Global complexity results

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Abstract

When no universal model for a set of structures exists at a given cardinal, then it is natural to ask how many structures of that cardinality it takes to embed them all. It is shown that at regular cardinals λ, this covering number (or complexity) for posets of size λ omitting certain large substructures can 2^λ for any reasonable value of 2^λ. These results can also be put into a global context where the complexity is determined at every regular cardinal simultaneously.[1]

1 Introduction

To clarify our use of universality, we begin with the basic definitions. An embedding (also called weak embedding) for ordered sets is an injective order-preserving map. A strong embedding is an embedding which also preserves incomparability. The range of a strong embedding is an isomorphic copy of the order in the domain. For graphs, an embedding is an injective function which preserves edges and a strong embedding is an injective function whose range is an induced subgraph.

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Given a set $\mathcal{A}_\lambda$ of structures each of size λ, a *(strong) universal model* for $\mathcal{A}_\lambda$ is an element of $\mathcal{A}_\lambda$ which (strongly) embeds all other structures in $\mathcal{A}_\lambda$. If there does not exist a universal model for $\mathcal{A}_\lambda$, then we consider its *complexity*, or the smallest size of a subfamily of $\mathcal{A}_\lambda$ which embeds the rest. This subfamily of structures is called a *universal family*. All of these notions have weak and strong counterparts depending on the type of embedding used.

In this paper, assume that λ, κ are regular, uncountable cardinals such that $\kappa \leq \lambda$.

We will now define notation for types of structures which will be discussed in this paper. As we would like to work with sets of structures rather than classes, we will assume that the structures are representatives under isomorphism. Let $\mathcal{C}(\lambda, \kappa)$ be the set of all posets of size λ which omit (i.e. do not contain) chains of size κ. Let $\mathcal{C}_a(\lambda, \kappa)$ be the set of all posets of size λ which omit antichains of size κ.

In section 3 it is shown to be consistent with ZFC that the complexity of $\mathcal{C}(\lambda, \kappa)$ and $\mathcal{C}_a(\lambda, \kappa)$ is 2^λ where the value of 2^λ can be any μ such that $\mathrm{cf}(\mu) > \lambda$. That is, μ-many posets will be added which do not embed into any other poset in the appropriate set. This will force the value for 2^λ to be μ. The posets built in this construction are well-founded, therefore, it is also consistent that the complexity for well-founded posets of size λ omitting chains or antichains of size κ is 2^λ where $2^\lambda = \mu$ as above.

Note that obtaining maximal weak complexity (i.e. 2^λ for structures of size λ) is optimal. That is, if there does not exist a small weakly universal family, then there certainly does not exist a strongly universal one.

In section 4, the above complexity results are shown to be consistent for all regular λ simultaneously. These use Easton products which destroy the GCH at all regular cardinals but in addition fix the complexity for posets as above.

Komjáth and Shelah in [3] give a similar localised result for graphs of size λ omitting $\aleph_1$-cliques for regular uncountable cardinals λ. Their methods were the basis of our methods for showing consistency results at one cardinal at a time, in particular the method of low sets is taken from [3]. That paper also gives another consistency result where the strong complexity for such graphs can be any $\nu \in [\lambda^+, 2^\lambda)$. The methods used for this result do not seem to apply to give strong complexity result for $\mathcal{C}(\lambda, \kappa)$ because of the difficulties imposed by transitivity.

The techniques in this paper also give a generalised statement for graphs of size λ omitting κ-cliques (for κ, λ as above). That is, we have the following results.

Theorem 1.1. Let $\mathcal{A}$ be the set of all (representatives under isomorphism of) graphs of size λ omitting κ-cliques. Let μ be a cardinal satisfying $\mathrm{cf}(\mu) > \lambda$. Then:

1. (Komjáth -Shelah) for any regular $\nu \in [\lambda^+, \mu]$, there exists a cardinal and cofinality preserving forcing extension such that GCH holds below λ, $2^\lambda = \mu$ and the complexity of $\mathcal{A}$ is ν.
2. there are forcing extensions such that the results above hold for all λ simultaneously.

The proof of (1) is omitted, but the proof of (2) can be found in section 4.

We will consider larger forcing conditions as the stronger ones. That is if p, p' are forcing conditions with $p \geq p'$ then we will say that p is stronger than p'. All the forcings P considered in this paper will have weakest elements, which we will call $\emptyset_P$.

The results in sections 2 and 4 are due to the second and third authors. The results in sections 3 are due to the first and third authors.

2 Weak λ-closure

If a forcing notion has the property that one can guarantee an upper bound to any increasing sequence of limit length $< \lambda$ by making careful choices of upper bounds at earlier limit stages, then we say that the forcing is weakly λ-closed. More precisely:

Definition 2.1. A forcing notion P is *weakly λ-closed*, if and only if there exists a function $F : P^{<\lambda} \to P$ such that for all limit $\tau < \lambda$ if $\langle p_\alpha : \alpha < \tau \rangle$ is a non-decreasing chain and satisfies the following:

$$\text{for all limit } \tau' < \tau \text{ we have } p_{\tau'} \geq F(\langle p_\alpha : \alpha < \tau' \rangle)$$

then the sequence $\langle p_\alpha : \alpha < \tau \rangle$ has an upper bound.

The notion of weak λ-closure is somewhat stronger than the familiar notion of strategic λ-closure. The two notions have similar properties, but the former is easier to work with in the context of products.

Fact 2.2.

1. If a poset is λ-closed then it is weakly λ-closed.
2. Weak λ-closure implies λ-distributivity. This is because a nondecreasing sequence of conditions can be built to meet any $< \lambda$ many dense subsets.
3. If $\theta \geq \lambda$ then weak θ-closure implies weak λ-closure.
4. In the definition of weak λ-closure, we may assume that if there is $\alpha < \tau$ such that $\alpha < \beta < \tau$ implies $p_\alpha = p_\beta$ then $F(\langle p_\alpha : \alpha < \tau' \rangle) = p_\alpha$ for all limit $\tau' < \tau$.

Claim 2.3. If P is a product forcing notion such that

$$P = \prod_{<\delta-\text{support}} \langle Q_i : i < \delta \rangle$$

for some regular $\delta \geq \lambda$ and Q_i is weakly λ-closed for all $i < \delta$, then P is weakly λ-closed.

Proof A condition $p_\beta \in P$ can be written as $\langle q^i_\beta : i < \delta \rangle$ where $q^i_\beta \in Q_i$ for all $i < \delta$. To witness the weak λ-closure of Q_i, there exists an appropriate $F_i : Q_i^{<\lambda} \to Q_i$. Let $F_P(\langle p_\beta : \beta < \tau \rangle) = \langle F_0(\langle q^0_\beta : \beta < \tau \rangle), F_1(\langle q^1_\beta : \beta < \tau \rangle) ... \rangle$. By Fact 2.2(4), this does not violate $< \delta$-support. $\square$

Claim 2.4. If $P = \langle (P_\alpha, \tilde{Q}_\alpha) : \alpha \in \text{Ord} \rangle$ is an Easton support iteration and $\tilde{Q}_\lambda$ is weakly λ-closed for all regular λ, then for regular λ, we have that $P(\geq \lambda)$ is weakly λ-closed.

Proof This is similar to the product, we must compose the functions from the Q_α's together. Let λ be a regular cardinal. We can write $p_\beta \in P(\geq \lambda)$ as $p_\beta = \langle p^i_\beta : i \in \mathrm{Ord}\rangle$ where each $p^i_\beta \in \underset{\sim}{Q}_i$. Let $F_P(\langle p_\beta : \beta < \tau\rangle) = \langle F_{\underset{\sim}{Q}_0}(\langle p^0_\beta : \beta < \tau\rangle), F_{\underset{\sim}{Q}_1}(\langle p^1_\beta : \beta < \tau\rangle)...\rangle$, again using Fact 2.2(4) to ensure Easton support. $\square$

Claim 2.5. If $P = \langle (P_\alpha, \underset{\sim}{Q}_\alpha) : \alpha < \mu\rangle$ is an iteration with $< \delta$-support for some regular cardinal $\delta \geq \lambda$ and cardinal μ and Q_α is weakly λ-closed for all $\alpha < \mu$, then P_α is weakly λ-closed.

Proof Similar to the proof of 2.4. $\square$

3 High complexity

In this section, we will show that the weak (and hence strong) complexity of $\mathcal{C}(\lambda, \kappa)$ can be 2^λ whose value will be forced to be μ for some $\mu > \lambda$.

Theorem 3.1. Assume that $M \models V = L$ and $\mu, \lambda, \kappa > \aleph_0$ are cardinals such that $\mathrm{cf}(\mu) > \lambda = \mathrm{cf}(\lambda)$ and $\kappa \leq \lambda$ is regular. Then in a cardinal and cofinality preserving forcing extension M^P, GCH holds below λ and the weak (and hence strong) complexity of $\mathcal{C}(\lambda, \kappa)$ is $2^\lambda = \mu$.

Remark 3.2. We actually do not need the full strength of $V = L$, but only GCH and $\square_\theta$ when $\lambda = \theta^+$ and $\mathrm{cf}(\theta) < \kappa$.

Proof First we will define and prove some lemmas about certain small subsets of λ in the ground model, called low subsets. We will use these facts in the definition of our forcing extension in order to prove that the extension preserves cofinalities.

When defining and working directly with these low sets, we will need three cases.

1. $\lambda = \kappa$
2. $\lambda > \kappa$, $\lambda = \theta^+$ and $\mathrm{cf}(\theta) < \kappa$

3. $\lambda > \kappa$ and case 2 does not apply.

For case 1 and case 3 the low subsets of λ are the nonempty bounded subsets of λ of size $< \kappa$ and of limit order type. For case 2 we will define low sets using $\Box_\theta$:

Assume we are in case 2. Jensen has shown in [2] that, a $\Box_\theta$-sequence exists in M. Fix a $\Box_\theta$-sequence $\langle C_\delta : \delta < \lambda,\ \delta \text{ limit}\rangle$ with the following properties:

1. $C_\delta \subseteq \delta$ is a club of δ,
2. if γ is a limit point of C_δ, then $C_\gamma = \gamma \cap C_\delta$,
3. $|C_\delta| < \theta$.

Let the increasing enumeration of each C_δ be $\langle c^\delta_\xi : \xi < \text{type}(C_\delta)\rangle$. Also, fix a sequence of cardinals $\{\theta_\gamma : \gamma < \text{cf}(\theta)\}$ strictly increasing to θ and an injective function $\phi_{\alpha,\beta} : [\alpha, \beta) \to \theta$ for each $\alpha < \beta < \lambda$.

For case 2, the low subsets of λ are nonempty sets $A \subset \lambda$ with the following properties:

1. $|A| < \kappa$ with limit order type
2. Let $\delta = \sup(A)$. There exists a $\gamma < \text{cf}(\theta)$ such that for all $\xi < \text{type}(C_\delta)$ and for all $a \in A$ with $c^\delta_\xi \leq a < c^\delta_{\xi+1}$, we have $\phi_{c^\delta_\xi, c^\delta_{\xi+1}}(a) < \theta_\gamma$.

We say that A is *witnessed by* γ.

The next two lemmas are the reasons that we needed the existence of the $\Box_\theta$-sequence. This method allows us to only have to deal with a small number of bounded subsets of λ. Consider in case 2 if we had simply taken all bounded subsets of λ of size $< \kappa$ of limit type as low. Then the number of low subsets of λ would be $\theta^{<\kappa} \geq \theta^{\text{cf}(\theta)} > \theta$ by König's Lemma, so Lemma 3.3 would be contradicted when $\alpha = \theta$ and $\lambda = \theta^+$.

Also, the coherence of the $\Box_\theta$-sequence will help us to shrink subsets of λ to a subset whose intersection with each of its limit point is low. This will be important as the low sets will be used to control the growth of chains when building sequences of increasing posets.

Lemma 3.3. Fix an ordinal $\alpha < \lambda$. For $\lambda > \kappa$, the number of low subsets of λ which are subsets of α is less than λ.

Proof For case 2, let $\lambda = \theta^+$ and let $\mathcal{A}_\gamma$ be the set of low subsets of λ contained in α which are witnessed by γ. The number of low subsets of λ contained in α is then $\sup\{|\mathcal{A}_\gamma| : \gamma < \mathrm{cf}(\theta)\}$. We will fix a $\gamma < \mathrm{cf}(\theta)$ and determine the size of $\mathcal{A}_\gamma$.

Let A be a set of size $< \kappa$ of limit type with $\delta = \sup(A)$ such that $\delta \leq \alpha$ and γ witnesses that A is low. The number of possibilities for A is the number of possible subsets of $\phi^{-1}_{c^\delta_\xi, c^\delta_{\xi+1}}(\theta_\gamma)$ of size $< \kappa$ times the number of intervals $[c^\delta_\xi, c^\delta_{\xi+1})$. We will calculate these values below.

For each $\xi < \mathrm{type}(C_\delta)$, let $X_\xi = \phi^{-1}_{c^\delta_\xi, c^\delta_{\xi+1}}[\theta_\gamma]$. Since $\phi_{c^\delta_\xi, c^\delta_{\xi+1}}$ is injective, we have $|X_\xi| \leq \theta_\gamma$ possibilities for $a \in X_\xi$. A low subset is a subset of $\bigcup_{\xi < \mathrm{type}(C_\delta)} Y_\xi$ of size $< \kappa$, where each $Y_\xi \subseteq X_\xi$. For each ξ, X_ξ has at most $2^{\theta_\gamma} \leq \theta_{\gamma+1}$ subsets, as GCH holds. Since $|C_\delta| < \theta$, certainly $\mathrm{type}(C_\delta) < \theta$ so there at most $\theta_{\gamma+1} \cdot \mathrm{type}(C_\delta) < \theta$ sets in $\mathcal{A}_\gamma$.

Hence, over all $\gamma < \mathrm{cf}(\theta)$, there are at most θ low subsets of α, which is less than λ.

One can see that for case 3, the number of low subsets of any $\alpha < \lambda$ is less than λ as the number of subsets of α of size $< \kappa$ of limit type is $|\alpha|^{<\kappa} < \lambda$. $\square$

Lemma 3.4. If $\lambda > \kappa$ and $B \subseteq \lambda$ has order type κ, then there exists a cofinal subset $B' \subseteq B$ such that if $\gamma < \sup(B')$ is a limit point of B' then $B' \cap \gamma$ is low.

Proof For case 2, let λ and B be as above and let $\delta = \sup(B)$. Let $\langle C_\delta : \delta < \lambda \text{ limit}\rangle$ be defined as above. Choose a cofinal subset $B' \subseteq B$ such that if $a, b \in B'$ then there exists c^δ_ξ such that $a \leq c^\delta_\xi < b$. Such a cofinal sequence exists as both B and C_δ are unbounded in δ. For all ξ such that there is $b \in B'$ with $c^\delta_\xi \leq b < c^\delta_{\xi+1}$, there is some $\alpha_\xi < \mathrm{cf}(\theta)$ such that

$\phi_{c^\delta_\xi, c^\delta_{\xi+1}}(b) < \theta_{\alpha_\xi}$, since $|B'| \le \kappa < \mathrm{cf}(\theta)$. Hence there is $\alpha < \mathrm{cf}(\theta)$ such that for all $b \in B'$ where $c^\delta_\xi \le b < c^\delta_{\xi+1}$, we have $\phi_{c^\delta_\xi, c^\delta_{\xi+1}}(b) < \theta_\alpha$.

Since B' is cofinal in B, it is clear that $\mathrm{type}(B')$ is a limit, in fact, the order type of B' is κ because that is the order type of B. Because of the cofinality of B' in B, we also know that $\sup(B') = \sup(B) = \delta$. If $\gamma < \delta$ is a limit point of B' then $|B' \cap \gamma| < \kappa$ (as the order type of B' is κ). Since γ is also a limit point for C_δ, the coherence property of $\square_\theta$ holds, namely $C_\gamma = \gamma \cap C_\delta$ and thus α witnesses that $B' \cap \gamma$ is low.

For case 3, let $B' = B$. If γ is a limit point of B', then $B' \cap \gamma$ is a subset with size $< \kappa$ of limit type. $\square$

We shall define our forcing notion $P = P(\lambda, \kappa, \mu)$ to be the $(< \lambda)$-support product of μ copies of $Q = Q(\lambda, \kappa)$, a forcing notion that will be defined below.

Let the elements of Q be $q = (\delta, X, \mathcal{A})$, where

1. $\delta < \lambda$ is an ordinal,
2. X is a poset $(X, \le_X)$ with domain δ which omits κ-chains and in which $\alpha \le_X \beta$ implies that $\alpha \le \beta$,
3. the conditions on $\mathcal{A}$ are as follows:
   (a) $\mathcal{A}$ is a family of low subsets A of λ such that $A \subseteq \delta$ and the size of $\mathcal{A}$ is $< \lambda$,
   (b) if $A \in \mathcal{A}$ is such that $\sup(A) \le x < \delta$, then $A \not\le_X x$.

We will assign the ordering of Q to be as follows. If $q = (\delta, X, \mathcal{A})$ and $q' = (\delta', X', \mathcal{A}')$ are in Q, then $q' \ge_Q q$ (i.e. q' is stronger than q) if and only if $\delta' \ge \delta$, with X a subposet of X' such that $X = X' \restriction \delta$ and $\mathcal{A} = \mathcal{A}' \cap [\delta]^{<\kappa}$. In addition, if $A \in \mathcal{A}$ and $a \in A$ then $a \not\le_{X'} \delta$.

Let us observe that this ordering is transitive: if $p_l = (\delta_l, X_l, \mathcal{A}_l)$ for $l < 3$ and $p_0 \le p_1, p_1 \le p_2$ hold, then to see that $p_0 \le p_2$ we have to see that for every $A \in \mathcal{A}_0$ and $a \in A$ we have $a \not\le_{X_2} \delta_0$. Since $p_0 \le p_1$ we have $a \not\le_{X_1} \delta_0$ but since $X_2 \restriction \delta_1 = X_1$ this implies $a \not\le_{X_2} \delta_0$.

Next we need to show that forcing with P preserves cardinalities and cofinalities. We start by defining upper bounds for sequences in Q.

Definition 3.5. Given $i^* < \lambda$ and $\bar{q} := \langle q_i = (\delta_i, X_i, \mathcal{A}_i) : i < i^* \rangle$ increasing in Q, we will define the *canonical upper bound* for $\bar{q}$ or $\mathrm{cub}(\bar{q})$ such that $\mathrm{cub}(\bar{q}) \geq q_i$ for all $i < i^*$. If i^* is a successor ordinal $j+1$, then let $\mathrm{cub}(\bar{q}) = q_j$. Assume now that i^* is a limit ordinal.

For $\lambda = \kappa$, let $\mathrm{cub}(\bar{q}) = (\delta, X, \mathcal{A})$ be such that $\delta = \sup\{\delta_i : i < i^*\}$ and let $X, \mathcal{A}$ be the union over all $i < i^*$ of their respective parts in q_i, (e.g. $X = \bigcup_{i<i^*} X_i$). No extra relations are added to X.

For $\lambda > \kappa$, let $\mathrm{cub}(\bar{q}) = (\delta, X, \mathcal{A})$ be such that δ, X are the union over all $i < i^*$ of their respective parts in q_i and let $\mathcal{A}$ be as follows. If $\mathrm{cf}(i^*) \geq \kappa$, take $\mathcal{A} = \bigcup_{i<i^*} \mathcal{A}_i$, otherwise we also include all low subsets of λ cofinal in δ.

We will show that the cub of an increasing sequence of elements of Q is in fact a condition in Q.

Claim 3.6. If $\alpha^* < \lambda$ and $\bar{q} := \langle q_\alpha = (\delta_\alpha, X_\alpha, \mathcal{A}_\alpha) : \alpha < \alpha^* \rangle$ is a sequence in Q such that for all limit $\alpha < \alpha^*$ we have q_α is the cub of $\langle q_\beta : \beta < \alpha \rangle$, then $\mathrm{cub}(\bar{q}) = (\delta, X, \mathcal{A}) \in Q$ is an upper bound of the sequence $\bar{q}$.

Proof The proof is by induction on α^*, assuming that the statement holds for all sequences of length less than α^*. In particular we may assume that $\bar{q}$ is increasing in Q.

If α^* is a successor ordinal then this is trivial, hence we shall now assume that α^* is a limit.

One can see that for all $\alpha < \alpha^*$ we have $\delta_\alpha < \lambda$ and X_α has domain δ_α, hence by the regularity of λ it follows that $\delta < \lambda$, and it is also clear that the domain of X is δ. We know that $\mathcal{A}$ is a family of low subsets of λ because all the sets that comprise each $\mathcal{A}_\alpha$ are low. If $\mathrm{cf}(\alpha^*) \geq \kappa$ then $|\mathcal{A}| < \lambda$ because $\alpha^* < \lambda$ and $|\mathcal{A}_\alpha| < \lambda$ for all $\alpha < \alpha^*$ as $q_\alpha \in Q$. If $\mathrm{cf}(\alpha^*) < \kappa$ then there are $< \lambda$ low sets cofinal in δ by Lemma 3.3, so we still have $|\mathcal{A}| < \lambda$.

For requirement 3b, if $\mathrm{cf}(\alpha^*) < \kappa$ and $A \in \mathcal{A}$, then either A is cofinal in δ and so $\sup(A) = \delta$ or A is also in $\mathcal{A}_\alpha$ for some $\alpha < \alpha^*$. Thus, in the former case there is no x such that $\sup(A) \leq x < \delta$, so that the requirement is trivially met. In the latter case the same can be seen by observing that for all $\beta' \in [\beta, \alpha^*)$ we have $A \in \mathcal{A}_{\beta'}$ and using that $\delta = \sup\{\delta_\alpha : \alpha < \alpha^*\}$. If $\mathrm{cf}(\alpha^*) \geq \kappa$, then for each $A \in \mathcal{A}$ there exists an $\alpha < \alpha^*$ such that $A \in \mathcal{A}_\alpha$ so the requirement is met similarly as above.

The only remaining difficulty that might arise is if we added a κ-chain in some poset X_α. Let $\alpha \leq \alpha^*$ be the minimal ordinal such that there exists a κ-chain T of X in δ_α. Note that the domain of T is necessarily cofinal in δ_α. By shrinking T if necessary we may assume that the order type of it is exactly κ.

We must have that $\mathrm{cf}(\alpha) = \kappa$ as this is the only point where such a chain could be added. If $\lambda = \kappa$, then this cannot occur as $\alpha^* < \lambda$. Assuming then that $\lambda > \kappa$, by Lemma 3.4, there exists a cofinal $T' \subseteq T$ such that if $\gamma < \delta_\alpha$ is a limit point of T', then $T' \cap \gamma$ is low. There exists a limit ordinal $\beta < \alpha$ such that δ_β is a limit point of T' and hence $\mathrm{cf}(\beta) \neq \kappa$ (since the order type of T' is κ). Such a β can be found as both the set of limit points of an unbounded subset of δ_α (namely T') and the set of $\{\delta_\beta : \beta < \alpha\}$ form clubs in δ_α, by the assumption on $\bar{q}$. By our construction $T' \cap \delta_\beta \in \mathcal{A}_\beta$ because it is low, so $T' \cap \delta_\beta$ could not have been extended to a κ-chain later by requirement 3b on q'_β for $\beta' > \beta$.

We still need to show that $q_\alpha \leq \mathrm{cub}(\bar{\mathrm{q}})$ for $\alpha < \alpha^*$. Note that when $\mathrm{cf}(\alpha^*) \geq \kappa$ we have $\mathcal{A} \cap [\delta_\alpha]^{<\kappa} = \mathcal{A}_\alpha$ because $[\delta_\alpha]^{<\kappa}$ does not include any Y cofinal in δ. The rest is clear from the definition of $\mathrm{cub}(\bar{q})$. $\square$

Lemma 3.7. *Q is weakly λ-closed.*

Proof The function F witnessing the weak closure will be the cub operator defined above. Suppose that we are given a sequence $\bar{q} = \langle q_\alpha = (\delta_\alpha, X_\alpha, \mathcal{A}_\alpha) : \alpha < \alpha^* \rangle$ in Q for some $\alpha^* < \lambda$, and this sequence satisfies that for all limit $\tau < \alpha^*$ we have $q_\tau \geq \mathrm{cub}(\langle q_\alpha : \alpha < \tau \rangle)$. If α^* is a successor ordinal $\alpha + 1$ then clearly q_α is an upper bound of the sequence, so let us assume that α^* is a limit. We can then define a sequence $\bar{q}'$ of length α^* such that for $q'_0 = q_0$ and $q_{\alpha+1} = q_\alpha$, while for α limit we have q'_α is the cub of $\langle q'_\beta : \beta < \alpha \rangle$. Then $\bar{q}$ is cofinal in $\bar{q'}$ and any upper bound for $\bar{q'}$ is an upper bound for $\bar{q}$. The existence of such an upper bound follows by Claim 3.6. $\square$

Note that by Claim 2.3, the forcing P is also weakly λ-closed.

Lemma 3.8. The forcing notion P is λ^+-cc.

Proof Suppose $\{p_\alpha : \alpha < \lambda^+\}$ is a sequence in P. We know that λ is regular and for all $\alpha < \lambda^+$ we have $|\alpha^{<\lambda}| < \lambda^+$. Also, for all $p \in P$, we know that $|\mathrm{ran}(p)| < \lambda$ since p has $< \lambda$ non-trivial elements. So, by the Δ-System Lemma, there exists an $X \subseteq \lambda^+$ such that $|X| = \lambda^+$ and $\{\mathrm{Dom}(p_\alpha) : \alpha \in X\}$ forms a Δ-system with some root r. Since $|Q| = \lambda$ and $\lambda^{<\lambda} = \lambda$, there are fewer than λ^+ possibilities for $p_\alpha \restriction r$ so there exists a $Y \subseteq X$ with $|Y| = \lambda^+$ such that for all $\alpha, \beta \in Y$, we have $p_\alpha \restriction r = p_\beta \restriction r$. Now for all $\alpha \in Y$, the p_α's are compatible, which shows that the original sequence could not have formed an antichain. $\square$

Let G_P be P-generic over M. If in $M[G_P]$ the complexity of $\mathcal{C}(\lambda, \kappa)$ is $< \mu$, then there exists a family $\mathcal{F}$ of posets that witnesses this. This subfamily is added by a $< \mu$ sized sub-product of P, call it R. (Note that R may be a cofinal sub-product of P in case μ is singular.) By the product lemma (see e.g. [4]), if we let G_R and $G_{P/R}$ be the generics for R and P/R respectively, then $M[G_R][G_{P/R}] = M[G_{P/R}][G_R] = M[G_P]$. It is enough to show that forcing with Q over $M[G_R]$ introduces a poset Y that cannot be embedded into any poset in $M[G_R]$ of size λ which omits κ-chains. Then we can take one copy of Q from P/R and we shall have in $M[G_P]$ that $\mathcal{F}$ is not a universal family. So let us show the required property of Q.

If $G \subseteq Q$ is generic for Q then let $Y = Y_G = \bigcup\{X : (\delta, X, \mathcal{A}) \in G\}$ be such that for all $x, y \in Y$ we have $x \leq_Y y$ if and only if for some X with $(\delta, X, \mathcal{A}) \in G$ we have $x \leq_X y$.

Lemma 3.9. For any generic G, the set $Y = Y_G$ as defined above is a poset of size λ which omits κ-chains.

Proof We start with a density argument to show that Y has size λ.

Claim 3.10. For every $\gamma < \lambda$ the set $D_\gamma = \{(\delta, X, \mathcal{A}) \in Q : \delta > \gamma\}$ is dense in Q.

Proof Given a condition $q = (\delta, X, \mathcal{A}) \in Q$, suppose that $\delta \leq \gamma$. Let Z be the poset extending X whose universe is $\gamma + 1$ such that for all ε in $[\delta, \gamma + 1)$ we have that ε is not comparable to any other element of Z.

Then Z is a poset which omits κ-chains. We wish to show that $q' = (\gamma + 1, Z, \mathcal{A})$ is a condition in D_γ and that it extends q.

If $A \in \mathcal{A}$ satisfies $\sup(A) \leq x \leq \gamma$ for some $x \in Z$ then either $x \in X$ so $A \not\leq_Z x$ or $x > \delta$ and again $A \not\leq_Z x$. Also, δ is not comparable with any other element of Z so taken together these observations show that q' is as required. $\square$

By the claim above, we know for any generic G that Y has size λ. It is also a poset by genericity. Now we will show that Y omits κ-chains.

If $\lambda = \kappa$, we will assume that $q \in Q$ and $\underset{\sim}{T}$ are such that $q \Vdash$ "$\underset{\sim}{T}$ is an increasing κ-chain in Y". We will choose an increasing sequence $\langle q_n : n < \omega \rangle$ such that $q_0 = q$ and let $q_{n+1} = (\delta_{n+1}, X_{n+1}, \mathcal{A}_{n+1})$ be such that $q_{n+1} \Vdash$ "$t_n \in \underset{\sim}{T}$ and $\delta_n < t_n < \delta_{n+1}$". This can be done because each $r = (\delta, X, \mathcal{A}) \in Q$ satisfies $r \Vdash$ "$\underset{\sim}{Y} \restriction \delta = X$". Hence, if it is also the case that $r \geq q$, then $r \Vdash$ "$\underset{\sim}{T}$ is a κ-chain in $\underset{\sim}{Y}$" and so $r \Vdash$ "$(\exists \alpha > \delta) \alpha \in \underset{\sim}{T}$". Applying this to $r = q_n$, we may find $q_{n'} \geq q_n$ and $t_n > \delta_n$ such that $q_{n'} \Vdash$ "$t_n \in \underset{\sim}{T}$". Then by extending $q_{n'}$ to q_{n+1} if necessary, we may by Claim 3.10 assume that $\delta_{n+1} > t_n$.

Let $\bar{t} = \{t_n : n < \omega\}$. Now let $q' = (\delta, X, \mathcal{A})$ be such that $\delta = \lim\{\delta_n : n < \omega\}$ with $X = \bigcup\{X_n : n < \omega\}$ and $\mathcal{A} = \bigcup\{\mathcal{A}_n : n < \omega\} \cup \{\bar{t}\}$. As $\sup\{t_n : n < \omega\}$ $= \sup\{\delta_n : n < \omega\}$ we know that $\sup(\bar{t}) = \delta$. Therefore, there does not exist x such that $\sup(\bar{t}) \leq_X x < \delta$ and thus requirement 3b holds for $\bar{t} \in \mathcal{A}$. The rest of the requirements hold by similar arguments to those in Claim 3.6. Thus we have that $q' \in Q$.

We claim that $q' \Vdash$ "$\underset{\sim}{T} \subseteq \delta$". This is equivalent to the statement that for all $r \geq q'$ with $r \in Q$, we have that r does not force "$T \not\subseteq \delta$". We will suppose otherwise and arrive at a contradiction.

Let $r = (\xi, Z, \mathcal{B})$ with $r \geq q'$ and let $\beta > \delta$ be such that $r \Vdash$ "$\beta \in \underset{\sim}{T}$". Without loss of generality $\xi > \beta$. We know that $\bar{t} \in \mathcal{A}$ so it is also the case that $\bar{t} \in \mathcal{B}$. Since $\delta < \beta$, we also must have that $\bar{t} \not\leq_Z \beta$ by requirement 3b of $\mathcal{B}$ for $r \in Q$. However, since r forces $\beta \in \underset{\sim}{T}$, r must force that $\bar{t} \leq_T \beta$ and $Z \subseteq Y$.

So we proved that $q' \Vdash$ "$\underset{\sim}{T} \subseteq \delta$". (Note that this does not imply that q' forces T to be countable as the elements of $\bar{t}$ are not necessarily the only elements of T). Since $\delta < \lambda = \kappa$, we proved that q' forces that $\underset{\sim}{T}$ has size $< \kappa$, which is a contradiction.

If $\lambda > \kappa$, then Q does not add any new κ-sequences since it is weakly λ-closed. Therefore, Q could not have added a κ-chain. $\square$

Lemma 3.11. The poset Y does not embed into any poset in the ground model which has size λ and omits κ-chains.

Proof Suppose that there exists $q \in Q$ such that $q \Vdash$ "$\underset{\sim}{f} : \underset{\sim}{Y} \to Z$ is an embedding" where Z in the ground model is a poset of size λ which omits κ-chains".

By induction on $\alpha < \kappa$, we will construct an increasing sequence $\langle q_\alpha = (\delta_\alpha, X_\alpha, \mathcal{A}_\alpha) : \alpha < \kappa \rangle$ and a function g as follows. Let $q_0 = q$ and let $q_{\alpha+1} \Vdash$ "$\underset{\sim}{f}(\delta_\alpha) = g(\alpha)$". For α a limit ordinal, define an intermediate condition $q'_\alpha = (\delta'_\alpha = \sup_{\beta<\alpha} \delta_\beta, X'_\alpha = \bigcup_{\beta<\alpha} X_\beta, \mathcal{A}'_\alpha = \bigcup_{\beta<\alpha} \mathcal{A}_\beta)$. It is easily seen that $q_\beta \leq q'_\alpha$ for all $\beta < \alpha$. Let $q_\alpha = (\delta_\alpha, X_\alpha, \mathcal{A}_\alpha)$ such that $\delta_\alpha = \delta'_\alpha + 1$ and $\mathcal{A}_\alpha = \mathcal{A}'_\alpha$. In X_α let $\delta_\beta \leq_{X_\alpha} \delta'_\alpha$ for $\beta < \alpha$, let $\delta'_\beta \leq_{X_\alpha} \delta'_\alpha$ for all $\beta < \alpha$ limit, let X_α inherit all the relations from X'_α, and finally take the transitive closure. Note that this does not affect X'_α as we are only adding pairs (x, y) in which $y = \delta'_\alpha$ and $\delta'_\alpha \notin X'_\alpha \cup \bigcup \mathcal{A}'_\alpha$.

We must see that q_α is a condition. The proof is by induction on the limit α.

It is easy to see that all the requirements for q_α to be in Q are met, except for requirement 3b which requires some argument.

Requirement 3b can fail for q_α if there exists $A \in \mathcal{A}_\alpha$ with $A \subseteq \delta_\alpha$ and $\sup(A) \leq x \leq \delta'_\alpha$ for some $x \in X_\alpha$ and $A \leq_{X_\alpha} x$. This breaks into three cases.

We cannot have that $\sup(A) \geq \delta'_\alpha$ as by the definition of q'_α for α limit, there are no sets $A \in \mathcal{A}_\alpha$ cofinal in δ'_α.

If $\sup(A) \leq x < \delta'_\alpha$ then there exists $\beta < \alpha$ such that $A \in \mathcal{A}_\beta$ and $x \in X_\beta$ as α is a limit. However, this contradicts q_β being a condition.

The last case to check is if $\sup(A) < \delta'_\alpha$ and $x = \delta'_\alpha$. We will assume this is the case and arrive at a contradiction. Nothing in X_α is connected to x

except $\{\delta_\beta : \beta < \alpha\}$, $\{\delta'_\beta : \beta < \alpha \text{ limit}\}$ and points below the δ_β's which we had to connect by transitivity. If $A \leq_{X_\alpha} x$ then for every $a \in A$ there must be $\delta_{\beta_a} = \delta_\beta$ such that $a \leq_{X_\alpha} \delta_\beta$, for some $\beta < \alpha$. On the other hand, since α is a limit ordinal and $\sup(A) < \delta'_\alpha$, then by the definition of $\mathcal{A}_\alpha$ there exists $\xi < \alpha$ such that $A \in \mathcal{A}_\xi$ and hence $\sup(A) \leq \delta_\xi$. Let ξ be the first such ordinal. By the fact that $\langle q_\beta : \beta < \alpha \rangle$ is increasing and by the definition of $\leq_Q$, we must have that for every $a \in A$ the inequality $\beta_a < \xi$ holds. The first possibility is that $\sup_{a \in A} \beta_a = \xi$. If this is the case then $\xi < \alpha$ is a limit. Then by the definition of q_ξ we have that for all $a \in A$, $\delta_{\beta_a} \leq_{X_\xi} \delta'_\xi$ and in particular $A \leq_{X_\xi} \delta'_\xi$. This contradicts the inductive hypothesis that q_ξ is a condition. Hence we must have $\beta \stackrel{\text{def}}{=} \sup_{a \in A} \beta_a < \xi$. Since for every $a \in A$ we have $a \leq_{X_\alpha} \delta_{\beta_a}$ then in particular $a \leq \delta_{\beta_a}$ as ordinals, and hence $\sup(A) \leq \delta_\beta$. However, $\mathcal{A}_\xi \cap [\delta_\beta]^{<\kappa} = \mathcal{A}_\beta$, so $A \in \mathcal{A}_\beta$, contradicting the choice of ξ.

Now that we have shown that q_α is a condition it is also easy to show that $q_\beta \leq_Q q_\alpha$ for all $\beta < \alpha$. Hence we can define the sequence $\langle q_\alpha : \alpha < \kappa \rangle$, but then $\{g(\alpha) : \alpha < \kappa \text{ limit}\}$ will be a κ-chain in Z, a contradiction. $\square$

We have now exhibited a poset of size λ which omits κ-chains in the extension and which does not embed into any such poset in $M[G_R]$. This shows that the complexity of $\mathcal{C}(\lambda, \kappa)$ is at least μ. It now only remains to show that $2^\lambda = \mu$ in M^P since 2^λ is the maximal complexity of a set of structures of size λ.

Each poset Y is a subset of λ and we introduced μ many of them. Hence in the extension $2^\lambda \geq \mu$.

Every subset of λ in M^P has a nice name, that is a name of the form $\{\langle \alpha, K_\alpha \rangle : \alpha < \lambda\}$ where each K_α is an antichain of P. By definition $|P| = \mu$ and we have shown that P is λ^+-cc, so it has at most $\mu^\lambda = \mu$ antichains. Hence, there are at most μ nice names for subsets of λ and thus $2^\lambda \leq \mu$. $\square$

Theorem 3.12. Assume that $M \models V = L$ and $\mu, \lambda, \kappa > \aleph_0$ are cardinals such that $\mathrm{cf}(\mu) > \lambda = \mathrm{cf}(\lambda)$ and $\kappa \leq \lambda$ regular. Then in a cofinality preserving forcing extension M^P, GCH holds below λ and the weak (and hence strong) complexity of $\mathcal{C}_a(\lambda, \kappa)$ is $2^\lambda = \mu$.

Proof The proof of this theorem is essentially the same as the one above except for one thing: Condition 3b in the definition of Q needs to be changed. The new version should read:

3b′. if $A \in \mathcal{A}$ and $\sup(A) \le x < \delta$ then A is comparable to x.

Then in the lemmas (in particular Lemmas 3.7, 3.9 and 3.11 and Claim 3.10), replace $A \le x$ with $A >< x$ and $A \not\le x$ with "A is comparable to x". $\square$

4 Global results

Theorem 4.1. Assume $V = L$. To each uncountable regular cardinal λ definably associate a regular uncountable cardinal $\kappa(\lambda) \le \lambda$ and a cardinal $F(\lambda)$ such that $\mathrm{cf}(F(\lambda)) > \lambda$ and $\lambda < \theta$ implies $F(\lambda) \le F(\theta)$. Then there exists a definable, ZFC-preserving and cofinality preserving class forcing notion P such that in L^P the complexity of $\mathcal{C}(\lambda, \kappa(\lambda))$ is $F(\lambda) = 2^\lambda$ for each regular uncountable λ.

Remark 4.2. As before, we do not need the full strength of $V = L$, but only GCH and $\square_\theta$ when $\lambda = \theta^+$ and $\mathrm{cf}(\theta) < \kappa(\lambda)$.

Proof We shall define a forcing notion $P(\lambda, F(\lambda), \kappa(\lambda))$ to be the $(< \lambda)$-support product of $F(\lambda)$ copies of $Q(\lambda, \kappa(\lambda))$, forcing notions that were defined in section 3. The relevant properties of these forcings are summarised below.

Fact 4.3.

1. The cardinality of $Q(\lambda, \kappa(\lambda))$ is λ.
2. If $\lambda = \kappa(\lambda)$, then $Q(\lambda, \kappa(\lambda))$ is $(< \kappa(\lambda))$-closed. If $\lambda > \kappa(\lambda)$, then forcing with $Q(\lambda, \kappa(\lambda))$ is weakly $\kappa(\lambda)$-closed. Similar statements hold for $P(\lambda, F(\lambda), \kappa(\lambda))$.
3. The forcing notion $P(\lambda, F(\lambda), \kappa(\lambda))$ is λ^+-cc.

P will be a product forcing notion of all $P(\lambda, F(\lambda), \kappa(\lambda))$ made with Easton support. Recall that Easton support means that the support is bounded at inaccessibles. We use the notation $P(< \lambda)$ to denote the part of the product made over $\theta < \lambda$ and $P(\geq \lambda)$ for the remainder of the product. Similarly, we use $G(< \lambda)$ and $G(\geq \lambda)$ for the corresponding generics.

Lemma 4.4. *$P(\geq \lambda)$ is weakly λ-closed.*

Proof Holds by Claim 2.4. $\square$

Lemma 4.5. *If λ is Mahlo or the successor of a regular cardinal, then $P(< \lambda)$ is λ-cc.*

Proof Works by standard Δ-system arguments. See Jech [1] for details. $\square$

Unfortunately, when λ is the successor of a singular cardinal or a non-Mahlo inaccessible, we only have that $P(< \lambda)$ is λ^+-cc.

Let G be P-generic over L.

Lemma 4.6. *$L[G]$ satisfies that the complexity of posets of size λ omitting $\kappa(\lambda)$ chains is $F(\lambda)$ for all regular uncountable λ.*

Proof Since P is a product, we can split $L[G] = L[G(\geq \lambda)][G(< \lambda)]$. By doing the larger forcing first, nothing is changed below λ. In particular, GCH and square still hold below λ and no new sequences of size $< \lambda$ were added, so forcing over $L[G(\geq \lambda)]$ is the same, from the point of view of the smaller forcing, as forcing over the ground model.

However, once we do the small forcing, we want to see that the complexity calculations achieved by large forcing are not changed by the small forcing. We can do this by anticipating a maximal antichain of conditions in the small forcing when we do the complexity proof for the large forcing.

If $G_Q \subseteq Q(\lambda, \kappa(\lambda))$ is generic for $Q(\lambda, \kappa(\lambda))$ then let $Y = Y(G_Q) = \bigcup\{X : (\delta, X, \mathcal{A}) \in G_Q\}$ be such that for all $x, y \in Y$ we have $x \leq_Y y$ iff for some X with $(\delta, X, \mathcal{A}) \in G_Q$ we have $x \leq_X y$. By Lemma 3.9, we know that for any generic G_Q, the set $Y = Y(G_Q)$ as defined above is a poset of size λ which omits $\kappa(\lambda)$-chains.

We also have that the poset Y does not embed into any ground model poset of size λ which omits $\kappa(\lambda)$-chains. We strengthen this argument below to show that Y does not embed into any poset of size λ which omits $\kappa(\lambda)$-chains and which exists in $L[G(<\lambda)]$. In particular, we will show that if there exists a condition in $P(\lambda, F(\lambda), \kappa(\lambda))$ which forces an embedding from Y into a poset in $L[G(<\lambda)]$, then we can find a sequence of conditions in $P(<\lambda) \times P(\lambda, F(\lambda), \kappa(\lambda))$ which "decides" the embedding.

By the product lemma we can look at conditions from $P(<\lambda) \times Q(\lambda, \kappa(\lambda))$ instead of $P(<\lambda) \times P(\lambda, F(\lambda), \kappa(\lambda))$.

Assume that there exists $q \in Q(\lambda, \kappa(\lambda))$ such that over the ground model $L^{P(<\lambda)}$, $q \Vdash$ "$\underset{\sim}{f} : \underset{\sim}{Y} \to Z$ is an embedding of $\underset{\sim}{Y}$ into some ground model poset Z of size λ which omits $\kappa(\lambda)$-chains".

By induction on $\alpha < \kappa(\lambda)$, we will construct an increasing sequence $\langle q_\alpha = (\delta_\alpha, X_\alpha, \mathcal{A}_\alpha) : \alpha < \kappa(\lambda)\rangle$ which, together with a maximal antichain of conditions from $P(<\lambda)$, "decides" the values of the embedding. We will deal with the cases where λ is Mahlo or the successor of a regular cardinal first, as the maximal antichains of $P(<\lambda)$ have size $<\lambda$.

Let $q_0 = q$. For α limit, let $\delta_\alpha = \sup\{\delta_\beta : \beta < \alpha\}$ with $X_\alpha = \bigcup\{X_\beta : \beta < \alpha\}$ and $\mathcal{A}_\alpha = \bigcup\{\mathcal{A}_\alpha : \beta < \alpha\}$. After each limit stage α, let q'_α be an intermediate condition $(\delta_\alpha + 1, X^+_\alpha, \mathcal{A}_\alpha)$ such that $X^+_\alpha = X_\alpha \cup \{\delta_\alpha\}$ and for $\beta < \alpha$ we have $\delta_\beta \leq_{X^+_\alpha} \delta_\alpha$ and take the transitive closure. Note that this does not affect X_α as we are only adding relations to δ_α and $\delta_\alpha \notin X_\alpha$. Then we will continue constructing $q_{\alpha+1}$ by extending q'_α as below.

For $\alpha+1$, we build the condition $q_{\alpha+1}$ by induction. Choose any $p_0 \in P(<\lambda)$ and $q^0_{\alpha+1}$ such that $(q^0_{\alpha+1}, p_0) \Vdash$"$\underset{\sim}{f}(\delta_\alpha) = g_{p_0}(\alpha)$" for some $g_{p_0}(\alpha) \in Z$. Then choose $p_1 \in P(<\lambda)$ and $q^1_{\alpha+1} \geq q^0_{\alpha+1}$ such that p_1 is incompatible with p_0 and $(q^1_{\alpha+1}, p_1) \Vdash$ "$\underset{\sim}{f}(\delta_\alpha) = g_{p_1}(\alpha)$" for some $g_{p_1}(\alpha) \in Z$. Note that by the choice of q we have that $(q, \emptyset)$ forces that $\underset{\sim}{f}$ is an embedding of the sort required, hence certainly (q^0_α, p_1) forces this. At limit ordinals γ, let $q^\gamma_{\alpha+1}$ be the $\mathrm{cub}(\langle q^\beta_{\alpha+1} : \beta < \gamma\rangle)$.

Continue choosing incompatible elements from $P(<\lambda)$ until a maximal antichain A_α is reached. Let $q_{\alpha+1} = \mathrm{cub}(\langle q^\beta_{\alpha+1} : \beta < \beta^*\rangle)$ for some $\beta^* < \lambda$, which is the length of the maximal antichain chosen.

By Lemma 3.11 all q_α are conditions in $Q(\lambda, \kappa(\lambda))$. Let $G(<\lambda)$ be $P(<\lambda)$-

generic and for each α choose $p_\alpha \in G(<\lambda) \cap A_\alpha$. Then $\langle g_{p_\alpha}(\alpha) : \alpha < \kappa(\lambda)\rangle$ is a $\kappa(\lambda)$-chain through Z in $V[G(<\lambda)]$, a contradiction.

The non-Mahlo inaccessible and successor of a singular cases can be further reduced to two subcases. The first is when $\kappa(\lambda) < \lambda$. When λ is the successor of a singular, we have that $\kappa(\lambda)^{++} < \lambda$ as $\kappa(\lambda)$ is regular and $\lambda = \mu^+$ for some singular limit cardinal μ and thus $\kappa(\lambda)$ must be less than μ. If λ is inaccessible, then it is a limit cardinal, so $\kappa(\lambda)^{++} < \lambda$.

Thus, the forcing can be split somewhere between $\kappa(\lambda)$ and λ, say at $\kappa(\lambda)^{++}$. Since $\kappa(\lambda)^{++}$ is the successor of a regular cardinal, $P(<\kappa(\lambda)^{++})$ is $\kappa(\lambda)^{++}$-cc and $P(\geq \kappa(\lambda)^{++})$ is $\kappa(\lambda)^{+++}$-distributive. We may proceed as above since the conditions q_α only need to be defined for $\alpha < \kappa(\lambda)$.

If λ is the successor of a singular or a non-Mahlo inaccessible and $\kappa(\lambda) = \lambda$, then we cannot cut the forcing below λ and $P(<\lambda)$ only has the λ^+-cc. We would still like to build the conditions q_α together with a maximal antichain on $P(<\lambda)$. The idea here is to dovetail the building of conditions so that at each stage α, the value for $g_p(\alpha)$ is decided for $<\lambda$ many p in a maximal antichain and the rest of the values are decided at later stages. This in itself would be easy enough, but we want this process not only to have size λ, but also order type λ, as we only have weak closure under sequences of length $<\lambda$.

To this end, we take an elementary submodel M of some large $H(\theta) = L_\theta$ which has size λ, includes all relevant information and is closed under $<\lambda$-sequences. Enumerate the elements of M in order type λ. So we can list all the elements of $P(<\lambda) \cap M$ as $\langle p_i : i < \lambda\rangle$.

Assume that there exists $q \in Q(\lambda, \lambda)$ such that $q \Vdash$ "$\underset{\sim}{f} : \underset{\sim}{Y} \to Z$ is an embedding of $\underset{\sim}{Y}$ into some ground model poset Z of size λ which omits λ-chains". We may assume that $q \in M$.

By induction on $\alpha < \lambda$, we will construct an increasing sequence $\langle q_\alpha = (\delta_\alpha, X_\alpha, \mathcal{A}_\alpha) : \alpha < \lambda\rangle$ which, together with a maximal antichain of conditions from $P(<\lambda)$, decides the values of the embedding. The entire construction will take place inside of M. In order to accomplish the dovetailing process, let $\alpha = \langle \alpha_0, \alpha_1\rangle$ be a pairing function on ordinals $<\lambda$.

Let $q_0 = q$ and at limit stages α, build q_α in the same way as in the case where

λ is the successor of a regular cardinal, including building the intermediate condition q'_α.

For $\alpha + 1$, we will build the condition $q^i_{\alpha+1}$ by induction on $i \leq \alpha$, and set $q_{\alpha+1} = q^\alpha_{\alpha+1}$. For $i = 0$ set $q^0_{\alpha+1} = \emptyset$. At successor stages $i + 1$, extend p_i and $q^i_{\alpha+1}$ inside M to p'_{i+1} and $q^{i+1}_{\alpha+1}$ such that $(p'_{i+1}, q^{i+1}_{\alpha+1}) \Vdash$ "$\underset{\sim}{f}(\delta_{\alpha_0}) = g_{p'_{i+1}}(\alpha_0)$" for some $g_{p'_{i+1}}(\alpha_0) \in Z$.

At limit stages i, let $q^i_{\alpha+1}$ be the cub of $\langle q^j_{\alpha+1} : j < i\rangle$.

Now we note that the conditions in M are predense in $P(< \lambda)$: Let $p^* = \langle r_\beta : \beta < \lambda\rangle$ be a condition in $P(< \lambda)$ (not necessarily in M), where we assume $r_\beta \in P(\beta)$. Let $p^* \restriction M = \langle r_\beta \restriction M : \beta \in M \cap \lambda\rangle$ where $r_\beta \restriction M = \langle s_\gamma \restriction M : \gamma \in \mathrm{supp}(r_\beta) \cap M\rangle$ and

$$\begin{aligned} s_\gamma \restriction M &= (\delta_\gamma \cap M, X_\gamma \restriction M, \mathcal{A}_\gamma \restriction M) \\ &= (\delta_\gamma, X_\gamma, \mathcal{A}_\gamma), \end{aligned}$$

well-defined as M is closed under $< \lambda$ sequences. Thus $p^* \restriction M \in P(< \lambda) \cap M$, and clearly $p^* \restriction M$ is compatible with p^*. By the same argument, any predense subset of $P(< \lambda) \cap M$ is predense in $P(< \lambda)$.

Now choose some generic G for $P(< \lambda)$. Let D_{α_0} be the set of all $p' \in M \cap P(< \lambda)$ such that (p', q_α) forces a value for $\underset{\sim}{f}(\delta_{\alpha_0})$ for some $\alpha < \lambda$. Then D_{α_0} is predense in $P(< \lambda)$ for all $\alpha_0 < \lambda$. Thus, for each α_0 we may choose $p'_{\alpha_0} \in G(< \lambda) \cap D_{\alpha_0}$. However, we have $\langle g_{p'_{\alpha_0}}(\alpha_0) : \alpha_0 < \lambda\rangle$ is a λ-chain through Z in $L[G(< \lambda)]$, a contradiction. $\square$

We have now exhibited a poset of size λ which omits $\kappa(\lambda)$-chains in the extension which does not embed into any such poset in the ground model. This shows that the complexity of $\mathcal{C}(\lambda, \kappa(\lambda))$ is $\geq F(\lambda)$. It now remains to be seen that $2^\lambda = F(\lambda)$ in the extension. It was shown in Section 3 that in each $M^{P(\lambda, F(\lambda), \kappa(\lambda))}$ the size of 2^λ is exactly $F(\lambda)$. Since in each forcing $P(\lambda, F(\lambda), \kappa(\lambda))$ no new subsets of λ were added, this fact remains true in M^P. $\square$

We can get a similar global result for the low complexity given in [3]. Even though the forcing at each regular cardinal is an iteration, the global forcing will still be in the form of a product.

Theorem 4.7. Assume GCH. To each uncountable regular cardinal λ definably associate cardinals $F(\lambda), \kappa(\lambda), \nu(\lambda)$ such that $\kappa(\lambda) \leq \lambda$ is regular, $\nu(\lambda) \in [\lambda^+, F(\lambda)]$ and $\mathrm{cf}(F(\lambda)), \mathrm{cf}(\nu(\lambda)) > \lambda$ and $\lambda < \theta$ implies $F(\lambda) \leq F(\theta)$. Then there exists a ZFC-preserving, cofinality preserving class forcing notion P such that in V^P the strong complexity of graphs of size λ which omit cliques of size $\kappa(\lambda)$ is $\nu(\lambda)$ and $F(\lambda) = 2^\lambda$ for each regular uncountable λ.

Proof Let $P(\lambda, \kappa(\lambda), \nu(\lambda), F(\lambda))$ be the iterated forcing notion of length $\nu(\lambda)$ in [3, Theorem 4]. Namely, this is the forcing which adds a universal family of size $\nu(\lambda)$ for graphs of size λ omitting $\kappa(\lambda)$-cliques where 2^λ is forced to be $F(\lambda)$. The forcing notion P will be the Easton support product of P_α for all $\alpha \in \mathrm{Ord}$ such that $P_\alpha = P(\alpha, \kappa(\alpha), \nu(\alpha), F(\alpha))$ if α is a regular uncountable cardinal and trivial otherwise.

Each forcing $P(\lambda, \kappa(\lambda), \nu(\lambda), F(\lambda))$ is weakly λ-closed and satisfies the λ^+-cc.

By Claim 2.4, $P(\geq \lambda)$ is weakly λ-closed for λ regular uncountable. Also, if λ is a successor of a regular or Mahlo then $P(< \lambda)$ has the λ-cc.

The only thing left to check is that we have not changed the complexity at each λ when doing the smaller forcing. This proceeds exactly like the proof of Theorem 4.1.

$\square$

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