κ
When can we obtain a *definable* wellorder of $H(\kappa^+)$?

Σ_n *definable wellorder of* $H(\kappa^+)$: Wellorder of $H(\kappa^+)$ which is Σ_n definable over $H(\kappa^+)$ with κ as a parameter

Remarks:
1. If n is at least 3, then κ can be eliminated, as $\{\kappa\}$ is Π_2 definable
2. If λ is a limit cardinal and $H(\kappa^+)$ has a definable wellorder for cofinally many $\kappa < \lambda$, then $H(\lambda)$ has a definable wellorder

Σ_n *definable wellorder of* $H(\kappa^+)$ *with parameters*: Wellorder of $H(\kappa^+)$ which is Σ_n definable over $H(\kappa^+)$ with *arbitrary* elements of $H(\kappa^+)$ as parameters

Definable Wellorders: Large cardinals and $H(\omega_1)$

The best situation:
$V = L \to$ Each $H(\kappa^+) = L_{\kappa^+}$ has a Σ_1 definable wellorder

Definable wellorders and Large Cardinals

$$H(\omega_1)$$

Σ_n definable wellorder of $H(\omega_1)$ (with parameters) $\sim$
Σ^1_{n+1} definable wellorder of the reals (with real parameters)

Theorem

(Mansfield) Σ^1_2 wellorder of the reals $\to$ every real belongs to L.
(Martin-Steel) A Σ^1_{n+2} wellorder of the reals is consistent with n Woodin cardinals but inconsistent with n Woodin cardinals and a measurable cardinal above them.

Definable Wellorders: Large Cardinals and $H(\omega_2)$

$H(\omega_2)$

A forcing is *small* iff it has size less than the least inacessible. Small forcings preserve all large cardinals.

Theorem

(Asperó-F) There is a small forcing which forces CH and a definable wellorder of $H(\omega_2)$.

The above wellorder is not Σ_1. In fact:

Theorem

(Woodin) Measurable Woodin cardinal + CH $\rightarrow$ there is no wellorder of the reals which is Σ_1 over $H(\omega_2)$.

However:

Definable Wellorders: Large cardinals and $H(\omega_2)$

Theorem

(Avraham-Shelah) There is a small forcing which forces $\sim$ CH and a wellorder of the reals which is Σ_1 over $H(\omega_2)$.

Question 1. Is there a small forcing which forces a Σ_2 wellorder of $H(\omega_2)$?

Definable Wellorders: Large cardinals and $H(\omega_2)$

About the proof of:

Theorem

(Asperó-F) There is a small forcing which forces CH and a definable wellorder of $H(\omega_2)$.

Two ingredients:

- Canonical function coding
- Strongly type-guessing coding (Asperó)

Definable Wellorders: Large cardinals and $H(\omega_2)$

Canonical funtion coding

For each $\alpha < \omega_2$ choose $f_\alpha : \omega_1 \to \alpha$ onto and define $g_\alpha : \omega_1 \to \omega_1$ by:

$$g_\alpha(\gamma) = \text{ordertype } f_\alpha[\gamma].$$

g_α is a "canonical function" for α.

Now code $A \subseteq \omega_2$ by $B \subseteq \omega_1$ as follows:

$$\alpha \in A \text{ iff } g_\alpha(\gamma) \in B \text{ for a club of } \gamma$$

Assuming GCH, the forcing to do this is ω-strategically closed and ω_2-cc.

Definable Wellorders: Large cardinals and $H(\omega_2)$

Asperó coding

A *club-sequence in* ω_1 *of height* τ is a sequence $\vec{C} = (C_\delta \mid \delta \in S)$ where $S \subseteq \omega_1$ is stationary and each C_δ is club in δ of ordertype τ.

$\vec{C}$ is *strongly type-guessing* iff for every club $C \subseteq \omega_1$ there is a club $D \subseteq \omega_1$ such that for all δ in $D \cap S$, ordertype$(C \cap C_\delta^+) = \tau$, where C_δ^+ denotes the set of successor elements of C_δ.

An ordinal γ is *perfect* iff $\omega^\gamma = \gamma$.

Lemma

(Asperó) Assume GCH. Let $B \subseteq \omega_1$. Then there is an ω-strategically closed, ω_2-cc forcing that forces: $\gamma \in B$ iff the γ-th perfect ordinal is the height of a strongly type-guessing club sequence.

Definable Wellorders: Large cardinals and $H(\omega_2)$

To prove:

Theorem

(Asperó-F) There is a small forcing which forces CH and a definable wellorder of $H(\omega_2)$.

Assume GCH. Write $H(\omega_2)$ as $L_{\omega_2}[A]$, $A \subseteq \omega_2$.

Use Canonical function coding to code A by $B \subseteq \omega_1$.

Use Asperó coding to code B definably over $H(\omega_2)$.

Problem: B only codes $H(\omega_2)$ of the ground model, not $H(\omega_2)$ of the extension!

Solution: Perform both codings "simultaneously". The forcing is a hybrid forcing: halfway between iteration and product.

Definable Wellorders: Large cardinals and $H(\kappa)$

$H(\kappa)$

Theorem

(Asperó-F) There is a class forcing which forces GCH, preserves all supercompact cardinals (as well as a proper class of n-huge cardinals for each n) and adds a definable wellorder of $H(\kappa^+)$ *for all regular* $\kappa \geq \omega_1$.

Corollary

There is a class forcing which forces GCH, preserves all supercompact cardinals (as well as a proper class of n-huge cardinals for each n) and adds a parameter-free definable wellorder of $H(\delta)$ *for all cardinals* $\delta \geq \omega_2$ *which are not successors of singulars.*

Successors of singulars? Σ_1 definable wellorders?

Definable Wellorders: Large cardinals and $H(\kappa)$

Successors of singulars:

Theorem

(Asperó-F) Suppose that there is a $j : L(H(\lambda^+)) \to L(H(\lambda^+))$ fixing λ, with critical point $< \lambda$. Then there is no definable wellorder of $H(\lambda^+)$ with parameters.

Question 2. Is there a small forcing that adds a definable wellorder of $H(\aleph_{\omega+1})$ with parameters?

Σ_1 definable wellorders:

Theorem

There is a class forcing which forces GCH, preserves all supercompact cardinals (as well as a proper class of n-huge cardinals for each n) and adds a Σ_1 definable wellorder of $H(\kappa^+)$ with parameters for all regular $\kappa \geq \omega_1$.

Definable Wellorders and Forcing Axioms

Question 3. Is there a small forcing that adds a Σ_1 definable wellorder of $H(\omega_3)$?

Definable wellorders and Forcing Axioms

$$H(\omega_1)$$

Theorem

MA is consistent with a Σ^1_3 wellorder of the reals.
(Caicedo-F) BPFA + $\omega_1 = \omega_1^L$ (which is consistent relative to a reflecting cardinal) implies that there is a Σ^1_3 wellorder of the reals.

Theorem

(Hjorth) Assume $\sim$ CH and every real has a $\#$. Then there is no Σ^1_3 wellorder of the reals.

Definable Wellorders and Forcing Axioms

Question 4. Does BPFA + $0^{\#}$ does not exist imply that there is a Σ^1_3 wellorder of the reals?
Question 5. Is BMM consistent with a projective wellorder of the reals? PFA is not.
Question 6. Is MA consistent with the *nonexistence* of a projective wellorder of the reals?

For $H(\omega_2)$:

Theorem

(Caicedo-Velickovic) BPFA + $\omega_1 = \omega_1^L$ implies that there is a Σ_1 definable wellorder of $H(\omega_2)$.

Theorem

(Larson) Relative to enough supercompacts, there is a model of MM with a definable wellorder of $H(\omega_2)$.

Definable Wellorders and Forcing Axioms

For larger $H(\kappa)$:

Theorem

MA is consistent with a definable wellorder of $H(\kappa^+)$ for all κ.
(Reflecting cardinal) BSPFA is consistent with a definable wellorder of $H(\kappa^+)$ for all κ.
(Enough supercompacts) MM is consistent with a definable wellorder of $H(\kappa^+)$ for all regular $\kappa \geq \omega_1$.

Definable Wellorders and Cardinal Characteristics

New context for definable wellorders: Cardinal Characteristics

Template iteration $\mathbb{T}$: A countable support, ω_2-cc iteration which adds a Σ^1_3 wellorder of the reals (and a Σ_1 wellorder of $H(\omega_2)$). It is not proper, but is S-proper for certain stationary $S \subseteq \omega_1$.

Broad project: Mix the template iteration with a variety of iterations for controlling cardinal characteristics.

Theorem

(V.Fischer - F) Each of the following is consistent with a Σ^1_3 wellorder of the reals: $\mathfrak{d} < \mathfrak{c}$, $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$, $\mathfrak{b} < \mathfrak{g}$.

$\mathfrak{b}$ = the bounding number, $\mathfrak{a}$ = the almost disjointness number, $\mathfrak{s}$ = the splitting number, $\mathfrak{g}$ = the groupwise density number

Definable Wellorders

The template iteration $\mathbb{T}$ is *gentle* (ω^ω bounding) but also *flexible* (it can be mixed with any countable support proper iteration of posets of size ω_1)

One can also ask for nicely definable *witnesses* to cardinal characteristics. A sample result:

Theorem

(F-Zdomskyy) It is consistent that $\mathfrak{a} = \omega_2$ and there is a Π^1_2 infinite maximal almost disjoint family.

Question 7. Is it consistent with $\mathfrak{a} = \omega_2$ that there is a Σ^1_2 infinite maximal almost disjoint family?

Definable Wellorders in other Contexts

Questions.

8. Is it consistent that for all infinite regular κ, GCH fails at κ and there is a definable wellorder of $H(\kappa^+)$?
9. Is the tree property at ω_2 consistent with a projective wellorder of the reals?
10. Is it consistent that the nonstationary ideal on ω_1 is saturated and there is a Σ^1_4 wellorder of the reals?
11. Is it consistent that GCH fails at a measurable cardinal κ and there is a definable wellorder of $H(\kappa^+)$?

Cardinal Characteristics at κ

Cardinal characteristics on ω is a vast subject.

Examples from Blass' survey:

$$\mathfrak{a}, \mathfrak{b}, \mathfrak{d}, \mathfrak{e}, \mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{m}, \mathfrak{p}, \mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{u}$$

These are all at most $\mathfrak{c}$, the cardinality of the continuum.

κ regular, uncountable. We consider analogues of some of the above for κ

$$\mathfrak{a}(\kappa), \mathfrak{b}(\kappa), \mathfrak{d}(\kappa) \ldots$$

Why?

Cardinal Characteristics at κ

Four reasons:

1. Higher iterated forcing

Card Chars ω / Countable support iterations $\equiv$
Card Chars κ / Higher support iterations

2. Large cardinal context: Card Chars at a measurable

3. Global behaviour as κ varies, Internal consistency

4. Solve problems at κ that are unsolved at ω

Illustrate with some examples

Cardinal Characteristics at κ

The Card Char 2^κ

Global behaviour

Theorem

(Corollary to Easton's Theorem) It is consistent that $2^\alpha = \alpha^{++}$ for all regular α.

Forcing used: *Easton product* of α-Cohen forcings Cohen(α, α^{++}).

Internal consistency

Theorem

(F-Ondrejović) Assuming that $0^\#$ exists, there is an inner model in which $2^\alpha = \alpha^{++}$ for all regular α.

Forcing used: *Reverse Easton iteration* of α-Cohen forcings.

Cardinal Characteristics at κ

Large cardinal context

Theorem

(Woodin) Assume that κ is hypermeasurable. Then in a forcing extension, κ is measurable and $2^\kappa = \kappa^{++}$.

Forcing used: Reverse Easton iteration of α-Cohen forcings, $\alpha \leq \kappa$, α inaccessible, followed by Cohen(κ^+, κ^{++}).

Cardinal Characteristics at κ

Now look at

The Card Char $\mathfrak{d}(\kappa)$

Global Behaviour

Theorem

(Cummings-Shelah) It is consistent that $\mathfrak{d}(\alpha) = \alpha^+ < 2^\alpha$ *for all regular* α.

Forcings used: α-Cohen product and α-Hechler iteration.

Cardinal Characteristics at κ

Large cardinal context

Theorem

(F-Thompson) Assume that κ is hypermeasurable. Then in a generic extension, κ is measurable and $\mathfrak{d}(\kappa) = \kappa^+ < 2^\kappa$.

Forcing used: Reverse Easton iteration of α-Sacks products, $\alpha \leq \kappa$, α inaccessible. Two interesting points:

i. If you try this with κ-Cohen and κ-Hechler then you need some supercompactness

ii. The proof is *easier* than Woodin's proof, which only gives $\kappa^+ < 2^\kappa$

Cardinal Characteristics at κ

Large Cardinal context together with Global Behaviour

Theorem

(F-Thompson) Assume that κ is hypermeasurable. Then in a generic extension, κ is measurable and $(\alpha) = \alpha^+ < 2^\alpha$ for all regular α.

Forcings used: (Reverse Easton iteration of) α-Sacks at inaccessible $\alpha \leq \kappa$, α-Cohen product followed by α-Hechler iteration at successors of non-inaccessibles, *something new* at α^+, α inacessible (α^+-Cohen product followed by a mixture of α-Sacks product and α^+-Hechler iteration).

Conclusion: Understanding $\mathfrak{d}(\kappa)$ in the Large cardinal setting requires a careful choice of forcings; mixing it with the Global Behaviour of $\mathfrak{d}(\alpha)$ requires the invention of new forcings

Cardinal Characteristics at κ

Remark. (F-Honzik) Easton's Theorem for 2^α has been worked out in the large cardinal setting. But:

Question 12. What Global Behaviours for $\mathfrak{d}(\alpha)$ are possible when there is a measurable cardinal?

The Card Char CofSym(α)

Sym(α) = group of permutations of α under composition.
CofSym(α) = least λ such that Sym(α) is the union of a strictly increasing λ-chain of subgroups.

Sharp and Thomas: CofSym(α) can be anything reasonable.

But its Global Behaviour is nontrivial!

Theorem

(Sharp-Thomas) (a) Suppose that $\alpha < \beta$ are regular and GCH holds. Then in a cofinality-preserving forcing extension, CofSym(α) = β.
(b) If CofSym(α) $> \alpha^+$ then CofSym(α^+) $\leq$ CofSym(α).

Question 13. Is it consistent that CofSym(ω) = CofSym(ω_1) = ω_3?

Cardinal Characteristics at κ

CofSym has been studied in the Large Cardinal setting:

Theorem

(F-Zdomskyy) Suppose that κ is hypermeasurable. Then in a forcing extension, κ is measurable and CofSym$(\kappa) = \kappa^{++}$.

Forcings used: Iteration of Miller(κ) (with continuous club-splitting) and a generalisation of Sacks(κ).
The proof also uses $\mathfrak{g}_{cl}(\kappa)$ (groupwise density number for *continuous* partitions).

Cardinal Characteristics at κ

$\mathfrak{a}(\kappa)$ and $\mathfrak{d}(\kappa)$

$\mathfrak{a}(\kappa) =$ minimum size of a (size at least κ) maximal almost disjoint family of subsets of κ

An old open problem:

Question 14. Does $\mathfrak{d}(\omega) = \omega_1$ imply $\mathfrak{a}(\omega) = \omega_1$?
But this is solved at uncountable cardinals!

Theorem

(Blass-Hyttinen-Zhang) For uncountable α, $\mathfrak{d}(\alpha) = \alpha^+$ implies $\mathfrak{a}(\alpha) = \alpha^+$

Cardinal Characteristics at κ

Are there other open questions for ω which can be solved for uncountable cardinals?

Question 15. Can $\mathfrak{p}(\kappa)$ be less than $\mathfrak{t}(\kappa)$? Maybe it will help to assume that κ is a large cardinal.

Question 16. Can $\mathfrak{s}(\kappa)$ be singular?

More open Questions.

17. (Without large cardinals) Is $\mathfrak{b}(\kappa) < \mathfrak{a}(\kappa)$ consistent for an uncountable κ?
18. Which Global Behaviours for $\mathfrak{b}(\alpha), \mathfrak{d}(\alpha)$ are internally consistent? Cummings-Shelah answered this for ordinary consistency.
19. (Without supercompactness) Can $\mathfrak{s}(\kappa)$ be greater than κ^+? Zapletal: Need (almost) a hypermeasurable.
20. Is it consistent that $\mathrm{CofSym}(\kappa) = \kappa^{+++}$ for a measurable κ?

Some models of PFA, BPFA

Let $\mathcal{C}$ be a class of forcings

FA($\mathcal{C}$) = Forcing Axiom for $\mathcal{C}$
For P in $\mathcal{C}$, can hit ω_1-many predense sets in P with a filter on P

BFA($\mathcal{C}$) = Bounded Forcing Axiom for $\mathcal{C}$
For P in $\mathcal{C}$, can hit ω_1-many predense sets of size $\leq \omega_1$ in P with a filter on P

PFA = FA(Proper) = Proper Forcing Axiom
BPFA = BFA(Proper) = Bounded Proper Forcing Axiom

Useful Fact. (Bagaria, Stavi-Väänänen) BPFA is equivalent to the Σ_1 elementarity of $H(\omega_2)^V$ in $H(\omega_2)^{V[G]}$ for proper P and P-generic G

Some models of PFA, BPFA

Theorem

(a) (Baumgartner) If there is a supercompact then PFA holds in a proper forcing extension.
(b) (Goldstern-Shelah) If there is a reflecting cardinal (i.e., a regular κ such that $H(\kappa) \prec_{\Sigma_2} V$) then BPFA holds in a proper forcing extension.

Some models of PFA, BPFA

Cardinal Minimality

V is *cardinal minimal* iff whenever M is an inner model with the correct cardinals (i.e., $\mathrm{Card}^M = \mathrm{Card}^V$) then $M = V$.

Local version: κ a cardinal. V is *κ-minimal* iff whenever M is an inner model with the correct cardinals $\leq \kappa$ then $H(\kappa)^M = H(\kappa)$.

Examples
L is trivially cardinal minimal.
Let x be κ-Sacks, κ-Miller or κ-Laver over L. Then $L[x]$ is *not* cardinal minimal.
Let $f : \kappa \to \kappa^+$ be a minimal collapse of κ^+ to κ over L. Then $L[f]$ is cardinal minimal.

More interesting examples: Core models

Some models of PFA, BPFA

Theorem

Let K be the core model for a measurable, hypermeasurable, strong or Woodin cardinal. Then K is cardinal minimal. In fact, K is κ-minimal for all $\kappa \geq \omega_2$.

ω_1-minimality fails for core models, and in fact whenever $0^\#$ exists:

Theorem

Suppose that $0^\#$ exists. Then V is not ω_1-minimal. In fact, there is an inner model M with the correct ω_1 which is a forcing extension of L.

Some models of PFA, BPFA

Another source of cardinal minimality: Models of forcing axioms

SPFA = FA(Semiproper) = Semiproper Forcing Axiom
BSPFA = BFA(Semiproper) = Bounded Semiproper Forcing Axiom

Theorem

(Velickovic) Suppose that SPFA holds. Then V is ω_2-minimal.

There is a related result for BPFA:

Theorem

(Caicedo-Velickovic) Suppose that BPFA holds. Then V is ω_2-minimal with respect to inner models satisfying BPFA: If M is an inner model satisfying BPFA with the correct ω_2 then $H(\omega_2)^M = H(\omega_2)$.

Some models of PFA, BPFA

Theorem

(a) Suppose that there is a supercompact. Then in some forcing extension, PFA holds and the universe is not ω_2-minimal.
(b) Suppose that there is a reflecting cardinal. Then in some forcing extension, BPFA holds and the universe is not ω_2-minimal.

The proofs are based on:

Lemma

(Collapsing to ω_2 with "finite conditions") Assume GCH. Suppose that κ is inaccessible and $\mathcal{S}$ denotes $[\kappa]^\omega$ of V. Then there is a forcing P such that:
(a) P forces $\kappa = \omega_2$.
(b) P is $\mathcal{S}$-proper, and hence preserves ω_1, in any extension of V in which $\mathcal{S}$ remains stationary.

Some models of PFA, BPFA

We sketch the proof of (a):

(a) Suppose that there is a supercompact. Then in some forcing extension, PFA holds and the universe is not ω_2-minimal.

κ supercompact.

Collapse κ to ω_2 with finite conditions, producing $V[F]$.

Perform Baumgartner's PFA iteration, but at stage $\alpha < \omega_2$, choose a forcing *in* $V[F \restriction \alpha, G_\alpha]$ which is $\mathcal{S}$-proper there; argue that it is also $\mathcal{S}$-proper in $V[F, G_\alpha]$. *Important:* Only use names from $V[F \restriction \alpha, G_\alpha]$, to keep the forcing small! *"Diagonal iteration"*

Verify that PFA (indeed FA($\mathcal{S} - \text{Proper}$)) holds in $V[F, G]$.

As $\kappa = \omega_2$ both in $V[F]$ and in $V[F, G]$, this shows that $V[F, G]$ is not ω_2-minimal.

Some models of PFA, BPFA

How to collapse an inaccessible κ to ω_2 with finite conditions?

Let $\# : [\kappa]^\omega \to \kappa$ be injective. P consists of all pairs $p = (A, S)$ such that:

1. A is a finite set of disjoint closed intervals $[\alpha, \beta]$, $\alpha \leq \beta < \kappa$, $\mathrm{cof}(\alpha) \leq \omega_1$.
2. S is a finite subset of $[\kappa]^\omega$ ("side conditions").
3. Technical.
4. Let F be the set of uncountable cofinality α for $[\alpha, \beta]$ in A, together with κ. The *height* of $x \in S$ is the least element of F greater than $\sup x$. Then:

i. (Closure under truncation) x in S, α in F implies $x \cap \alpha$ in S.
ii. (Almost an $\in$-chain) If $x, y \in S$ have the same height then $\#(x) \in y$, $\#(y) \in x$ or $x = y$.

Some models of PFA, BPFA

The forcing is κ-cc and adds a club in κ consisting only of ordinals of cofinality $\leq \omega_1$. So κ becomes ω_2.

Questions.
21. Suppose that BSPFA holds. Then is V ω_2-minimal with respect to inner models satisfying BSPFA?
22. Is there a forcing which collapses an inaccessible to ω_3 "with finite conditions"?