Cardinal Characteristics and Definability

I write Σ^1_n, Π^1_n, Δ^1_n for these classes *without parameters*

A wellorder of the reals is Σ^1_n iff it is Δ^1_n iff it is Π^1_n
$x \not\leq y$ iff ($y \leq x$ and $y \neq x$).

(Gödel) $V = L \to \Delta^1_2$ wellorder

(Mansfield) Δ^1_2 wellorder $\to$ $V = L$ for reals

(Harrington) Δ^1_3 wellorders are consistent with large continuum

(SDF) Δ^1_3 wellorders are consistent with $\mathrm{MA} + \mathfrak{c} = \omega_2$

(Caicedo-SDF) $\mathrm{BPFA} + \omega_1 = \omega_1^L \to \Delta^1_3$ wellorder

Large cardinals $\to$ There is no Δ^1_n wellorder, $n < \omega$

Cardinal Characteristics and Definability

Question: Can we bring Δ^1_n definability into the study of cardinal characteristics?

Theorem

(Fischer-SDF) Each of the following is consistent with a Δ^1_3 wellorder: $\mathfrak{d} < \mathfrak{c}$, $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$, $\mathfrak{b} < \mathfrak{g}$.

$\mathfrak{a}, \mathfrak{b}, \mathfrak{d}, \mathfrak{g}, \mathfrak{s} =$ almost-disjointness, bounding, dominating, groupwise-density, splitting numbers

Cardinal Characteristics and Definability

All three results have the same proof strategy:

Template Iteration: A countable support ω_2-iteration of S-proper forcings (for some stationary $S \subseteq \omega_1$) which is ω^ω bounding, forces a Δ^1_3 wellorder and allows for complete freedom to insert additional proper forcings of size ω_1 into the iteration.

For $\mathfrak{d} < \mathfrak{c}$: Use the basic template
For $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$: Insert proper almost ω^ω bounding forcings due to Shelah to kill MAD families and push up the splitting number
For $\mathfrak{b} < \mathfrak{g}$: Insert Miller forcings to push up $\mathfrak{g}$

The idea of the Template Iteration:

1. *Club Coding:* Code a wellorder by adding clubs to selectively killing the stationarity of certain stationary subsets of ω_1 (no reals added).
2. *Sacks Coding:* Code these clubs by reals using a variant of Sacks forcing (to ensure ω^ω-boundedness).
3. *David's Trick:* Make the coding Δ^1_3.
4. *Stationarity Preservation:* Verify that no stationary set "unintentionally" loses its stationary, to ensure that the intended Club Coding works.

Cardinal Characteristics and Definability

Write the Template Iteration as $P = (P_\alpha \mid \alpha < \omega_2)$

Club Coding

Fix a stationary subset S of ω_1 and a sequence $(S_\alpha \mid \alpha < \omega_2)$ of almost disjoint stationary subsets of ω_1, each disjoint from S

At iteration stage α (α limit):
Let $<_\alpha$ denote the canonical wellorder of the reals in $L[G_\alpha]$
Choose two reals $x <_\alpha y$ and force to code the pair (x, y):

$n \in x \to S_{\alpha+2n}$ is nonstationary
$n \in y \to S_{\alpha+2n+1}$ is nonstationary

A condition is a countable closed subset of ω_1 disjoint from the $S_{\alpha+2n}$ for $n \in x$ and from the $S_{\alpha+2n+1}$ for $n \in y$
The forcing is S-proper

Cardinal Characteristics and Definability

Sacks Coding

Suppose $V = L[A]$ where A is a subset of $\omega_1 = \omega_1^L$

Define countable ordinals μ_i as follows:

$\mu_i =$ least $\mu > \sup_{j<i} \mu_j$ such that $L_\mu[A \cap i] \vDash \mathrm{ZF}^-$ and i is countable in L_μ

A real R *codes* A *below* i iff for $j < i$:

$$j \in A \text{ iff } L_{\mu_j}[A \cap j, R] \vDash \mathrm{ZF}^-$$

For T a Sacks tree, $|T| =$ least i such that $T \in L_{\mu_i}[A \cap i]$

Now force with Sacks trees T such that:

$$R \text{ a branch through } T \to R \text{ codes } A \text{ below } |T|$$

By absoluteness, this holds for all R in the generic extension

Cardinal Characteristics and Definability

We need to check that forcing with Sacks coding behaves nicely

Fact 1. If T is a condition and $|T| < i$ then T has an extension T^* with $|T^*| = i$.

Proof: Suppose $|T| = j$, $i = j + 1$.
Let $\mathcal{A}_j$ denote $L_{\mu_j}[A \cap j]$; so T belongs to $\mathcal{A}_j$.

If j belongs to A then thin T to T^* so that each branch of T^* is T-generic over $\mathcal{A}_j$. Then R a branch through $T^* \rightarrow \mathcal{A}_j[R] \models \mathrm{ZF}^-$.

Otherwise let R_0 be a real in $\mathcal{A}_i$ coding μ_j and thin T to T^* so that branches through T^* go right at the $2n$-th splitting level of T iff n belongs to R_0. Then:

R a branch through $T^* \rightarrow R_0 \in \mathcal{A}_j[T, R] = \mathcal{A}_j[R] \rightarrow \mathcal{A}_j[R] \not\models \mathrm{ZF}^-$

The case of limit i uses fusion.

Also using fusion:

Fact 2. Sacks Coding is proper and ω^ω bounding.

The effect of Sacks Coding is to add a real R, the unique branch through all T in the generic, such that for all $i < \omega_1$:

$$i \in A \text{ iff } \mathcal{A}_i[R] \vDash \mathrm{ZF}^-$$

and therefore $V = L[A]$ is contained in $L[R]$.

David's Trick

I won't say much about this (it is Tricky!). The idea is to strengthen

"For some α there is a club C disjoint from $S_{\alpha+2n}$ for $n \in x$ and from $S_{\alpha+2n+1}$ for $n \in y$"

to:

"There is a real R such that in all ZF^- models M containing R satisfying $\omega_1 = \omega_1^L$, $M \vDash$ There is an α and a club C disjoint from $S^M_{\alpha+2n}$ for $n \in x$ and from $S^M_{\alpha+2n+1}$ for $n \in y$"
where $S^M_{\alpha+2n}, S^M_{\alpha+2n+1} = S_{\alpha+2n}, S_{\alpha+2n+1}$ interpreted in M.

This Σ^1_3 statement about (x, y) is the source of the Δ^1_3 wellorder.

Cardinal Characteristics and Definability

Stationary Preservation

Our iteration does the following: If $<$ is the canonical wellorder of the reals in $L[G]$ then:

$x < y \rightarrow$ For some limit $\alpha < \omega_2$ there is a club C disjoint from $S_{\alpha+2n}$ for $n \in x$ and from $S_{\alpha+2n+1}$ for $n \in y$

To get a definable wellorder we need the converse; i.e. we need to show that the stationarity of $S_{\alpha+2n}$ or $S_{\alpha+2n+1}$ was not "unintentionally" killed.

Stationary Preservation (continued)

For this we argue as follows:

Suppose that p forces that $S_{\alpha+2n}$ "should" remain stationary (because n does not belong to x where the pair (x, y) is considered at stage α). Then the iteration below p is a countable support iteration of $S_{\alpha+2n}$-proper forcings and therefore is $S_{\alpha+2n}$-proper. But p forces the generic to be also generic for this iteration, so p forces stationary-preservation for $S_{\alpha+2n}$.

We now have all the ingredients to show:

The Template Iteration is S-proper for some stationary S, is ω^ω bounding, is ω_2-cc, forces $\sim$ CH and adds a Δ^1_3 wellorder of the reals. Moreover we can mix into the iteration any size ω_1 proper forcings and preserve these properties with the exception of ω^ω boundedness.

Cardinal Characteristics and Definability

Cardinal Characteristics and Δ^1_3 wellorders

Now we mix in specific additional proper forcings to control cardinal characteristics.

Example 1: $\mathfrak{d} < \mathfrak{c}$

No mixing is required; the basic Template Iteration is ω^ω bounding so does not increase $\mathfrak{d}$.

Example 3: $\mathfrak{b} < \mathfrak{g}$

Miller forcing is almost ω^ω-bounding. The countable support limit of S-proper, almost ω^ω-bounding forcings is weakly bounding, i.e. does not increase $\mathfrak{b}$. By an argument of Blass, Miller forcing pushes up $\mathfrak{g}$.

Cardinal Characteristics and Definability

Example 2: $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$

We need two forcings of Shelah. Assume CH.
(a) A proper, almost ω^ω-bounding forcing $\mathcal{Q}$ of size ω_1 which adds a real not split by any ground model real.
(b) A proper, almost ω^ω-bounding forcing $\mathcal{R}$ of size ω_1 which kills a given MAD family of the ground model.

By inserting the forcings $\mathcal{Q}$ and $\mathcal{R}$ into the Template Iteration we keep $\mathfrak{b} = \omega_1$ but push up both $\mathfrak{a}$ and $\mathfrak{s}$ to ω_2.

Projective MAD Families

We consider MAD families of subsets of ω.

(Mathias) There is no Σ^1_1 MAD family

(Miller) In L, there is a Π^1_1 MAD family

(Kastermans-Steprans-Zhang + Kurilic) There is a Π^1_1 MAD family in L that remains Π^1_1 and MAD after adding any number of Cohen reals.

So it is consistent with $\sim$ CH to have a Π^1_1 MAD family of size ω_1.

Projective MAD Families

Question: Is it consistent to have projective MAD families when there are no MAD families of size ω_1?

Theorem

(SDF-Zdomskyy) It is consistent with $\mathfrak{b} = \omega_2$ to have a Π^1_2 MAD family.

As $\mathfrak{b} \leq \mathfrak{a}$ it follows that it is consistent to have a Π^1_2 MAD family when there are no MAD families of size ω_1.

Projective MAD Families

Very brief proof sketch:

Insert proper forcings that add a dominating real into the Template Iteration; this guarantees $\mathfrak{b} = \omega_2$.
At limit stage α look at the α-th real x_α and add it to $\mathcal{F}_\alpha =$ the part of the MAD family built so far, if x_α is almost disjoint from $\mathcal{F}_\alpha$.
Using Club Coding and David's Trick, witness $x_\alpha \in \mathcal{F}$ or $x_\alpha \notin \mathcal{F}$ in a Π^1_2 way; this ensures that $\mathcal{F}$ is Δ^1_3.

The above is straightforward.

The hard part: Actually only put x_α into $\mathcal{F}$ which "witness themselves"; this ensures that $\mathcal{F}$ will be Π^1_2.

Projective MAD Families

Question: Can we have projective MAD families when $\mathfrak{b} > \omega_2$?

Theorem

(Aristotle-Fischer-SDF-Zdomskyy) It is consistent with $\mathfrak{b} = \mathfrak{c} = \omega_3$ to have a Π^1_2 MAD family and a Δ^1_3 wellorder of the reals.

Of course we now need to use *finite support* iteration, to avoid collapsing ω_2.

Projective MAD Families

Another very brief proof sketch:

We begin with a sequence $(S_\alpha \mid \alpha < \omega_3)$ of almost disjoint stationary subsets of $\omega_2 \cap \mathrm{Cof}(\omega_1)$, each disjoint from some fixed stationary $S \subseteq \omega_2 \cap \mathrm{Cof}(\omega_1)$

Let $\mathcal{P}_0$ be the product with size ω_1 support of forcings to add clubs C_α disjoint from S_α

Next let $\mathcal{P}_1$ be the product with countable support of forcings to code each C_α by a subset X_α of ω_1. (So 2^{ω_1} now equals ω_3.)

Finally we iterate with finite support to add reals R_β, $\beta < \omega_3$, which code certain of the X_α's using σ-centered almost disjoint coding. At limit stage β we look at the β-th pair of reals $x < y$ and force R_β to code $X_{\beta+2n}$ for n in x and $X_{\beta+2n+1}$ for n in y. Using David's Trick, this will give a Δ^1_3 wellorder provided nonstationarity is not "unintenionally" coded by a real.

Projective MAD Families

Obtaining a Π^1_2 MAD family with $\mathfrak{b} = \omega_3$ is accomplished by mixing in Hechler reals and the methods of the previous theorem.

To show that nonstationarity is not "unintentionally" coded, it is important that the wellorder of the reals be determined just by the generic reals being added, and not by the generic as a whole.

This means that our technique is limited to handling countable objects in the iteration. So the following remain open:

Is Martin's Axiom consistent with a Δ^1_3 wellorder and $\mathfrak{c} = \omega_3$?

Can one separate cardinal characteristics with a Δ^1_3 wellorder and $\mathfrak{c} = \omega_3$?

(Also: Is it consistent to have a Σ^1_2 MAD family when there are no MAD families of size ω_1?)

The Last Slide

Question: Can one introduce definability into the study of cardinal characteristics at an uncountable cardinal?

Theorem

(SDF-Honzik) Assume the consistency of a weak compact hypermeasurable cardinal. Then it is consistent for GCH to fail at a measurable κ *with a definable wellorder of* $H(\kappa^+)$*; the same holds for* $\aleph_\omega$ *with* $\aleph_\omega$ *strong limit.*